Doubly geometric processes and applications

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Abstract

The geometric process has attracted extensive research attention from authors in reliability mathematics since its introduction. However, it possesses some limitations, which include that: (1) it can merely model stochastically increasing or decreasing inter-arrival times of recurrent event processes, and (2) it cannot model recurrent event processes where the inter-arrival time distributions have varying shape parameters. Those limitations may prevent it from a wider application in the real world.

In this paper, we extend the geometric process to a new process, the *doubly geometric process*, which overcomes the above two limitations. Probability properties are derived and two methods of parameter estimation are given. Application of the proposed model is presented: one is on fitting warranty claim data and the other is to compare the performance of the doubly geometric process with the performance of other widely used models in fitting real world datasets, based on the corrected Akaike information criterion.

Keywords: failure process modelling, geometric processes, recurrent events, Poisson processes, maintenance.

1 Introduction

1.1 Motivation

Since its introduction by Lam (1988), the geometric process (GP) has attracted extensive research attention. A considerable bulk of research on the GP, including more than 200 papers and one monograph (Lam, 2007), has been published. For example, the GP has been applied in system

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reliability analysis (Yuan and Meng, 2011; Jain and Gupta, 2013), maintenance policy optimisation (Zhang, et.al, 2002; Liu and Huang, 2010; Wang, 2011; Zhang, et.al, 2013), warranty cost analysis (Chukova, et. al., 2005), modelling of the outbreak of an epidemic disease (Chan, et.al., 2006), and modelling of electricity prices (Chan, et al., 2014). In the meantime, some authors propose extended models to overcome the limitations of the GP (Finkelstein, 1993; Wang and Pham, 1996; Braun, et. al., 2005; Chan, et.al., 2006; Wu and Clements-Croome, 2006).

The GP is a stochastic process that is defined as (Lam, 1988): a sequence of random variables $\{X_k, k = 1, 2, \dots\}$ is a GP if the cdf (cumulative distribution function) of X_k is given by $F(a^{k-1}t)$ for $k = 1, 2, \dots$ and a is a positive constant.

As can be seen, the distinction between the GP and the renewal process lies in the fact that the inter-arrival times of the renewal process have the same distribution $F(t)$ over k's and the inter-arrival times of the GP have a cdf $F(a^{k-1}t)$, which changes over k's. In some scenarios such as reliability mathematics, this distinction makes the GP more attractive in application as it can model the failure process of ageing or deteriorating systems, which may have decreasing working times between failures.

While the GP is an important model and has been widely used in solving problems in various research areas, its scope is still limited and does not fit the purposes of various empirical studies. First, this model is not suitable for a stochastic process in which the inter-arrival times may need to be modelled by distributions with varying shape parameters. Second, it can merely describe stochastically increasing or decreasing stochastic processes. This paper aims to propose a new process that can overcome those two limitations and to study its probabilistic properties.

1.2 The geometric process and related work

This section introduces the GP and discusses its limitations in detail. We begin with an important definition on stochastic order.

Definition 1 *Stochastic order (p. 404 in Ross (1996)). Assume that X and Y are two random variables. If for every real number r, the inequality*

$$P(X \geq r) \geq P(Y \geq r)$$

holds, then X is stochastically greater than or equal to Y, or $X \geq_{st} Y$. Equivalently, Y is stochastically less than or equal to X, or $Y \leq_{st} X$.

From Definition 1, one can define the monotonicity of a stochastic process: Given a stochastic process $\{X_k, k = 1, 2, ...\}$, if $X_k \leq_{st} X_{k+1}$ ($X_k \geq_{st} X_{k+1}$) for $k = 1, 2, ...$, then $\{X_k, k = 1, 2, ...\}$ is said stochastically to be increasing (decreasing).

Lemma 1 *(p. 405 in Ross (1996)) Assume that X and Y are two random variables, then*

$$X \geq_{st} Y \text{ if only if } \mathbb{E}[u(X)] \geq \mathbb{E}[u(Y)],$$

for all increasing functions $u(.)$.

Lam proposes the definition of the GP, as shown below (Lam, 1988).

Definition 2 *(Lam, 1988) Given a sequence of non-negative random variables $\{X_k, k = 1, 2, \ldots\}$, if they are independent and the cdf of X_k is given by $F(a^{k-1}x)$ for $k = 1, 2, \ldots,$ where a is a positive constant, then $\{X_k, k = 1, 2, \cdots\}$ is called a geometric process (GP).*

We refer to the random variable X_k as the kth inter-arrival time in what follows.

Remark 1 *From Definition 1 and Lemma 2, we have the following results.*

- *If $a > 1$, then $\{X_k, k = 1, 2, \cdots\}$ is stochastically decreasing.*
- *If $a < 1$, then $\{X_k, k = 1, 2, \cdots\}$ is stochastically increasing.*
- *If $a = 1$, then $\{X_k, k = 1, 2, \cdots\}$ is a renewal process (RP).*
- *If $\{X_k, k = 1, 2, \ldots\}$ is a GP and X_1 follows the Weibull distribution, then the shape parameter of X_k for $k = 2, 3, \ldots$ remains the same as that of X_1. This observation is not specific to the Weibull distribution and holds for many other distributions with a scale and shape parameter such as the Gamma distribution.*

The GP offers an alternative process to model recurrent event processes. For example, in reliability mathematics, the renewal process (RP) and the non-homogeneous Poisson process (NHPP) are two widely used stochastic processes. The RP is normally used to model working times of a system if the system is renewed (or replaced with new and identical items upon failures) and the NHPP is used to model working times of a system where a repair restores the system to the status just before the failure happened, i.e., the repair is a minimal repair. Those assumptions of the RP and the NHPP may be too stringent in real applications. On the other hand, repairing a given item may have a limited number of methods, which implies that repair effect on the item is not random (Kijima, 1989). Meanwhile, the reliability of the item may decrease over time. Considering those facts, time between failures may therefore become shorter and shorter. The GP can model time between failures of such items.

Meanwhile, some authors either proposed similar definitions to that of the GP (Finkelstein, 1993; Wang and Pham, 1996) or made an attempt to extend the GP (Braun, et. al., 2005; Wu and Clements-Croome, 2006; Lam, 2007). Those different versions can be unified as: they replace a^{k-1} with $g(k)$, where $g(k)$ is a function of k and is defined differently by different authors, as discussed below.

For a sequence of non-negative random variables $\{X_k, k = 1, 2, \dots\}$, different consideration has been laid on the distribution of X_k, as illustrated in the following (in chronological order).

(i) Finkelstein (1993) proposes a process, named the *general deteriorating renewal process*, in which the distribution of X_k is $F_k(x)$, where $F_{k+1}(x) \leq F_k(x)$. A more specific model is defined such that $F_k(x) = F(a_k x)$ where $1 = a_1 \leq a_2 \leq a_3 \leq \dots$ and a_k are parameters. In this model, $g(k) = a_k$.

(ii) Wang and Pham (1996) defines a quasi-renewal process, which assumes $X_1 = W_1$, $X_2 = aW_2$, $X_3 = a^2 W_3, \dots$, and the W_k are independently and identically distributed and $a > 0$ is constant. Here, $g(k) = a^{1-k}$.

(iii) Braun, et. al. (2005) proposes a variant, which assumes that the distribution of X_k is $F_k(x) = F(k^{-a}x)$, or $g(k) = k^{-a}$. The authors proved that the expected number of event counts before a given time, or analogously, the Mean Cumulative Function (MCF) (or, the renewal function), tends to infinite for the decreasing GP. As such, they propose the process as a complement.

(iv) Wu and Clements-Croome (2006) set $g(k) = \alpha a^{k-1} + \beta b^{k-1}$, where α, β, a and b are parameters. Their intention is to extend the GP to model more complicated failure patterns such as the bathtub shaped failure patterns.

