GAP LOCALIZATION OF TE-MODES BY ARBITRARILY WEAK DEFECTS

B.M.BROWN, V.HOANG, M.PLUM, M.RADOSZ AND I.WOOD

Abstract. This paper considers the propagation of TE-modes in photonic crystal waveguides. The waveguide is created by introducing a linear defect into a periodic background medium. Both the periodic background problem and the perturbed problem are modelled by a divergence type equation. A feature of our analysis is that we allow discontinuities in the coefficients of the operator, which is required to model many photonic crystals. It is shown that arbitrarily weak perturbations introduce spectrum into the spectral gaps of the background operator.

1. Introduction

Electromagnetic waves in periodically structured media, such as photonic crystals and metamaterials, are a subject of ongoing interest. Typically, the propagation of waves in such media exhibit *band-gaps* (see [26, 30]), i.e. intervals on the frequency or energy axis where propagation is forbidden. Mathematically, these correspond to gaps in the spectrum of the operator describing a problem with periodic background medium. The existence of these gaps for certain choices of material coefficients was proved in [11, 19, 23] and in [20] for the full Maxwell case. Using layer potential techniques this question has been studied in [3, 4, 5].

In this paper, we consider the propagation of TE-polarized waves in photonic crystals. TE-polarization (transverse electric) here means that the direction of the electric field is confined to a plane perpendicular to the direction of propagation. When the periodicity is perturbed by point or line defects, localization may take place in band gaps, analogous to the situation in solid-state physics and semiconductor devices. The use of line defects in photonic crystals has been proposed in the context of wave guide applications. The gap localization gives rise to *guided modes* which decay exponentially into the bulk structure and propagate along the direction of the line defect. For this reason, it is of great importance to know whether a given line defect produces gap modes.

Is it possible to give rigorous sufficient conditions which imply localization in gaps? In particular, does localization also occur when arbitrarily weak defects are introduced? Here, "weakness" either means a perturbation of small magnitude in the material coefficients or a perturbation of finite magnitude, but small lateral extent. The latter are particularly interesting for optical applications, since defects are usually created by inserting materials with differing dielectric constant ε into the photonic crystal structure.

Weak localization results are quite different from results for sufficiently strong defects (like for example [1, 17, 18, 25, 31, 32, 34]) and there are surprisingly few of them in the literature. The first rigorous results on weak gap localization for periodic Schrödinger operators were given by Parzygnat et al. in [39]. Brown et al. showed weak gap localization in [8] for a periodic Helmholtz-type operator corresponding to TM-mode polarization. We also refer to the paper of Parzygnat et al. [39] for a thorough discussion of the literature on strong and weak localization for Schrödinger operators. We note that we consider the situation of perturbations to the material coefficients. Using different methods to ours, many localisation results have been obtained for perturbations of a periodic geometry, for example in the setting of the coupling of two waveguides through narrow windows, weak localisation results for the Helmholtz equation

B.M. Brown and I. Wood thank the British EPSRC for support under EP/I038217/1. V. Hoang expresses his gratitude to the Deutsche Forschungsgemeinschaft (DFG) for continued support through grants FOR 5156/1-1 and FOR 5156/1-2. He also acknowledges partial support by NSF grant DMS 1412023 and support by NSF grant DMS-1614797. M. Plum gratefully acknowledges financial support by the Deutsche Forschungsgemeinschaft (DFG) through CRC 1173.

were obtained in [33, 40]. For further localisation results due to perturbations of the geometry see e.g. [6, 10, 12, 16, 35, 36, 37].

For TE-polarized waves in periodic media, the problem is very challenging and we present, for the first time, conditions ensuring weak gap localization. The method we present here relies on [8], but the proofs are considerably more difficult due to the different structure of the operator. The chief difficulty is the fact that here the perturbation is of the same order as the principal part of the differential operator. Moreover, we will be working with operators (2.1) with non-smooth coefficients $\varepsilon(\mathbf{x})$, requiring a sophisticated functional analytic setting.

We remark that for the case of a 3D photonic crystal, periodic in two directions and invariant in the third, the spectral problem for Maxwell's equation decomposes into two problems (see [19, Section 7.1] for details), one for so-called TM-modes, studied by four of the authors in in [9], and one for TE-modes which we study here. Therefore the combination of the results in this paper with those from [9] shows the generation of guided modes by arbitrarily week perturbations for the full Maxwell problem in this kind of photonic crystal.

2. Problem statement

We consider the propagation of electromagnetic waves in a non-magnetic, inhomogeneous medium described by a varying dielectric function $\varepsilon(\mathbf{X})$ with $\mathbf{X} = (x, y, z)$. Assuming that the magnetic field $\mathbf{H}$ has the form $\mathbf{H} = H(x, y)\hat{\mathbf{z}}$, where $\hat{\mathbf{z}}$ denotes the unit vector in the z-direction, we look for time-harmonic solutions to Maxwell's equations. This leads to the equation

$$-\nabla \cdot \frac{1}{\varepsilon(\mathbf{x})} \nabla H = \lambda H \tag{2.1}$$

for the $\mathbf{z}$-component H of the magnetic field. Note that in the context of polarized waves, we assume that all fields and constitutive functions depend only on $\mathbf{x} = (x, y)$. The periodic background medium is characterised by $\varepsilon_0(\mathbf{x})$, where for simplicity we assume that the unit square $[0, 1]^2$ is a cell of periodicity.

Polarized waves are commonly studied in the mathematical theory of wave propagation. One reason is of course mathematical: regarding certain questions, a treatment of the full Maxwell's equations would be too complex. From a physics point of view, polarized waves propagate in a two-dimensional photonic crystal ([26], Chapter 5). For example, one can imagine a structure built up periodically from dielectric rods which are aligned with the $\hat{\mathbf{z}}$-axis and extend far in the $\hat{\mathbf{z}}$-direction. For waves propagating in a plane far from the end of the rods, and with propagation direction in the $\hat{\mathbf{x}}$-$\hat{\mathbf{y}}$-plane, (2.1) is a reasonable model. We refer to [26] for a thorough discussion of the underlying physics.

2.1. **Line defects.** Let $\hat{\mathbf{x}} = (1, 0)$ and $\hat{\mathbf{y}} = (0, 1)$. We now introduce a line defect, which we assume to be aligned in the $\hat{\mathbf{x}}$-direction and to preserve the periodicity in this direction. In addition, the defect is assumed to be localized in the $\hat{\mathbf{y}}$-direction. The new system is therefore described by a dielectric function $\varepsilon_1(\mathbf{x})$, periodic in $\hat{\mathbf{x}}$-direction, i.e.

$$\varepsilon_1(\mathbf{x} + m\hat{\mathbf{x}}) = \varepsilon_1(\mathbf{x}) \quad (m \in \mathbb{Z}) \tag{2.2}$$

and there exists some $R > 0$ such that $\varepsilon_1(\cdot, y)$ may differ from $\varepsilon_0(\cdot, y)$, if $|y| < R$ and equals $\varepsilon_0(\cdot, y)$ if $|y| > R$.

Since the system is still periodic in the $\hat{\mathbf{x}}$-direction, we can apply Bloch's theorem [38, 29] to reduce our problem to a problem on the strip $\Omega := (0, 1) \times \mathbb{R}$. Thus, the generalized eigenfunctions of (2.1) have the form $e^{ik_x x}\psi^{(k_x)}(\mathbf{x})$, where $k_x \in [-\pi, \pi]$, $\psi^{(k_x)}$ is periodic in the $\hat{\mathbf{x}}$-direction and satisfies

$$-(\nabla + ik_x\hat{\mathbf{x}}) \cdot \left[\varepsilon_1(\mathbf{x})^{-1}(\nabla + ik_x\hat{\mathbf{x}})\psi^{(k_x)}\right] = \lambda\psi^{(k_x)}. \tag{2.3}$$

Equivalently, we may introduce a function $u^{(k_x)}$ by setting

$$u^{(k_x)}(\mathbf{x}) = e^{ik_x x}\psi^{(k_x)}. \tag{2.4}$$

The $u^{(k_x)}$ now solve the equation

$$-\nabla \cdot \left[\varepsilon_1(\mathbf{x})^{-1}\nabla u^{(k_x)}\right] = \lambda u^{(k_x)}. \tag{2.5}$$

and now formally satisfy the k_x-quasiperiodic boundary conditions

$$\begin{aligned} u^{(k_x)}(1,y) &= e^{ik_x} u^{(k_x)}(0,y), \\ \varepsilon_1^{-1}\partial_x u^{(k_x)}(1,y) &= e^{ik_x}\varepsilon_1^{-1}\partial_x u^{(k_x)}(0,y) \end{aligned} \tag{2.6}$$

relating the two boundaries of the strip Ω. In stating the boundary conditions, one has to be slightly careful, since we will be working with a nonsmooth function ε_1. We will make the boundary conditions more precise in Section 4 below. For our purposes, it is slightly more convenient to use (2.5), since unlike in (2.3), the differential operator is not changed. Note that the boundary condition (2.6) now depends on k_x.

Suppose now (Λ_0, Λ_1) is a band gap of the unperturbed system (2.1) with $\varepsilon = \varepsilon_0$. We will give conditions which ensure that localized modes appear in the interval below Λ_1 under arbitrarily weak perturbation. The unperturbed system is periodic with respect to two directions, and the application of Bloch's theorem leads to the usual Bloch functions $\psi_s(\mathbf{x}, k_x, k_y)$ and corresponding band functions $\lambda_s(k_x, k_y)$ with $s \in \mathbb{N}$, $\mathbf{x} \in [0,1]^2$ and $(k_x, k_y) \in [-\pi, \pi]^2$, see, e.g. [8] for more details. Let $M \in \mathbb{N}$ be such that Λ_1 is the minimum of the M-th band function and let $\mathbf{k}^0 = (k_x^0, k_y^0)$ be a value of the quasi momentum at which Λ_1 is attained, i.e.

$$\lambda_M(\mathbf{k}^0) = \Lambda_1. \tag{2.7}$$

We note that the minimum is attained; for more details see [8, Proposition 3.2]. For simplicity, we assume that $\lambda_M(k_x^0, k_y) \neq \Lambda_1$ for all k_y different from k_y^0. We intend to deal with the more general case in forthcoming work. We note that due to analyticity of the function $k_y \mapsto \lambda_M(k_x^0, k_y)$ in a complex neighbourhood of the interval $[-\pi, \pi]$ (see e.g. [28, Theorem VII.3.9]), we have

$$\lambda_M(k_x^0, k_y) \leq \Lambda_1 + \alpha|k_y - k_y^0|^2 \tag{2.8}$$

close to k_y^0, for some $\alpha > 0$. (This also holds if $k_y^0 = \pm\pi$, due to the periodic boundary behaviour of $\lambda_M(k_x^0, \cdot)$).

One of the main features of this paper is that we do not require the functions ε_i to be continuous. The smoothness we require of the ε_i is merely that $\varepsilon_i \in L^\infty$. This is motivated by physical applications, where, to produce the typical band-gap spectrum, ε_0 is usually piecewise constant. See, for instance, [11, 19, 20]. Moreover, we make the following assumptions on the perturbation:

(i) $\varepsilon_i \geq c_0 > 0$ for some constant c_0 and $i = 0, 1$.

