$\mathbb{Z}_N$ graded discrete Lax pairs and integrable difference equations

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Abstract

We introduce a class of $\mathbb{Z}_N$ graded discrete Lax pairs, with $N \times N$ matrices, linear in the spectral parameter. We give a classification scheme for such Lax pairs and the associated integrable lattice systems. We present two potential forms and completely classify the generic case. Many well known examples belong to our scheme for $N = 2$, so many of our systems may be regarded as generalisations of these. Even at $N = 3$, several new integrable systems arise. A decomposable case gives rise to interesting coupled systems of lower dimensional equations.

Many of our equations are mutually compatible, so can be used together to form "coloured" lattices.

Keywords: Discrete integrable system, lattice equations, Lax pair, Bäcklund transformation, $3D$ consistency.

1 Introduction

The classification of discrete integrable systems is still rather primitive when compared with that of integrable PDEs. The most famous is the ABS classification of integrable equations on quad-graphs [2], but this is only for scalar equations, associated with 2×2 matrix Lax pairs. Multi-component generalisations and scalar equations with 3×3 Lax pairs are rather sporadic [1, 12, 14, 15, 17].

In this paper we introduce an $N \times N$ spectral problem, linear in the spectral parameter and with $\mathbb{Z}_N$ grading, which gives rise to a large class of discrete integrable systems. In the 2×2 specialisation, these include some well known examples (discrete potential, modified and Schwarzian KdV equations, Hirota's KdV equation and the discrete sine-Gordon equation). The 3×3 case includes the discrete versions of the Boussinesq and modified Boussinesq equations [14]. The $N \times N$ case includes multi-component generalisations of them all. Many completely new examples arise for $N \geq 3$. An important aspect of this paper is that we both synthesise and generalise a number of well known low-dimensional examples.

In section 2, we present the basic algebraic framework used throughout the paper. One of the most important results is to provide a classification scheme which enables us to place all our examples in a coherent framework. There are two basic cases: the *indecomposable* (*coprime case*) and the *decomposable* (*non-coprime case*). The indecomposable case splits further into *generic* and *degenerate* cases. The decomposable case gives rise to interesting coupled systems of lower dimensional equations.

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In section 3 we give the classification of indecomposable systems. The *generic* case naturally gives rise to two *potential* formulations. The *quotient potential* case includes the two dimensional examples of the discrete modified KdV equation and Hirota's discrete sine-Gordon equation. For $N=3$ we obtain the well-known "discrete modified Boussinesq equation", but also derive several completely new examples. In particular the system

$$\begin{aligned} \phi^{(0)}_{m+1,n+1} &= \left(\frac{\alpha\,\phi^{(1)}_{m+1,n}-\beta\phi^{(1)}_{m,n+1}}{\alpha\,\phi^{(0)}_{m+1,n}\phi^{(1)}_{m,n+1}-\beta\,\phi^{(0)}_{m,n+1}\phi^{(1)}_{m+1,n}}\right)\frac{1}{\phi^{(0)}_{m,n}}, \\ \phi^{(1)}_{m+1,n+1} &= \left(\frac{\alpha\,\phi^{(0)}_{m,n+1}-\beta\phi^{(0)}_{m+1,n}}{\alpha\,\phi^{(0)}_{m+1,n}\phi^{(1)}_{m,n+1}-\beta\,\phi^{(0)}_{m,n+1}\phi^{(1)}_{m+1,n}}\right)\frac{1}{\phi^{(1)}_{m,n}}, \end{aligned}$$

arises and can be *decoupled* to a nine-point scalar equation for either of the functions involved in it (see (3.12)) and can be *reduced* to a simple scalar equation on a quadrilateral (see (3.13)). This last reduction can be generalised to a $k=\frac{N-1}{2}$ component system, with an $N\times N$ Lax pair:

$$\phi^{(i)}_{m+1,n+1}\phi^{(i)}_{m,n}=\frac{1}{\phi^{(k-i-2)}_{m+1,n}\phi^{(k-i-2)}_{m,n+1}}\left(\frac{\phi^{(k-i-2)}_{m+1,n}+\phi^{(k-i-2)}_{m,n+1}}{\phi^{(k-i-1)}_{m+1,n}+\phi^{(k-i-1)}_{m,n+1}}\right),\quad \text{for}\quad i=0,1,\ldots,k-1,$$

which is <u>not</u> $3D$ consistent.

On the other hand, the *additive potential* case includes the *discrete potential KdV* and *discrete Schwarzian KdV* equations for $N=2$, together with several new systems at $N=3$, including the system

$$\begin{aligned} \chi^{(0)}_{m+1,n+1} &= \frac{(\chi^{(0)}_{m+1,n}-\chi^{(1)}_{m,n})\chi^{(1)}_{m+1,n}-(\chi^{(0)}_{m,n+1}-\chi^{(1)}_{m,n})\chi^{(1)}_{m,n+1}}{\chi^{(0)}_{m+1,n}-\chi^{(0)}_{m,n+1}}, \\ \chi^{(1)}_{m+1,n+1} &= \chi^{(0)}_{m,n}+\frac{1}{\chi^{(1)}_{m+1,n}-\chi^{(1)}_{m,n+1}}\left(\frac{\alpha^3}{\chi^{(0)}_{m+1,n}-\chi^{(1)}_{m,n}}-\frac{\beta^3}{\chi^{(0)}_{m,n+1}-\chi^{(1)}_{m,n}}\right), \end{aligned}$$

which can be decoupled for either of the variables to the nine point scalar discrete Boussinesq equation.

The final part of this section deals with a degenerate form of our Lax pair, which includes multicomponent generalisations of Hirota's KdV equation (see (3.39)).

In section 4, we introduce the *decomposable* case. Here we can change the basis to put our matrices in a block form, with a corresponding grouping of variables, giving rise to (generally) *coupled* systems of lower dimensional equations.

Many of our systems are pairwise compatible, so can be used to build two and three dimensional consistent lattices, some of which have unusual features, giving rise to non-standard *initial value problems*. This is discussed in section 5.

A number of open problems are discussed in the conclusions.

2 $\mathbb{Z}_N$-Graded Lax Pairs

In this section we introduce the general framework for what follows in the paper. We first introduce the idea of $\mathbb{Z}_N$-grading and *level structure*. We then introduce the general Lax pair and the corresponding discrete system. There is a considerable amount of redundancy, so we introduce an *equivalence relation*, which enables us to classify all systems belonging to our framework. As a consequence of the *level structure*, our systems fall naturally into two categories: *coprime* (indecomposable) and *non-coprime* (decomposable) cases. The *coprime* case further separates into the *generic* and the *degenerate* subcases.

We use the following convention for upper indices in parentheses: the notation $a \in \mathbb{Z}_N$ implies that a is an integer such that $0 \leq a \leq N-1$ and (*by definition*) summation in $\mathbb{Z}_N$ is taken $\bmod N$. The notation $u^{(i)}_{m,n}$ denotes that $u^{(i)}$ is located at lattice point (m,n). The symbol $\delta_{i,j}$ will denote the usual Kroneker delta. The symbols $\mathcal{S}_m$ and $\mathcal{S}_n$ will respectively denote the shifts in the m and n directions: $\mathcal{S}_m u^{(i)}_{m,n} = u^{(i)}_{m+1,n}$ and $\mathcal{S}_n u^{(i)}_{m,n} = u^{(i)}_{m,n+1}$, with $\Delta_m = \mathcal{S}_m - 1$ and $\Delta_n = \mathcal{S}_n - 1$ the corresponding *differences*.

2.1 $\mathbb{Z}_N$-Grading

To introduce $\mathbb{Z}_N$-grading, we need:

Definition 2.1 (Matrix Ω)**.** *The $N \times N$ matrix Ω is defined by*

$$(\Omega)_{i,j} \;=\; \delta_{j-i,1} + \delta_{i-j,N-1},$$

which will be said to have level 1*.*

This is a cyclic matrix, satisfying $\Omega^N = \mathrm{I}_N$ (the $N \times N$ identity matrix), so defines a grading, with

$$(\Omega^k)_{i,j} \;=\; \delta_{j-i,k} + \delta_{i-j,N-k}, \quad 0 \leq k \leq N-1,$$

having *level* k. Ω^N has *level* 0. It also follows that $(\Omega^k)^{-1} = \Omega^{N-k}$.

Definition 2.2 (A level k matrix)**.** *An $N \times N$ matrix A of the form*

$$A \;=\; \mathrm{diag}\left(a^{(0)}, \ldots, a^{(N-1)}\right) \Omega^k$$

will be said to have level k, written $lev(A) = k$.

It can be seen that if $(N,k) = 1$, then $A^N = \left(\prod_{i=0}^{N-1} a^{(i)}\right) \mathrm{I}_N$.

If B is another $N \times N$ matrix of a certain level, then

$$lev(AB) \;=\; lev(BA) \;=\; lev(A) + lev(B) \;(\mathrm{mod} N).$$

Let us denote by R the ring of the $N \times N$ matrices, and by R_k the set of all $N \times N$ matrices of level k. Then, it follows that R is a $\mathbb{Z}_N$-graded ring as it is the direct sum decomposition

$$\mathrm{R} \;=\; \bigoplus_{k \in \mathbb{Z}_N} \mathrm{R}_k\,, \qquad \mathrm{R}_k \mathrm{R}_\ell \subseteq \mathrm{R}_{k+\ell}.$$

2.2 The General Lax Pair

We employ the above $\mathbb{Z}_N$ grading of R to build some particular classes of discrete Lax pairs, whose matrices belong to the polynomial ring $\mathrm{R}[\lambda]$.

Specifically, we consider a pair of matrix equations of the form

$$\Psi_{m+1,n} \;=\; L_{m,n}\,\Psi_{m,n} \;\equiv\; \left(U_{m,n} \,+\, \lambda\,\Omega^{\ell_1}\right)\,\Psi_{m,n}, \quad lev\,(U_{m,n}) = k_1 \neq \ell_1\,, \tag{2.1a}$$

$$\Psi_{m,n+1} \;=\; M_{m,n}\,\Psi_{m,n} \;\equiv\; \left(V_{m,n} \,+\, \lambda\,\Omega^{\ell_2}\right)\,\Psi_{m,n}, \quad lev\,(V_{m,n}) = k_2 \neq \ell_2\,, \tag{2.1b}$$

which is characterised by the quadruple $(k_1, \ell_1; k_2, \ell_2)$. We refer to it as *the level structure* of system (2.1) and derive necessary conditions for the system (2.1) to be compatible.

Since matrices U, V and Ω are independent of λ, the compatibility condition of (2.1),

$$L_{m,n+1}M_{m,n} \;=\; M_{m+1,n}L_{m,n}, \tag{2.2}$$

splits into the system

$$\begin{aligned} U_{m,n+1}V_{m,n} \;&=\; V_{m+1,n}U_{m,n}\,, && \text{(2.3a)}\\ U_{m,n+1}\Omega^{\ell_2} - \Omega^{\ell_2}U_{m,n} \;&=\; V_{m+1,n}\Omega^{\ell_1} - \Omega^{\ell_1}V_{m,n}\,. && \text{(2.3b)} \end{aligned}$$

It is obvious that both sides of equation (2.3a) have the same level $k_1 + k_2$. On the other hand, the left and right hand sides of equation (2.3b) have respective levels $k_1 + \ell_2$ and $k_2 + \ell_1$. Hence, compatibility condition (2.3b) yields nontrivial equations for the entries of matrices U and V if and only if

$$k_1 + \ell_2 \;\equiv\; k_2 + \ell_1 \;(\mathrm{mod} N). \tag{2.4}$$

Definition 2.3 (The quadruple $\mathcal{Q}_N$). *We denote the set of quadruples* $(k_1, \ell_1; k_2, \ell_2)$ *which satisfy condition (2.4) as*

$$\mathcal{Q}_N \;=\; \left\{(k_1, \ell_1; k_2, \ell_2) \in \mathbb{Z}_N^4 \,\middle|\, k_1 \neq \ell_1,\; k_2 \neq \ell_2,\; k_1 + \ell_2 \equiv k_2 + \ell_1 \;(\mathrm{mod} N)\right\},$$

Remark 2.4. *This condition implies that* $\ell_2 - k_2 \equiv \ell_1 - k_1\,(\mathrm{mod} N)$*, so the greatest common divisors* $(N, \ell_1 - k_1)$ *and* $(N, \ell_2 - k_2)$ *are equal.*

Definition 2.5 (Coprime Case)**.** *We define the Lax pair (2.1) to be coprime if* $(N, \ell_i - k_i) = 1$*. Otherwise, we refer to it as non-coprime.*

It is easily seen that in the coprime case, we have

$$\det(L_{m,n}) = \prod_{i=0}^{N-1} u^{(i)}_{m,n} - (-\lambda)^N, \quad \text{and} \quad \det(M_{m,n}) = \prod_{i=0}^{N-1} v^{(i)}_{m,n} - (-\lambda)^N. \tag{2.5}$$

2.2.1 The General Discrete System

Supposing that condition (2.4) is satisfied, let

$$U_{m,n} \;=\; \mathrm{diag}\left(u^{(0)}_{m,n}, \ldots, u^{(N-1)}_{m,n}\right)\Omega^{k_1}\,, \quad V_{m,n} \;=\; \mathrm{diag}\left(v^{(0)}_{m,n}, \ldots, v^{(N-1)}_{m,n}\right)\Omega^{k_2}. \tag{2.6}$$

In view of (2.6), equations (2.3) can be written explicitly as

$$\begin{aligned} u^{(i)}_{m,n+1}v^{(i+k_1)}_{m,n} \;&=\; v^{(i)}_{m+1,n}u^{(i+k_2)}_{m,n}\,, \quad i \in \mathbb{Z}_N, && \text{(2.7a)}\\ u^{(i)}_{m,n+1} - u^{(i+\ell_2)}_{m,n} \;&=\; v^{(i)}_{m+1,n} - v^{(i+\ell_1)}_{m,n}\,, \quad i \in \mathbb{Z}_N, && \text{(2.7b)} \end{aligned}$$

or, in a solved form, as

$$u^{(i)}_{m,n+1} \;=\; \frac{u^{(i+\ell_2)}_{m,n} - v^{(i+\ell_1)}_{m,n}}{u^{(i+k_2)}_{m,n} - v^{(i+k_1)}_{m,n}}\,u^{(i+k_2)}_{m,n}\,, \quad v^{(i)}_{m+1,n} \;=\; \frac{u^{(i+\ell_2)}_{m,n} - v^{(i+\ell_1)}_{m,n}}{u^{(i+k_2)}_{m,n} - v^{(i+k_1)}_{m,n}}\,v^{(i+k_1)}_{m,n}\,, \tag{2.8}$$

assuming that $u^{(i)}_{m,n} \neq v^{(j)}_{m,n}$ for all $i, j \in \mathbb{Z}_N$.