(v) Chan, et.al. (2006) extends the GP to the threshold GP: A stochastic process $\{Z_n, n = 1, 2, ...\}$ is said to be a threshold geometric process (threshold GP), if there exists real numbers $a_i > 0, i = 1, 2, ...$, and integers $\{1 = M_1 < M_2 < \dots\}$ such that for each $i = 1, 2, ...$, $\{a_i^{n-M_i} Z_n, M_i \leq n < M_{i+1}\}$ forms a renewal process.

Apparently, the model proposed in Finkelstein (1993) has a limitation in common: there is a need to estimate a large number of parameters, which may be problematic in real applications as a large number of failure data are needed to estimate the parameters. It should be noted that it is notoriously difficult to collect a large number of failure data in practice.

1.3 Comments on the geometric process and its extensions

While the GP is an important model and widely used, its scope is still limited and does not fit the purposes of various empirical studies due to the following two limitations.

- *Invariance of the shape parameter.* Suppose the cdf $F_k(x)$ of X_k in the GP have a scale parameter and a shape parameter. Then, all of the above discussed GP-like variants and extensions implicitly make an assumption: the processes merely change the scale parameter of $F_k(x)$, but keep their shape parameter constant over k's. In other words, none of the existing GP-like processes can model a recurrent event process whose shape parameter of $F_k(x)$ changes over k. To elaborate, let us take the Weibull distribution as an example. Assume that the cdf of X_1 is $F(x) = 1 - e^{-(\frac{x}{\theta_1})^{\theta_2}}$. Then according to the GP-like processes, the cdf of X_k is $F(g(k)x) = 1 - \exp\{-(\frac{x}{\theta_1 g^{-1}(k)})^{\theta_2}\}$. That is, the scale parameter $\theta_1 g^{-1}(k)$ is a function of k and it changes over k's, but the shape parameter θ_2 is independent of k and remains constant over different k's. This assumption may be too stringent and should be relaxed for a wider application. To this end, one may assume a natural extension of the GP, in which X_k has a cdf $F(g(k)x^{h(k)})$, where $h(k)$ is a function of k and the parameters in $h(k)$ are estimable. As a result, in the Weibull distribution case, for example, the inter-arrival times, X_k's, may be fitted with cdf $F(g(k)x^{h(k)}) = 1 - \exp\{-(\frac{x}{(\theta_1 g^{-1}(k))^{1/h(k)}})^{\theta_2 h(k)}\}$.

  A similar description of the above paragraph is the invariance of the CV (coefficient of variation). Assume that $\{X_1, X_2, ...\}$ follows the GP. Denote $\lambda_{11} = \mathbb{E}[X_1]$ and $\lambda_{21} = \mathbb{E}[X_1^2] - \lambda_{11}^2$. Then it is easy to obtain the expected value and the variance of X_k: $\mathbb{E}[X_k] = a^{(1-k)}\lambda_{11}$ and $\mathbb{V}[X_k] = a^{(2-2k)}\lambda_{21}$, respectively. The coefficient of variation (CV) of X_k is therefore given by $\gamma_k = \frac{\sqrt{\mathbb{V}[X_k]}}{\mathbb{E}[X_k]} = \sqrt{\lambda_{21}}/\lambda_{11}$, which suggests that the CVs are independent of k and keep constant over k's.

  An example of such a process with varying shape parameters in $F_k(x)$ can be found in Chan, et.al. (2006), in which X_k are the number of daily infected cases of an epidemic disease (i.e., the severe acute respiratory syndrome) in Hong Kong in 2003 are assumed to be independent and follow the threshold geometric process, in which $F_k(x)$ have different shape parameters for $k = 1, 2, ...$

- *Monotonicity of the GP.* From Remark 1, the GP $\{X_k, k = 1, 2, \dots\}$ change monotonously. That is, it can merely model the processes with increasing or decreasing inter-arrival times, or renewal processes. It is known, however, that the inter-arrival time processes of some real-world systems may exhibit non-monotonous failure patterns. For those systems, using the GP to model their failure processes is apparently inappropriate.

1.4 Contribution and importance of this work

This paper proposes a new stochastic process, the *doubly geometric process (DGP)*, which makes contribution to the literature in the following aspects.

- First, the DGP can model recurrent event processes where $F_k(x)$'s have different shape parameters over k's, which can be done by neither the GP-like models nor other repair models such as reduction of age models discussed in Doyen and Gaudoin (2004). One may note that the DGP differs from the research that treats the parameters in a lifetime distribution as functions of time (Zuo et al., 1999).

- Second, the DGP can model not only monotonously increasing or decreasing stochastic processes, but also processes with complicated failure intensity functions such as the bathtub shaped curves and the upside-down bathtub shaped curves, as can be seen from examples shown in Fig. 1. Noteworthily, although the models proposed by Wu and Clements-Croome (2006) and Chan, et.al. (2006) can also model complicated failure intensity functions, they assume that $F_k(x)$'s have constant shape parameters over k's and they need more parameters than the DGP (i.e., the DGP needs 2 parameters whereas the models proposed by Wu and Clements-Croome (2006) and Chan, et.al. (2006) need at least 3 parameters).

- Third, as Braun, et. al. (2005) points out, the GP has a limitation that *it only allows for logarithmic growth or explosive growth*. The DGP can overcome this limitation.

One may also notice that, in recent years, many authors have devoted considerable effort on developing novel methods to model repair processes, see Wu and Scarf (2015), for example. The current paper can of course be regarded as a new contribution to the literature of modelling repair processes.

The paper has important managerial implications, as it provides a more flexible model for wider application than the GP. Although this paper uses cases from reliability engineering, its results and discussion can also be applied to analyse other recurrent events. Such applications can be found in scientific studies, medical research, marketing research, etc, just as the GP can be used to model recurrent events such as the outbreaks of diseases (Chan, et.al., 2006) and the electricity price (Chan, et al., 2014).

1.5 Overview

The rest of the paper is structured as follows. Section 2 introduces the DGP and discusses its probabilistic properties. Section 3 proposes methods of parameter estimation. Section 4 compares the performance of the DGP with that of other models based on datasets collected from the real-world. We finish with a conclusion and future work in Section 5.

2 A doubly geometric process and its probabilistic properties

In this section, we propose the following definition and then discuss its statistical properties.

Definition 3 *Given a sequence of non-negative random variables* $\{X_k, k = 1, 2, \dots\}$*, if they are independent and the cdf of* X_k *is given by* $F(a^{k-1}x^{h(k)})$ *for* $k = 1, 2, \dots,$ *where* a *is a positive constant,* $h(k)$ *is a function of* k *and the likelihood of the parameters in* $h(k)$ *has a known closed form, and* $h(k) > 0$ *for* $k \in \mathbb{N}$*, then* $\{X_k, k = 1, 2, \cdots\}$ *is called a doubly geometric process (DGP).*

In the above definition, for the sake of simplicity, we call the process as *doubly geometric process* since the process can include two geometric processes: $\{a^{k-1}, k = 1, 2, \dots\}$ is a geometric series and $\{h(k), k = 1, 2, \dots\}$ can be a geometric series.

We refer to a^{k-1} as the scale impact factor and $h(k)$ as the shape impact factor. It should be noted that the cdf of X_1 is $F(x)$.

Remark 2 *Similar to the definition of the quasi-renewal process given by Wang and Pham (1996), one may give an alternative definition of Definition 3 as: assume* $X_1 = W_1$, $X_2 = (a^{-1}W_2)^{1/h(1)}$, $\dots$, $X_k = (a^{1-k}W_2)^{1/h(k)}, \dots$ *and the* W_k *are i.i.d., then the process* $\{X_k, k = 1, 2, \dots\}$ *is called a doubly geometric process.*

Although the extension from the GP to the DGP seems quite natural, it may create difficulties in mathematical derivation. For example, deriving some probability properties of the DGP becomes much more complicating than that of the GP, it is difficult to derive a closed-form of the MCF for the DGP whereas an explicit iteration equation of the MCF for the GP can be derived.