(ii) The perturbation is nonnegative, i.e.

$$\varepsilon_1(\mathbf{x}) - \varepsilon_0(\mathbf{x}) \geq 0. \tag{2.9}$$

(iii) There exists a ball D such that $\varepsilon_1 - \varepsilon_0 > 0$ on D.

We are now in a position to state our main result.

Theorem 1. *In addition to (i), (ii) and (iii), assume that*

$$\left\|\frac{\varepsilon_1}{\varepsilon_0}\right\|_\infty \left\|\frac{1}{\varepsilon_1} - \frac{1}{\varepsilon_0}\right\|_\infty < \frac{\Lambda_1 - \Lambda_0}{\Lambda_0 + 1}. \tag{2.10}$$

Then weak localization takes place, i.e. the problem

$$-\nabla \cdot \varepsilon_1(\mathbf{x})^{-1}\nabla u^{(k_x^0)} = \lambda u^{(k_x^0)}, \quad \mathbf{x} \in \Omega = (0,1) \times \mathbb{R} \tag{2.11}$$

has a nontrivial k_x^0-quasiperiodic solution $u^{(k_x^0)} \in L^2(\Omega)$ for some $\Lambda_0 < \lambda < \Lambda_1$.

Remark 1. (1) *In fact, we have a slightly weaker condition for localization. As can be seen from the proof,* (2.10) *can be replaced by*

$$\left\| \frac{1}{\varepsilon_1} - \frac{1}{\varepsilon_0} \right\|_\infty < \frac{\Lambda_1 - \Lambda_0}{\|\mathfrak{G}_1\|_{H^{-1}\to H^1}(\Lambda_0 + 1)}, \tag{2.12}$$

where $\mathfrak{G}_1$ *is the Green's operator introduced by* (4.3)*. From the proof of Lemma 1, we have* $\|\mathfrak{G}_1\|_{H^{-1}\to H^1} \le \left\|\frac{\varepsilon_1}{\varepsilon_0}\right\|_\infty$*, thus* (2.10) *implies* (2.12)*.*

(2) *Condition* (2.10) *is satisfied for sufficiently weak perturbations, so arbitrarily weak perturbations induce spectrum into the gap.*

(3) *Note that* $u^{(k_x^0)} \in L^2(\Omega)$ *precisely expresses the type of localization we expect in the context of line defects, i.e. the eigensolutions* $u^{(k_x^0)}$ *decay in the direction perpendicular to the line defect, whereas they are* k_x^0*-quasi periodic in the* $\hat{\mathbf{x}}$*-direction. This is different from the localization by point defects: these induce defect eigenfunctions that are square integrable over the whole space.*

(4) *Our method extends to a more general situation: Under the assumptions of the theorem, for any fixed wavenumber* k_x *we can create spectrum inside the spectral gaps of the corresponding operator* $L_0 = L_0(k_x)$ *introduced in Section 3. Here, we have chosen* k_x^0 *such that the additional spectrum is created in a gap of the full unperturbed system, i.e. the system before reduction to quasiperiodic boundary conditions.*

3. The periodic Green's function

In this section, we recall the mathematical formalism needed and introduce the operators to be studied first in the L^2-setting. The discussion is preliminary, since we shall introduce realizations of the same operators in negative Sobolev spaces later on. This is required to apply the perturbation theory to nonsmooth coefficients. However, it will introduce more cumbersome notation and technicalities, so we use a simpler setting in this section to illuminate the ideas behind our approach. We therefore postpone the precise definition of the operators L_0, L_1 and for now just think of them as self-adjoint realizations of the formal differential expressions

$$L_0 = -\nabla \cdot \frac{1}{\varepsilon_0(\mathbf{x})}\nabla, \; L_1 = -\nabla \cdot \frac{1}{\varepsilon_1(\mathbf{x})}\nabla$$

acting on functions satisfying k_x^0-quasiperiodic boundary conditions (2.6).

As our technique is based on exploiting the Bloch representation of the Green's functions (or, equivalently, the resolvent operators), combined with a variational approach, we first review the definition and properties of the Green's function. The Green's function $G_0(\mathbf{x}, \mathbf{x}')$ (see e.g. [13]) satisfying

$$(L_0 + 1)\; G_0(\mathbf{x}, \mathbf{x}') = \delta(\mathbf{x} - \mathbf{x}'). \tag{3.1}$$

We note that the Green's function in (3.1) is then also subject to k_x^0-quasiperiodic boundary conditions:

$$G_0(\mathbf{x} + m\hat{\mathbf{x}}, \mathbf{x}') = e^{ik_x^0 m} G_0(\mathbf{x}, \mathbf{x}') \tag{3.2}$$

for all integers m. It is very useful to have a representation of $G_0(\mathbf{x}, \mathbf{x}')$ in terms of eigenfunctions, i.e. Bloch waves. We shall now derive such a representation.

The set of Bloch waves $\psi_s(\mathbf{x}, \mathbf{k})$ is known to form a complete system in the space of square-integrable functions defined on the whole space. Likewise, the Bloch functions $\psi_s(\mathbf{x}, k_x^0, k_y)$ with the x-component of the quasimomentum fixed, form a complete system in the space of square-integrable functions on the strip Ω. This means that any such function f can be expanded in terms of Bloch waves:

$$f(\mathbf{x}) = \frac{1}{\sqrt{2\pi}} \sum_s \int_{-\pi}^{\pi} \left\langle Uf(\cdot, k_y), \psi_s(\cdot, k_x^0, k_y)\right\rangle \psi_s(\mathbf{x}, k_x^0, k_y)\; dk_y \tag{3.3}$$

where U denotes the Floquet transform in the $\hat{\mathbf{y}}$-direction and the series converges in the L^2-sense. Here, $\left\langle Uf(\cdot, k_y), \psi_s(\cdot, k_x^0, k_y)\right\rangle$ is the L^2-inner product over the unit square $[0, 1]^2$.

Note that in (3.3), we only integrate over the k_y-component of the quasi-momentum.To simplify notation, we will also write $\lambda_s(k_y) := \lambda_s(k_x^0, k_y)$ and $\psi_s(\mathbf{x}, k_y) = \psi_s(\mathbf{x}, k_x^0, k_y)$ in the following.

As in [13, Chapter 1], (3.3) immediately implies the following representation:

$$G_0(\mathbf{x}, \mathbf{x}') = \frac{1}{2\pi} \sum_s \int_{-\pi}^{\pi} \frac{\psi_s(\mathbf{x}', k_y)\psi_s(\mathbf{x}, k_y)}{\lambda_s(k_y) + 1} \; dk_y. \tag{3.4}$$

Formula (3.4) is extremely powerful. As we shall show, it allows us to analyze rigorously the interaction of the defect with the Bloch waves of the unperturbed system. It is convenient to write $G_0 := (L_0 + 1)^{-1}$, i.e.

$$(G_0 f)(\mathbf{x}) = \int_\Omega G_0(\mathbf{x}, \mathbf{x}') f(\mathbf{x}') \; d\mathbf{x}'.$$

Then G_0 is a symmetric and positive operator in $L^2(\Omega)$. We also need to introduce the analogous Green's operator for L_1 (subject to k_x^0-quasiperiodic boundary conditions in the $\hat{\mathbf{x}}$-direction). Since the spectrum of the differential operator L_1 is contained in the positive half-axis, $G_1 := (L_1 + 1)^{-1}$ exists and is a symmetric positive operator in $L^2(\Omega)$. Note that the coefficients of L_1 describe the perturbed system and have no periodicity in the $\hat{\mathbf{y}}$-direction. As a consequence, G_1 cannot be expressed in terms of Bloch waves, as in (3.4).

We will show that the essential spectra of L_0 and L_1 coincide, so any new spectrum introduced in the gap can only consist of eigenvalues. The key idea of our approach is then to transform the eigenvalue problem (2.11) into an eigenvalue problem for the Green's operator G_1. In fact, suppose that $u = u^{(k_x^0)} \neq 0$ solves (2.11) together with the boundary condition (2.6), i.e.

$$(L_1 - \lambda)u = 0.$$

It is easy to see that this is equivalent to

$$G_1 u = \mu u, \tag{3.5}$$

where

$$\mu = (\lambda + 1)^{-1}. \tag{3.6}$$

Thus the eigenvalue problem consisting of (2.11) and (2.6) can be transformed into the eigenvalue problem (3.5), with λ and μ related by (3.6). For $\lambda \in (\Lambda_0, \Lambda_1)$ we have $\mu \in ((\Lambda_1 + 1)^{-1}, (\Lambda_0 + 1)^{-1})$. We may apply the same reasoning to L_0 yielding a similar relation between the spectra of L_0 and G_0. Recall that our goal is to show the existence of spectrum of the perturbed operator L_1 in a spectral gap of the unperturbed operator. Since the perturbation changes the principal part of the operator, it is not easy to compare L_0 and L_1 directly. Instead our approach will be to compare the spectra of G_0 and G_1. A standard device to accomplish this is to use the operator identity

$$G_0 - G_1 = G_1(L_1 - L_0)G_0.$$

This will work if $\varepsilon_1, \varepsilon_0$ are both sufficiently smooth, since then the domains of L_0, L_1 will be the same. However, if $\varepsilon_1, \varepsilon_0$ are only L^∞, a precise characterization of the operator domains is much harder and $G_0 = (L_0 + 1)^{-1}$ might not map into the domain of L_1. For this reason, we construct realisations of the operators in negative Sobolev spaces. These realisations will have the same domain for both perturbed and unperturbed coefficients, so the problem is avoided.

4. The operator theoretic formulation

In this paper we shall only assume $\varepsilon_0, \varepsilon_1 \in L^\infty$ with a positive lower bound. As stated above, we will work in negative Sobolev spaces to overcome the lack of smoothness of the coefficients. In particular, rather than study the operators G_0 and G_1 directly, we will consider their realisations in the space $H_{qp}^{-1}(\Omega)$, introduced below, denoted by $\mathfrak{G}_0$ and $\mathfrak{G}_1$, respectively.

Recall that we work with quasimomentum k_x^0 fixed. First we introduce the space of quasi-periodic H^1-functions on Ω

$$H_{qp}^1(\Omega) := \{u \in H_{\mathrm{loc}}^1(\mathbb{R}^2) : u|_\Omega \in H^1(\Omega) \text{ and } u(\mathbf{x} + (m, 0)) = e^{ik_x^0 m} u(\mathbf{x}), m \in \mathbb{Z}, \mathbf{x} \in \mathbb{R}^2\}.$$

For $u, v \in H^1_{qp}(\Omega)$ consider the sesquilinear form

$$B_0[u,v] = \int_\Omega \left(\frac{1}{\varepsilon_0(\mathbf{x})} \nabla u \overline{\nabla v} + u\overline{v} \right) \, d\mathbf{x}.$$

The standard theory [28] of quadratic forms now allows us to associate a unique self-adjoint operator with B_0 (its realisation in $L^2(\Omega)$), whose domain is a subset of the space form domain H^1_{qp}. This operator will be denoted by $L_0 + 1$, which we use as a definition of L_0. The operator analogously constructed from the bilinear form where ε_0 is replaced by ε_1 is L_1.