If we add equations (2.7b) we obtain the *discrete local conservation law*

$$\Delta_n\left(\sum_{i=0}^{N-1} u_{m,n}^{(i)}\right) = \Delta_m\left(\sum_{i=0}^{N-1} v_{m,n}^{(i)}\right). \tag{2.9}$$

This could be obtained directly from the matrix equation (2.2) by first multiplying by $\Omega^{-k_1-\ell_2}$ and then taking the trace. The matrix equation (2.2) also implies the following equation for the determinants:

$$\det(L_{m,n+1})\,\det(M_{m,n}) = \det(M_{m+1,n})\,\det(L_{m,n}), \tag{2.10}$$

which is polynomial in λ, so implies a separate condition for each coefficient. The precise form of these determinants depends upon the choice of quadruple $(k_1, \ell_1; k_2, \ell_2)$.

In the *coprime* case we have

$$\Delta_n(a) = \Delta_m(b), \quad \text{where} \quad a = \prod_{i=0}^{N-1} u_{m,n}^{(i)}, \quad b = \prod_{i=0}^{N-1} v_{m,n}^{(i)},$$

and $\mathcal{S}_n(a)b = \mathcal{S}_m(b)a$, which together imply

$$\Delta_n(a) = 0 \quad \text{and} \quad \Delta_m(b) = 0, \tag{2.11}$$

This can also be seen from looking directly at the form of equations (2.8). The quantities a and b, defined above, play an important part in this paper.

The non-coprime case is discussed in Section 2.3.

2.2.2 Equivalent Lax Pairs

We consider two transformations which give rise to equivalent Lax pairs:

1. The interchange of lattice variables $(m, n) \mapsto (n, m)$ is a point transformation which corresponds to the interchange of matrices $L_{m,n}$ and $M_{m,n}$. Algebraically, this point transformation corresponds to the interchange of pairs (k_1, ℓ_1) and (k_2, ℓ_2) which does not affect condition (2.4).

2. Consider the gauge transformation of Lax pair (2.1) with the constant matrix G defined as

   $$(G)_{i,j} = \delta_{i+j,2} + \delta_{i+j,N+2}, \quad \text{for } 1 \le i, j \le N \tag{2.12}$$

   which satisfies $G^{-1} = G$.

   Since $G\Omega G^{-1} = \Omega^{-1} = \Omega^{N-1}$, this switches level k matrices with level $N - k$ matrices (mod$N$, so level 0 stay as level 0).

   Applying this gauge transformation to Lax pair (2.1), we will derive a system with level structure $(N - k_1, N - \ell_1; N - k_2, N - \ell_2)$. For the discrete system (2.7), this gauge transformation corresponds to a permutation of the dependent variables,

   $$\left(u_{m,n}^{(i)}, v_{m,n}^{(i)}\right) \mapsto \left(u_{m,n}^{(N-i)}, v_{m,n}^{(N-i)}\right). \tag{2.13}$$

   Again, condition (2.4) is satisfied for the quadruple $(N-k_1, N-\ell_1; N-k_2, N-\ell_2)$ provided that it holds for $(k_1, \ell_1; k_2, \ell_2)$.

Remark 2.6. *The choice of* $(G)_{i,j}$ *is not unique. We can define*

$$G^{(p)} = \Omega^{N-p} G^{(0)} \Omega^p, \quad \text{for} \quad 0 \le p \le N-1, \quad \text{where} \quad G^{(0)} = G, \tag{2.14}$$

each of which satisfies

$$\left(G^{(p)}\right)^{-1} = G^{(p)} \quad \text{and} \quad G^{(p)} \Omega G^{(p)} = \Omega^{-1} = \Omega^{N-1}.$$

We have added an extra rotation of the variables, which means that (2.13) is replaced by

$$\left(u^{(i)}_{m,n}\,,\, v^{(i)}_{m,n}\right) \mapsto \left(u^{(N+2p-i)}_{m,n}\,,\, v^{(N+2p-i)}_{m,n}\right), \tag{2.15}$$

with $\left(u^{(p)}_{m,n}\,,\, v^{(p)}_{m,n}\right)$ *being fixed.*

We use this, with $p = N-1$ *in our discussion of "self-dual" systems in Section 3.1.3.*

Using the above transformations we define the following equivalence relation among Lax pairs (2.1).

Definition 2.7 (Equivalence relation)**.** *Two discrete Lax pairs with levels* $(k_1, \ell_1; k_2, \ell_2) \in \mathcal{Q}_N$ *and* $(k_1', \ell_1'; k_2', \ell_2') \in \mathcal{Q}_N$ *are equivalent, and we write*

$$\left(k_1', \ell_1'; k_2', \ell_2'\right) \sim (k_1, \ell_1; k_2, \ell_2)\ ,$$

if one quadruple can be mapped to the other by applying either of the following transformations.

$$\begin{array}{lcl} \mathcal{T}_1 & : & (a,b;c,d) \mapsto (c,d;a,b) \\ \mathcal{T}_2 & : & (a,b;c,d) \mapsto (N-a, N-b; N-c, N-d)\,. \end{array} \tag{2.16}$$

Otherwise, they will be called inequivalent.

We can now reduce the problem of classification of Lax pairs to the classification of inequivalent classes of quadruples $(k_1, \ell_1; k_2, \ell_2)$ in the quotient space $\mathcal{Q}_N / \sim$.

2.3 The Greatest Common Divisor $(N, \ell - k) = p$

For this section, we define $\mathbf{u} = (u^{(0)}, u^{(1)}, \ldots, u^{(N-1)})$ and $\mathcal{D}_N^{\mathbf{u}} = \text{diag}(u^{(0)}, u^{(1)}, \ldots, u^{(N-1)})$ and denote the $N \times N$ matrix Ω by Ω_N. Our Lax matrix L is of the form

$$L = \mathcal{D}_N^u\, \Omega_N^k + \lambda\, \Omega_N^\ell = \left(\mathcal{D}_N^u + \lambda\, \Omega_N^{\ell-k}\right) \Omega_N^k. \tag{2.17}$$

For this section we are just writing (k, ℓ) instead of (k_1, ℓ_1) and we are suppressing the dependence on (m, n). We now consider the consequence of $(N, \ell - k) = p \neq 1$. We have integers q, r, such that $N = pq$, $\ell - k = pr$, with $(q, r) = 1$.

Definition 2.8 (Permutation matrix)**.** *Let the permutation matrix* P *be such that*

$$(P)_{ij} = \begin{cases} 1 & \text{when } (i,j) \in \{(n + (m-1)q, m + (n-1)p),\ 1 \le m \le p, 1 \le n \le q\}, \\ 0 & \text{otherwise.} \end{cases}$$

Defining $\mathbf{u}_i = (u^{(i)}, u^{(i+p)}, \ldots, u^{(i+p(q-1))})$, $i = 0, \ldots, p-1$ and $\mathbf{u}^b = (\mathbf{u}_0, \ldots, \mathbf{u}_{p-1})$, we have

$$\left(\mathbf{u}^b\right)^T = P\,\mathbf{u}^T \quad \text{and therefore} \quad \mathbf{u}^b = \mathbf{u}P^T = \mathbf{u}P^{-1}.$$

Thus, for any matrix A, the matrix PAP^{-1} has a $p \times p$ block structure, with each block a $q \times q$ matrix. The components of the (i,j) block are $\{A_{i+(l-1)p,j+(m-1)p}\}_{l,m=1}^{q}$.

Let $\mathcal{D}_q^{\mathbf{u}_i} = \text{diag}\left(u^{(i)}, u^{(i+p)}, \ldots, u^{(i+p(q-1))}\right)$, a $q \times q$ diagonal matrix. Then

$$P\mathcal{D}_N^u P^{-1} = \text{diag}\left(\mathcal{D}_q^{\mathbf{u}_0}, \ldots, \mathcal{D}_q^{\mathbf{u}_{p-1}}\right),$$

and $\omega_p = P\Omega_N P^{-1}$ has a $p \times p$ block structure, with $(\omega_p)_{i,i+1} = I_q$, $i = 1, \ldots, p-1$ and $(\omega_p)_{p,1} = \Omega_q$. We then have

$$\omega_p^p = \text{diag}\left(\Omega_q, \ldots, \Omega_q\right), \quad \text{so} \;\; \omega_p^{\ell-k} = \omega_p^{pr} = \text{diag}\left(\Omega_q^r, \ldots, \Omega_q^r\right).$$

We can piece these formulae together in

Proposition 2.9 (Block Structure when $(N, \ell-k) = p \neq 1$)**.** *Let the permutation matrix P be defined as above and L be given by (2.17). Then*

$$PLP^{-1} = \text{diag}\left(L^{(0)}, \ldots, L^{(p-1)}\right)\omega_p^k, \tag{2.18}$$

where

$$L^{(i)} = \mathcal{D}_q^{\mathbf{u}_i} + \lambda\Omega_q^r \quad \textit{and} \quad N = pq, \;\; \ell - k = pr.$$

We then have

$$\det(L) = (-1)^{(N-1)k}\prod_{i=0}^{p-1}\left(a_i - (-\lambda)^q\right), \quad \textit{where} \;\; a_i = \prod_{j=0}^{q-1} u^{(i+jp)}.$$

Note that this determinant follows from that of each (coprime) block, with

$$\det(L^{(i)}) = a_i - (-\lambda)^q \quad \text{and} \quad \det(\omega_p) = \det(\Omega_N) = (-1)^{N-1}.$$

2.3.1 The Lax Pair when $(N, \ell_i - k_i) = p$

For the Lax pair (2.1), we have seen that k_i, ℓ_i must satisfy (2.4). As a consequence, $(N, \ell_1 - k_1) = (N, \ell_2 - k_2) = p$. Suppose $p \neq 1$. Then the *same* permutation matrix P transforms *both* L and M to the form described in Proposition 2.9, with the *same* p, q, r, but k_1 and k_2 may be distinct. Therefore the determinants have the *same* structure:

$$\det(L_{m,n}) = (-1)^{(N-1)k_1}\prod_{i=0}^{p-1}\left(a_i - (-\lambda)^q\right), \quad \text{where} \;\; a_i = \prod_{j=0}^{q-1} u_{m,n}^{(i+jp)}, \tag{2.19a}$$

$$\det(M_{m,n}) = (-1)^{(N-1)k_2}\prod_{i=0}^{p-1}\left(b_i - (-\lambda)^q\right) \quad \text{where} \;\; b_i = \prod_{j=0}^{q-1} v_{m,n}^{(i+jp)}. \tag{2.19b}$$

where

$$b_i - (-\lambda)^q = |M^{(i)}| = |\mathcal{D}_q^{\mathbf{v}_i} + \lambda\Omega_q^r|.$$

The formula (2.11) then implies that *symmetric functions* of a_i and of b_i are constants of the motion.

In fact, if we use $\hat{a}_i$ to denote a_i when evaluated at $(m, n+1)$ and $\tilde{b}_i$ to denote b_i when evaluated at $(m+1, n)$, then we have

$$\hat{a}_i = a_{i+k_2} \quad \text{and} \quad \tilde{b}_i = b_{i+k_1},$$

where $i+k_1$ and $i+k_2$ are taken $\bmod p$. These are simple permutations on finite sets. Symmetric polynomials of their orbits are first integrals.

Remark 2.10. *When $p = 1$ we just have that*

$$a_0 = \prod_{j=0}^{N-1} u^{(j)} \quad \textit{and} \quad b_0 = \prod_{j=0}^{N-1} v^{(j)}$$

are constants in n and m respectively. This is easy to see by looking at the form of equations (2.8), since the product

$$\prod_{i=0}^{N-1} \left(\frac{u_{m,n}^{(i+\ell_2)} - v_{m,n}^{(i+\ell_1)}}{u_{m,n}^{(i+k_2)} - v_{m,n}^{(i+k_1)}} \right) = 1.$$

This is a consequence of equation (2.4), which implies that $\ell_2 - \ell_1 = k_2 - k_1$, so the product in the denominator is just a re-ordering of that in the numerator.

2.4 Classification Problem

Summarising the above analysis, we can formulate the classification problem of Lax pairs as follows.

For every dimension N, find all the equivalence classes in the quotient space $\mathcal{Q}_N/\sim$. For the representatives of the classified equivalence classes, consider the corresponding Lax pairs (2.1) and analyse the resulting systems (2.7) depending on whether or not N and $\ell_i - k_i$, $i = 1, 2$, are coprime.

1. *The coprime case:* $\big(N, \ell_1 - k_1\big) = \big(N, \ell_2 - k_2\big) = 1$.
   This involves Lax pairs which satisfy

   $$\prod_{j=0}^{N-1} u_{m,n}^{(j)} = a, \quad \prod_{j=0}^{N-1} v_{m,n}^{(j)} = b, \quad a, b \in \mathbb{C}. \tag{2.20}$$

   The above relations allow us to express one function from each set in terms of the remaining ones. The coprime case is further subdivided into:

   - *The generic subcase :* $ab \neq 0$,
   - *The degenerate subcase :* $a \neq 0$, $b = 0$.

   Lax pairs with $a = 0$, $b \neq 0$ are equivalent to the above degenerate case by a change of independent variables. Finally, the fully degenerate case $a = b = 0$ is empty.

2. *The non-coprime case:* $\big(N, \ell_1 - k_1\big) = \big(N, \ell_2 - k_2\big) = p > 1$
   This case corresponds to the Lax pairs discussed in section (2.3.1), which allows us to reduce the number of functions in each set by p.

   The fundamental variables now form $q \times q$ blocks, which have various degrees of coupling/decoupling, depending upon the values of k_1 and k_2.

3 The Coprime Case

We analyse system (2.7) for the coprime case with $\big(N, \ell_i - k_i\big) = 1$.