Remark 3 *From Definition 3, it follows the results below.*

(i) If $h(k) = 1$*, then* $\{X_k, k = 1, 2, \cdots\}$ *reduces to the geometric process.*

(ii) Denote $\lambda_{1k} = \mathbb{E}[X_1^{h^{-1}(k)}] = \int_0^\infty x^{h^{-1}(k)} f(x)dx$ *and* $\lambda_{2k} = \mathbb{E}[X_1^{2h^{-1}(k)}] = \int_0^\infty x^{2h^{-1}(k)} f(x)dx$*, where* $f(x) = \partial F(x)/\partial x$ *exists and* $h^{-1}(k) = \frac{1}{h(k)}$*. Assume that* $\mathbb{E}[X_1^{h^{-1}(k)}] < \infty$ *and* $\mathbb{E}[X_1^{2h^{-2}(k)}] < \infty$*. Then it is easy to obtain the expected value and the variance of* X_k*:* $\mathbb{E}[X_k] = a^{(1-k)h^{-1}(k)}\lambda_{1k}$ *and* $\mathbb{V}[X_k] = a^{(2-2k)h^{-1}(k)}\lambda_{2k} - \lambda_{1k}^2$ *for* $k = 1, 2, \dots$*.*

(iii) If X_1 follows the exponential distribution and

(a) if $\{X_k, k = 1, 2, \dots\}$ follows the GP, then X_k (for $k = 2, 3, \dots$) follows the exponential distribution with different rate parameters from that of X_1,

(b) if $\{X_k, k = 1, 2, \dots\}$ follows the DGP, then X_k (for $k = 2, 3, \dots$) follows the Weibull distribution,

(iv) If $\{X_k, k = 1, 2, \dots\}$ follows the DGP and X_1 follows the Weibull distribution, then X_k (for $k > 1$) follows the Weibull distribution with different shape and scale parameters from those of X_1.

If we assume that $\{X_1, X_2, ...\}$ follows the DGP. Then from (ii) in Remark 3, the coefficient of variation (CV) of X_k is $\gamma_k = \frac{\sqrt{\mathbb{V}[X_k]}}{\mathbb{E}[X_k]} = \frac{\sqrt{a^{(2-2k)h^{-1}(k)}\lambda_{2k} - \lambda_{1k}^2}}{a^{(1-k)h^{-1}(k)}\lambda_{1k}}$, which implies that the CVs change over k's. Hence, we can make the following conclusion.

Lemma 2 *Suppose that $\{X_k, k = 1, 2, \cdots\}$ is a GP, then the coefficient of variation (CV) of X_k changes over k's.*

Now a question arisen is the selection of the forms of $h(k)$. In what follows, we investigate the DGP with the $h(k)$ defined below:

$$h(k) = (1 + \log(k))^b, \tag{1}$$

where log is the logarithm with base 10 and b is a parameter.

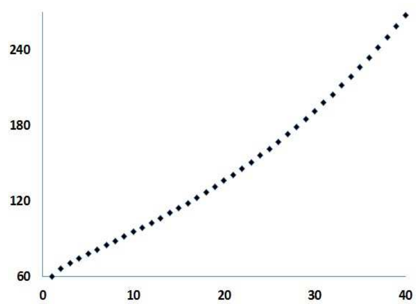
2.1 Probabilistic properties of the DGP with $h(k) = (1 + \log(k))^b$

In this entire section, i.e., Section 2.1, we assume $h(k) = (1 + \log(k))^b$.

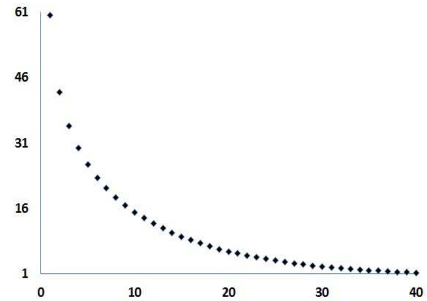
The reason that we select $h(k) = (1 + \log(k))^b$ is: we have fit the DGP with different $h(k)$, which are b^{k-1}, $b^{\log(k)}$, and $1 + b\log(k)$, on ten real-world datasets (see Section 4) and found that the DGP with $h(k) = (1 + \log(k))^b$ outperforms the processes with the other three $h(k)$'s. In real applications, it is suggested that other form of $h(k)$ may also be investigated and selected once a comparison on the performance of difference $h(k)$ has been made.

In selecting $h(k)$, one may set some conditions, for example, $h(1) = 1$ and $h(k) > 0$ for $k = 1, 2, \dots$.

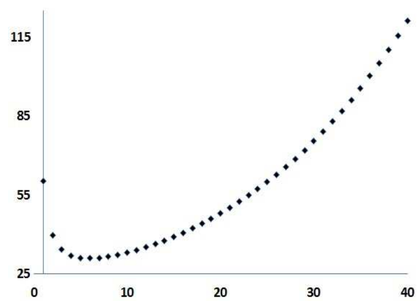
Unlike the GP that can only be either stochastically increasing or stochastically decreasing, the DGP can model more flexible processes, as shown in the four examples in Figure 1.



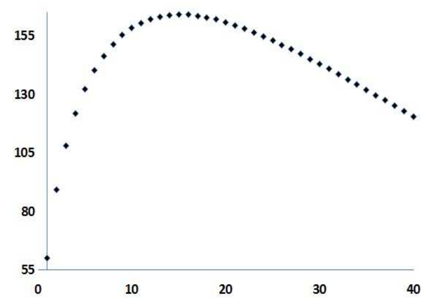
(a) $a = 0.97, b = -0.05, \theta_1 = 40$ and $\theta_2 = 0.6$.



(b) $a = 1.1, b = 0.2, \theta_1 = 40$ and $\theta_2 = 0.6$.



(c) $a = 0.92, b = 0.4, \theta_1 = 40$ and $\theta_2 = 0.6$.



(d) $a = 1.02, b = -0.3, \theta_1 = 40$ and $\theta_2 = 0.6$.

Figure 1: DGPs with different parameter settings.

Proposition 1 *Given a DGP* $\{X_k, k = 1, 2, \dots\}$,

(i) if $0 < a < 1$, $P(X_1 > 1) = 1$, *and* $b < 0$, *then* $\{X_k, k = 1, 2, \dots\}$ *is stochastically increasing.*

(ii) if $a > 1$, $P(0 < X_1 < 1) = 1$, *and* $b < 0$, *then* $\{X_k, k = 1, 2, \dots\}$ *is stochastically decreasing.*

(iii) if $0 < a < 1$, $P(0 < X_1 < 1) = 1$, *and* $0 < b < 4.898226$, *then* $\{X_k, k = 1, 2, \dots\}$ *is stochastically increasing.*

(iv) if $a > 1$, $P(X_1 > 1) = 1$, *and* $0 < b < 4.898226$, *then* $\{X_k, k = 1, 2, \dots\}$ *is stochastically decreasing.*

Proposition 2 *Given a DGP* $\{X_k, k = 1, 2, \dots\}$ *with* $h(k) = (1 + \log(k))^b$, *if* $(1 + \log(k+1))^{-b}(\log(y) - k\log(a)) + (1 + \log(k))^{-b}((k-1)\log(a) - \log(y))$ *varies between negative and positive values, then the DGP is not stochastically monotonous over k's, where y represents all the possible values on* X_k *(for* $k = 1, 2, \dots$*).*

Stochastic ageing properties are widely discussed in the reliability literature. For example, $F(t)$ is IFR (Increasing Failure Rate) if $\frac{f(t)}{\bar{F}(t)}$ is increasing in t for all $t \geq 0$, where $f(t) = \frac{dF(t)}{dt}$

and $\bar{F}(t) = 1 - F(t)$. With regard to the stochastic ageing properties of the DGP, we have the following proposition.