The functions in the domain of these self-adjoint operator satisfy the k_x-quasiperiodic boundary conditions

$$\begin{aligned} u^{(k_x^0)}(1,y) &= e^{ik_x^0} u^{(k_x^0)}(0,y), \\ \varepsilon_0^{-1} \partial_x u^{(k_x^0)}(1,y) &= e^{ik_x^0} \varepsilon_0^{-1} \partial_x u^{(k_x^0)}(0,y) \end{aligned} \tag{4.1}$$

Here the first equation in (4.1) has to be interpreted in the sense of the H^1-trace, and in the second equation of (4.1) both sides are the normal trace of the vector field $\varepsilon_0^{-1}\nabla u^{(k_x^0)}$, whose divergence is in $L^2(\Omega)$. Note that if the coefficient ε_0 is smooth, say C^1, (4.1) will hold in a classical, pointwise sense.

For our purposes, the L^2-realisations are not sufficient, since below we will alter the coefficient ε_0 in a possibly non-smooth way, which may change the domain of its L^2-realisation. We therefore proceed to construct a realisation of the operator in H^{-1}.

As ε_0 is bounded and bounded away from zero, we can introduce a new scalar product on $H^1_{qp}(\Omega)$ given by

$$\langle u, v \rangle_{H^1_{qp}(\Omega)} := B_0[u,v]$$

which is equivalent to the usual scalar product in $H^1(\Omega)$. When there is no danger of confusion, we denote the associated norm $\|\cdot\|_{H^1}$.

Definition 1. *Let $H^{-1}_{qp}(\Omega)$ denote the dual space of $H^1_{qp}(\Omega)$. Let $\phi : H^1_{qp}(\Omega) \to H^{-1}_{qp}(\Omega)$ be defined by*

$$\langle \phi[u], \varphi \rangle = B_0[u, \varphi] \quad \text{for all} \;\; u, \varphi \in H^1_{qp}(\Omega) \tag{4.2}$$

where the $\langle \cdot, \cdot \rangle$-notation indicates the dual pairing, i.e. $\langle w, \varphi \rangle$ is the action of the linear functional w on the function $\overline{\varphi}$. We shall also use $w[\varphi]$ to denote the dual pairing.

ϕ is an isometric isomorphism, and hence the scalar product on $H^{-1}_{qp}(\Omega)$ given by

$$\langle u, v \rangle_{H^{-1}_{qp}(\Omega)} := \langle \phi^{-1}u, \phi^{-1}v \rangle_{H^1_{qp}(\Omega)}$$

induces a norm which coincides with the usual operator sup-norm on $H^{-1}_{qp}(\Omega)$.

After this preparation, we now introduce the realisations of L_0 and G_0 in $H^{-1}_{qp}(\Omega)$.

Proposition 1. *We define an operator $\mathfrak{L}_o : D(\mathfrak{L}_o) \to H^{-1}_{qp}(\Omega)$ by $D(\mathfrak{L}_o) := H^1_{qp}(\Omega) \subset H^{-1}_{qp}(\Omega)$ and*

$$\mathfrak{L}_o u := \phi u - u.$$

Then $\mathfrak{L}_o + 1$ is bijective and both $\mathfrak{L}_o$ and $\mathfrak{G}_o := (\mathfrak{L}_o + 1)^{-1}$ are self-adjoint.

Proof. For $u, v \in H^1_{qp}(\Omega)$,

$$\begin{aligned} \langle (\mathfrak{L}_o + 1)u, v \rangle_{H^{-1}} &= \langle \phi^{-1}(\mathfrak{L}_o + 1)u, \phi^{-1}v \rangle_{H^1} = \langle u, \phi^{-1}v \rangle_{H^1} = \overline{\langle \phi^{-1}v, u \rangle_{H^1}} \\ &= \overline{B_0[\phi^{-1}v, u]} = \overline{\langle v, u \rangle} = \overline{\langle v, u \rangle_{L^2}} = \langle u, v \rangle_{L^2}\,; \end{aligned}$$

the last line follows by (4.2). Thus $\mathfrak{L}_o + 1$ is symmetric.

Since ϕ is bijective it follows that $\mathfrak{L}_o + 1$ is bijective, thus $(\mathfrak{L}_o + 1)^{-1} : H^{-1}_{qp}(\Omega) \to H^{-1}_{qp}(\Omega)$ is defined on the whole space and is also symmetric. Therefore, $\mathfrak{G}_o = (\mathfrak{L}_o + 1)^{-1}$ is self-adjoint. Hence $\mathfrak{L}_o + 1$, and so $\mathfrak{L}_o$ itself is self-adjoint. ☐

Remark 2. (1) *The map ϕ corresponds to the operator $\mathfrak{L}_o + 1$ and $\phi^{-1} : H^{-1}_{qp}(\Omega) \to H^1_{qp}(\Omega)$ acts in the same way as $\mathfrak{G}_o : H^{-1}_{qp}(\Omega) \to H^{-1}_{qp}(\Omega)$.*

(2) *We remind the reader of the standard embedding of* $L^2(\Omega)$ *in* $H^{-1}_{qp}(\Omega)$*: a function* $f \in L^2(\Omega)$ *acts on* $v \in H^1_{qp}(\Omega)$ *via* $f[v] = \langle f, v\rangle_{L^2}$.

(3) *From the definitions of* ϕ *and* $\mathfrak{L}_\mathfrak{o}$ *follows the useful identity*

$$\langle u, v\rangle_{H^{-1}} = \langle \mathfrak{G}_\mathfrak{o} u, \mathfrak{G}_\mathfrak{o} v\rangle_{H^1} = \langle u, \mathfrak{G}_\mathfrak{o} v\rangle_{L^2} \quad \textit{for} \quad u \in L^2(\Omega), v \in H^{-1}_{qp}(\Omega).$$

(4) *We note that just as in* [7, Section 5]*, the* L^2*- and* H^{-1}*-spectra coincide:*

$$\sigma(L_0) = \sigma(\mathfrak{L}_\mathfrak{o}).$$

Let $\mu \in ((\Lambda_1 + 1)^{-1}, (\Lambda_0 + 1)^{-1})$. Then by the previous remark, $1/\mu \in \rho(\mathfrak{L}_\mathfrak{o} + 1)$, so

$$(I - \mu(\mathfrak{L}_\mathfrak{o} + 1))^{-1} = (I - \mu {\mathfrak{G}_\mathfrak{o}}^{-1})^{-1}$$

is well defined and maps $H^{-1}_{qp}(\Omega)$ bijectively onto $H^1_{qp}(\Omega)$. The operator $(I - \mu {\mathfrak{G}_\mathfrak{o}}^{-1})^{-1}$ is the solution operator to the problem

$$\langle u, \varphi\rangle_{L^2} - \mu \int_\Omega \left(\tfrac{1}{\varepsilon_0}\nabla u \overline{\nabla\varphi} + u\overline{\varphi}\right) d\mathbf{x} = f[\varphi], \qquad \text{for all } \varphi \in H^1_{qp}(\Omega)$$

for a given $f \in H^{-1}_{qp}(\Omega)$.

We now introduce the solution operator for the perturbed problem. Let $\mathfrak{G}_\mathfrak{1}$ be the operator defined on $H^{-1}_{qp}(\Omega)$ such that for given $f \in H^{-1}_{qp}(\Omega)$ the function $u = \mathfrak{G}_\mathfrak{1} f$ is the unique solution in $H^1_{qp}(\Omega)$ to

$$B_1[u, \varphi] := \int_\Omega \left[\frac{1}{\varepsilon_1}\nabla u \overline{\nabla\varphi} + u\overline{\varphi}\right] dx = f[\varphi] \quad \text{for all } \varphi \in H^1_{qp}(\Omega). \tag{4.3}$$

We see that $\mathfrak{G}_\mathfrak{1}$ is well-defined, since it can be constructed via a form in the same way as $\mathfrak{G}_\mathfrak{o}$, noting that the norms in the H^{-1}_{qp}-spaces constructed from both sesquilinear forms B_0 and B_1 are equivalent. Note that $\mathfrak{G}_\mathfrak{o}|_{L^2} = G_0$ and $\mathfrak{G}_\mathfrak{1}|_{L^2} = G_1$, which are both symmetric operators in L^2. Moreover, again, as in [7, Section 5], the L^2- and H^{-1}-spectra coincide: $\sigma(G_1) = \sigma(\mathfrak{G}_\mathfrak{1})$. We also denote $\mathfrak{L}_\mathfrak{1} = {\mathfrak{G}_\mathfrak{1}}^{-1} - 1$.

We conclude the section with the proof of some more simple properties of $\mathfrak{G}_\mathfrak{1}$ and $\mathfrak{G}_\mathfrak{o}$ which will be useful later. Recall that by assumption, $\varepsilon_1 \geq \varepsilon_0$.

Lemma 1. $\mathfrak{G}_\mathfrak{1} : H^{-1}_{qp}(\Omega) \to H^1_{qp}(\Omega)$ *is bounded with* $\|\mathfrak{G}_\mathfrak{1}\|_{H^{-1}\to H^1} \leq \left\|\frac{\varepsilon_1}{\varepsilon_0}\right\|_\infty$.

Proof. Let $f \in H^{-1}_{qp}(\Omega)$ and $u = \mathfrak{G}_\mathfrak{1} f$. Choose $\varphi = u$ in (4.3). It then follows that

$$\begin{aligned}\|u\|^2_{H^1} &= \int \left(\frac{1}{\varepsilon_0}|\nabla u|^2 + |u|^2\right) dx \leq \left\|\frac{\varepsilon_1}{\varepsilon_0}\right\|_\infty \int \left(\frac{1}{\varepsilon_1}|\nabla u|^2 + |u|^2\right) dx \\ &= \left\|\frac{\varepsilon_1}{\varepsilon_0}\right\|_\infty f[u] \leq \left\|\frac{\varepsilon_1}{\varepsilon_0}\right\|_\infty \|f\|_{H^{-1}} \|u\|_{H^1},\end{aligned}$$

where $\left\|\frac{\varepsilon_1}{\varepsilon_0}\right\|_\infty$ is bounded by assumption. □

Lemma 2. *For* $w \in H^{-1}_{qp}(\Omega)$, $\quad w[\mathfrak{G}_i w] \geq 0$ *for* $i = 0, 1$.