For the *generic case*, with $ab \neq 0$, relations (2.20) imply that none of the functions u and v can be identically zero. This allows us to express functions u and v in terms of some potentials so that either equations (2.7a) or equations (2.7b) hold identically. Sections 3.1 and 3.2 analyse these two basic cases. We give a list of all inequivalent systems in 2 and 3 dimensions and extend some of these to N dimensions. For $N = 2$ we recover many of the standard scalar systems, including the discrete modified KdV and potential KdV equations, the discrete Schwarzian KdV (see [13] and references therein) and the Hirota's discrete sine-Gordon equation (see [8] and references therein). In Section 3.3 we discuss the Bäcklund relation between the two potential cases.

We finish this section by discussing the *degenerate case*, for which $a \neq 0, b = 0$. This means that at least one of the functions $v^{(i)}_{m,n}$ must be identically zero. The degenerate case is generally much more complex, so we cannot yet give a full classification. After some general calculations, we restrict ourselves to some examples, the simplest of which is the Hirota KdV equation. We generalise this to N dimensions and give a scalar reduction in each case.

3.1 Quotient Potentials

Equations (2.7a) hold identically if we set

$$u^{(i)}_{m,n} = \alpha \frac{\phi^{(i)}_{m+1,n}}{\phi^{(i+k_1)}_{m,n}}, \quad v^{(i)}_{m,n} = \beta \frac{\phi^{(i)}_{m,n+1}}{\phi^{(i+k_2)}_{m,n}}, \quad i \in \mathbb{Z}_N, \tag{3.1}$$

where, from (2.20), $a = \alpha^N$, $b = \beta^N$. Equations (2.7b) then take the form

$$\alpha \left(\frac{\phi^{(i)}_{m+1,n+1}}{\phi^{(i+k_1)}_{m,n+1}} - \frac{\phi^{(i+\ell_2)}_{m+1,n}}{\phi^{(i+\ell_2+k_1)}_{m,n}} \right) = \beta \left(\frac{\phi^{(i)}_{m+1,n+1}}{\phi^{(i+k_2)}_{m+1,n}} - \frac{\phi^{(i+\ell_1)}_{m,n+1}}{\phi^{(i+\ell_1+k_2)}_{m,n}} \right), \quad i \in \mathbb{Z}_N, \tag{3.2}$$

and their solved form (2.8) is written as

$$\phi^{(i)}_{m+1,n+1} = \frac{\phi^{(i+k_1)}_{m,n+1}\phi^{(i+k_2)}_{m+1,n}}{\phi^{(i+k_1+\ell_2)}_{m,n}} \left(\frac{\alpha\, \phi^{(i+\ell_2)}_{m+1,n} - \beta\, \phi^{(i+\ell_1)}_{m,n+1}}{\alpha\, \phi^{(i+k_2)}_{m+1,n} - \beta\, \phi^{(i+k_1)}_{m,n+1}} \right), \quad i \in \mathbb{Z}_N. \tag{3.3}$$

In this potential form, the Lax pair (2.1) can be written

$$\begin{aligned} \Psi_{m+1,n} &= \big(\alpha \boldsymbol{\phi}_{m+1,n} \Omega^{k_1} \boldsymbol{\phi}^{-1}_{m,n} + \lambda \Omega^{\ell_1}\big) \Psi_{m,n}, \\ \Psi_{m,n+1} &= \big(\beta \boldsymbol{\phi}_{m,n+1} \Omega^{k_2} \boldsymbol{\phi}^{-1}_{m,n} + \lambda \Omega^{\ell_2}\big) \Psi_{m,n}, \end{aligned} \tag{3.4a}$$

where

$$\boldsymbol{\phi}_{m,n} := \operatorname{diag}\Big(\phi^{(0)}_{m,n}, \ldots, \phi^{(N-1)}_{m,n}\Big) \quad \text{and} \quad \det\big(\boldsymbol{\phi}_{m,n}\big) = \prod_{i=0}^{N-1} \phi^{(i)}_{m,n} = 1. \tag{3.4b}$$

We can then show that the Lax pair (3.4) is compatible if and only if the system (3.2) holds.

Using the equivalence relation $\sim$ we can determine the inequivalent systems in every dimension N. However, in the quotient potential case we have a further equivalence relation at our disposal:

Proposition 3.1. *We denote system (3.2) by $\mathcal{R}(\phi^{(i)}; k_1, \ell_1, \alpha; k_2, \ell_2, \beta)$. Consider also the system which follows from (3.2) by interchanging indices k_i and ℓ_i and replacing $(\phi^{(i)}, \alpha, \beta)$ with $(\widetilde{\phi}^{(i)}, \tilde{\alpha}, \tilde{\beta})$, i.e. system $\mathcal{R}(\widetilde{\phi}^{(i)}; \ell_1, k_1, \tilde{\alpha}; \ell_2, k_2, \tilde{\beta})$. Then,*

1. *Solutions of systems $\mathcal{R}(\phi^{(i)}; k_1, \ell_1, \alpha; k_2, \ell_2, \beta)$ and $\mathcal{R}(\widetilde{\phi}^{(i)}; \ell_1, k_1, \tilde{\alpha}; \ell_2, k_2, \tilde{\beta})$ are related by the point transformation*

$$\mathcal{I} \,:\, \left\{ \phi^{(i)}_{m,n} \widetilde{\phi}^{(i)}_{m,n} \,=\, 1\,, \quad \alpha\,\tilde{\alpha} \,=\, 1, \quad \beta\,\tilde{\beta} \,=\, 1 \right\}.$$

2. *The Lax pairs (3.4) for systems $\mathcal{R}(\phi^{(i)}; k_1, \ell_1, \alpha; k_2, \ell_2, \beta)$ and $\mathcal{R}(\widetilde{\phi}^{(i)}; \ell_1, k_1, \tilde{\alpha}; \ell_2, k_2, \tilde{\beta})$ are related by the gauge transformation $\widetilde{\Psi}_{m,n} = \alpha^{-m}\beta^{-n}\lambda^{-m-n}\boldsymbol{\phi}^{-1}_{m,n}\Psi_{m,n}$, along with the above point transformation and the inversion $\lambda \mapsto \lambda^{-1}$.*

Thus, two Lax pairs of the form (3.4) with level structures $(k_1, \ell_1; k_2, \ell_2)$ and $(\ell_1, k_1; \ell_2, k_2)$ can be considered equivalent as they are related by a point transformation. Taking into consideration this observation, we present the inequivalent systems in two and three dimensions. In particular, in the two-dimensional case there exist only two classes, and we find only four inequivalent systems when $N = 3$.

3.1.1 Integrable Systems for $N = 2$

When $N = 2$, there exist only two inequivalent classes of Lax pairs (3.4) and corresponding equations.

Equivalence class $[(0, 1; 0, 1)]$

$$\alpha\,\left(\phi_{m,n}\phi_{m,n+1} - \phi_{m+1,n}\phi_{m+1,n+1}\right) \,-\, \beta\,\left(\phi_{m,n}\phi_{m+1,n} - \phi_{m,n+1}\phi_{m+1,n+1}\right) \,=\, 0, \tag{3.5}$$

where $\phi_{m,n} = \phi^{(0)}_{m,n} = 1/\phi^{(1)}_{m,n}$. This equation is known as the discrete modified KdV equation, or H3 with $\delta = 0$ (see [2, 13]).

Equivalence class $[(0, 1; 1, 0)]$

$$\alpha\,\left(\phi_{m,n}\phi_{m+1,n+1} - \phi_{m+1,n}\phi_{m,n+1}\right) \,-\, \beta\,\left(\phi_{m,n}\phi_{m+1,n}\phi_{m,n+1}\phi_{m+1,n+1} - 1\right) \,=\, 0, \tag{3.6}$$

where $\phi_{m,n} = \phi^{(0)}_{m,n} = 1/\phi^{(1)}_{m,n}$. This is Hirota's discrete sine-Gordon equation, see [8] and references therein.

3.1.2 Integrable Systems for $N = 3$

There are only four inequivalent systems when $N = 3$. We write the corresponding systems (3.2) with only 2 components, by using the substitution

$$\left(\phi^{(0)}_{m,n}\,,\,\phi^{(1)}_{m,n}\,,\,\phi^{(2)}_{m,n}\right) \,\mapsto\, \left(\frac{1}{\phi^{(0)}_{m,n}}\,,\,\phi^{(1)}_{m,n}\,,\,\frac{\phi^{(0)}_{m,n}}{\phi^{(1)}_{m,n}}\right), \tag{3.7}$$

which incorporates the constraint $\phi^{(0)}_{m,n}\phi^{(1)}_{m,n}\phi^{(2)}_{m,n} = 1$.

Equivalence class $[(0,1;0,1)]$

$$\phi^{(0)}_{m+1,n+1} = \left(\frac{\alpha\,\phi^{(0)}_{m,n+1} - \beta\,\phi^{(0)}_{m+1,n}}{\alpha\,\phi^{(1)}_{m+1,n} - \beta\,\phi^{(1)}_{m,n+1}}\right)\phi^{(1)}_{m,n}\,, \tag{3.8a}$$

$$\phi^{(1)}_{m+1,n+1} = \left(\frac{\alpha\,\phi^{(0)}_{m+1,n}\phi^{(1)}_{m,n+1} - \beta\,\phi^{(0)}_{m,n+1}\phi^{(1)}_{m+1,n}}{\alpha\,\phi^{(1)}_{m+1,n} - \beta\,\phi^{(1)}_{m,n+1}}\right)\frac{\phi^{(1)}_{m,n}}{\phi^{(0)}_{m,n}}\,. \tag{3.8b}$$

This is the three dimensional analog of equation (3.5) and it is a well known example of a two-component discrete integrable system. It can be decoupled for either of the functions involved in it to a nine-point scalar equation known as the modified Boussinesq equation [14]. The hierarchies of its symmetries and conservation laws were studied in [19].

Equivalence class $[(0,1;1,2)]$

$$\phi^{(0)}_{m+1,n+1} = \left(\frac{\alpha\,\phi^{(0)}_{m,n+1}\phi^{(1)}_{m+1,n} - \beta}{\alpha\,\phi^{(0)}_{m+1,n} - \beta\,\phi^{(1)}_{m+1,n}\phi^{(1)}_{m,n+1}}\right)\frac{\phi^{(0)}_{m,n}}{\phi^{(1)}_{m,n}}\,, \tag{3.9a}$$

$$\phi^{(1)}_{m+1,n+1} = \left(\frac{\alpha\,\phi^{(1)}_{m,n+1} - \beta\,\phi^{(0)}_{m+1,n}\phi^{(0)}_{m,n+1}}{\alpha\,\phi^{(0)}_{m+1,n} - \beta\,\phi^{(1)}_{m+1,n}\phi^{(1)}_{m,n+1}}\right)\phi^{(0)}_{m,n}\,. \tag{3.9b}$$

This is a new integrable system (but see also [11]) which cannot be decoupled to a (local) scalar equation on a bigger stencil for any of the variables.

Equivalence class $[(0,1;2,0)]$

$$\phi^{(0)}_{m+1,n+1} = \left(\frac{\alpha\,\phi^{(0)}_{m+1,n}\phi^{(0)}_{m,n+1} - \beta\phi^{(1)}_{m+1,n}}{\alpha - \beta\,\phi^{(0)}_{m+1,n}\phi^{(1)}_{m,n+1}}\right)\frac{1}{\phi^{(0)}_{m,n}}\,, \tag{3.10a}$$

$$\phi^{(1)}_{m+1,n+1} = \left(\frac{\alpha\,\phi^{(1)}_{m+1,n}\phi^{(1)}_{m,n+1} - \beta\phi^{(0)}_{m,n+1}}{\alpha - \beta\,\phi^{(0)}_{m+1,n}\phi^{(1)}_{m,n+1}}\right)\frac{1}{\phi^{(1)}_{m,n}}\,. \tag{3.10b}$$

This is another new integrable system (but see also [11]) which occurs only in three dimensions. Again, this system cannot be decoupled to a scalar equation on a bigger stencil for either $\phi^{(0)}$ or $\phi^{(1)}$.

Equivalence class $[(1,2;1,2)]$

$$\phi^{(0)}_{m+1,n+1} = \left(\frac{\alpha\,\phi^{(1)}_{m+1,n} - \beta\phi^{(1)}_{m,n+1}}{\alpha\,\phi^{(0)}_{m+1,n}\phi^{(1)}_{m,n+1} - \beta\,\phi^{(0)}_{m,n+1}\phi^{(1)}_{m+1,n}}\right)\frac{1}{\phi^{(0)}_{m,n}}\,, \tag{3.11a}$$

$$\phi^{(1)}_{m+1,n+1} = \left(\frac{\alpha\,\phi^{(0)}_{m,n+1} - \beta\phi^{(0)}_{m+1,n}}{\alpha\,\phi^{(0)}_{m+1,n}\phi^{(1)}_{m,n+1} - \beta\,\phi^{(0)}_{m,n+1}\phi^{(1)}_{m+1,n}}\right)\frac{1}{\phi^{(1)}_{m,n}}\,. \tag{3.11b}$$

This is a new integrable system which can be decoupled to a nine-point scalar equation for either of the functions involved in it. This scalar equation for $\phi^{(0)}$ can be written as

$$\left(\frac{\alpha\phi^{(0)}_{m+1,n+1} - \beta\phi^{(0)}_{m+2,n}}{\alpha\phi^{(0)}_{m,n+2} - \beta\phi^{(0)}_{m+1,n+1}}\right)\left(\frac{\alpha^2\phi^{(0)}_{m+1,n+1} - \beta^2\phi^{(0)}_{m,n+2}}{\alpha^2\phi^{(0)}_{m+2,n} - \beta^2\phi^{(0)}_{m+1,n+1}}\right) = \frac{\mathcal{M}_{m,n}\mathcal{M}_{m,n+1}\mathcal{M}_{m+1,n+1}}{\mathcal{N}_{m,n}\mathcal{N}_{m+1,n}\mathcal{N}_{m+1,n+1}}\,, \tag{3.12a}$$

where

$$\mathcal{M}_{m,n} := \alpha\phi^{(0)}_{m,n}\phi^{(0)}_{m+1,n}\phi^{(0)}_{m+1,n+1} + \beta, \quad \mathcal{N}_{m,n} := \beta\phi^{(0)}_{m,n}\phi^{(0)}_{m,n+1}\phi^{(0)}_{m+1,n+1} + \alpha\,, \tag{3.12b}$$

while the corresponding equation for $\phi^{(1)}$ follows from (3.12) by replacing $\phi^{(0)}_{m,n}$ with $1/\phi^{(1)}_{m,n}$.