Proposition 3 *Suppose $\{X_k, k = 1, 2, \ldots\}$ follows the DGP. If $b > 0$ and $F(t)$ is IFR, then the cdf $F_k(t)$ of X_k is IFR.*

Suppose $\{X_k, k = 1, 2, \ldots\}$ follows the DGP, denote $S_n \equiv \sum_{k=1}^{n} X_k$ with $S_0 \equiv 0$. Then the distribution of S_n is

$$\begin{aligned} P(S_n \le t) &= P(S_{n-1} + X_n \le t) \\ &= \int_0^t F^{(n-1)}(t-u) dF_n(u) \\ &= \int_0^t F^{(n-1)}(t-u) \left(a^{n-1}(1+\log(n))^b u^{(1+\log(n))^b - 1} f(a^{n-1} u^{(1+\log(n))^b}) \right) du \\ &= \int_0^{a^{n-1} t^{(1+\log(n))^b}} F^{(n-1)}(t - a^{(1-n)(1+\log(n))^{-b}} v^{(1+\log(n))^{-b}}) f(v) dv, \end{aligned} \tag{2}$$

where $F^{(0)}(t) = 1$ and $F^{(n)}(t) \equiv P(S_n \le t)$. Let $N(t) = \max\{n : S_n \le t\}$, then the MCF, $m(t)$, is given by

$$m(t) = \mathbb{E}[N(t)] = \sum_{n=1}^{\infty} P(S_n \le t). \tag{3}$$

Denote

$$m_1(t) = \sum_{n=1}^{\infty} P(\sum_{k=1}^{n} Y_k \le t), \tag{4}$$

where $\{Y_k : k \ge 1\}$ is a renewal process with $Y_k > 0$ and the cdf of the inter-arrival times is $F(x)$ (which has the same as the cdf of X_1). Then, equivalently, $m_1(t)$ is the MCF of the ordinary renewal process $\{N_1(t) : t \ge 0\}$ with $N_1(t) \equiv \max\{n : \sum_{k=1}^{n} Y_k \le t\}$. For $\{Y_k : k \ge 1\}$, $m_1(t) = F(t) + \int_0^t m_1(t-y) dF(y)$, as can be seen in many textbooks of stochastic processes (for example, see Ross (1996)).

Unlike the MCF, $m_1(t)$, for the ordinary renewal process where an iteration equation can be given, deriving an iteration equation for $m(t)$ defined in Eq. (3) seems not an easy task. In real applications, numerical analysis may be sought. For example, on the four examples used in Figure 1, we run the Monte Carlo simulation for 2000 times and estimate the values of the MCF for each example. Figure 2 shows the values of the MCF of the four examples with the parameter settings shown in Figure 1.

Below, the lower bounds or the upper bounds are given for two scenarios.

Proposition 4 *(i) Given that $m_1(t)$ and $m(t)$ are defined in Eq. (3) and Eq. (4), respectively,*

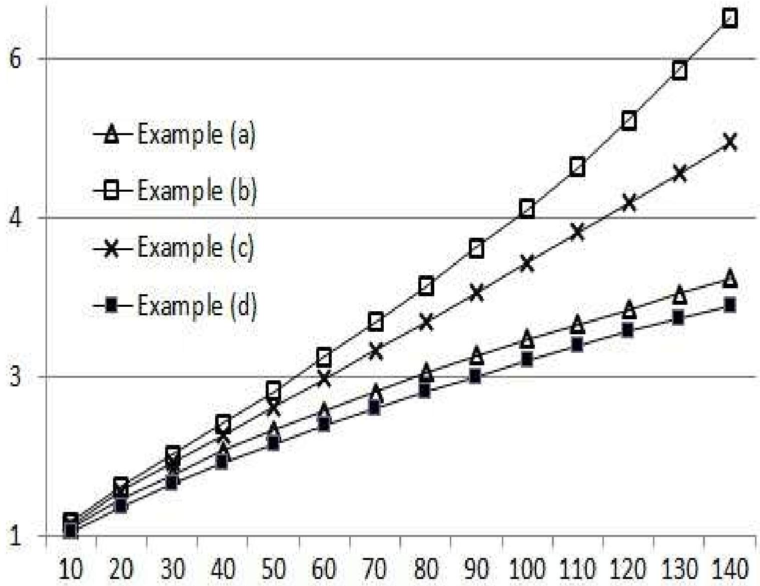


Figure 2: The MCF, $m(t)$, of the four examples shown in Figure 1.

if $\{X_k, k = 1, 2, \dots\}$ *is stochastically non-decreasing, then*

$$m(t) \le m_1(t). \tag{5}$$

(ii) Suppose that $\{X_k, k = 1, 2, \dots\}$ *follows the DGP and* $P(X_k < c) = 1$ *for* $k = 1, 2, \dots$ *and* c *is a positive real number. Denote* $\Lambda_n = \sum_{k=1}^{n} \mathbb{E}[X_k]$ *and* $\sigma^2 = \frac{1}{n}\sum_{k=1}^{n} \mathbb{V}[X_k]$. *Assume that* $\{X_k, k = 1, 2, \dots\}$ *is stochastically non-increasing and* $t > \lim_{n\to\infty} \Lambda_n (< +\infty)$, *then*

$$m(t) \ge \max\left\{ m_1(t), \sum_{n=1}^{\infty} \left[1 - \exp\left(-\frac{n\sigma^2}{c^2} H\left(\frac{ct - c\Lambda_n}{n\sigma^2}\right) \right) \right] \right\}. \tag{6}$$

The following proposition compares the MCFs of the GP and the DGP.

Proposition 5 *Suppose that* $\{X_k^g, k = 1, 2, \cdots\}$ *is a GP with* $X_k^g \sim F(a^{k-1}x)$ *and* $\{X_k^d, k = 1, 2, \cdots\}$ *is a DGP with* $X_k^d \sim F(a^{k-1}x^{(1+\log(k))^b})$. *Denote* $m^g(t) = \sum_{n=1}^{\infty} P(\sum_{k=1}^{n} X_k^g \le t)$ *and* $m^d(t) = \sum_{n=1}^{\infty} P(\sum_{k=1}^{n} X_k^d \le t)$. *Then,*

(i) $m^g(t) > m^d(t)$ *if* $0 < a < 1$, $b < 0$ *and* $P(X_1 > 1) = 1$, *or if* $a > 1$, $b > 0$ *and* $P(0 < X_1 < 1) = 1$.

(ii) $m^g(t) < m^d(t)$ *if* $0 < a < 1$, $b > 0$ *and* $P(X_1 > 1) = 1$, *or if* $a > 1$, $b < 0$ *and* $P(0 < X_1 < 1) = 1$.

The following proposition compares the MCFs of two DGPs.

Proposition 6 *Suppose that* $\{X_k^{d_1}, k = 1, 2, \cdots\}$ *with* $X_k^{d_1} \sim F(a_1^{k-1} x^{(1+\log(k))^{b_1}})$ *is a DGP and* $\{X_k^{d_2}, k = 1, 2, \cdots\}$ *with* $X_k^{d_2} \sim F(a_2^{k-1} x^{(1+\log(k))^{b_2}})$ *is a DGP. Denote* $m^{d_1}(t)$ $= \sum_{n=1}^{\infty} P(\sum_{k=1}^{n} X_k^{d_1} \leq t)$ *and* $m^{d_2}(t) = \sum_{n=1}^{\infty} P(\sum_{k=1}^{n} X_k^{d_2} \leq t)$.

(i) If $a_1 = a_2$ *and* $b_1 > b_2$,

- $m^{d_1}(t) < m^{d_2}(t)$ *if* $a > 1$ *and* $P(0 < X_1 < 1) = 1$,
- $m^{d_1}(t) > m^{d_2}(t)$ *if* $0 < a < 1$ *and* $P(X_1 > 1) = 1$.

(ii) $m^{d_1}(t) < m^{d_2}(t)$ *if* $b_1 = b_2$ *and* $a_1 > a_2$.