Proof. Choose a sequence $(w_n) \in (L^2(\Omega))^\mathbb{N}$ such that $w_n \to w$ in $H^{-1}_{qp}(\Omega)$. By continuity of $\mathfrak{G}_i : H^{-1}_{qp}(\Omega) \to H^1_{qp}(\Omega)$, we have $\mathfrak{G}_i w_n \to \mathfrak{G}_i w$ in $H^1_{qp}(\Omega)$, so $w_n[\mathfrak{G}_i w_n] \to w[\mathfrak{G}_i w]$. Furthermore,

$$w_n[\mathfrak{G}_i w_n] = \int w_n \overline{G_i w_n} \geq 0,$$

since $G_i \geq 0$ as operators in L^2. □

5. Birman-Schwinger-type reformulation

An essential feature of our approach is to first perform a Birman-Schwinger-type reformulation of the problem. In this way, we bring the unperturbed Green's operator into play. We will show below (see Lemma 8) that $\mathfrak{G}_1 - \mathfrak{G}_0$ is compact as an operator in $H^{-1}_{qp}(\Omega)$. Hence the spectra of $\mathfrak{G}_0$ and $\mathfrak{G}_1$ can only differ by eigenvalues.

The eigenvalue problem for our original operator, $(L_1 - \lambda)u = 0$ with $\lambda \in (\Lambda_0, \Lambda_1)$, is equivalent to

$$(\mathfrak{G}_1 - \mu)u = 0, \quad u \in H^{-1}_{qp}(\Omega) \tag{5.1}$$

for $\mu = (\lambda+1)^{-1} \in ((\Lambda_1+1)^{-1}, (\Lambda_0+1)^{-1})$. Then (5.1) implies that each eigenfunction u lies in $H^1_{qp}(\Omega)$ and so

$$\begin{aligned}(\mathfrak{G}_1 - \mu)u = 0 \quad &\Leftrightarrow \quad (\mathfrak{G}_0 - \mu)u + (\mathfrak{G}_1 - \mathfrak{G}_0)u = 0\\ &\Leftrightarrow \quad (I - \mu{\mathfrak{G}_0}^{-1})u + ({\mathfrak{G}_0}^{-1}\mathfrak{G}_1 - I)u = 0\\ &\Leftrightarrow \quad u + (I - \mu{\mathfrak{G}_0}^{-1})^{-1}({\mathfrak{G}_0}^{-1}\mathfrak{G}_1 - I)u = 0, \end{aligned} \tag{5.2}$$

where the last equivalence follows, as $(I - \mu{\mathfrak{G}_0}^{-1})^{-1} : H^{-1}_{qp}(\Omega) \to H^1_{qp}(\Omega)$ is bijective.

In order to proceed, it is most convenient to modify the equation (5.2) by suitably projecting out the null space of the operator

$$K = ({\mathfrak{G}_0}^{-1}\mathfrak{G}_1 - I) : H^{-1}_{qp}(\Omega) \to H^{-1}_{qp}(\Omega).$$

Set $\mathcal{K} = \overline{\operatorname{ran} K}$ and let $P : H^{-1}_{qp}(\Omega) \to \mathcal{K}$ be the orthogonal projection on $\mathcal{K}$. On $\mathcal{K}$, we introduce a new inner product given by

$$\langle f, g\rangle_{\mathcal{K}} := \langle Kf, g\rangle_{H^{-1}}\,. \tag{5.3}$$

We will show the symmetry and definiteness of this inner product in the Appendix. Applying P to (5.2) and noting writing $u = Pu + (1-P)u$ in (5.2) we obtain the following equation

$$v + A_\mu v = 0,\ \ v \in \mathcal{K} \tag{5.4}$$

where

$$A_\mu := P(I - \mu{\mathfrak{G}_0}^{-1})^{-1}K : \mathcal{K} \to \mathcal{K}. \tag{5.5}$$

(5.4) is equivalent to (5.2): the existence of a nontrivial solution of (5.2) implies the existence of a nontrivial solution of (5.4) and vice versa.

Next we need to show that A_μ is self-adjoint in $\mathcal{K}$ and that its spectrum consists only of eigenvalues. What follows now is a sucession of Lemmas, culminating in Proposition 2.

Lemma 3. *$(I - \mu{\mathfrak{G}_0}^{-1})^{-1}$ is symmetric in $H^{-1}_{qp}(\Omega)$ for $\mu \in ((\Lambda_1+1)^{-1}, (\Lambda_0+1)^{-1})$.*

Proof. This is obvious, since $\mathfrak{L}_0$ is a self-adjoint operator in $H^{-1}_{qp}(\Omega)$. □

Lemma 4. *We have the estimates*

(i)
$$\|K\| \le \|\mathfrak{G}_1\|_{H^{-1}\to H^1} \left\| \frac{1}{\varepsilon_0} - \frac{1}{\varepsilon_1}\right\|_\infty$$

(ii) *and for $u \in \mathcal{K}$,*
$$\|Ku\|^2_{H^{-1}} \le \|K\|\,\|u\|^2_{\mathcal{K}}\,.$$

(iii) *Moreover, if $\delta := \left\|\frac{1}{\varepsilon_0} - \frac{1}{\varepsilon_1}\right\|_\infty < 1/\|\mathfrak{G}_0\|_{H^{-1}\to H^1}$, then*
$$\|\mathfrak{G}_1\|_{H^{-1}\to H^1} \le \frac{\|\mathfrak{G}_0\|_{H^{-1}\to H^1}}{1 - \delta\,\|\mathfrak{G}_0\|_{H^{-1}\to H^1}}.$$

Proof. The identity $K = (\mathfrak{L}_0 + 1)\mathfrak{G}_1 - I = (\mathfrak{L}_0 - \mathfrak{L}_1)\mathfrak{G}_1$ implies for $u \in H^{-1}_{qp}(\Omega)$ and $\varphi \in H^1_{qp}(\Omega)$,

$$Ku[\varphi] = \int_\Omega \left(\frac{1}{\varepsilon_0} - \frac{1}{\varepsilon_1}\right) \nabla \mathfrak{G}_1 u \nabla \overline{\varphi}\, d\mathbf{x}. \tag{5.6}$$

Therefore,

$$|Ku[\varphi]| \leq \left\| \frac{1}{\varepsilon_0} - \frac{1}{\varepsilon_1} \right\|_\infty \|\mathfrak{G}_1 u\|_{H^1} \|\varphi\|_{H^1}\,, \tag{5.7}$$

proving (i). By Lemma 17, we know that K is symmetric with respect to the H^{-1}_{qp} inner product, and by Lemma 18 we have $K \geq 0$. So $\|K\|$ is given by the supremum of

$$\frac{\langle Ku, u\rangle_{H^{-1}_{qp}}}{\langle u, u\rangle_{H^{-1}_{qp}}}$$

over all $u \neq 0$. Setting $u = \sqrt{K}v$, we get

$$\|K\| \geq \frac{\langle Kv, Kv\rangle_{H^{-1}_{qp}}}{\langle Kv, v\rangle_{H^{-1}_{qp}}} = \frac{\|Kv\|_{H^{-1}_{qp}}}{\|v\|_{\mathcal{K}}}$$

for all v such that $\langle Kv, v\rangle_{H^{-1}_{qp}} = \|v\|^2_{\mathcal{K}} \neq 0$. This gives us (ii).
Finally, as $\mathfrak{G}_1 - \mathfrak{G}_0 = \mathfrak{G}_0 K$, using (i) we have

$$\|\mathfrak{G}_1\|_{H^{-1}\to H^1} \leq \|\mathfrak{G}_0\|_{H^{-1}\to H^1}(1 + \|K\|) \leq \|\mathfrak{G}_0\|_{H^{-1}\to H^1}\left(1 + \delta \|\mathfrak{G}_1\|_{H^{-1}\to H^1}\right)$$

and rearranging gives (iii). □

Note in particular that this means that for small perturbations, the only dependence of the bound for $\|Ku\|_{H^{-1}}$ on the perturbation ε_1 is through the term $\left\|\frac{1}{\varepsilon_0} - \frac{1}{\varepsilon_1}\right\|_\infty$.

Lemma 5. *Equation* (5.1) *has a non-trivial solution u iff -1 is an eigenvalue of A_μ.*

Proof. Let $v = Pu$, where u is a solution of (5.1). Applying P to (5.2) shows that v solves

$$v + A_\mu v = 0. \tag{5.8}$$

Conversely, one easily checks that a solution $v \neq 0$ of (5.8) gives a solution $u \neq 0$ of the original problem (5.1): we just have to set $u = -(I - \mu{\mathfrak{G}_0}^{-1})^{-1}Kv$ and use (5.2), noting that $Pu = v$, so $Ku = Kv$. □

Lemma 6. *A_μ is symmetric in $\mathcal{K}$.*

Proof. Let $u, v \in \mathcal{K}$. Then

$$\begin{aligned}\langle A_\mu u, v\rangle_{\mathcal{K}} = \langle KA_\mu u, v\rangle_{H^{-1}} = \langle A_\mu u, Kv\rangle_{H^{-1}} &= \left\langle P(I - \mu{\mathfrak{G}_0}^{-1})^{-1}Ku, Kv\right\rangle_{H^{-1}} \\ = \left\langle (I - \mu{\mathfrak{G}_0}^{-1})^{-1}Ku, Kv\right\rangle_{H^{-1}} &= \left\langle Ku, (I - \mu{\mathfrak{G}_0}^{-1})^{-1}Kv\right\rangle_{H^{-1}} = \langle u, A_\mu v\rangle_{\mathcal{K}}\,,\end{aligned}$$

where we have used Lemma 17 and Lemma 3. □

In the following recall that $\varepsilon_1 = \varepsilon_0$ if $|y| > R$.

Lemma 7. *Let H^{-1}_{cs} denote the space of distributions in $H^{-1}_{qp}(\Omega)$ with compact support in the $\hat{\mathbf{y}}$-direction, the support being contained in $[0, 1] \times [-R, R]$. Then* $\operatorname{ran} K \subseteq H^{-1}_{cs}$.

Proof. Let $f \in \operatorname{ran} K$, i.e. there exists $g \in H^{-1}_{qp}(\Omega)$ such that $f = Kg = (\mathfrak{G_0}^{-1}\mathfrak{G_1} - I)g$. Then for any $\varphi \in H^1_{qp}(\Omega)$ we have

$$\begin{aligned} f[\varphi] = (\mathfrak{G_0}^{-1}\mathfrak{G_1} g)[\varphi] - g[\varphi] &= \int_\Omega \left(\frac{1}{\varepsilon_0}\nabla \mathfrak{G_1} g \overline{\nabla\varphi} + \mathfrak{G_1} g\overline{\varphi}\right) - g[\varphi] \\ &= \int_\Omega \left(\frac{1}{\varepsilon_0} - \frac{1}{\varepsilon_1}\right)\nabla \mathfrak{G_1} g\overline{\nabla\varphi} + \int_\Omega \left(\frac{1}{\varepsilon_1}\nabla \mathfrak{G_1} g\overline{\nabla\varphi} + \mathfrak{G_1} g \overline{\varphi}\right) - g[\varphi] \\ &= \int_\Omega \left(\frac{1}{\varepsilon_0} - \frac{1}{\varepsilon_1}\right)\nabla \mathfrak{G_1} g \overline{\nabla\varphi}. \end{aligned}$$

Observe that the second integral in the second line of the calculation is just $((\mathfrak{L_1}+1)\mathfrak{G_1} g)\,[\varphi]$. It is therefore clear that f vanishes on all functions φ supported outside the support of $\frac{1}{\varepsilon_0} - \frac{1}{\varepsilon_1}$. ☐

We are now in a position to establish that the spectrum of A_μ consists only of eigenvalues. We first show that $\mathfrak{G_1} - \mathfrak{G_0}$ enjoys the same property and thus that existence of any non-trivial solution to (5.1) implies existence of an isolated eigenvalue of $\mathfrak{G_1}$.