There is an interesting reduction of system (3.11) to the integrable equation [12]

$$u_{m,n}u_{m+1,n+1}\left(u_{m+1,n} + u_{m,n+1}\right) + 1 = 0. \tag{3.13}$$

Specifically, system (3.11) reduces to equation (3.13) by setting

$$\phi^{(0)}_{m,n} \;=\; \phi^{(1)}_{m,n} \;=\; \frac{-1}{2^{1/3}u_{m,n}}, \quad \beta \;=\; -\alpha. \tag{3.14}$$

In particular, in view of the above reduction and setting for convenience $\alpha = 2^{1/3}$, the Lax pair for system (3.11) becomes

$$\Psi_{m+1,n} \;=\; \begin{pmatrix} 0 & \frac{1}{u_{m+1,n}} & \lambda \\ \lambda & 0 & \frac{1}{u_{m,n}} \\ -2u_{m,n}u_{m+1,n} & \lambda & 0 \end{pmatrix} \Psi_{m,n},$$

$$\Psi_{m,n+1} \;=\; \begin{pmatrix} 0 & \frac{-1}{u_{m,n+1}} & \lambda \\ \lambda & 0 & \frac{-1}{u_{m,n}} \\ 2u_{m,n}u_{m,n+1} & \lambda & 0 \end{pmatrix} \Psi_{m,n},$$

providing us with another Lax pair for equation (3.13), compared with the one in [12].

Remark 3.2 (General N)**.** *Apart from the exceptional case (3.10), each of these examples can be extended to general N. In particular, the equivalence class $[(0,1;0,1)]$ is extended to*

$$\phi^{(i)}_{m+1,n+1} \;=\; \frac{\phi^{(i)}_{m,n+1}\phi^{(i)}_{m+1,n}}{\phi^{(i+1)}_{m,n}}\left(\frac{\alpha\phi^{(i+1)}_{m,n+1} - \beta\phi^{(i+1)}_{m+1,n}}{\alpha\phi^{(i)}_{m,n+1} - \beta\phi^{(i)}_{m+1,n}}\right), \quad i = 0,\ldots,N-3, \tag{3.15a}$$

$$\phi^{(N-2)}_{m+1,n+1} \;=\; \left(\frac{\phi^{(N-2)}_{m,n+1}\phi^{(N-2)}_{m+1,n}}{\alpha\phi^{(N-2)}_{m,n+1} - \beta\phi^{(N-2)}_{m+1,n}}\right) H_{m,n}\left(\frac{\alpha}{H_{m,n+1}} - \frac{\beta}{H_{m+1,n}}\right), \tag{3.15b}$$

where $H_{m,n} := \prod_{j=0}^{N-2}\phi^{(j)}_{m,n}$, by using constraint (3.4b) to replace $\phi^{(N-1)}_{m,n}$ in terms of the remaining potentials.

This is equivalent to the $N-$component nonlinear superposition formula for the two dimensional Toda lattice, given in [9] (see also [4]).

3.1.3 The Self-Dual Case

In the case when $(k_2,\ell_2) = (k_1,\ell_1) = (k,\ell)$, we may combine the two equivalence relations, $\mathcal{T}_2$ of Section 2.2.2 and that of Proposition 3.1. The action on the parameters (k,ℓ) is given by

$$(k,\ell) \mapsto (N-k, N-\ell) \mapsto (N-\ell, N-k) = (\bar{k},\bar{\ell}).$$

Since $(k,\ell) \mapsto (\bar{k},\bar{\ell})$ is an involution, we refer to such systems as *dual*. The *self-dual* case is when $(\bar{k},\bar{\ell}) = (k,\ell)$, giving $k+\ell = N$. In particular, we consider the case with

$$k+\ell = N, \quad \ell - k = 1 \quad \Rightarrow \quad k = \frac{N-1}{2}, \quad \ell = \frac{N+1}{2}, \tag{3.16}$$

so we require that N is odd.

Combining the formula (2.15), with $p = 2k$, and Proposition 3.1, we have $\widetilde{\phi}^{(i)}_{m,n}\phi^{(2k-1-i)}_{m,n} = 1$. The self-dual case admits the reduction $\widetilde{\phi}^{(i)}_{m,n} = \phi^{(i)}_{m,n}$, when $\alpha = -\beta$ ($= 1$, without loss of generality), which we write as

$$\phi^{(i+k)}_{m,n}\phi^{(k-1-i)}_{m,n} = 1, \quad i = 0, \dots, k-1.$$

The condition $\prod_{i=0}^{N-1} \phi^{(i)}_{m,n} = 1$ then implies $\phi^{(N-1)}_{m,n} = 1$. Therefore the matrices $U_{m,n}$ and $V_{m,n}$ are built from k components:

$$\begin{aligned} U_{m,n} \;=\; \operatorname{diag}\Bigg(& \phi^{(0)}_{m+1,n}\phi^{(k-1)}_{m,n}, \dots, \phi^{(k-1)}_{m+1,n}\phi^{(0)}_{m,n}, \frac{1}{\phi^{(k-1)}_{m+1,n}}, \frac{1}{\phi^{(0)}_{m,n}\phi^{(k-2)}_{m+1,n}}, \dots \\ & \dots, \frac{1}{\phi^{(k-2)}_{m,n}\phi^{(0)}_{m+1,n}}, \frac{1}{\phi^{(k-1)}_{m,n}}\Bigg)\Omega^k, \end{aligned}$$

with $V_{m,n}$ given by the same formula, but with $(m+1, n)$ replaced by $(m, n+1)$. In this case the system (3.3) reduces to

$$\phi^{(i)}_{m+1,n+1}\phi^{(i)}_{m,n} = \frac{1}{\phi^{(k-i-2)}_{m+1,n}\phi^{(k-i-2)}_{m,n+1}}\left(\frac{\phi^{(k-i-2)}_{m+1,n} + \phi^{(k-i-2)}_{m,n+1}}{\phi^{(k-i-1)}_{m+1,n} + \phi^{(k-i-1)}_{m,n+1}}\right), \quad \text{for} \quad i = 0, 1, \dots, k-1. \tag{3.17}$$

Remark 3.3. *This reduction has $\frac{N-1}{2}$ components and is represented by an $N \times N$ Lax pair, but is not 3D consistent.*

When $N = 3$, we just recover the reduction (3.13).

Example 3.4 (The case $N = 5$ with $(k, \ell) = (2, 3)$)**.** Here we have two components $\phi^{(0)}_{m,n}, \phi^{(1)}_{m,n}$, with $\phi^{(2)}_{m,n} = \frac{1}{\phi^{(1)}_{m,n}}$, $\phi^{(3)}_{m,n} = \frac{1}{\phi^{(0)}_{m,n}}$, $\phi^{(4)}_{m,n} = 1$, and

$$\phi^{(0)}_{m+1,n+1}\phi^{(0)}_{m,n} = \frac{1}{\phi^{(0)}_{m+1,n}\phi^{(0)}_{m,n+1}}\left(\frac{\phi^{(0)}_{m+1,n} + \phi^{(0)}_{m,n+1}}{\phi^{(1)}_{m+1,n} + \phi^{(1)}_{m,n+1}}\right),$$

$$\phi^{(1)}_{m+1,n+1}\phi^{(1)}_{m,n}\,(\phi^{(0)}_{m+1,n} + \phi^{(0)}_{m,n+1}) = 2.$$

3.2 Additive Potentials

Equations (2.7b) hold identically if we set

$$u^{(i)}_{m,n} = \chi^{(i)}_{m+1,n} - \chi^{(i+\ell_1)}_{m,n}, \quad v^{(i)}_{m,n} = \chi^{(i)}_{m,n+1} - \chi^{(i+\ell_2)}_{m,n}, \quad i \in \mathbb{Z}_N. \tag{3.18}$$

Equations (2.7a) then take the form

$$\frac{\left(\chi^{(i)}_{m+1,n+1} - \chi^{(i+\ell_1)}_{m,n+1}\right)}{\left(\chi^{(i+k_2)}_{m+1,n} - \chi^{(i+k_2+\ell_1)}_{m,n}\right)} = \frac{\left(\chi^{(i)}_{m+1,n+1} - \chi^{(i+\ell_2)}_{m+1,n}\right)}{\left(\chi^{(i+k_1)}_{m,n+1} - \chi^{(i+k_1+\ell_2)}_{m,n}\right)}, \tag{3.19}$$

for $i \in \mathbb{Z}_N$, and their solved form (2.8) is written as

$$\chi^{(i)}_{m+1,n+1} = \frac{\chi^{(i+k_1)}_{m,n+1}\chi^{(i+\ell_1)}_{m,n+1} - \chi^{(i+k_2)}_{m+1,n}\chi^{(i+\ell_2)}_{m+1,n} - \chi^{(i+k_1+\ell_2)}_{m,n}(\chi^{(i+\ell_1)}_{m,n+1} - \chi^{(i+\ell_2)}_{m+1,n})}{\chi^{(i+k_1)}_{m,n+1} - \chi^{(i+k_2)}_{m+1,n}}. \tag{3.20}$$

In this potential form, the Lax pair (2.1) can be written

$$\begin{aligned}\Psi_{m+1,n} &= \Big(\left(\boldsymbol{\chi}_{m+1,n} - \Omega^{\ell_1}\boldsymbol{\chi}_{m,n}\Omega^{-\ell_1}\right)\Omega^{k_1} + \lambda\Omega^{\ell_1}\Big)\Psi_{m,n},\\ \Psi_{m,n+1} &= \Big(\left(\boldsymbol{\chi}_{m,n+1} - \Omega^{\ell_2}\boldsymbol{\chi}_{m,n}\Omega^{-\ell_2}\right)\Omega^{k_2} + \lambda\Omega^{\ell_2}\Big)\Psi_{m,n},\end{aligned} \tag{3.21}$$

where

$$\boldsymbol{\chi}_{m,n} := \operatorname{diag}\left(\chi^{(0)}_{m,n}, \dots, \chi^{(N-1)}_{m,n}\right).$$

We can then show that the Lax pair (3.21) is compatible if and only if the system (3.19) holds.

In this case, conditions (2.20) become the first integrals

$$\prod_{i=0}^{N-1}\left(\chi^{(i)}_{m+1,n} - \chi^{(i+\ell_1)}_{m,n}\right) = \alpha^N, \quad \prod_{i=0}^{N-1}\left(\chi^{(i)}_{m,n+1} - \chi^{(i+\ell_2)}_{m,n}\right) = \beta^N, \tag{3.22}$$

where we have set $a = \alpha^N$, $b = \beta^N$. Hence it is not always possible to reduce the number of potentials χ (in *local* terms) by employing these.

Remark 3.5. *There do exist cases where we are able to reduce the number of potentials in system (3.19) but not in Lax pair (3.21). In those cases, one may derive another local Lax pair for the reduced system which will not belong to the class we consider in this paper.*

3.2.1 Integrable Systems for $N = 2$

Here there are three inequivalent integrable systems.

Equivalence class $[(0,1;0,1)]$

Using constraints (3.22), we can replace either $\chi^{(0)}$ or $\chi^{(1)}$, to obtain

$$\left(\chi_{m+1,n+1} - \chi_{m,n}\right)\left(\chi_{m+1,n} - \chi_{m,n+1}\right) = \alpha^2 - \beta^2, \tag{3.23}$$

which is the discrete potential KdV or H1 equation [2, 13].

Equivalence class $[(0,1;1,0)]$

In this case, the corresponding system cannot be decoupled. It is omitted here as it follows from (3.19) by setting $k_1 = \ell_2 = 0$ and $\ell_1 = k_2 = 1$. In the same way, its first integrals

$$\left(\chi^{(0)}_{m+1,n} - \chi^{(1)}_{m,n}\right)\left(\chi^{(1)}_{m+1,n} - \chi^{(0)}_{m,n}\right) = \alpha^2, \quad \left(\chi^{(0)}_{m,n+1} - \chi^{(0)}_{m,n}\right)\left(\chi^{(1)}_{m,n+1} - \chi^{(1)}_{m,n}\right) = \beta^2,$$

follow from (3.22).

Equivalence class $[(1,0;1,0)]$
This is another case where the first integrals can be used to eliminate one of the two variables. This leads to the Schwarzian KdV equation [13] (ie Q1 of [2], with $\delta = 0$):

$$\alpha^2\left(\chi_{m,n} - \chi_{m,n+1}\right)\left(\chi_{m+1,n} - \chi_{m+1,n+1}\right) - \beta^2\left(\chi_{m,n} - \chi_{m+1,n}\right)\left(\chi_{m,n+1} - \chi_{m+1,n+1}\right) = 0. \tag{3.24}$$

3.2.2 Integrable Systems for $N = 3$

In the three-dimensional case $N = 3$ there exist six inequivalent classes which can be divided into two categories.

1. The first category contains two equivalence classes, namely $[(0,1;0,1)]$ and $[(1,0;1,0)]$, for which we can reduce the number of potentials by employing the corresponding first integrals. More precisely, using the first integrals (3.22), one can choose to eliminate potential $\chi^{(0)}_{m,n}$, and derive a two-component system for the remaining two potentials, denoted here by $P(\chi^{(1)}_{m,n}, \chi^{(2)}_{m,n})$. In fact, working in the same way, the elimination of $\chi^{(1)}_{m,n}$ yields system $P(\chi^{(2)}_{m,n}, \chi^{(0)}_{m,n})$, whereas the elimination of $\chi^{(2)}_{m,n}$ results to system $P(\chi^{(0)}_{m,n}, \chi^{(1)}_{m,n})$. In this sense, the first integrals may be regarded as a periodic map of the corresponding two-component systems $P(\chi^{(j)}_{m,n}, \chi^{(j+1)}_{m,n})$, $j \in \mathbb{Z}_3$. Moreover, for both classes, the corresponding two-component systems can be decoupled further to nine-point scalar equations, and, we may interpret the two-component systems as maps of the corresponding scalar equations.

2. The second category contains classes $[(0,1;1,2)]$, $[(0,1;2,0)]$, $[(1,2;1,2)]$ and $[(1,2;2,0)]$. The common characteristic of these classes is that the corresponding systems involve three potentials, the number of which cannot be reduced using the first integrals (3.22). To the best of our knowledge, these systems are new. Also, the equivalence classes $[(0,1;2,0)]$ and $[(1,2;2,0)]$, as well as the corresponding discrete systems, exist only in three dimensions, as a consequence of (2.4).