(iii) $m^{d_1}(t) > m^{d_2}(t)$ *if* $a_2 > a_1 > 1$, $b_1 > b_2$, *and* $P(X_1 > 1) = 1$.

(iv) $m^{d_1}(t) < m^{d_2}(t)$ *if* $0 < a_1 < a_2 < 1$, $b_1 > b_2$, *and* $P(X_1 < 1) = 1$.

Proposition 1 shows the monotonicity property of the DGP, but it has not shown the convergence of the DGP in probability. The following property addresses this issue.

Proposition 7 *Given a DGP* $\{X_k, k = 1, 2, \ldots\}$,

(i) if $0 < a < 1$, *then then* X_k *converges to infinity in probability as* $k \to \infty$,

(ii) if $a > 1$, *then* X_k *converges to zero in probability as* $k \to \infty$.

2.2 Discussion

We make the following discussion.

- *On the scale impact factor* $g(k)$ *and the shape impact factor* $h(k)$. Although we only discussed the DGP in which the scale impact factor is set to $g(k) = a^{k-1}$, $g(k)$ may also be replaced with other forms of functions such as those proposed in Finkelstein (1993); Braun, et. al. (2005); Wu and Clements-Croome (2006); Chan, et.al. (2006). The function $h(k) = (1 + \log(k))^b$ in Eq. (1) can be replaced with any other functions of k, for example, $h(k) = b^{k-1}$, or $h(k) = b^{\log(k)}$ etc. However, the propositions of DGPs with different $g(k)$ and $h(k)$ are discussed in the following bullet.

- *On the propositions.* Among the propositions discussed in Section 2.1, Proportion 4 holds for any $g(k)$ and $h(k) > 0$ as both $g(k)$ and $h(k) > 0$ are not involved in the proof process of Proposition 4. But the other propositions are discussed for the case where $g(k) = a^{k-1}$ and $h(k) = (1 + \log(k))^b$.

3 Estimation of the parameters in the DGP

In this section, we discuss two methods of estimation of the parameters in the DGP.

3.1 Least squares method

For the geometric process, Lam (1992) develops a method, which is a least squares method, to estimate the parameters in the GP. With a similar method, we estimate the parameters in the DGP in this section.

Suppose that a process $\{X_k, k = 1, 2, \dots\}$ follows the DGP with $X_k \sim F(a^{k-1}x^{(1+\log(k))^b})$. Let

$$Z_k = a^{k-1}X_k^{(1+\log(k))^b}. \tag{7}$$

Then $\{Z_k, k = 1, 2, \dots\}$ follows an ordinary renewal process. Given observations x_k of X_k (for $k = 1, 2, \dots$), from Eq. (7), we can have

$$\mu = a^{k-1}x_k^{(1+\log(k))^b} + e_k \tag{8}$$

where $\mu = \mathbb{E}[Z_k]$ and e_k are i.i.d. random variables each having mean 0 and a constant variance.

When $b \neq 0$, it is not possible to linearise model (8) by means of a suitable transformation, that is, model (8) is intrinsically nonlinear.

For given observations x_k of X_k (with $k = 1, 2, ..., N_0$), one can minimise the following sum of the squares of the errors to estimate the parameters a, b and μ.

$$(\hat{\mu}, \hat{a}, \hat{b}) = \arg\min_{\mu,a,b} \sum_{k=1}^{N_0} \left(x_k - (\mu a^{1-k})^{(1+\log(k))^{-b}} \right)^2. \tag{9}$$

Obviously, there is no general closed-form solution for $\hat{\mu}$, $\hat{a}$, and $\hat{b}$, one needs therefore pursue nonlinear programming methods to solve the problem.

The reader is referred to Theorem 2.1 in page 24 in the book by Seber and Wild (2003) for obtaining the asymptotic distributions of $(\hat{\mu}, \hat{a}, \hat{b})$.

3.2 Maximum likelihood method

Suppose that one observes N systems starting from time 0 until time T. Assume that system j ($j = 1, 2, \dots, N$) has failed for N_j times at time points $s_{j,k}$ with $k = 0, 1, \dots, N_j$. Let $s_{j,0} = 0$. Then the working times of system j are $s_{j,1} - s_{j,0}$, $s_{j,2} - s_{j,1}$, $\dots$, $s_{j,N_j} - s_{j,N_j-1}$, and $T - s_{j,N_j}$, respectively. Denote $x_{j,i} = s_{j,i} - s_{j,i-1}$ for $i = 1, 2, ..., N_j$ and $x_{j,N_j+1} = T - s_{j,N_j}$.

Then, for the DGP with $h(k) = (1 + \log(k))^b$, the likelihood function is given by

$$\begin{aligned} L(a, b, \boldsymbol{\theta}) &= \prod_{j=1}^{N} \left\{ \left[1 - F(a^{N_j}(x_{j,N_j})^{(1+\log(N_j+1))^b})\right] \prod_{k=1}^{N_j} f_k(x_{j,i}) \right\} \\ &= \prod_{j=1}^{N} \Bigg\{ \left[1 - F(a^{N_j}(x_{j,N_j})^{(1+\log(N_j+1))^b})\right] \\ &\quad \times \prod_{k=1}^{N_j} \left[a^{k-1}(1 + \log(k))^b (x_{j,i})^{(1+\log(k))^b - 1} f(a^{k-1}(x_{j,i})^{(1+\log(k))^b})\right] \Bigg\}, \end{aligned} \tag{10}$$

where $\prod_{k=1}^{N_j} \bullet = 1$ for $N_j = 0$, $\boldsymbol{\theta}$ is the vector of the parameters of distribution $F(x)$.

Maximising the above likelihood function, we can obtain $\hat{a}$, $\hat{b}$, and $\hat{\boldsymbol{\theta}}$, which are the estimates of the corresponding parameters, respectively. That is

$$(\hat{a}, \hat{b}, \hat{\boldsymbol{\theta}}) = \arg\max_{a,b,\boldsymbol{\theta}} L(a, b, \boldsymbol{\theta}). \tag{11}$$

Denote $\boldsymbol{\vartheta} = (a, b, \boldsymbol{\theta})$, where $\vartheta_1 = a$, $\vartheta_2 = b$. The Fisher information matrix $I_{N_0}(\hat{a}, \hat{b}, \hat{\boldsymbol{\theta}})$ can then be calculated by $I_{N_0}(\hat{a}, \hat{b}, \hat{\boldsymbol{\theta}}) = -\mathbb{E}\left(\frac{\partial^2 \log L(a,b,\boldsymbol{\theta})}{\partial\vartheta_i \partial\vartheta_j}\right)|_{\boldsymbol{\vartheta}=(\hat{a},\hat{b},\hat{\boldsymbol{\theta}})}$, which can be used to estimate the asymptotic variance-covariance matrix of $(\hat{a}, \hat{b}, \hat{\boldsymbol{\theta}})$. In this paper, the Fisher information matrix will be used to calculate the standard deviations of the estimated parameters.

Obviously, there is no general closed-form solution in Eq. (10) for the MLE of $\hat{a}$, $\hat{b}$, and $\hat{\boldsymbol{\theta}}$.

4 Applications of the DGP

In Section 4.1 and Section 4.2, two case studies based on real-world datasets are conducted to compare the performance of the DGP with $h(k) = (1 + \log(k))^b$, in terms of the corrected Akaike information criterion, or AICc for short.

- For the least squares method, model performance is measured by the root mean squared error (RMSE)= $\sqrt{\frac{1}{N_0}\sum_{k=1}^{N_0}(x_k - \hat{x}_k)}$, where $\hat{x}_k$ is the estimate of the x_k.

- For the maximum likelihood method, model performance is measured with the AICc value, $N_0 \ln(L) + 2p + \frac{2p(p+1)}{n-p+1}$, where p is the number of parameters in the model and L is the maximised likelihood. The reader is referred to Burnham and Anderson (2004) for more discussion on the AICc. The value $2p + \frac{2p(p+1)}{n-p+1}$ in the AICc value is a penalty term that is proportional to the number p of parameters in a model.