Lemma 8. *$\mathfrak{G_1} - \mathfrak{G_0}$ is compact as an operator in $H^{-1}_{qp}(\Omega)$. Hence, the essential spectra of $\mathfrak{G_0}$ and $\mathfrak{G_1}$ coincide.*

Proof. We show compactness of $\mathfrak{G_1} - \mathfrak{G_0}$. The coincidence of the essential spectra then follows from standard results, see, e.g. [14]. We have the following mappings:

$$\mathfrak{G_1} - \mathfrak{G_0} = \mathfrak{G_0} K : H^{-1}_{qp}(\Omega) \xrightarrow{K} H^{-1}_{cs} \xrightarrow{\mathfrak{G_0}} H^1_{qp}(\Omega, e^{\gamma|y|}) \overset{c}{\hookrightarrow} H^{-1}_{qp}(\Omega),$$

where $\gamma > 0$ and $H^1_{qp}(\Omega, e^{\gamma|y|})$ is the space of functions $u \in H^1_{qp}(\Omega)$ such that $e^{\gamma|y|}u(x,y) \in H^1_{qp}(\Omega)$ with norm $\|u\|_{H^1_{qp}(\Omega,e^{\gamma|y|})} := \left\|e^{\gamma|y|}u\right\|_{H^1}$. The last embedding is compact (see the Appendix). It remains to show that $\mathfrak{G_0} : H^{-1}_{cs} \to H^1_{qp}(\Omega, e^{\gamma|y|})$, and that it is bounded. Let $f \in H^{-1}_{cs}$ and $u = \mathfrak{G_0} f$, i.e. for all $\varphi \in H^1_{qp}(\Omega)$ we have

$$f[\varphi] = \int_\Omega \frac{1}{\varepsilon_0}\nabla u\overline{\nabla\varphi} + u\overline{\varphi}.$$

Set $\omega = e^{\gamma|y|}u$ and $\varphi = e^{\gamma|y|}\psi$ where ψ is compactly supported. Then

$$f(e^{\gamma|\cdot|}\psi) = \int_\Omega \frac{1}{\varepsilon_0}\left(\nabla - \gamma\frac{y}{|y|}\hat{\mathbf{y}}\right)\omega\left(\nabla + \gamma\frac{y}{|y|}\hat{\mathbf{y}}\right)\overline{\psi} + \omega\overline{\psi} = \int_\Omega \left(\frac{1}{\varepsilon_0}\nabla\omega\overline{\nabla\psi} + \omega\overline{\psi}\right) + \gamma S_\gamma\omega[\psi]$$

where we have set $S_\gamma : H^1_{qp}(\Omega) \to H^{-1}_{qp}(\Omega)$,

$$S_\gamma\omega[\psi] := \int_\Omega \frac{1}{\varepsilon_0}\left(-\frac{y}{|y|}\overline{\frac{\partial\psi}{\partial y}}\omega + \frac{y}{|y|}\frac{\partial\omega}{\partial y}\overline{\psi} - \gamma\omega\overline{\psi}\right). \tag{5.9}$$

Now, $f \circ e^{\gamma|\cdot|} = (\mathfrak{L_0} + 1)\omega + \gamma S_\gamma\omega \in H^{-1}_{cs}$, as f is. Hence, $\mathfrak{G_0}(f \circ e^{\gamma|\cdot|}) = \omega + \mathfrak{G_0}(\gamma S_\gamma\omega) = (I + \mathfrak{G_0}\gamma S_\gamma)\omega$. For small $|\gamma|$ this can be inverted by the Neumann series, so $\omega = (I + \mathfrak{G_0}\gamma S_\gamma)^{-1}\mathfrak{G_0}(f \circ e^{\gamma|\cdot|})$ and

$$\|\omega\|_{H^1} \le \left\|(I + \mathfrak{G_0}\gamma S_\gamma)^{-1}\right\|_{H^1\to H^1}\left\|\mathfrak{G_0}(f\circ e^{\gamma|\cdot|})\right\|_{H^1}$$

Thus $\left\|e^{\gamma|\cdot|}u\right\|_{H^1} \le C_\gamma \|f\|_{H^{-1}}$. ☐

Proposition 2. *$A_\mu : \mathcal{K} \to \mathcal{K}$ is a compact, self-adjoint operator on $\mathcal{K}$.*

Proof. We rewrite the operator as $A_\mu = P\mathfrak{G_0}^{-1}(I - \mu\mathfrak{G_0}^{-1})^{-1}\mathfrak{G_0} K$. By Lemma 4, the operator $K : \mathcal{K} \to H^{-1}_{qp}(\Omega)$ is bounded , and it maps into H^{-1}_{cs} by Lemma 7. Since, again using Lemma 4, we have

$$\|Pu\|^2_{\mathcal{K}} = \langle KPu, Pu\rangle_{H^{-1}} = \langle Ku, u\rangle_{H^{-1}} \le \|Ku\|_{H^{-1}}\|u\|_{H^{-1}} \le C\|u\|^2_{H^{-1}},$$

the operator $P : H_{qp}^{-1}(\Omega) \to \mathcal{K}$ is bounded. As $\mu^{-1} \in \rho(\mathfrak{L}_o + 1)$, the operator $I - \mu(\mathfrak{L}_o + 1) : H_{qp}^{1}(\Omega) \to H_{qp}^{-1}(\Omega)$ is onto, and bounded, and hence also $(I - \mu{\mathfrak{G}_o}^{-1})^{-1} : H_{qp}^{-1}(\Omega) \to H_{qp}^{1}(\Omega)$ is continuous and, as in the proof of Lemma 8, we have the following mapping properties

$$\mathcal{K} \overset{K}{\to} H_{cs}^{-1} \underset{compact}{\overset{\mathfrak{G}_o}{\to}} H_{qp}^{-1}(\Omega) \overset{(I-\mu{\mathfrak{G}_o}^{-1})^{-1}}{\longrightarrow} H_{qp}^{1}(\Omega) \overset{{\mathfrak{G}_o}^{-1}}{\to} H_{qp}^{-1}(\Omega) \overset{P}{\to} \mathcal{K}.$$

Thus, $A_\mu : \mathcal{K} \to \mathcal{K}$ is compact. □

6. Existence of spectrum for weak perturbations

We next estimate the eigenvalues of A_μ using variational methods. Lemma 5 enables us to study our spectral problem by applying variational methods to the equation (5.8). As a mathematical subtlety, note that $\mathcal{K}$ is in general not complete with $\langle \cdot, \cdot \rangle_{\mathcal{K}}$ as an inner product. However, this does not affect our arguments, since the spectral theory of symmetric compact operators is applicable on Pre-Hilbert spaces (see [22]).

It follows from our analysis below that (at least) for some μ in the spectral gap $\big((\Lambda_1 + 1)^{-1}, (\Lambda_0 + 1)^{-1}\big)$, the operator A_μ has a negative eigenvalue. Our strategy consists in following $\kappa(\mu)$, the most negative eigenvalue of the operator A_μ, as μ varies. $\kappa(\mu)$ can be characterized by

$$\kappa(\mu) = \min_{u \neq 0} \frac{\langle u, A_\mu u \rangle_{\mathcal{K}}}{\langle u, u \rangle_{\mathcal{K}}}. \tag{6.1}$$

We prove below that $\kappa(\mu)$ is monotonically increasing in μ and continuous, in the range $(\Lambda_1 + 1)^{-1} < \mu < (\Lambda_0 + 1)^{-1}$, and that $\kappa(\mu)$ goes to $-\infty$ as μ approaches $(\Lambda_1 + 1)^{-1}$ from the right. At the same time, we will find a $\tilde{\mu}$ to the right of $(\Lambda_1 + 1)^{-1}$ for which $-1 < \kappa(\tilde{\mu})$, provided (2.10) is satisfied. Hence $\kappa(\mu) = -1$ holds necessarily for some μ, i.e. A_μ has -1 as an eigenvalue and (5.8) has a non-trivial solution.

Lemma 9. *For μ in the spectral gap $\big((\Lambda_1 + 1)^{-1}, (\Lambda_0 + 1)^{-1}\big)$ we have that $\mu \mapsto \kappa(\mu)$ is continuous and increasing.*

Proof. As $\mu \mapsto A_\mu$ is norm-continuous, we have that for $\mu \in \big((\Lambda_1 + 1)^{-1}, (\Lambda_0 + 1)^{-1}\big)$ and $\widetilde{\varepsilon} > 0$, there exists $\delta > 0$ such that $|\mu - \widetilde{\mu}| < \delta$ implies, for every $u \in \mathcal{K}$,

$$\big| \langle A_\mu u, u \rangle_{\mathcal{K}} - \big\langle A_{\widetilde{\mu}} u, u \big\rangle_{\mathcal{K}} \big| \le \widetilde{\varepsilon} \, \|u\|_{\mathcal{K}}^2 .$$

Thus

$$\frac{\big\langle A_{\widetilde{\mu}} u, u \big\rangle_{\mathcal{K}}}{\|u\|_{\mathcal{K}}^2} \le \frac{\langle A_\mu u, u \rangle_{\mathcal{K}}}{\|u\|_{\mathcal{K}}^2} + \widetilde{\varepsilon},$$

and therefore $\kappa(\widetilde{\mu}) \le \kappa(\mu) + \widetilde{\varepsilon}$ by (6.1). Similarly, we obtain the reverse inequality. Together these imply continuity of $\mu \mapsto \kappa(\mu)$.