Equivalence class $[(0,1;0,1)]$

Consider the discrete system (3.19) and first integrals (3.22) with $(k_1, \ell_1; k_2, \ell_2) = (0,1;0,1)$. We can employ the first integrals in order to eliminate one of the potentials from the discrete system and derive a two-component system for the remaining potentials. Following the above discussion, we eliminate $\chi^{(2)}$ to obtain a system for $(\chi^{(0)}_{m,n}, \chi^{(1)}_{m,n})$:

$$\begin{aligned}
\chi^{(0)}_{m+1,n+1} &= \frac{(\chi^{(0)}_{m+1,n} - \chi^{(1)}_{m,n})\chi^{(1)}_{m+1,n} - (\chi^{(0)}_{m,n+1} - \chi^{(1)}_{m,n})\chi^{(1)}_{m,n+1}}{\chi^{(0)}_{m+1,n} - \chi^{(0)}_{m,n+1}}, \\
\chi^{(1)}_{m+1,n+1} &= \chi^{(0)}_{m,n} + \frac{1}{\chi^{(1)}_{m+1,n} - \chi^{(1)}_{m,n+1}} \left(\frac{\alpha^3}{\chi^{(0)}_{m+1,n} - \chi^{(1)}_{m,n}} - \frac{\beta^3}{\chi^{(0)}_{m,n+1} - \chi^{(1)}_{m,n}} \right),
\end{aligned} \tag{3.25}$$

which is a new integrable system.

On the other hand, using the first integrals we are not able to eliminate the potential $\chi^{(2)}_{m,n}$ from Lax pair (3.21). However, employing the three-dimensional consistency of system (3.25), we are able to construct the following Lax pair for system (3.25), which does not belong in the class of Lax pairs considered in this paper:

$$\begin{aligned}
\Psi_{m+1,n} &= \begin{pmatrix} \chi^{(1)}_{m+1,n} - \chi^{(0)}_{m,n} & 0 & -1 \\ -1 & \chi^{(0)}_{m+1,n} - \chi^{(1)}_{m,n} & 0 \\ F_{m,n} & \lambda & \chi^{(0)}_{m+1,n} - \chi^{(0)}_{m,n} \end{pmatrix} \Psi_{m,n}\,, \\
\Psi_{m,n+1} &= \begin{pmatrix} \chi^{(1)}_{m,n+1} - \chi^{(0)}_{m,n} & 0 & -1 \\ -1 & \chi^{(0)}_{m,n+1} - \chi^{(1)}_{m,n} & 0 \\ G_{m,n} & \lambda & \chi^{(0)}_{m,n+1} - \chi^{(0)}_{m,n} \end{pmatrix} \Psi_{m,n},
\end{aligned}$$

where

$$F_{m,n} := \frac{\alpha^3}{\chi^{(0)}_{m+1,n} - \chi^{(1)}_{m,n}} - (\chi^{(0)}_{m+1,n} - \chi^{(0)}_{m,n})(\chi^{(1)}_{m+1,n} - \chi^{(0)}_{m,n}),$$
$$G_{m,n} := \frac{\beta^3}{\chi^{(0)}_{m,n+1} - \chi^{(1)}_{m,n}} - (\chi^{(0)}_{m,n+1} - \chi^{(0)}_{m,n})(\chi^{(1)}_{m,n+1} - \chi^{(0)}_{m,n}).$$

Remark 3.6 (Reduction to the discrete Boussinesq equation)**.** *This system can be decoupled for either of the variables to the nine point scalar equation known as discrete Boussinesq equation [14].*

Equivalence class $[(1,0;1,0)]$

Working as with the previous equivalence class, we can derive a two-component system for any pair of potentials corresponding to the discrete system (3.19) and its first integrals (3.22) with $N = 3$ and $(k_1, \ell_1; k_2, \ell_2) = (1, 0; 1, 0)$. The system can be written in the following form.

$$\begin{aligned}
\chi^{(0)}_{m+1,n+1} &= \frac{\chi^{(0)}_{m+1,n}\Delta_m(\chi^{(1)}_{m,n}) - \chi^{(0)}_{m,n+1}\Delta_n(\chi^{(1)}_{m,n})}{\chi^{(1)}_{m+1,n} - \chi^{(1)}_{m,n+1}},\\
\chi^{(1)}_{m+1,n+1} &= \frac{\alpha^3\chi^{(1)}_{m+1,n}\,\Delta_n(\chi^{(0)}_{m,n})\,\Delta_n(\chi^{(1)}_{m,n}) - \beta^3\chi^{(1)}_{m,n+1}\,\Delta_m(\chi^{(0)}_{m,n})\,\Delta_m(\chi^{(1)}_{m,n})}{\alpha^3\,\Delta_n(\chi^{(0)}_{m,n})\,\Delta_n(\chi^{(1)}_{m,n}) - \beta^3\,\Delta_m(\chi^{(0)}_{m,n})\,\Delta_m(\chi^{(1)}_{m,n})}.
\end{aligned} \tag{3.26}$$

A Lax pair for system (3.26) is of the form (3.21), with $\chi^{(2)}$ replaced as discussed above. The hierarchies of its symmetries and conservation laws were studied in [20].

3.2.3 Integrable Systems for General N

The two equivalence classes we discussed above can be defined in any dimension N and we can always eliminate one of the potentials from the discrete systems using the first integrals.

Equivalence class $[(0,1;0,1)]$

The following $(N-1)$-component system is related to this equivalence class.

$$\begin{aligned}
\chi^{(i)}_{m+1,n+1} &= \frac{(\chi^{(i)}_{m,n+1} - \chi^{(i+1)}_{m,n})\chi^{(i+1)}_{m,n+1} - (\chi^{(i)}_{m+1,n} - \chi^{(i+1)}_{m,n})\chi^{(i+1)}_{m+1,n}}{\chi^{(i)}_{m,n+1} - \chi^{(i)}_{m+1,n}}, \quad i = 0,\ldots,N-3,\\
\chi^{(N-2)}_{m+1,n+1} &= \chi^{(0)}_{m,n} + \frac{1}{\chi^{(N-2)}_{m+1,n} - \chi^{(N-2)}_{m,n+1}}\left(\frac{\alpha^N}{X} - \frac{\beta^N}{Y}\right).
\end{aligned} \tag{3.27}$$

where $X = \prod_{j=0}^{N-3}(\chi^{(j)}_{m+1,n} - \chi^{(j+1)}_{m,n})$ and $Y = \prod_{j=0}^{N-3}(\chi^{(j)}_{m,n+1} - \chi^{(j+1)}_{m,n})$.

It can be derived from the compatibility condition of the Lax pair

$$\Psi_{m+1,n} = \left(\boldsymbol{\alpha}_{m,n} - \Omega^{N-1} + J_{m,n}\right)\Psi_{m,n},$$
$$\Psi_{m,n+1} = \left(\boldsymbol{\beta}_{m,n} - \Omega^{N-1} + K_{m,n}\right)\Psi_{m,n},$$

where $\boldsymbol{\alpha}_{m,n}$ and $\boldsymbol{\beta}_{m,n}$ are $N \times N$ diagonal matrices with entries

$$(\boldsymbol{\alpha}_{m,n})_{i,i} = (1-\delta_{i,N})\left(\chi^{(N-i-1)}_{m+1,n} - \chi^{(N-i)}_{m,n}\right) + \delta_{i,N}\left(\chi^{(0)}_{m+1,n} - \chi^{(0)}_{m,n}\right),$$

$$(\boldsymbol{\beta}_{m,n})_{i,i} = (1-\delta_{i,N})\left(\chi^{(N-i-1)}_{m,n+1} - \chi^{(N-i)}_{m,n}\right) + \delta_{i,N}\left(\chi^{(0)}_{m,n+1} - \chi^{(0)}_{m,n}\right),$$

with all upper indices being considered $\mathrm{mod}(N-1)$, and

$$(J_{m,n})_{i,j} = \delta_{i,N}\delta_{j,1}X_{m,n} + \delta_{i,N}\delta_{j,N-1}(\lambda+1),$$

$$(K_{m,n})_{i,j} = \delta_{i,N}\delta_{j,1}Y_{m,n} + \delta_{i,N}\delta_{j,N-1}(\lambda+1),$$

where $X_{m,n}$ and $Y_{m,n}$ are determined by the requirement

$$\det\left(\boldsymbol{\alpha}_{m,n} - \Omega^{N-1} + J_{m,n}\right) = \lambda + \alpha^N, \qquad \det\left(\boldsymbol{\beta}_{m,n} - \Omega^{N-1} + K_{m,n}\right) = \lambda + \beta^N.$$

Equivalence class $[(1,0;1,0)]$

With $(k_1,\ell_1;k_2,\ell_2) = (1,0;1,0)$, one may eliminate potential $\chi^{(N-1)}_{m,n}$ from system (3.19), using first integrals (3.22), to derive

$$\chi^{(i)}_{m+1,n+1} = \frac{\chi^{(i)}_{m+1,n}(\chi^{(i+1)}_{m+1,n} - \chi^{(i+1)}_{m,n}) - \chi^{(i)}_{m,n+1}(\chi^{(i+1)}_{m,n+1} - \chi^{(i+1)}_{m,n})}{\chi^{(i+1)}_{m+1,n} - \chi^{(i+1)}_{m,n+1}}, \quad i = 0,\dots,N-3,$$

$$\chi^{(N-2)}_{m+1,n+1} = \frac{\alpha^N \chi^{(N-2)}_{m+1,n} A_{m,n} - \beta^N \chi^{(N-2)}_{m,n+1} B_{m,n}}{\alpha^N A_{m,n} - \beta^N B_{m,n}},$$

where

$$A_{m,n} := \prod_{j=0}^{N-2}(\chi^{(j)}_{m,n+1} - \chi^{(j)}_{m,n}), \quad B_{m,n} := \prod_{j=0}^{N-2}(\chi^{(j)}_{m+1,n} - \chi^{(j)}_{m,n}).$$

A Lax pair follows from (3.21) using (3.22).

3.3 Bäcklund Transformations between Potential Forms

The introduction of two different sets of potentials allows us to derive a Bäcklund transformation between systems (3.2) and (3.19), which follows from the combination of relations (3.1) and (3.18). If we denote systems (3.2) and (3.19) respectively by $\mathcal{R}(\phi^{(i)};k_1,\ell_1,\alpha;k_2,\ell_2,\beta)$ and $\mathcal{A}(\chi^{(i)};k_1,\ell_1;k_2,\ell_2)$, then we have:

Proposition 3.7. *The system of equations*

$$\alpha\,\frac{\phi^{(i)}_{m+1,n}}{\phi^{(i+k_1)}_{m,n}} = \chi^{(i)}_{m+1,n} - \chi^{(i+\ell_1)}_{m,n}, \qquad \beta\,\frac{\phi^{(i)}_{m,n+1}}{\phi^{(i+k_2)}_{m,n}} = \chi^{(i)}_{m,n+1} - \chi^{(i+\ell_2)}_{m,n},$$

which we call $\mathbb{B}_{RA}(\phi,\chi;k_1,\ell_1,\alpha;k_2,\ell_2,\beta)$*, defines a Bäcklund transformation between systems* $\mathcal{R}(\phi^{(i)};k_1,\ell_1,\alpha;k_2,\ell_2,\beta)$ *and* $\mathcal{A}(\chi^{(i)};k_1,\ell_1;k_2,\ell_2)$*.*

Finally, if we combine Propositions 3.1 and 3.7, we have

Proposition 3.8. *Consider systems $\mathcal{A}(\chi^{(i)}; k_1, \ell_1; k_2, \ell_2)$ and $\mathcal{A}(\widetilde{\chi}^{(i)}; \ell_1, k_1; \ell_2, k_2)$ with $k_i + \ell_i \neq N$. Solutions of one system are mapped to solutions of the other through the Bäcklund transformation*

$$\mathbb{B}_{AA}(\chi, \widetilde{\chi}) = \left\{ \begin{array}{l} \left(\chi^{(i)}_{m+1,n} - \chi^{(i+\ell_1)}_{m,n}\right)\left(\widetilde{\chi}^{(i)}_{m+1,n} - \widetilde{\chi}^{(i+k_1)}_{m,n}\right) = \frac{\phi^{(i+\ell_1)}_{m,n}}{\phi^{(i+k_1)}_{m,n}} = \frac{\widetilde{\phi}^{(i+k_1)}_{m,n}}{\widetilde{\phi}^{(i+\ell_1)}_{m,n}}\,, \\ \\ \left(\chi^{(i)}_{m,n+1} - \chi^{(i+\ell_2)}_{m,n}\right)\left(\widetilde{\chi}^{(i)}_{m,n+1} - \widetilde{\chi}^{(i+k_2)}_{m,n}\right) = \frac{\phi^{(i+\ell_2)}_{m,n}}{\phi^{(i+k_2)}_{m,n}} = \frac{\widetilde{\phi}^{(i+k_2)}_{m,n}}{\widetilde{\phi}^{(i+\ell_2)}_{m,n}} \end{array} \right\},$$

where the auxiliary functions ϕ and $\widetilde{\phi}$ are related to χ and $\widetilde{\chi}$ by the Bäcklund transformations $\mathbb{B}_{RA}(\phi, \chi; k_1, \ell_1, \alpha; k_2, \ell_2, \beta)$ and $\mathbb{B}_{RA}\left(\widetilde{\phi}, \widetilde{\chi}; \ell_1, k_1, \alpha^{-1}; \ell_2, k_2, \beta^{-1}\right)$, respectively.

Remark 3.9. *In section 2.4 we proposed a classification scheme of Lax pairs and corresponding integrable difference equations, in terms of their level structures. Propositions 3.1, 3.7 and 3.8 place these systems within families related with each other by Bäcklund transformations. Schematically, it is encoded in the following diagram:*

$$\begin{array}{ccccc} \mathcal{R}(\phi^{(i)}; k_1, \ell_1, \alpha; k_2, \ell_2, \beta) & \rightleftarrows & \mathcal{I} & \rightleftarrows & \mathcal{R}\left(\widetilde{\phi}^{(i)}; \ell_1, k_1, \frac{1}{\alpha}; \ell_2, k_2, \frac{1}{\beta}\right) \\ \uparrow\downarrow & & & & \uparrow\downarrow \\ \mathbb{B}_{RA} & & & & \mathbb{B}_{RA} \\ \uparrow\downarrow & & & & \uparrow\downarrow \\ \mathcal{A}(\chi^{(i)}; k_1, \ell_1; k_2, \ell_2) & \leftrightarrows & \mathbb{B}_{AA} & \leftrightarrows & \mathcal{A}(\widetilde{\chi}^{(i)}; \ell_1, k_1; \ell_2, k_2) \end{array}$$

3.4 The Degenerate Case

The final class of coprime system discussed in our classification scheme of Section 2.4 is the degenerate case, for which $b = 0$. From conditions (2.20) we set

$$u^{(N-1)}_{m,n} = a \prod_{i=0}^{N-2} \frac{1}{u^{(i)}_{m,n}}\,, \quad v^{(N-1)}_{m,n} = 0\,. \tag{3.28}$$

If $v^{(N-1)}_{m,n}$ is the only zero component, then equation (2.7a) for $i = N-1$ implies that $k_1 = 0$. Hence $\ell_1 \neq 0$ and $\ell_2 \equiv k_2 + \ell_1 \pmod N$, which follows from the consistency condition (2.4).