4.1 Estimating the number of warranty claims

Table 1 shows warranty claim data that were collected from a networking card manufacturer. The manufacturer ships a certain number of items to its retailers on a month basis and then the warranty agency manages warranty claims. The exact number of the items sold in a shipment is unknown to the warranty agency. It includes the number of warranty claims in consecutive 12 months on 20 shipments. For example, the underlined number 8 in month 2 and shipment 3 means that 8 2-month-old items that were claimed were from shipment 3 (or they were shipped in month 3). The last column shows the CV of the warranty claims in each month.

Figure 3 illustrates the coefficient of variation (CV) on the warranty claims over the 12 months. As can be seen, the CV values show an increasing trend. Following Lemma 2, it is more appropriate to use the DGP to fit the data than the GP.

We fit the data with the nonparametric method by solving the problem for the DGP:

$$(\hat{\mu}, \hat{a}, \hat{b}) = \arg\min_{\mu,a,b} \sum_{i=1}^{20} \sum_{k=1}^{12} \left(x_{k,i} - (\mu a^{1-k})^{(1+\log(k))^{-b}} \right)^2 \tag{12}$$

where $x_{k,i}$ is the number of warranty claims of k-month-old items that are shipped in month i. Similarly, the parameters of the GP are estimated. For the DGP model, $\hat{\mu} = 9.19(3.495)$, $\hat{a} = 1.00232(0.114)$ and $\hat{b} = 0.250(0.739)$ (the values in the brackets are the estimate errors of the corresponding estimates). The AICc values are $\text{AICc}_{\text{DGP}} = 630.090$ and $\text{AICc}_{\text{GP}} = 630.242$, which suggests that the DGP outperforms the GP.

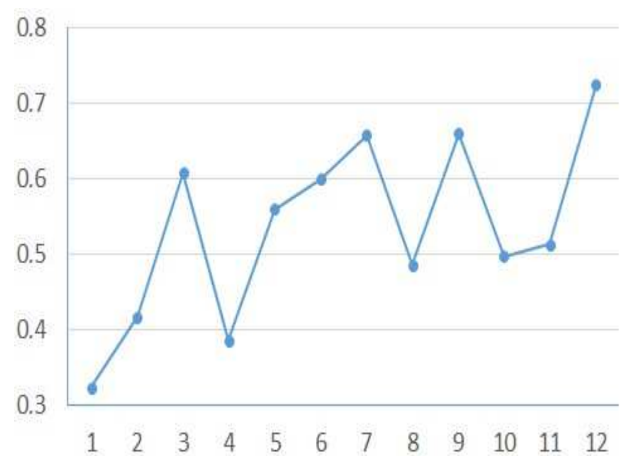


Figure 3: Change of the CVs over 12 months.

Table 1: Time between warranty claims of 22 identical items (unit: day).

Shipments / Months	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	CV
1	10	8	13	7	8	16	9	6	7	15	11	9	13	7	9	6	13	10	9	5	0.323
2	7	4	$\underline{8}$	6	9	6	1	8	8	9	11	10	10	9	7	8	1	3	9	12	0.417
3	11	7	15	3	4	3	3	13	9	13	6	4	3	5	5	6	3	2	8	5	0.607
4	8	3	12	6	7	6	11	9	9	7	10	7	8	11	6	5	8	5	6	17	0.385
5	4	3	4	2	8	6	7	15	7	9	10	5	2	6	4	14	3	7	10	13	0.559
6	11	8	5	10	4	5	7	8	1	6	11	1	3	4	3	9	4	5	16	13	0.599
7	7	7	22	3	5	14	12	5	4	7	9	4	4	6	17	4	13	3	6	5	0.658
8	11	8	4	5	4	12	6	10	3	4	8	3	5	12	9	10	3	11	4	4	0.486
8	4	3	16	7	1	8	3	6	1	5	6	4	4	12	5	2	4	5	5	6	0.660
10	2	5	9	4	3	10	11	8	1	12	8	6	10	7	2	3	9	10	6	9	0.497
11	5	4	8	4	7	12	1	9	5	8	4	7	3	2	3	5	13	8	7	6	0.513
12	4	5	2	6	1	7	6	10	4	3	12	2	2	17	4	13	6	1	9	5	0.724

4.2 Modelling time-between-failure data

4.2.1 The datasets

Two datasets published in Kumar and Klefsjö (1992); Ascher and Feingold (1984) are used in this section. Both datasets are collected from the real world and are time-between-failures. The names and the sample sizes of the datasets are shown in Table 2, where N_0 is the sample size. Kumar and Klefsjö (1992) develop a power-law-based non-homogeneous Poisson process (PL-NHPP) model on dataset 1 and Lam (2007) develops geometric process models and PL-NHPP models on dataset 2, which allow us to compare the performance of the DGP with their results.

Table 2: The datasets, including TBF(Time between failures).

No.	Dataset	N_0	References
1	Hydraulic system (LHD3)	25	Kumar and Klefsjö (1992)
2	Propulsion diesel engine failure data	71	Ascher and Feingold (1984)

In the following, we compare the performance of the models that are estimated with the least squares and the maximum likelihood estimation methods, respectively.

4.2.2 Model comparison

Definition 3 assumes that $\{X_k, k = 1, 2, \dots\}$ in the DGP are independent. We therefore use the Box-Ljung test to check the hypothesis that a given series of data is independent (Ljung and Box, 1978). Applying the Box-Ljung test on datasets 1 and 2, the result fails to reject the null hypothesis that observations in datasets 1 and 2 are independent at the 5% level of significance.

On the two datasets listed in Table 2, we use both the least squares method and the maximum likelihood method to estimate the parameters and then compare the performance of the DGP with the GP.

With the least squares method, both the DGP and the GP are estimated and their RMSE values are denoted by RMSE_{DGP} and RMSE_{GP}, respectively. The estimated parameters and their standard deviations (which are shown in brackets under the estimated parameters), and the RMSE values of both the DGP and the GP are shown in Table 3. As can been seen, the RMSE values of the DGP on each dataset is smaller than the RMSE values of the GP, based on which one can conclude the DGP outperforms the GP on both datasets.

Table 3: Comparison of the performance of the GP and the DGP based on the least squares method.

	Parameters of the DGP			Parameters of the GP			
No.	$\hat{a}$	$\hat{b}$	$\hat{\mu}$	$\hat{a}$	$\hat{\mu}$	RMSE_{DGP}	RMSE_{GP}
1	0.944	0.499	531.406	1.0382	209.841	<u>111.729</u>	144.431
	(0.0559)	(0.174)	(109.390)	(0.0315)	(67.652)		
2	0.909	0.488	147.624	0.972	56.702	<u>65.670</u>	69.810
	(0.0607)	(0.280)	(62.664)	(0.0181)	(20.486)		

Suppose $F(t) = 1 - e^{-(\frac{t}{\theta_1})^{\theta_2}}$. With the maximum likelihood method, we use the DGP, the GP, the PL-NHPP to fit the two datasets, and denote their corresponding AICc values as AICc_{DGP}, AICc_{GP}, and AICc_{PL}, respectively. The number of the parameters (i.e., a, b, θ_1, θ_2) in the DGP and the number of the parameters (i.e., a, θ_1, θ_2) in the GP are 4 and 3, respectively, i.e., $p = 4$ for the DGP and $p = 3$ for the GP. The number of the parameters in the PL-NHPP is 2 (i.e., $p = 2$). The results are shown in Table 4. The estimated parameters and their standard deviations (which are shown in brackets under the estimated parameters) of the DGP are also given in the table. On the rest comparison, the AICc values of the DGP are the smallest.