We next consider monotonicity. Let $u \in \mathcal{K}$,

$$(\Lambda_1 + 1)^{-1} < \widetilde{\mu} < \mu < (\Lambda_0 + 1)^{-1}$$

and $N := \big\langle (A_\mu - A_{\widetilde{\mu}}) u, u \big\rangle_{\mathcal{K}}$. Then using Lemma 17,

$$\begin{aligned} N &= \big\langle P \left[(I - \mu{\mathfrak{G}_o}^{-1})^{-1} - (I - \widetilde{\mu}{\mathfrak{G}_o}^{-1})^{-1} \right] K u, u \big\rangle_{\mathcal{K}} \\ &= \big\langle \left[(I - \mu{\mathfrak{G}_o}^{-1})^{-1} - (I - \widetilde{\mu}{\mathfrak{G}_o}^{-1})^{-1} \right] K u, K u \big\rangle_{H^{-1}} . \end{aligned}$$

Let (v_n) be a sequence in $L^2(\Omega)$ such that $v_n \to Ku$ in $H_{qp}^{-1}(\Omega)$. Then

$$\begin{aligned} N &= \lim_{n \to \infty} \big\langle \left[(I - \mu{\mathfrak{G}_o}^{-1})^{-1} - (I - \widetilde{\mu}{\mathfrak{G}_o}^{-1})^{-1} \right] v_n, v_n \big\rangle_{H^{-1}} \\ &= \lim_{n \to \infty} \big\langle G_0 \left[(I - \mu{\mathfrak{G}_o}^{-1})^{-1} - (I - \widetilde{\mu}{\mathfrak{G}_o}^{-1})^{-1} \right] v_n, v_n \big\rangle_{L^2} . \end{aligned}$$

As $v_n \in L^2$ we can use the representation of G_0 and the resolvents in terms of the Bloch functions, so from (3.3) and (3.4), we have

$$N = \lim_{n\to\infty} \int_{-\pi}^{\pi} \sum_s \frac{1}{\lambda_s(k)+1}\left[\frac{1}{1-\mu(\lambda_s(k)+1)} - \frac{1}{1-\widetilde{\mu}(\lambda_s(k)+1)}\right] |\langle Uv_n(k), \psi_s(k)\rangle_{L^2}|^2\, dk.$$

Since

$$\frac{1}{\lambda_s(k)+1}\left[\frac{1}{1-\mu(\lambda_s(k)+1)} - \frac{1}{1-\widetilde{\mu}(\lambda_s(k)+1)}\right] = \frac{\mu-\widetilde{\mu}}{(1-\mu(\lambda_s(k)+1))(1-\widetilde{\mu}(\lambda_s(k)+1))} > 0,$$

we have $N > 0$. Thus, by (6.1), the map $\mu \mapsto \kappa(\mu)$ is monotonically increasing. □

We next seek both lower and upper bounds on (6.1).

6.1. Lower bound.

Lemma 10. *For all $u \in \mathcal{K}$ and $\mu \in \left((\Lambda_1+1)^{-1}, (\Lambda_0+1)^{-1}\right)$ we have*

$$\langle A_\mu u, u\rangle_{\mathcal{K}} \geq \frac{1}{1-\mu(\Lambda_1+1)} \|Ku\|^2_{H^{-1}}.$$

Proof. Let $(v_n) \in (L^2(\Omega))^{\mathbb{N}}$ such that $v_n \to Ku \in H^{-1}_{qp}(\Omega)$. Then, as in the proof of Lemma 6,

$$\langle A_\mu u, u\rangle_{\mathcal{K}} = \left\langle (I-\mu\mathfrak{G}_0{}^{-1})^{-1}Ku, Ku\right\rangle_{H^{-1}}.$$

Using the expansions in terms of Bloch functions (3.3) and (3.4), we have

$$\begin{aligned}\langle A_\mu u, u\rangle_{\mathcal{K}} &= \left\langle (I-\mu\mathfrak{G}_0{}^{-1})^{-1}Ku, Ku\right\rangle_{H^{-1}} = \lim_{n\to\infty}\left\langle (I-\mu\mathfrak{G}_0{}^{-1})^{-1}v_n, v_n\right\rangle_{H^{-1}}\\ &= \lim_{n\to\infty}\left\langle \phi^{-1}(I-\mu\mathfrak{G}_0{}^{-1})^{-1}v_n, \phi^{-1}v_n\right\rangle_{H^1} = \lim_{n\to\infty}\left\langle G_0(I-\mu\mathfrak{G}_0{}^{-1})^{-1}v_n, v_n\right\rangle_{L^2}\\ &= \lim_{n\to\infty}\sum_s \int_{-\pi}^{\pi} \frac{1}{1-\mu(\lambda_s(k)+1)}\cdot\frac{1}{\lambda_s(k)+1}\,|\langle Uv_n, \psi_s\rangle|^2\, dk.\end{aligned}$$

Next, let M be the index introduced in (2.7). Then, as all terms in the series with $s < M$ are non-negative, we have

$$\begin{aligned}\langle A_\mu u, u\rangle_{\mathcal{K}} &\geq \lim_{n\to\infty}\sum_{s\geq M}\int_{-\pi}^{\pi} \frac{1}{1-\mu(\lambda_s(k)+1)}\cdot\frac{1}{\lambda_s(k)+1}\,|\langle Uv_n, \psi_s\rangle|^2\, dk\\ &\geq \frac{1}{1-\mu(\Lambda_1+1)}\lim_{n\to\infty}\sum_{s\geq M}\int_{-\pi}^{\pi}\frac{1}{\lambda_s(k)+1}\,|\langle Uv_n, \psi_s\rangle|^2\, dk.\end{aligned}$$

Since $\dfrac{1}{1-\mu(\Lambda_1+1)} < 0$, we can now add the missing bands back in to get

$$\begin{aligned}\langle A_\mu u, u\rangle_{\mathcal{K}} &\geq \frac{1}{1-\mu(\Lambda_1+1)}\lim_{n\to\infty}\sum_s\int_{-\pi}^{\pi}\frac{1}{\lambda_s(k)+1}\,|\langle Uv_n, \psi_s\rangle|^2\, dk\\ &= \lim_{n\to\infty}\frac{1}{1-\mu(\Lambda_1+1)}\left\langle \phi^{-1}v_n, v_n\right\rangle_{L^2}\\ &= \lim_{n\to\infty}\frac{1}{1-\mu(\Lambda_1+1)}\|v_n\|^2_{H^{-1}} = \frac{1}{1-\mu(\Lambda_1+1)}\|Ku\|^2_{H^{-1}},\end{aligned}$$

as required. □

From this, Lemma 4 (ii) and (iii) easily lead to

$$\frac{\langle A_\mu u, u\rangle_{\mathcal{K}}}{\|u\|_{\mathcal{K}}^2} \geq \frac{\|\mathfrak{G}_1\|_{H^{-1}\to H^1} \left\| \frac{1}{\varepsilon_0} - \frac{1}{\varepsilon_1} \right\|_\infty}{1-\mu(\Lambda_1+1)}. \tag{6.2}$$

Corollary 1. *If* $\left\| \frac{1}{\varepsilon_0} - \frac{1}{\varepsilon_1} \right\|_\infty$ *satisfies the estimate (2.12), then there exists* $\mu \in \left(\frac{1}{\Lambda_1+1}, \frac{1}{\Lambda_0+1}\right)$ *such that* $\frac{\langle A_\mu u, u\rangle_{\mathcal{K}}}{\|u\|_{\mathcal{K}}^2} \geq c > -1$ *for all* $u \in \mathcal{K}$.

Proof. For $\mu \to (\Lambda_0+1)^{-1}$, the right hand side of (6.2) tends to a limit, which is greater than -1 by (2.12). ☐

Inequality (6.2) also shows that for a *fixed* μ in the spectral gap, the size of the perturbation has to reach a threshold before it is possible for μ to lie in the spectrum of $\mathfrak{G}_1$.

6.2. **Upper bound.** In this part, we show that the minimum of the Rayleigh quotient of A_μ diverges to $-\infty$ as μ approaches $\frac{1}{\Lambda_1+1}$ from above (Lemma 16). To do this, we have to determine a suitable test function (Lemma 12) and show that we can approximate it appropriately by smoother functions (Lemmas 13, 14 and 15). In a first step, to bring the interaction with the gap edge into play, we use the edge Bloch wave ψ_M. Here, M is as introduced in (2.7). We recall our assumption that there exists a ball D such that $\varepsilon_1 - \varepsilon_0 > 0$ on D.

Lemma 11. $(\mathfrak{L}_0 - \mathfrak{L}_1)\psi_M(\cdot, k_y^0) \neq 0$.

Proof. Assume $(\mathfrak{L}_0 - \mathfrak{L}_1)\psi_M(\cdot, k_y^0) = 0$. Then

$$[(\mathfrak{L}_0 - \mathfrak{L}_1)\psi_M(\cdot, k_y^0)][\psi_M(\cdot, k_y^0)] = 0, \text{ so } \int_\Omega \left(\frac{1}{\varepsilon_0} - \frac{1}{\varepsilon_1}\right) |\nabla \psi_M(\cdot, k_y^0)|^2 = 0$$

and $\nabla\psi_M(\cdot, k_y^0) = 0$ on D. Hence $L_0\psi_M(\cdot, k_y^0) = 0$ on D. Together with $(L_0 - \Lambda_1)\psi_M(\cdot, k_y^0) = 0$ on Ω, this gives $\psi_M(\cdot, k_y^0) = 0$ on D and by unique continuation $\psi_M(\cdot, k_y^0) \equiv 0$ (see [2]). ☐

Remark 3. *The condition we require for our results is* $(\mathfrak{L}_0 - \mathfrak{L}_1)\psi_M(\cdot, k_y^0) \neq 0$. *We make the assumption on* $\varepsilon_1 - \varepsilon_0$ *instead, as this can be checked from the data.*

Lemma 12. *There exists* $u \in \mathcal{K}$ *such that* $[(\mathfrak{L}_0 - \mathfrak{L}_1)\psi_M(\cdot, k_y^0)][\mathfrak{G}_1 u] \neq 0$.

Proof. As $\mathfrak{G}_1 : H_{qp}^{-1}(\Omega) \to H_{qp}^1(\Omega)$ is surjective, by Lemma 11 there exists $\tilde{u} \in H_{qp}^{-1}(\Omega)$ such that $[(\mathfrak{L}_0 - \mathfrak{L}_1)\psi_M(\cdot, k_y^0)][\mathfrak{G}_1 \tilde{u}] \neq 0$. Set $u = P\tilde{u}$, then

$$\begin{aligned}
[(\mathfrak{L}_0 - \mathfrak{L}_1)\psi_M(\cdot, k_y^0)][\mathfrak{G}_1 u] &= \int_\Omega \left(\frac{1}{\varepsilon_0} - \frac{1}{\varepsilon_1}\right) \nabla\psi_M(\cdot, k_y^0)\overline{\nabla\mathfrak{G}_1 u} \\
&= \overline{[(\mathfrak{L}_0 - \mathfrak{L}_1)\mathfrak{G}_1 u][\psi_M(\cdot, k_y^0)]} \;=\; \overline{[Ku][\psi_M(\cdot, k_y^0)]} \\
&= \overline{[KP\tilde{u}][\psi_M(\cdot, k_y^0)]} \;=\; \overline{[K\tilde{u}][\psi_M(\cdot, k_y^0)]}.
\end{aligned}$$

Reversing all the steps with u replaced by $\tilde{u}$, we get

$$[(\mathfrak{L}_0 - \mathfrak{L}_1)\psi_M(\cdot, k_y^0)][\mathfrak{G}_1 u] = [(\mathfrak{L}_0 - \mathfrak{L}_1)\psi_M(\cdot, k_y^0)][\mathfrak{G}_1 \tilde{u}] \neq 0,$$

which completes the proof. ☐

From now on, u will always denote the test function in $\mathcal{K}$ given in Lemma 12. In considering the Rayleigh quotient for our test function, expressions involving $Ku = (\mathfrak{L}_0 - \mathfrak{L}_1)\mathfrak{G}_1 u$ will arise. To be able to make use of the resolvent representation via Bloch waves in $L^2(\Omega)$, we need to regularize Ku. First, define

$$Tu = \left(\frac{1}{\varepsilon_0} - \frac{1}{\varepsilon_1}\right) \nabla\mathfrak{G}_1 u$$

and extend Tu quasi-periodically from Ω to $\mathbb{R}^2$. Next, we introduce a mollifier $(\chi_n)_{n\geq 0}$ with support in $[0,1]^2$ and set

$$u_n = Tu * \chi_n \quad \text{and} \quad v_n = -\operatorname{div}(Tu * \chi_n).$$

Then $u_n|_\Omega, v_n|_\Omega$ are supported on Ω_n, a neighbourhood of $[0,1]\times[-R,R]$ in Ω and with U denoting the Floquet-Bloch transform in the $\hat{\mathbf{y}}$-direction, we have for sufficiently large n that

$$Uu_n(x,y,k) = \frac{1}{2\pi}\sum_{m\in\mathbb{Z}} e^{ikm} u_n(x,y-m) \;=\; \frac{1}{2\pi}\sum_{|m|\leq R+1} e^{ikm} u_n(x,y-m). \tag{6.3}$$

Similarly,

$$Uv_n(x,y,k) \;=\; \frac{1}{2\pi}\sum_{|m|\leq R+1} e^{ikm} u_n(x,y-m). \tag{6.4}$$

In particular, in both cases, the sum is finite.