Using (2.7a) to eliminate $v^{(i)}_{m+1,n}$, equation (2.7b) implies

$$v^{(i+\ell_1)}_{m,n} = u^{(i+\ell_2)}_{m,n} - u^{(i)}_{m,n+1} + \frac{u^{(i)}_{m,n+1}}{u^{(i+\ell_2-\ell_1)}_{m,n}} v^{(i)}_{m,n} \tag{3.29}$$

With $i = N-1$, we have

$$v^{(\ell_1-1)}_{m,n} = u^{(\ell_2-1)}_{m,n} - u^{(N-1)}_{m,n+1}.$$

Using (3.29) for the inductive step, we can show that

$$v^{(q\ell_1-1)}_{m,n} = u^{(\ell_2+(q-1)\ell_1-1)}_{m,n} - u^{(N-1)}_{m,n+1} P_q, \tag{3.30}$$

where

$$P_{q+1} = \frac{u^{(q\ell_1-1)}_{m,n+1}}{u^{(\ell_2+(q-1)\ell_1-1)}_{m,n}} P_q, \quad \text{with} \quad P_1 = 1.$$

This leads to the general formula given in (3.31) below.

Proposition 3.10. *Let* $(0,\ell_1;k_2,\ell_2)\in\mathcal{Q}_N$ *with* $\big(N,\ell_1\big)=\big(N,k_2-\ell_2\big)=1$. *Consider the system (2.1), with* $U_{m,n}=\boldsymbol{u}_{m,n}$, $V_{m,n}=\boldsymbol{v}_{m,n}\Omega^{k_2}$ *where*

$$\boldsymbol{u}_{m,n} := \operatorname{diag}\left(u^{(0)}_{m,n},\dots,u^{(N-1)}_{m,n}\right) \quad \textit{with} \quad \det(\boldsymbol{u}_{m,n}) = \prod_{i=0}^{N-1} u^{(i)}_{m,n} = a,$$

and $\boldsymbol{v}_{m,n} := \operatorname{diag}\left(v^{(0)}_{m,n},\dots,v^{(N-2)}_{m,n},0\right)$. *Then with*

$$\begin{aligned} v^{(\ell_1-1)}_{m,n} &= u^{(\ell_2-1)}_{m,n} - u^{(N-1)}_{m,n+1}, \\ v^{(q\ell_1-1)}_{m,n} &= u^{(\ell_2+(q-1)\ell_1-1)}_{m,n} - u^{(N-1)}_{m,n+1}\prod_{r=1}^{q-1}u^{(r\ell_1-1)}_{m,n+1}\prod_{s=0}^{q-2}\frac{1}{u^{(\ell_2+s\ell_1-1)}_{m,n}}, \quad q=2,\dots,N-1, \end{aligned} \tag{3.31}$$

the system (2.7a) leads to a system of equations for the components $u^{(i)}$.

We now present inequivalent integrable systems for $N=2$ and $N=3$, and give the system which corresponds to the level structure $(0,1;0,1)$ for any dimension N. Whilst our general discussion has concentrated on the case $v^{(N-1)}=0$, with other components nonzero, we also present some cases, for $N=3$, for which $v^{(1)}_{m,n}=v^{(2)}_{m,n}=0$.

3.4.1 Integrable Systems for $N=2$

In two dimensions, there exists only one nontrivial system:

Level Structure $[(0,1;0,1)]$

We set

$$u^{(0)}_{m,n} = u_{m,n}, \quad u^{(1)}_{m,n} = \frac{a}{u_{m,n}}, \quad v^{(0)}_{m,n} = u_{m,n} - \frac{a}{u_{m,n+1}},$$

and the resulting equation is Hirota's KdV equation,

$$\frac{a}{u_{m+1,n+1}} + u_{m,n+1} = u_{m+1,n} + \frac{a}{u_{m,n}}. \tag{3.32}$$

Remark 3.11. *The system which follows from the Lax pair with structure* $(0,1;1,0)$ *can be easily shown to be reducible to an ordinary difference equation.*

3.4.2 Integrable Systems for $N=3$

In three dimensions, only two systems arise.

Equivalence class $[(0,1;0,1)]$

In this case the system may be considered as the two-component analogue of Hirota's KdV equation (3.32), for the components $u^{(0)}_{m,n}$, $u^{(1)}_{m,n}$. We have

$$u^{(2)}_{m,n} = \frac{a}{u^{(0)}_{m,n}u^{(1)}_{m,n}}, \quad v^{(0)}_{m,n} = u^{(0)}_{m,n} - \frac{a}{u^{(0)}_{m,n+1}u^{(1)}_{m,n+1}}, \quad v^{(1)}_{m,n} = u^{(1)}_{m,n} - \frac{a}{u^{(0)}_{m,n}u^{(1)}_{m,n+1}},$$

together with the system

$$\frac{a}{u^{(0)}_{m+1,n+1}u^{(1)}_{m+1,n+1}} + u^{(0)}_{m,n+1} = u^{(0)}_{m+1,n} + \frac{a}{u^{(0)}_{m,n}u^{(1)}_{m,n+1}}\,, \tag{3.33a}$$

$$\frac{a}{u^{(0)}_{m+1,n}u^{(1)}_{m+1,n+1}} + u^{(1)}_{m,n+1} = u^{(1)}_{m+1,n} + \frac{a}{u^{(0)}_{m,n}u^{(1)}_{m,n}}\,. \tag{3.33b}$$

This example admits a reduction, with $v^{(1)}_{m,n} = 0$, corresponding to $u^{(0)}_{m,n} = \frac{a}{u^{(1)}_{m,n}u^{(1)}_{m,n+1}}$. In this case, (3.33b) holds identically, whilst (3.33a) takes the form of a six point equation

$$u^{(1)}_{m,n} + \frac{a}{u^{(1)}_{m+1,n}u^{(1)}_{m+1,n+1}} = u^{(1)}_{m+1,n+2} + \frac{a}{u^{(1)}_{m,n+1}u^{(1)}_{m,n+2}}. \tag{3.34}$$

Remark 3.12. *Another reduction, with $v^{(0)}_{m,n} = 0$, corresponds to $u^{(1)}_{m,n} = \frac{a}{u^{(0)}_{m,n}u^{(0)}_{m,n-1}}$, in which case system (3.33) reduces again to equation (3.34), but for $u^{(0)}$. System (3.33) and its reductions (3.34) were derived first in [6] in a different context.*

Equivalence class $[(0,1;1,2)]$

In this case we introduce variables $u_{m,n}$ and $v_{m,n}$ by

$$u^{(0)}_{m,n} = u_{m,n}v_{m,n}, \quad u^{(1)}_{m,n} = \frac{1}{u_{m,n}}, \quad u^{(2)}_{m,n} = \frac{a}{v_{m,n}}\,,$$

$$v^{(0)}_{m,n} = \frac{1}{u_{m,n}} - \frac{a}{v_{m,n+1}}\,, \quad v^{(1)}_{m,n} = a\left(\frac{1}{v_{m,n}} - u_{m,n}u_{m,n+1}\right),$$

to derive the system

$$u_{m,n}v_{m,n} + \frac{a}{v_{m+1,n}} = \frac{1}{u_{m,n+1}} + au_{m+1,n}\,u_{m+1,n+1}\,, \tag{3.35a}$$

$$u_{m,n+1}v_{m,n+1} + \frac{a}{v_{m+1,n+1}} = \frac{1}{u_{m+1,n}} + au_{m,n}\,u_{m,n+1}\,. \tag{3.35b}$$

Noting that the left hand sides of these equations are related by a shift in the n direction, we can derive an equation for the single component u:

$$au_{m+1,n+1}u_{m+1,n+2} + \frac{1}{u_{m,n+2}} = \frac{1}{u_{m+1,n}} + au_{m,n}u_{m,n+1}\,. \tag{3.36}$$

Equation (3.35a) is then a first order, "driven" difference equation for $v_{m,n}$.

Remark 3.13. *The further reduction $v^{(1)}_{m,n} = 0$ corresponds to $v_{m,n} = \frac{1}{u_{m,n}\,u_{m,n+1}}$, after which (3.35a) is identically satisfied, whilst (3.35b) takes the form of (3.36). Similarly, the choice $v^{(0)}_{m,n} = 0$ corresponds to $v_{m,n} = au_{m,n-1}$ and reduces system (3.35) to equation (3.36) again. Equation (3.36) is related to*

$$u_{m+1,n}u_{m,n+1}\left(u_{m,n} + u_{m+1,n+1}\right) + \frac{1}{a} = 0,$$

found in [12], which, up to inversion of one of the lattice directions, is equation (3.13). Finally, equation (3.36) is also related to equation (3.34) by the point transformation $(u_{i,j}, a) \to (1/u_{-i,j}, 1/a)$.

Equivalence class $[(0,1;2,0)]$

As in the two-dimensional case, the resulting system can be reduced to a scalar ordinary difference equation, so is not considered here.

3.4.3 Integrable Systems for General N

The equivalence class $[(0,1;0,1)]$ is defined for any dimension N. From Proposition 3.10 with $k_2=0$ and $\ell_1=\ell_2=1$, we find that

$$v^{(i)}_{m,n} = u^{(i)}_{m,n} - a\prod_{r=0}^{i-1}\frac{1}{u^{(r)}_{m,n}}\prod_{s=i}^{N-2}\frac{1}{u^{(s)}_{m,n+1}}, \tag{3.37}$$

in view of which we derive the system

$$u^{(i)}_{m,n}u^{(i)}_{m+1,n} - \frac{au^{(i)}_{m,n}}{\prod_{r=0}^{i-1}u^{(r)}_{m+1,n}\prod_{s=i}^{N-2}u^{(s)}_{m+1,n+1}} = u^{(i)}_{m,n}u^{(i)}_{m,n+1} - \frac{au^{(i)}_{m,n+1}}{\prod_{r=0}^{i-1}u^{(r)}_{m,n}\prod_{s=i}^{N-2}u^{(s)}_{m,n+1}}, \tag{3.38}$$

where $i\in\mathbb{Z}_{N-1}$.

This system can be reduced further by setting $v^{(i)}=0$ for $i\neq 0$, after which all other variables can be written in terms of $u^{(1)}_{m,n}$ and its shifts. Setting $u_{m,n}=\frac{1}{u^{(1)}_{m,n}}$ and employing $a\mapsto\frac{1}{a}$, we obtain

$$\frac{a}{u_{m+1,n+N-1}} + \prod_{i=1}^{N-1}u_{m,n+i} = \prod_{i=0}^{N-2}u_{m+1,n+i} + \frac{a}{u_{m,n}}, \tag{3.39}$$

which obviously involves $2N$ points. The above equation coincides with Hirota's KdV equation (3.32) for $N=2$, and, up to the inversion of u and a, becomes equation (3.34) when $N=3$.

4 The Non-Coprime Case

Our classification of Section 2.4 finished with the *non-coprime case*, which may be considered as representing a coupling between coprime systems. This follows by the block structure described in Proposition 2.9. In this short section we give some examples to illustrate this structure.

It should be emphasised that the permutation matrix P depends only upon the greatest common divisor, $p=(N,\ell_i-k_i)$, so all matrices with the same p can *simultaneously* be put in block form. We can see from formula (2.18) that when k_1 (respectively k_2) is a multiple of p, then L (respectively M) takes block diagonal form. If *both L and M* take block diagonal form, then the system decouples into p copies of the same $q-$component system. Otherwise, the system is organised as a coupling between the $q-$vectors $\mathbf{u}_i=(u^{(i)},u^{(i+p)},\dots,u^{(i+p(q-1))})$, $i=0,\dots,p-1$.

Example 4.1 ($N=4$, $\ell_i-k_i\equiv 2$)**.** Using the notation of Section 2.3, we have $p=2, q=2, r=1$. Inequivalent choices of (k_1,ℓ_1) are $(0,2),(1,3)$ or $(2,0)$, in which case we have respectively

$$L=\begin{pmatrix} L^{(0,1)}_{02} & 0\\ 0 & L^{(0,1)}_{13}\end{pmatrix},\quad L=\begin{pmatrix} 0 & L^{(0,1)}_{02}\\ L^{(1,0)}_{13} & 0\end{pmatrix},\quad L=\begin{pmatrix} L^{(1,0)}_{02} & 0\\ 0 & L^{(1,0)}_{13}\end{pmatrix},$$

where $L_{ij}^{(k,\ell)}$ is the 2×2 matrix of level structure (k,ℓ) and depending on variables $(u_{m,n}^{(i)}, u_{m,n}^{(j)})$. We must also choose M to have one of these structures.

If we choose *both* L *and* M to be block-diagonal, then components $(0,2)$ and $(1,3)$ *decouple*. For example, with level structure $(0,2;0,2)$, we can use (3.1) to define $(\phi_{m,n}^{(0)}, \alpha_0, \beta_0)$, with $\phi_{m,n}^{(2)} = (\phi_{m,n}^{(0)})^{-1}$ and $(\phi_{m,n}^{(1)}, \alpha_1, \beta_1)$, with $\phi_{m,n}^{(3)} = (\phi_{m,n}^{(1)})^{-1}$ to obtain two copies of the modified KdV equation (3.5).