In addition to the independence test conducted before, to test the assumption that the DGP can model datasets 1 and 2, we use the Cramér-von-Mises test to test the null hypotheses that $\{\hat{a}^{k-1}X_k^{(1+\log(k))^{\hat{b}}}, k = 1, \ldots, N_0\}$ on datasets 1 and 2 follow the Weibull distribution, respectively. We conduct the hypothesis testing with a R-package *EWGoF* (Krit, 2014). The results fail to reject the null hypotheses at the 5% level of significance.

4.3 Comparison between different forms of $h(k)$

In the preceding sections, we set $h(k) = (1 + \log(k))^b$ in Definition 3. By setting other forms of $h(k)$ such as $h(k) = b^{k-1}$, $h(k) = b^{\log(k)}$, or $h(k) = 1 + b\log(k)$, one can define other forms of

Table 4: Comparison of the performance of the GP and the DGP based on the maximum likelihood method.

	Estimated Parameters of the DGP				Estimated Parameters of the GP					
No.	$\hat{a}$	$\hat{b}$	$\hat{\theta}_1$	$\hat{\theta}_2$	$\hat{a}$	$\hat{\theta}_1$	$\hat{\theta}_2$	$\mathrm{AICc}_{\mathrm{DGP}}$	$\mathrm{AICc}_{\mathrm{GP}}$	$\mathrm{AICc}_{\mathrm{PL}}$
1	0.884	0.638	449.165	0.789	1.0147	168.807	1.0287	301.376	304.182	311.851
	(0.0938)	(0.352)	(337.92)	(0.227)	(0.0230)	(58.139)	(0.159)			
2	0.899	0.502	147.636	0.964	0.983	73.070	1.295	318.030	319.445	323.094
	(0.0714)	(0.349)	(103.569)	(0.281)	(0.0151)	(19.461)	(0.182)			

the DGP. To differentiate them, we refer to the processes with $h(k) = (1 + \log(k))^b$, $h(k) = b^{k-1}$, $h(k) = b^{\log(k)}$ and $h(k) = 1 + b\log(k)$ as $\mathrm{DGP}_{\mathrm{log1}}$, $\mathrm{DGP}_{\mathrm{exp}}$, $\mathrm{DGP}_{\mathrm{log2}}$, and $\mathrm{DGP}_{\mathrm{log3}}$, respectively. Similarly, one can estimate parameters a and b of the $\mathrm{DGP}_{\mathrm{exp}}$, $\mathrm{DGP}_{\mathrm{log2}}$, and $\mathrm{DGP}_{\mathrm{log3}}$ with either the least squares or the maximum likelihood estimation method. We have compared the AICc values of the $\mathrm{DGP}_{\mathrm{log1}}$ with the AICc values of the rest three models on the ten datasets and found that the AICc value of the $\mathrm{DGP}_{\mathrm{log}}$ on each dataset is smaller than those of the other three models, respectively, which implies that the DGP with $h(k) = (1 + \log(k))^b$ outperforms. That is the reason that we investigated the GDP with $h(k) = (1 + \log(k))^b$ in this paper.

5 Conclusion and future work

This paper proposed a new stochastic process, the doubly geometric process (DGP), which extends the geometric process (GP). The DGP can overcome three limitations inherent in the GP. The paper discussed probabilistic properties of the DGP with $h(k) = (1 + \log(k))^b$, compared the mean cumulative functions between the DGP and other processes, and then proposed methods of estimation of the parameters in the DGP.

The paper also applied the DGP to fit two inter-arrival time datasets collected from the real world and then compared its performance with the performance of other models. It is found that the DGP outperforms the other models on those datasets. This has practical implications for lifecycle costing, for example.

As the DGP is a new model, there are plenty of questions waiting for answers. Those questions include, for example, what are the differences between the DGP and the other models in terms of the application of the DGP in reliability mathematics? Before we fit a given dataset with the DGP, how can we test if the dataset agrees with the DGP? To answer those questions will be our future work.

Acknowledgements

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Appendix

Proof of Proposition 1.

Let $u(x)$ denote a non-decreasing function. With Lemma 1, X_k is stochastically increasing if $\mathbb{E}[u(X_{k+1})] - \mathbb{E}[u(X_k)] > 0$ and let $y = a^{k-1}x^{(1+\log(k))^b}$, then we can obtain

$$\begin{aligned}
\mathbb{E}[u(X_{k+1})] - \mathbb{E}[u(X_k)] &= \int_0^{+\infty} u(x)dF(a^k x^{(1+\log(k+1))^b}) - \int_0^{+\infty} u(x)dF(a^{k-1}x^{(1+\log(k))^b}) \\
&= \int_0^{+\infty} \Big(u(a^{-k(1+\log(k+1))^{-b}} y^{(1+\log(k+1))^{-b}}) \\
&\quad - u(a^{(1-k)(1+\log(k))^{-b}} y^{(1+\log(k))^{-b}}) \Big) \, dF(y) \\
&> 0. \qquad (13)
\end{aligned}$$

Let y represent all the possible values on X_k (for $k = 1, 2, \ldots$). Hence, $\mathbb{E}[u(X_{k+1})] - \mathbb{E}[u(X_k)] > 0$ if $u(a^{-k(1+\log(k+1))^{-b}} y^{(1+\log(k+1))^{-b}}) - u(a^{(1-k)(1+\log(k))^{-b}} y^{(1+\log(k))^{-b}}) > 0$. As $u(.)$ is a non-decreasing function, $u(a^{-k(1+\log(k+1))^{-b}} y^{(1+\log(k+1))^{-b}}) - u(a^{(1-k)(1+\log(k))^{-b}} y^{(1+\log(k))^{-b}}) > 0$ iff

$$\frac{a^{-k(1+\log(k+1))^{-b}} y^{(1+\log(k+1))^{-b}}}{a^{(1-k)(1+\log(k))^{-b}} y^{(1+\log(k))^{-b}}} = a^{-k(1+\log(k+1))^{-b}+(k-1)(1+\log(k))^{-b}} y^{(1+\log(k+1))^{-b}-(1+\log(k))^{-b}} > 1. \qquad (14)$$

From equality (14), we have

- if $b < 0$, then $-k(1+\log(k+1))^{-b} + (k-1)(1+\log(k))^{-b} < 0$ and $(1+\log(k+1))^{-b} - (1+\log(k))^{-b} > 0$. That implies,

  (i) if $0 < a < 1$, $P(X_1 > 1) = 1$, and $b < 0$, the inequality (14) holds. Then $\{X_k, k = 2, 3, \ldots\}$ is stochastically increasing, and

  (ii) if $a > 1$, $P(0 < X_1 < 1) = 1$, and $b < 0$, the greater-than sign in the inequality (14) should be changed to the smaller-than sign. Then $\{X_k, k = 2, 3, \ldots\}$ is stochastically decreasing.

- On the other hand, if $b > 0$, $(1+\log(k+1))^{-b} - (1+\log(k))^{-b} < 0$. But if $b > 0$, $-k(1+\log(k+1))^{-b} + (k-1)(1+\log(k))^{-b}$ may be positive or negative, which can be equivalently expressed as

$$\frac{k-1}{k} < \left(\frac{1+\log(k)}{1+\log(k+1)}\right)^b \tag{15}$$

may hold and

$$\frac{k-1}{k} > \left(\frac{1+\log(k)}{1+\log(k+1)}\right)^b \tag{16}$$

may hold as well.