We now show that this gives us the desired smooth approximation of Ku.

Lemma 13. *$v_n \to Ku$ in $H^{-1}_{qp}(\Omega)$.*

Proof. Let $\varphi \in H^1_{qp}(\Omega)$. Then

$$v_n[\varphi] = \int_{\Omega_n} v_n\overline{\varphi} = \int_{\Omega_n}\left(\left(\frac{1}{\varepsilon_0}-\frac{1}{\varepsilon_1}\right)\nabla\mathfrak{G}_1 u * \chi_n\right)\overline{\nabla\varphi} = \int_{\Omega_n}(Tu*\chi_n)\,\overline{\nabla\varphi},$$

where the boundary term in the integration by parts vanishes, as all functions satisfy quasiperiodic boundary conditions in the $\hat{\mathbf{x}}$-direction. On the other hand,

$$Ku[\varphi] = \int_{(0,1)\times(-R,R)}\left(\frac{1}{\varepsilon_0}-\frac{1}{\varepsilon_1}\right)\nabla\mathfrak{G}_1 u\overline{\nabla\varphi} = \int_{(0,1)\times(-R,R)} Tu\overline{\nabla\varphi}.$$

Hence,

$$|(Ku-v_n)[\varphi]| = \left|\int_\Omega (Tu - Tu*\chi_n)\,\overline{\nabla\varphi}\right| \;\leq\; \|Tu - Tu*\chi_n\|_{L^2}\,\|\varphi\|_{H^1}\,.$$

As $Tu - Tu*\chi_n \to 0$ in $L^2(\Omega)$, we see that $v_n \to Ku$ in $H^{-1}_{qp}(\Omega)$. □

Before finally considering the Rayleigh quotient, we need two more auxilliary results.

Lemma 14. *$Uu_n(\cdot,k)\to U(Tu)(\cdot,k)$ uniformly in k in $L^2((0,1)^2)$ as $n\to\infty$.*

Proof. We consider the expression for Uu_n from (6.3) and note that

$$UTu(x,y,k) \;=\; \frac{1}{2\pi}\sum_{|m|\leq R} e^{ikm} Tu(x,y-m).$$

Clearly, we have that $(Tu*\chi_n)(\cdot,\cdot-m)|_{(0,1)^2} \to Tu(\cdot,\cdot-m)|_{(0,1)^2}$ in $L^2((0,1)^2)$ for $|m|\leq R$ and $\left\|e^{\mp ik}u_n(x,y\pm(R+1))\right\|_{L^2} = \|u_n(x,y\pm(R+1))\|_{L^2} \to 0$ uniformly in k. □

Lemma 15. *There exist $c>0, \delta>0$ and $N\in\mathbb{N}$ such that*

$$|\langle Uv_n(\cdot,k),\psi_M(\cdot,k)\rangle|^2 \geq c$$

for all $|k-k_y^0|<\delta$ and $n>N$.

Proof. Integrating by parts, we have

$$\langle Uv_n(\cdot,k),\psi_M(\cdot,k)\rangle = \int_{(0,1)^2} Uu_n(\cdot,k)\cdot\overline{\nabla\psi_M(\cdot,k)} \;\to\; \int_{(0,1)^2} UTu(\cdot,k)\cdot\overline{\nabla\psi_M(\cdot,k)},$$

where, by Lemma 14 the convergence is uniform in k. Now, in view of the location of the support of the functions, and using the quasi-periodicity of ψ_M,

$$\begin{aligned}\int_{(0,1)^2} UTu(\cdot,k)\cdot\overline{\nabla\psi_M(\cdot,k)} &= \frac{1}{2\pi}\sum_{|m|\le R}\int_{(0,1)^2} e^{ikm}Tu(x,y-m)\overline{\nabla\psi_M(x,y,k)}\\ &= \frac{1}{2\pi}\int_{(0,1)\times(-R,R)} Tu(x,z)\overline{\nabla\psi_M(x,z,k)}\\ &= \frac{1}{2\pi}\int_\Omega\left(\frac{1}{\varepsilon_0}-\frac{1}{\varepsilon_1}\right)\nabla\mathfrak{G}_1 u\cdot\overline{\nabla\psi_M(\cdot,k)}\\ &= \frac{1}{2\pi}\overline{[(\mathfrak{L}_0-\mathfrak{L}_1)\psi_M(\cdot,k)]\,[\mathfrak{G}_1 u]}.\end{aligned}$$

By Lemma 12, this is non-zero at $k=k_y^0$. Consider the map $k\mapsto[(\mathfrak{L}_0-\mathfrak{L}_1)\psi_M(\cdot,k)]\,[\mathfrak{G}_1 u]$. This is continuous in k and so there exists $\delta>0$ such that it is non-zero for all $|k-k_y^0|<\delta$. Uniformity of the convergence then proves the result for all $n>N$ for some $N\in\mathbb{N}$. □

Lemma 16. *For the test function u given in Lemma 12 we have $\langle A_\mu u,u\rangle\to-\infty$ as $\mu\to\dfrac{1}{\Lambda_1+1}$.*

Proof. As in the proof of Lemma 10, and using Lemma 13,

$$\begin{aligned}\langle A_\mu u,u\rangle &= \lim_{n\to\infty}\sum_s\int_{-\pi}^{\pi}\frac{1}{1-\mu(\lambda_s(k)+1)}\frac{1}{\lambda_s(k)+1}\left|\langle Uv_n,\psi_s\rangle_{L^2}\right|^2dk\\ &\le \lim_{n\to\infty}\sum_{s\le M}\int_{-\pi}^{\pi}\frac{1}{1-\mu(\lambda_s(k)+1)}\frac{1}{\lambda_s(k)+1}\left|\langle Uv_n,\psi_s\rangle_{L^2}\right|^2dk\end{aligned}$$

Now, for μ near $(\Lambda_1+1)^{-1}$,

$$\begin{aligned}&\sum_{s<M}\int_{-\pi}^{\pi}\frac{1}{1-\mu(\lambda_s(k)+1)}\frac{1}{\lambda_s(k)+1}\left|\langle Uv_n,\psi_s\rangle_{L^2}\right|^2dk\\ &\le\frac{1}{1-\mu(\Lambda_0+1)}\sum_{s<M}\int_{-\pi}^{\pi}\frac{1}{\lambda_s(k)+1}\left|\langle Uv_n,\psi_s\rangle_{L^2}\right|^2dk\\ &\le\frac{1}{1-\mu(\Lambda_0+1)}\sum_{s}\int_{-\pi}^{\pi}\frac{1}{\lambda_s(k)+1}\left|\langle Uv_n,\psi_s\rangle_{L^2}\right|^2dk\\ &\le C\,\|v_n\|^2_{H^{-1}}\to C\,\|Ku\|^2_{H^{-1}}\le C\,\|u\|^2_{\mathcal{K}},\end{aligned}$$

where we have again used Lemma 13 and the last estimate follows by Lemma 4 (iii). We are left with the contribution from the M-band which we divide up into integration over two disjoint regions: Let δ be as in Lemma 15 and $B_\delta(k_y^0)$ denote the ball of radius δ around k_y^0. Then

$$\int_{[-\pi,\pi]\setminus B_\delta(k_y^0)}\frac{1}{1-\mu(\lambda_M(k)+1)}\frac{1}{\lambda_M(k)+1}\left|\langle Uv_n,\psi_M\rangle_{L^2}\right|^2\ dk\le 0$$

and using that for $|k-k_y^0|<\delta$ we have $\dfrac{1}{\lambda_M(k)+1}\ge c_1>0$ (by choosing a smaller δ, if necessary) and $\left|\langle Uv_n,\psi_M\rangle_{L^2}\right|^2\ge c$ by Lemma 15, we have

$$\int_{B_\delta(k_y^0)}\frac{1}{1-\mu(\lambda_M(k)+1)}\frac{1}{\lambda_M(k)+1}\left|\langle Uv_n,\psi_M\rangle_{L^2}\right|^2\ dk\le C_\delta\int_{B_\delta(k_y^0)}\frac{dk}{1-\mu(\lambda_M(k)+1)}.$$

Now observe that from (2.8) we have

$$\int_{B_\delta(k_y^0)} \frac{dk}{1-\mu(\lambda_M(k)+1)} \;=\; -\int_{B_\delta(k_y^0)} \frac{dk}{\mu(\lambda_M(k)+1)-1}$$
$$\le -\int_{B_\delta(k_y^0)} \frac{dk}{\mu(\Lambda_1+1+\alpha|k-k_y^0|^2)-1}$$
$$\le -(\Lambda_1+1)^{-1}\int_{B_\delta(k_y^0)} \frac{dk}{\mu-1/(\Lambda_1+1)+\alpha|k-k_y^0|^2/((\Lambda_1+1)(\Lambda_0+1))}.$$

The last integral has a nonnegative integrand and has the form

$$\int_{B_\delta(k_y^0)} \frac{dk}{\eta+c_1|k-k_y^0|^2} \tag{6.5}$$

with c_1 a positive constant and $\eta = \mu - 1/(\Lambda_1+1) \to 0$ as $\mu \to \dfrac{1}{\Lambda_1+1}$. The expression (6.5) is larger than

$$\int_{|k-k_y^0|\le\delta_1} \frac{dk}{\eta+c_1|k-k_y^0|^2} \ge \frac{2\delta_1}{\eta+c_1\delta_1^2}$$

for any $0 < \delta_1 \le \delta$. By setting $\delta_1^2 = \eta$, we see that the integral diverges as $\eta \to 0$. Thus finally, $\langle A_\mu u, u\rangle \to -\infty$ as $\mu \to 1/(\Lambda_1+1)$. □

Combining the results of Lemma 9, Lemma 10 and Lemma 16, we obtain our main result, Theorem 1, from the Intermediate Value Theorem. In particular, any arbitrarily weak perturbation induces spectrum into the gap.