On the other hand, if we choose level structure $(1,3;1,3)$, then equation (2.2) yields a coupled system

$$\begin{aligned} L_{02}^{(0,1)}(\mathbf{u}_{m,n+1})M_{13}^{(1,0)}(\mathbf{v}_{m,n}) &= M_{02}^{(0,1)}(\mathbf{v}_{m+1,n})L_{13}^{(1,0)}(\mathbf{u}_{m,n}), \\ L_{13}^{(1,0)}(\mathbf{u}_{m,n+1})M_{02}^{(0,1)}(\mathbf{v}_{m,n}) &= M_{13}^{(1,0)}(\mathbf{v}_{m+1,n})L_{02}^{(0,1)}(\mathbf{u}_{m,n}). \end{aligned}$$

If we use (3.1) to rewrite the equations in quotient potential form, and set $\phi_{m,n}^{(0)} = 1/\phi_{m,n}^{(2)} = \phi_{m,n}$ and $1/\phi_{m,n}^{(1)} = \phi_{m,n}^{(3)} = \psi_{m,n}$, we obtain the following coupled two-component system.

$$\phi_{m,n}\phi_{m+1,n+1} = \frac{\alpha\psi_{m+1,n} - \beta\psi_{m,n+1}}{\alpha\psi_{m,n+1} - \beta\psi_{m+1,n}}, \quad \psi_{m,n}\psi_{m+1,n+1} = \frac{\alpha\phi_{m,n+1} - \beta\phi_{m+1,n}}{\alpha\phi_{m+1,n} - \beta\phi_{m,n+1}}.$$

This system can be decoupled for either of the functions involved in it to the following five-point equation.

$$\begin{aligned} &\left(\frac{\alpha\phi_{m+1,n+1} - \beta\phi_{m+2,n}}{\beta\phi_{m+1,n+1} - \alpha\phi_{m+2,n}}\right)\left(\frac{\alpha\phi_{m+1,n+1} - \beta\phi_{m,n+2}}{\beta\phi_{m+1,n+1} - \alpha\phi_{m,n+2}}\right) = \\ &\qquad\qquad \left(\frac{\alpha\phi_{m,n}\phi_{m+1,n+1} + \beta}{\beta\phi_{m,n}\phi_{m+1,n+1} + \alpha}\right)\left(\frac{\alpha\phi_{m+1,n+1}\phi_{m+2,n+2} + \beta}{\beta\phi_{m+1,n+1}\phi_{m+2,n+2} + \alpha}\right), \end{aligned}$$

which could be interpreted as a discrete version of the modified Hirota-Satsuma equation, which has a 4×4 matrix Lax pair (see section 3 of [5]).

Example 4.2 ($N=6$, $\ell_i - k_i \equiv 2$)**.** Using the notation of Section 2.3, we have $p=2, q=3, r=1$. Inequivalent choices of (k_1,ℓ_1) are $(0,2),(1,3),(2,4),(3,5),(4,0)$ or $(5,1)$. The choices $(0,2),(2,4)$ and $(4,0)$ give block diagonal forms for L. The choices $(1,3)$ and $(3,5)$, respectively, give the following forms for L:

$$L = \begin{pmatrix} 0 & L_{024}^{(0,1)} \\ L_{135}^{(1,2)} & 0 \end{pmatrix}, \qquad L = \begin{pmatrix} 0 & L_{024}^{(1,2)} \\ L_{135}^{(2,0)} & 0 \end{pmatrix},$$

where $L_{abc}^{(k,\ell)}$ is the 3×3 Lax matrix of level structure (k,ℓ) and depending on variables $u_{m,n}^{(a)}, u_{m,n}^{(b)}$ and $u_{m,n}^{(c)}$. To obtain a coupled system, we could choose L to be one of these three structures, with M being any of the six.

For example, the choice $(1,3;1,3)$ leads to the system

$$\begin{aligned} L_{024}^{(0,1)}(\mathbf{u}_{m,n+1})M_{135}^{(1,2)}(\mathbf{v}_{m,n}) &= M_{024}^{(0,1)}(\mathbf{v}_{m+1,n})L_{135}^{(1,2)}(\mathbf{u}_{m,n}), \\ L_{135}^{(1,2)}(\mathbf{u}_{m,n+1})M_{024}^{(0,1)}(\mathbf{v}_{m,n}) &= M_{135}^{(1,2)}(\mathbf{v}_{m+1,n})L_{024}^{(0,1)}(\mathbf{u}_{m,n}), \end{aligned}$$

whilst the choice $(1,3;3,5)$ leads to the system

$$\begin{aligned} L_{024}^{(0,1)}(\mathbf{u}_{m,n+1})M_{135}^{(2,0)}(\mathbf{v}_{m,n}) &= M_{024}^{(1,2)}(\mathbf{v}_{m+1,n})L_{135}^{(1,2)}(\mathbf{u}_{m,n}), \\ L_{135}^{(1,2)}(\mathbf{u}_{m,n+1})M_{024}^{(1,2)}(\mathbf{v}_{m,n}) &= M_{135}^{(2,0)}(\mathbf{v}_{m+1,n})L_{024}^{(0,1)}(\mathbf{u}_{m,n}). \end{aligned}$$

The explicit form of the latter, in quotient potential form, with $\phi^{(0)}_{m,n} = 1/\varphi^{(0)}_{m,n}$, $\phi^{(2)}_{m,n} = \psi^{(0)}_{m,n}$, $\phi^{(4)}_{m,n} = \varphi^{(0)}_{m,n}/\psi^{(0)}_{m,n}$ and $\phi^{(1)}_{m,n} = 1/\psi^{(1)}_{m,n}$, $\phi^{(3)}_{m,n} = \varphi^{(1)}_{m,n}$, $\phi^{(5)}_{m,n} = \psi^{(1)}_{m,n}/\varphi^{(1)}_{m,n}$, is

$$\varphi^{(0)}_{m,n}\varphi^{(0)}_{m+1,n+1} = \frac{\alpha\varphi^{(1)}_{m+1,n}\psi^{(1)}_{m,n+1} - \beta}{\alpha\psi^{(1)}_{m+1,n} - \beta\varphi^{(1)}_{m+1,n}\varphi^{(1)}_{m,n+1}}, \qquad \psi^{(0)}_{m,n}\psi^{(0)}_{m+1,n+1} = \frac{\alpha\varphi^{(1)}_{m,n+1} - \beta\psi^{(1)}_{m+1,n}\psi^{(1)}_{m,n+1}}{\alpha\psi^{(1)}_{m+1,n} - \beta\varphi^{(1)}_{m+1,n}\varphi^{(1)}_{m,n+1}},$$

$$\varphi^{(1)}_{m,n}\varphi^{(1)}_{m+1,n+1} = \frac{\alpha\varphi^{(0)}_{m,n+1}\psi^{(0)}_{m+1,n} - \beta}{\alpha\psi^{(0)}_{m,n+1} - \beta\varphi^{(0)}_{m+1,n}\varphi^{(0)}_{m,n+1}}, \qquad \psi^{(1)}_{m,n}\psi^{(1)}_{m+1,n+1} = \frac{\alpha\varphi^{(0)}_{m+1,n} - \beta\psi^{(0)}_{m+1,n}\psi^{(0)}_{m,n+1}}{\alpha\psi^{(0)}_{m,n+1} - \beta\varphi^{(0)}_{m+1,n}\varphi^{(0)}_{m,n+1}},$$

which should be compared with system (3.9).

Example 4.3 ($N = 6$, $\ell_i - k_i \equiv 3$)**.** Using the notation of Section 2.3, we have $p = 3, q = 2, r = 1$. Inequivalent choices of (k_1, ℓ_1) are $(0,3), (1,4), (2,5), (3,0), (4,1)$ or $(5,2)$. The choices $(0,3)$ and $(3,0)$ give block diagonal forms for L. The choices $(1,4)$ and $(2,5)$, for example, give the following forms for L:

$$L = \begin{pmatrix} 0 & L^{(0,1)}_{03} & 0 \\ 0 & 0 & L^{(0,1)}_{14} \\ L^{(1,0)}_{25} & 0 & 0 \end{pmatrix}, \quad L = \begin{pmatrix} 0 & 0 & L^{(0,1)}_{03} \\ L^{(1,0)}_{14} & 0 & 0 \\ 0 & L^{(1,0)}_{25} & 0 \end{pmatrix},$$

where $L^{(k,\ell)}_{ab}$ is the 2×2 Lax matrix of level structure (k, ℓ) and depending on variables $u^{(a)}_{m,n}$ and $u^{(b)}_{m,n}$. The choice $(2,5;2,5)$ leads to the system

$$\begin{aligned} L^{(0,1)}_{03}(\mathbf{u}_{m,n+1})M^{(1,0)}_{25}(\mathbf{v}_{m,n}) &= M^{(0,1)}_{03}(\mathbf{v}_{m+1,n})L^{(1,0)}_{25}(\mathbf{u}_{m,n}), && (4.1a)\\ L^{(1,0)}_{14}(\mathbf{u}_{m,n+1})M^{(0,1)}_{03}(\mathbf{v}_{m,n}) &= M^{(1,0)}_{14}(\mathbf{v}_{m+1,n})L^{(0,1)}_{03}(\mathbf{u}_{m,n}), && (4.1b)\\ L^{(1,0)}_{25}(\mathbf{u}_{m,n+1})M^{(1,0)}_{14}(\mathbf{v}_{m,n}) &= M^{(1,0)}_{25}(\mathbf{v}_{m+1,n})L^{(1,0)}_{14}(\mathbf{u}_{m,n}), && (4.1c) \end{aligned}$$

Writing the equations in potential form (3.1), with $\phi^{(0)}_{m,n} = 1/\phi^{(3)}_{m,n} = \psi^{(0)}_{m,n}, \phi^{(4)}_{m,n} = 1/\phi^{(1)}_{m,n} = \psi^{(1)}_{m,n}, \phi^{(2)}_{m,n} = 1/\phi^{(5)}_{m,n} = \psi^{(2)}_{m,n}$, we obtain the system

$$\psi^{(i)}_{m+1,n+1} = \left(\frac{\alpha\psi^{(i+2)}_{m,n+1} - \beta\psi^{(i+2)}_{m+1,n}}{\alpha\psi^{(i+2)}_{m+1,n} - \beta\psi^{(i+2)}_{m,n+1}}\right)\psi^{(i+1)}_{m,n}, \quad i \in \mathbb{Z}_3, \tag{4.2}$$

which is a coupled discrete MKdV system, reducing to (3.5) when all components are equal.

On the other hand, the choice $(1,4;2,5)$ leads to the system

$$\begin{aligned} L^{(0,1)}_{03}(\mathbf{u}_{m,n+1})M^{(1,0)}_{14}(\mathbf{v}_{m,n}) &= M^{(0,1)}_{03}(\mathbf{v}_{m+1,n})L^{(1,0)}_{25}(\mathbf{u}_{m,n}), && (4.3a)\\ L^{(0,1)}_{14}(\mathbf{u}_{m,n+1})M^{(1,0)}_{25}(\mathbf{v}_{m,n}) &= M^{(1,0)}_{14}(\mathbf{v}_{m+1,n})L^{(0,1)}_{03}(\mathbf{u}_{m,n}), && (4.3b)\\ L^{(1,0)}_{25}(\mathbf{u}_{m,n+1})M^{(0,1)}_{03}(\mathbf{v}_{m,n}) &= M^{(1,0)}_{25}(\mathbf{v}_{m+1,n})L^{(0,1)}_{14}(\mathbf{u}_{m,n}). && (4.3c) \end{aligned}$$

Writing the equations in potential form (3.1), with $\phi^{(0)}_{m,n} = 1/\phi^{(3)}_{m,n} = \psi^{(0)}_{m,n}, \phi^{(4)}_{m,n} = 1/\phi^{(1)}_{m,n} = \psi^{(1)}_{m,n}, \phi^{(2)}_{m,n} = 1/\phi^{(5)}_{m,n} = \psi^{(2)}_{m,n}$, we now obtain the system

$$\psi^{(i)}_{m,n}\psi^{(i)}_{m+1,n+1} = \frac{\alpha - \beta\psi^{(i+1)}_{m,n+1}\psi^{(i+2)}_{m+1,n}}{\alpha\psi^{(i+1)}_{m,n+1}\psi^{(i+2)}_{m+1,n} - \beta}, \quad i \in \mathbb{Z}_3, \tag{4.4}$$

which is a coupled system of Hirota's discrete sine-Gordon equations, reducing to (3.6) when all components are equal.

The above generalisations of the discrete MKdV and of Hirota's discrete sine-Gordon equations, with this 2×2 block structure, are easily extended to an arbitrary number of components.

5 Building the Lattice

We now consider some interesting lattices which can be built out of our systems. Of course, we may take a copy of the *same* $L-M$ pair around *each* quadrilateral of the planar lattice. However, in this section we wish to describe other, more exotic, lattices, formed by taking *different* systems around adjacent quadrilaterals. Such systems have received much attention recently [10, 7].

These are most conveniently described in terms of quotient potentials.

5.1 The General Case

Here we exploit the condition (2.4) and build lattices for which the number $\ell_i - k_i$, calculated in each quadrilateral, is *fixed* $(\text{mod} N)$. This can be done by assigning to every edge of the lattice a Lax matrix with a certain level structure and by taking into account the following two rules:

1. Lax matrices on opposite edges have the same level structure and are related by shifts.

2. On every elementary quadrilateral, level structures assigned to the edges belong in $\mathcal{Q}_N$.

Each quadrilateral will then have a specific choice of $(k_1, \ell_1; k_2, \ell_2)$, subject only to $\ell_i - k_i$ being fixed $(\text{mod} N)$. To determine which combinations give *inequivalent* systems, we must take into account the equivalence relations of Definition 2.7 and Proposition 3.1. As a result only *two* inequivalent systems arise when $N = 2$ (Figure 1) and only *four* when $N = 3$. In this way, every

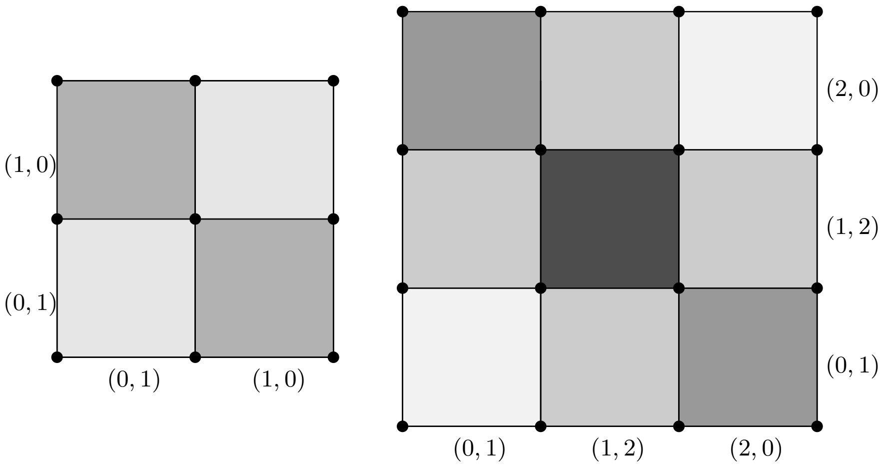

Figure 1: *Opposite edges carry matrices with exactly the same structure. Here it is demonstrated the simplest configuration for the case of $N = 2$ (left), which corresponds to a black-white lattice [21], and the case of $N = 3$ (right). Quadrilaterals with the same colour carry equivalent integrable systems.*

quadrilateral carries a different discrete system deriving from the compatibility condition of the corresponding Lax pair.