If b is small ($b = 1$, for example), then inequality (15) holds. If b is large, then inequality (16) holds (this is because $\left(\frac{1+\log(k)}{1+\log(k+1)}\right)^b \to 0$ for $b \to \infty$). Since $\left(\frac{1+\log(k)}{1+\log(k+1)}\right)^b$ is a decreasing function of b, we can find a value of b, denoted as b_0, which satisfies: if $0 < b < b_0$, then inequality (15) always holds for any k. Taking the logarithm on both sides of inequality (15) and then dividing both sides by $\log(1 + \log(k)) - \log(1 + \log(k + 1))$, then inequality (15) becomes

$$\frac{\log(k-1)-\log(k)}{\log(1+\log(k))-\log(1+\log(k+1))} < b. \tag{17}$$

Let $b_0 = \min\limits_k \{\frac{\log(k-1)-\log(k)}{\log(1+\log(k))-\log(1+\log(k+1))}, k = 2, 3, \ldots\}$. One can obtain $b_0 = 4.898226$. If $0 < b < b_0$, then $-k(1 + \log(k + 1))^{-b} + (k - 1)(1 + \log(k))^{-b} < 0$ and $(1 + \log(k + 1))^{-b} - (1 + \log(k))^{-b} < 0$, the inequality (14) holds. That implies

(iii) if $0 < a < 1$, $0 < b < b_0$, and $P(0 < X_1 < 1) = 1$, then $\{X_k, k = 2, 3, \ldots\}$ is stochastically increasing, and

(iv) if $a > 1$, $0 < b < b_0$, and $P(X_1 > 1) = 1$, then $\{X_k, k = 2, 3, \ldots\}$ is stochastically decreasing.

This completes the proof. □

Proof of Proposition 2. Denote

$$U = a^{-k(1+\log(k+1))^{-b}+(k-1)(1+\log(k))^{-b}} y^{(1+\log(k+1))^{-b}-(1+\log(k))^{-b}}. \tag{18}$$

Similar to the proof of Proposition 1, if $\log(U) = (1 + \log(k + 1))^{-b}(\log(y) - k\log(a)) + (1 + \log(k))^{-b}((k - 1)\log(a) - \log(y))$ varies between negative and positive values, the left hand side of Eq. (14) changes between $(0, 1)$ and $(1, +\infty)$. That is, the process $\{X_k, k = 1, 2, \ldots\}$ is stochastically non-monotonous.

This completes the proof. □

Proof of Proposition 3.

$$f_k(t) = a^{k-1}(1+\log(k))^b t^{(1+\log(k))^b - 1} f(a^{k-1} t^{(1+\log(k))^b}). \tag{19}$$

Denote $r(t) = \frac{f(t)}{\bar{F}(t)}$. We have,

$$\begin{aligned} r_k(t) &= \frac{f_k(t)}{\bar{F}_k(t)} \\ &= \frac{a^{k-1}(1+\log(k))^b t^{(1+\log(k))^b - 1} f(a^{k-1} t^{(1+\log(k))^b})}{\bar{F}(a^{k-1} t^{(1+\log(k))^b})} \\ &= a^{k-1}(1+\log(k))^b t^{(1+\log(k))^b - 1} r(a^{k-1} t^{(1+\log(k))^b}), \end{aligned} \tag{20}$$

If $b > 0$, then $t^{(1+\log(k))^b}$ is increasing in t. Since $r(t)$ is an increasing function in t, $r_k(t)$ is increasing in t. □

Proof of Proposition 4.

If $\{X_k, k = 1, 2, \ldots\}$ is stochastically non-decreasing, for every real numbers r_0 and r_1, we have $P(X_k > r_0) \geq P(X_1 > r_0)$, or $P(X_k < r_0) \leq P(X_1 < r_0)$. Then we have $P(\sum_{i=1}^n X_k < r_1) \leq P(\sum_{i=1}^n Y_i < r_1)$, which implies that inequality $m(t) \leq m_1(t)$ holds.

Similarly, we can prove that $m(t) \geq m_1(t)$ if $\{X_k, k = 1, 2, \ldots\}$ is stochastically non-decreasing.

According to Bennett's inequality (Bennett, 1962) below,

$$P\left(\sum_{k=1}^n (X_k - \mathbb{E}[X_k]) > t\right) \leq \exp\left(-\frac{n\sigma^2}{c^2} H(\frac{ct}{n\sigma^2})\right), \tag{21}$$

where $H(u) = (1+u)\ln(1+u) - u$, we can obtain

$$P\left(S_n < t\right) \geq 1 - \exp\left(-\frac{n\sigma^2}{c^2} H(\frac{ct - c\Lambda_n}{n\sigma^2})\right). \tag{22}$$

Hence,

$$m(t) \geq \sum_{n=1}^{\infty} \left[1 - \exp\left(-\frac{n\sigma^2}{c^2} H(\frac{ct - c\Lambda_n}{n\sigma^2})\right)\right]. \tag{23}$$

This completes the proof. □

Proof of Proposition 5.

In the following, we prove (i).

According to Definition 1, if $X_k^g <_{st} X_k^d$, we have $m^g(t) > m^d(t)$. For a given non-decreasing function $u(x)$, with Lemma 1, $X_k^g <_{st} X_k^d$ if $\mathbb{E}[u(X_k^g)] < \mathbb{E}[u(X_k^d)]$. Since

$$\begin{aligned}\mathbb{E}[u(X_k^g)] - \mathbb{E}[u(X_k^d)] &= \int_0^{+\infty} u(x)dF(a^{k-1}x) - \int_0^{+\infty} u(x)dF(a^{k-1}x^{(1+\log(k))^b}) \\ &= \int_0^{+\infty} \left(u(a^{(1-k)}y) - u(a^{(1-k)(1+\log(k))^{-b}}y^{(1+\log(k))^{-b}})\right) dF(y), \quad (24)\end{aligned}$$

$\mathbb{E}[u(X_k^g)] < \mathbb{E}[u(X_k^d)]$ if $u(a^{(1-k)}y) < u(a^{(1-k)(1+\log(k))^{-b}}y^{(1+\log(k))^{-b}})$. As $u(.)$ is a non-decreasing function, $u(a^{(1-k)}y) < u(a^{(1-k)(1+\log(k))^{-b}}y^{(1+\log(k))^{-b}})$ holds if

$$\frac{a^{(1-k)}y}{a^{(1-k)(1+\log(k))^{-b}}y^{(1+\log(k))^{-b}}} = a^{(1-k)(1-(1+\log(k))^{-b})}y^{1-(1+\log(k))^{-b}} < 1. \quad (25)$$

Inequality (25) holds if either of the following conditions is true,

- if $0 < a < 1$, $b < 0$ and $P(X_1 > 1) = 1$,
- if $a > 1$, $b > 0$ and $P(0 < X_1 < 1) = 1$.

Similarly, the other bullet (ii) can be established.

This completes the proof. □

Proof of Proposition 6.

Similar to the proof for Proposition 5, Proposition 6 can be established. □

Proof of Proposition 7.

- For any given $M > 0$,

$$\lim_{k\to\infty} P(|X_k| < M) = \lim_{k\to\infty} P(0 < X_k < M) = \lim_{k\to\infty} P(X_1 < a^{k-1}M^{(1+\log(k))^b}). \quad (26)$$

  If $0 < a < 1$, then $\lim_{k\to\infty} a^{k-1}M^{(1+\log(k))^b} = 0$. Since X_1 is non-negative, $\lim_{k\to\infty} P(X_1 < a^{k-1}M^{(1+\log(k))^b}) = 0$, or $\lim_{k\to\infty} P(|X_k| < M) = 0$. That is, X_k converges to infinity in probability as $k \to \infty$.

- For any given $\varepsilon > 0$,

$$\lim_{k\to\infty} P(|X_k| > \varepsilon) = \lim_{k\to\infty} P(X_k > \varepsilon) = 1 - \lim_{k\to\infty} P(X_1 \le \varepsilon) = 1 - \lim_{k\to\infty} P(X_k \le a^{k-1}\varepsilon^{(1+\log(k))^b}). \quad (27)$$

  If $a > 1$, then $\lim_{k\to\infty} a^{k-1}\varepsilon^{(1+\log(k))^b} = \infty$. That implies $\lim_{k\to\infty} P(X_1 \le a^{k-1}\varepsilon^{(1+\log(k))^b}) = 1$, or $\lim_{k\to\infty} P(|X_k| > \varepsilon) = 0$. That is, X_k converges to zero in probability as $k \to \infty$.

This completes the proof. □

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