7. Concluding Remarks and open problems

We provided a sufficient rigorous criterion for localization in gaps by arbitrarily weak line defects, for the case of TE-polarized electromagnetic waves. We arrive at our results by comparing the Green's operators of the perturbed and unperturbed systems. While Green's functions techniques have been a part of the theoretical physics literature for a long time (see e.g. [13]), our method combines Green's functions and variational methods. For example, we do not use series expansion of the difference of the operators G_0 and G_1 to get approximations, and the variational approach avoids in an elegant way the need to control the remainder terms.

The method presented here is, in principle, also applicable to both the case when the band edge under consideration is degenerate and to the full Maxwell equations, at the expense of greater technical complexity. We plan to deal with these in forthcoming work.

There are many interesting questions in this area, e.g. what happens in geometries which are finite, at least in one direction, such as a photonic crystal slab (see, e.g. [27]), where effects of the boundary need to be taken into account?

Another open problem is the following. We know now sufficient conditions to create gap modes which are localized in the $\hat{\mathbf{y}}$-direction centering on the line defect. If the modes were additionally localized in the $\hat{\mathbf{x}}$-direction, we would have a bound state of the operator $-\nabla\cdot\varepsilon_1^{-1}\nabla$ on the whole of $\mathbb{R}^2$. This would go against physical intuition, since then a localized standing wave would exist, which impedes wave propagation in $\hat{\mathbf{x}}$ direction.

It would be desirable to show that there are no modes that are localized in the $\hat{\mathbf{x}}$-direction, i.e. the perturbation creates truly guided modes. This would equivalently mean, that there is no flat band created in the gap (for a discussion, see [31]). The absence of bound states for periodic Helmholtz operators with line defects has been proven in [24]. However, to show absence of bound states for periodic divergence type operators seems to be extremely difficult. For periodic operators with sufficiently smooth coefficients, this question is investigated addressed in [21].

Appendix A. Definiteness of K

In this section of the Appendix, we show that the bilinear form defined by $\langle \cdot,\cdot\rangle_{\mathcal{K}}$ is indeed an inner product on $\mathcal{K}$. This is a consequence of the following Lemmas.

Lemma 17. *K is symmetric.*

Proof. Let $u, v \in H^{-1}_{qp}(\Omega)$ and choose sequences (v_n) and (u_m) in $L^2(\Omega)$ such that $v_n \to v$ and $u_m \to u$ in $H^{-1}_{qp}(\Omega)$. We first note that

$$\begin{aligned}\langle Ku, v\rangle_{H^{-1}} &= \left\langle \phi^{-1}Ku, \phi^{-1}v\right\rangle_{H^1} = \left\langle \mathfrak{G}_0 Ku, \phi^{-1}v\right\rangle_{H^1}\\ &= \left\langle (\mathfrak{G}_1 - \mathfrak{G}_0)u, \phi^{-1}v\right\rangle_{H^1} \;=\; \overline{v[(\mathfrak{G}_1 - \mathfrak{G}_0)u]}.\end{aligned} \tag{A.1}$$

Now, using the convergence in $H^{-1}_{qp}(\Omega)$ and the symmetry of the G_i, $i = 0, 1$, in $L^2(\Omega)$ we get

$$\begin{aligned}v[(\mathfrak{G}_1 - \mathfrak{G}_0)u] &= \lim_{n\to\infty} v_n[(\mathfrak{G}_1 - \mathfrak{G}_0)u] \;=\; \lim_{n\to\infty} \langle v_n, (\mathfrak{G}_1 - \mathfrak{G}_0)u\rangle_{L^2}\\ &= \lim_{n\to\infty}\lim_{m\to\infty} \langle v_n, (G_1 - G_0)u_m\rangle_{L^2} \;=\; \lim_{n\to\infty}\lim_{m\to\infty} \langle (G_1 - G_0)v_n, u_m\rangle_{L^2}\\ &= \lim_{n\to\infty}\lim_{m\to\infty} \overline{u_m[(G_1 - G_0)v_n]} \;=\; \lim_{n\to\infty} \overline{u[(\mathfrak{G}_1 - \mathfrak{G}_0)v_n]} = \overline{u[(\mathfrak{G}_1 - \mathfrak{G}_0)v]}.\end{aligned}$$

By a similar calculation to (A.1), this equals $\langle Kv, u\rangle_{H^{-1}} = \overline{\langle u, Kv\rangle}_{H^{-1}}$, proving the result. □

Lemma 18. *For $u \in H^{-1}_{qp}(\Omega)$, $\langle Ku, u\rangle_{H^{-1}} \geq 0$.*

Proof. From the proof of Lemma 17 we have

$$\langle Ku, u\rangle_{H^{-1}} = u[(\mathfrak{G}_1 - \mathfrak{G}_0)u].$$

Moreover, $(\mathfrak{L}_0 + 1)\mathfrak{G}_0 u = u \in H^{-1}_{qp}(\Omega)$. Using(4.3) with $f = (\mathfrak{L}_1 + 1)\mathfrak{G}_0 u$ and $\varphi = \mathfrak{G}_1 u$, we get

$$\begin{aligned}((\mathfrak{L}_1 + 1)\mathfrak{G}_0 u)\,[\mathfrak{G}_1 u] &= \int_\Omega \left(\frac{1}{\varepsilon_1}\nabla\mathfrak{G}_0 u\overline{\nabla\mathfrak{G}_1 u} + \mathfrak{G}_0 u\overline{\mathfrak{G}_1 u}\right) d\mathbf{x}\\ &= \overline{((\mathfrak{L}_1 + 1)\mathfrak{G}_1 u)\,[\mathfrak{G}_0 u]} = \overline{u[\mathfrak{G}_0 u]} \;=\; u[\mathfrak{G}_0 u],\end{aligned}$$

where the last step follows from Lemma 2. Combining these three equalities, we get

$$\begin{aligned}\langle Ku, u\rangle_{H^{-1}} &= u[(\mathfrak{G}_1 - \mathfrak{G}_0)u] \;=\; u[\mathfrak{G}_1 u] - u[\mathfrak{G}_0 u]\\ &= ((\mathfrak{L}_0 + 1)\mathfrak{G}_0 u)[\mathfrak{G}_1 u] - ((\mathfrak{L}_1 + 1)\mathfrak{G}_0 u)[\mathfrak{G}_1 u]\\ &= -\left((\mathfrak{L}_1 - \mathfrak{L}_0)\mathfrak{G}_0 u\right)[\mathfrak{G}_1 u]\\ &= ((\mathfrak{L}_1 - \mathfrak{L}_0)\mathfrak{G}_0 u)[(\mathfrak{G}_1(\mathfrak{L}_1 - \mathfrak{L}_0)\mathfrak{G}_0 - \mathfrak{G}_0)u]\\ &= ((\mathfrak{L}_1 - \mathfrak{L}_0)\mathfrak{G}_0 u)\,[\mathfrak{G}_1(\mathfrak{L}_1 - \mathfrak{L}_0)\mathfrak{G}_0 u] - ((\mathfrak{L}_1 - \mathfrak{L}_0)\mathfrak{G}_0 u)\,[\mathfrak{G}_0 u].\end{aligned}$$

The first term is non-negative by Lemma 2. Also,

$$-\left((\mathfrak{L}_1 - \mathfrak{L}_0)\mathfrak{G}_0 u\right)[\mathfrak{G}_0 u] = \int_\Omega \left(\frac{1}{\varepsilon_0} - \frac{1}{\varepsilon_1}\right)|\nabla\mathfrak{G}_0 u|^2 \geq 0,$$

implying $\langle Ku, u\rangle_{H^{-1}} \geq 0$. □

Lemma 19. *$\langle \cdot,\cdot\rangle_{\mathcal{K}}$ is positive definite on $\mathcal{K}$.*

Proof. Suppose $\langle u, u\rangle_{\mathcal{K}} = \langle Ku, u\rangle_{H^{-1}} = 0$ for some $u \in \mathcal{K}$. As $K \geq 0$, we can define $K^{1/2}$ as a selfadjoint operator in $H^{-1}_{qp}(\Omega)$ and get $K^{1/2}u = 0$, implying that $Ku = 0$. Thus $u \in \mathcal{K} \cap \ker K$ giving $u = 0$. □

Appendix B. Compact embedding of $H^1_{qp}(\Omega, e^{\gamma|y|})$ in $L^2(\Omega)$

In this appendix, we briefly sketch the compact embedding of $H^1_{qp}(\Omega, e^{\gamma|y|})$ in $L^2(\Omega)$ for $\gamma > 0$. For any $f \in H^1_{qp}(\Omega, e^{\gamma|y|})$,

$$\int_{\Omega, |y|\geq R} |f|^2 \leq e^{-\gamma R} \int_{\Omega, |y|\geq R} e^{\gamma|y|}|f|^2 \leq e^{-\gamma R}\|f\|^2_{H^1_{qp}(\Omega, e^{\gamma|y|})}. \tag{B.1}$$

Let $f_j = f_j^{(1)}$ be a bounded sequence in $H^1_{qp}(\Omega, e^{\gamma|y|})$. Let $\Omega_p := (0,1)\times(-p,p)$ for any $p \in \mathbb{N}$. Since $H^1_{qp}(\Omega_p)$ embeds compactly into $L^2(\Omega_p)$, we may extract from $(f_j^{(1)})$ a subsequence $(f_j^{(2)})$ converging in $L^2(\Omega_1)$ and from $(f_j^{(2)})$ a subsequence converging in $L^2(\Omega_2)$ and so forth. We claim that the diagonal sequence $(f_p^{(p)})$ is Cauchy in $L^2(\Omega)$. This is seen as follows: given any $\varepsilon > 0$, determine first a p_0 so large that

$$\left(\int_{\Omega, |y|\geq p_0} |f_p^{(p)}|^2 dy\right)^{1/2} \leq \frac{\varepsilon}{3}.$$

for all $p \geq p_0$, using (B.1). Now determine a $p_1 \geq p_0$ so large that $\|f_p^{(p_0)} - f_q^{(p_0)}\|_{L^2(\Omega_{p_0})} \leq \varepsilon/3$ for all $p, q \geq p_1$. Since $(f_p^{(p)})_{p\geq p_0}$ is a subsequence of $(f_j^{(p_0)})$, we have for $p, q \geq p_1$

$$\|f_p^{(p)} - f_q^{(q)}\|_{L^2(\Omega)} \leq \|f_p^{(p)} - f_q^{(q)}\|_{L^2(\Omega_{p_0})} + \|f_p^{(p)}\|_{L^2(\Omega\setminus\Omega_{p_0})} + \|f_q^{(q)}\|_{L^2(\Omega\setminus\Omega_{p_0})} \leq \varepsilon.$$

Acknowledgements. *The authors would like to thank Prof. Carlos Kenig for pointing out the paper* [2] *on unique continuation. The authors are grateful to the anonymous reviewer for a careful reading of the manuscript and helpful comments which greatly improved our presentation.*

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