We can then introduce a third direction. Above a square with $(k_1, \ell_1; k_2, \ell_2)$, with $\ell_2 - k_2 \equiv \ell_1 - k_1 \, (\text{mod} N)$ we can place a third direction, with level structure (k_3, ℓ_3), such that $\ell_3 - k_3 \equiv \ell_1 - k_1 \, (\text{mod} \, N)$. Above the lattices of Figure 1 we therefore build respectively four or nine cubes. For each cube, opposite faces have the same level structure (and therefore the same *planar* system of equations). For the examples of Figure 1 all cubes either have *four* equivalent faces (with the remaining two being different) or *six* equivalent faces. Whatever the choice of (k_3, ℓ_3), the cubes can be consistently placed above the planar lattice (with common faces having the same level structure). Around each cube the array of systems are consistent. Since opposite faces have the same level structure, the "ground floor" planar lattice is reproduced on the "first floor", so the procedure can be repeated. In this way, we build a $3D$ consistent lattice.

This $3D$ consistency is another manifestation of the integrability of these systems. In fact, the two equations corresponding to the "vertical faces" of the cube can be interpreted as a Bäcklund transformation for system on the "horizontal face". In this way, two copies of system (3.9) serve as a Bäcklund transformation for system (3.11).

5.2 The Non-Coprime Case

We saw in Section 4 that when $(N, \ell_i - k_i) = p \neq 1$, then the discrete system (2.7) takes the form of a coupled system, involving $q \times q$ matrices, where $N = pq$. We can, of course, follow the general procedure described above.

For example, with $N = 6$ and $\ell_i - k_i \equiv 3$, we may consider a lattice that *looks like* the two-dimensional case of Figure 1, but with 3 components at each vertex. We just replace $(0, 1)$ and $(1, 0)$ by $(1, 4)$ and $(2, 5)$ to obtain the coupled discrete modified KdV (dMKdV) system (4.2) in the light coloured squares and the coupled discrete sine-Gordon (dSG) system (4.4) in the dark squares.

However, we wish to introduce a different lattice, more closely reflecting the coupled system in the non-coprime case. We describe the procedure within the context of Example 4.3.

Example 5.1 (Coupled dMKdV and dSG Systems for $N = 6$)**.** We note that for the dMKdV case, compatibility conditions (4.1) define three "elementary quadrilaterals", with edges labelled by the appropriate (k, ℓ, a, b), shown in Figure 2. The first and second of these share a common edge, indicated by $M_{03}^{(0,1)}$ and similarly for the second and third. This is also true for the right edge of the third quadrilateral and the left edge of the first. Similar identifications can be made between top and bottom edges, so it is possible to construct consistent arrays of elementary quadrilaterals to form a lattice. We can draw similar elementary quadrilaterals for any of the coupled systems for which $\ell_i - k_i \equiv 3$.

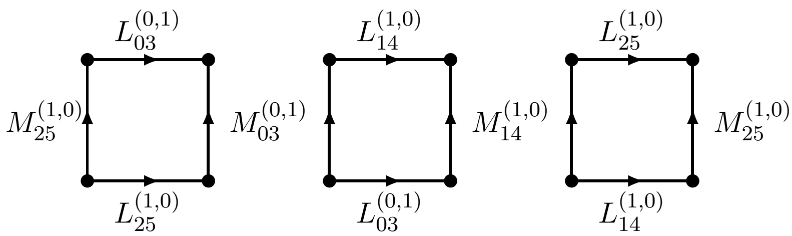

Figure 2: Elementary quadrilaterals for the coupled MKdV System

When written in terms of the quotient potentials, the equations depicted by the three elementary quadrilaterals of Figure 2 are precisely equations (4.2) for $i = 0, 1, 2$ respectively. Notice that, from the structure of these equations, each vertex of the quadrilateral is associated with a *specific variable* $\psi^{(i)}$, $i \in \{0, 1, 2\}$. Similarly, each of the three elementary quadrilaterals corresponding to equations (4.3) gives just one component of equations (4.4).

The elementary quadrilaterals can then be drawn in a very simple way, just indicating the specific variables which correspond to the particular quadrilateral. These are depicted in Figure 3 (for both the dMKdV and dSG cases), where nine such elementary quadrilaterals are shown in the unique consistent configuration (for each choice of $i \in \{0, 1, 2\}$). This configuration is then extended periodically to the whole plane.

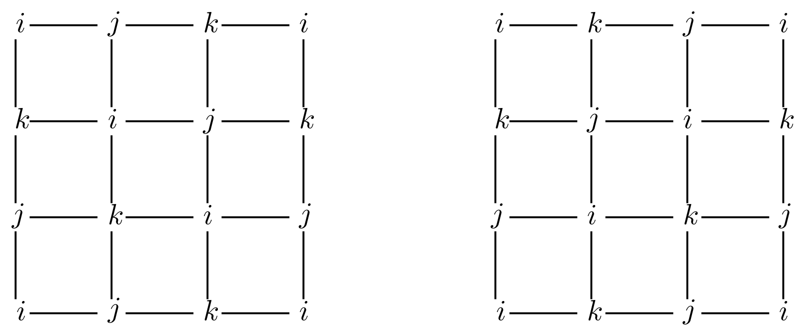

Figure 3: *Patterns on the lattice with* $(i, j, k) \in \{(0, 1, 2), (1, 2, 0), (2, 0, 1)\}$ *: The first involves a copy of modified KdV on every quadrilateral, whereas, in the second pattern, every quadrilateral carries a sine-Gordon equation.*

Remark 5.2 (Interlacing Columns)**.** *We can also interlace columns of the dMKdV lattice with columns of the dSG lattice. The only constraint is that right vertices of the left column must correspond to the left vertices of the right column.*

Corresponding to each choice of $i \in \{0, 1, 2\}$ there is also a unique $3D$ consistent cube, shown (for both the dMKdV and dSG cases) in Figure 4. Each face of the dMKdV cube corresponds to an dMKdV equation. It is not possible to build a consistent cube from just dSG faces. Two of the faces must be of dMKdV form (we already saw this in the 1 component case).

For the dMKdV case, opposite faces are related by the shift $i \mapsto i + 1 \,(\bmod 3)$. Starting with a planar dMKdV lattice in the “horizontal plane”, there is a unique configuration of dMKdV cubes with such “bottom faces”. On the “first floor” we now have the shifted planar lattice (with $i \mapsto i + 1\,(\text{mod}3)$), whilst on the “second floor” we have the twice shifted planar lattice (with $i \mapsto i + 2\,(\text{mod}3)$). Since the whole process is periodic, the next shift gives back the original “ground floor” lattice. In this way, we build a unique $3D$ consistent lattice.

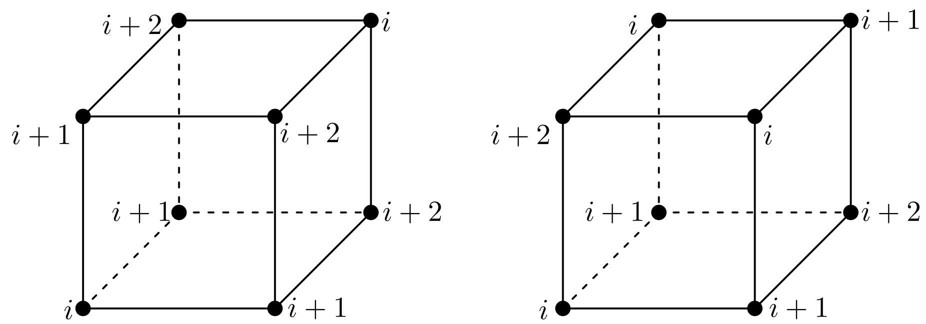

Figure 4: *The first cube involves only copies of the modified KdV equation, whereas the second cube carries two copies of mKdV (bottom and top faces) and four copies of the sine-Gordon equation.*

For the dSG case we can choose the top and bottom faces to be of dMKdV type. We are then obliged to use a slightly different form of the dSG equation, corresponding to interchanging the positions of $\psi^{(i+1)}$ and $\psi^{(i+2)}$ in the i^{th} equation. This corresponds to the involution $\tilde{f}_{i,j} = f_{1-i,1-j}$, for any function f located at vertex (i, j) in the planar lattice.

Starting with an dMKdV planar lattice in the “horizontal plane”, there is a unique configuration of dSG cubes with such “bottom faces”. On the “first floor” we now have the shifted planar lattice (with $i \mapsto i+2\,(\text{mod}3)$), whilst on the "second floor" we have the twice shifted planar lattice (with $i \mapsto i+4 \equiv i+1\,(\text{mod}3)$). Since the whole process is periodic, the next shift gives back the original "ground floor" lattice. In this way, we build a unique $3D$ consistent lattice. The $3\times3\times3$ cube, with bottom and top faces in the configuration of the dMKdV lattice shown in Figure 3 has side faces with the dSG pattern of Figure 3, but oriented so that vertices match at the base.

We can then place this side face on top of the planar dSG lattice of Figure 3, and the $3\times3\times3$ cube above this planar dSG lattice is the unique consistent cube (exactly the one we could have built directly).

In this way we obtain two $3D$ lattices which consistently support a mixture of dMKdV and dSG equations on quadrilateral faces.

5.3 The Initial Value Problem

The initial value problem for the standard planar lattice with a *single system* over the entire plane is just a multi-component version of the scalar case. Since the equations can be solved for evolution in any direction, we can set initial conditions on (for example) a staircase. This statement does not change if we introduce the "multicoloured" lattices of Section 5.1. The extension to the $3D$ lattice is standard.

However, for the lattices constructed in Section 5.2, the situation is different. Since the equations can still be solved for evolution in any direction, we can again set initial conditions on (for example) an arbitrary staircase. However, since each vertex only involves a subset of the systems components, we *cannot* determine the values of the other components at this vertex.

To be specific, suppose we consider the dMKdV system (4.2) and the corresponding planar lattice shown in Figure 3. For simplicity, we consider the equation in the first quadrant (m,n), with $m \geq 0$, $n \geq 0$ and set *initial conditions* on the axes. It can be seen from Figure 3 that we have three possible configurations of initial conditions, depicted in Figure 5 by the black vertices, where $(i,j,k) \in \{(0,1,2),(1,2,0),(2,0,1)\}$. We can then use the equations to calculate specific values on the next (white) "L" shape (a similar configuration, but with index i shifted by 2 in the diagonal direction). Each iteration involves this shift by 2, so after 3 iterations we return to the same configuration, as depicted by the grey vertices. In this way we build a unique array of the letters (i,j,k), in the pattern given in Figure 3, and thus build *three* different lattices, depending upon the choice of values for these letters.

Remark 5.3. *The standard lattice, with the whole coupled MKdV system on each quadrilateral, is obtained by superposition of these three lattices.*

Remark 5.4. *Since we can calculate the value of $\psi^{(0)}$ (say) on a specific sub-array of vertices, we conjecture that a higher order equation exists for $\psi^{(0)}$ alone. In fact, it follows from the symmetry of the system that each component $\psi^{(i)}$ would satisfy the same higher order equation.*

6 Conclusions & discussion

In this paper we considered the class of discrete Lax pairs (2.1) and a classification problem for such systems. We were naturally led to considering *coprime* vs *non-coprime* cases and focussed mainly in the analysis of the coprime case. We were also naturally led to considering *generic* vs *degenerate* cases. All aspects of the generic case can be systematically analysed, but the degenerate systems requires a case by case analysis, so their classification is far from complete.

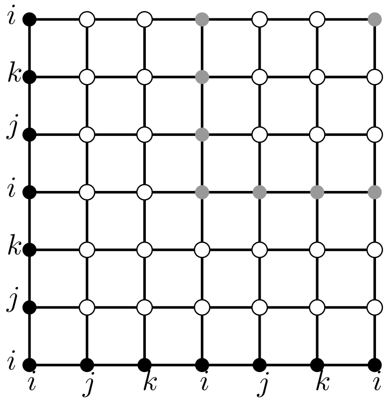

Figure 5: *Patterns on the lattice and initial value problems with* $(i, j, k) \in \{(0,1,2),(1,2,0),(2,0,1)\}$ *: Every black vertex carries the initial value of the corresponding variable, e.g. the left bottom vertex carries the initial value of* $\psi^{(i)}$*. This initial pattern repeats after three diagonal steps leading to the updated gray vertices.*

The *generic coprime* case has two natural descriptions in terms of potential functions. These are related through a Bäcklund transformation, but some well known, low dimensional, examples fit naturally into each description. We presented all the inequivalent systems in two and three dimensions. As well as the very well known systems, we found several new ones, including (3.9), (3.10), (3.11), (3.17) and (3.25). The degenerate case lead to Hirota's KdV equation and its multi-component generalisations (3.39). Reductions of the three dimensional systems resulted in new $2-$component integrable systems (3.33) and (3.35), with further reductions leading to integrable scalar equations (3.34) and (3.36) which are defined on six-point stencils, which may be considered as the discrete analogue of higher order hyperbolic partial differential equations [3].

In Section 4 we presented some examples of *non-coprime* systems, which give a mechanism for coupling coprime systems. In particular, we presented coupled systems of the discrete MKdV equation and of Hirota's discrete sine-Gordon equation. In Section 5 we discussed the problem of building $2D$ lattices with a mixture of equations in adjacent quadrilaterals and also the extension to $3D$ consistent systems.

There are a number of open questions. We have found several *reductions* to lower dimensional systems (such as (3.17) and (3.39)) but have no systematic way of analysing these. The connection with well known reduced PDEs, such as the Sawada-Kotera and Hirota-Satsuma equations is also not clear. In this paper, we restricted our Lax pairs to be *linear* in λ. Similar Lax pairs, polynomial in λ would be interesting to consider.

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