Homotopy of Rational Maps and the Quantization of Skyrmions

Steffen Krusch*

Department of Pure Mathematics
School of Mathematics, University of Leeds, Leeds LS2 9JT
England

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Abstract

The Skyrme model is a classical field theory which models the strong interaction between atomic nuclei. It has to be quantized in order to compare it to nuclear physics. When the Skyrme model is semi-classically quantized it is important to take the Finkelstein-Rubinstein constraints into account. The aim of this paper is to show how to calculate these FR constraints directly from the rational map ansatz using basic homotopy theory. We then apply this construction in order to quantize the Skyrme model in the simplest approximation, the zero mode quantization. This is carried out for up to 22 nucleons, and the results are compared to experiment.

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*S.Krusch@maths.leeds.ac.uk

1 Introduction

The Skyrme model is a classical model for the strong interaction between atomic nuclei, [1]. Any field configuration in this model is labelled by a topological winding number which can be interpreted as the baryon number B. In order to compare the Skyrme model to experiments it has to be quantized. Since a canonical quantization of the model is not possible because it is non-renormalizable as a field theory, the model can only be quantized approximately.

In [2, 3] Adkins *et al.* quantized the rotational and translational zero modes of the $B = 1$ Skyrmion and obtained good agreement with experiment. A subtle point is that the Skyrme model, as a classical field theory, allows for quantizing Skyrmions as fermions. This is because the configuration space is not simply connected [4]. For the $SU(2)$ Skyrme model there is a choice whether to quantize a Skyrmion as a boson or a fermion when B is odd. Yet, it has to be quantized as a boson if B is even, [5]. Witten showed in Refs. [6, 7] that in the $SU(3)$ model the $B = 1$ Skyrmion has to be quantized as fermion if the number of colours is odd, yet no such restriction applies to the $SU(2)$ Skyrme model.

In order to quantize Skyrmions with higher baryon number it is important to understand the classical solutions. Well-separated Skyrmions can be described reasonably well by the product ansatz, however when Skyrmions coalesce this ansatz fails. In fact, the static solutions of the Skyrme model have a surprisingly rich structure. Braaten *et al.* showed in Ref. [8] that the $B = 2$ Skyrmion is a torus, the $B = 3$ Skyrmion is a tetrahedron and the $B = 4$ Skyrmion is a cube. Battye and Sutcliffe calculated the Skyrmions up to $B = 9$, [9] and later to $B = 22$, [10], and also found shell-like structures with discrete symmetries.

The $B = 2$ Skyrmion with toroidal symmetry was quantized in Refs. [11, 12] using the zero mode quantization. Later, the approximation was improved by taking massive modes into account, [13]. The $B = 3$ Skyrmion was first quantized in [14]. Then the cubically symmetric $B = 4$ Skyrmion was quantized in [15]. Irwin has performed a zero mode quantization for $B = 4$ to $B = 9$, [16], where the monopole moduli space was used as an approximation for the Skyrmion moduli space. For all the even baryon numbers he found that the predicted quantum numbers of the ground state agreed with nuclear physics. Yet, his findings disagreed with experiments for $B = 5$, 7 and 9.

The aim of this paper is to construct the Finkelstein-Rubinstein constraints directly from the rational map ansatz, [17]. With this ansatz it is possible to understand the symmetries of Skyrmions. Effectively, it gives a finite dimensional approximation to the configuration space which is more tractable than the infinite dimensional configuration space but still retains important topological properties. Moreover, the ansatz plays a major role in constructing Skyrmions. To date, both the numerical configurations and the relevant rational maps are known up to $B = 22$, [10, 18]. In this paper we apply these results to calculate the quantum ground states up to $B = 22$ using zero mode quantization.

This paper is organised as follows. In Sect. 2 we briefly review the Skyrme model and discuss its topology in some detail. The topology, in particular the fundamental group of configuration space, is very important for understanding the Finkelstein-Rubinstein constraints for quantizing scalar fields as fermions [4]. We describe this construction and its implications on zero mode quantization. In Sect. 3 we recall the rational map ansatz [17] and show that it can be viewed as a suspension. This enables us to calculate the fundamental group of configuration space directly from rational maps. Applying some theorems on the fundamental group of rational maps we prove a simple formula to calculate the homotopy class of a loop generated by a combined rotation and isorotation. In Sect. 4 we discuss how to use group theory to find the ground states in the Skyrme model and present the results of our calculations. We end with a conclusion.

2 The Topology of the Skyrme Model

The Skyrme model is a classical field theory of mesons. The basic field is the $SU(2)$ valued field $U(\mathbf{x}, t)$ where $\mathbf{x} \in \mathbb{R}^3$. The static solutions of the Skyrme model can be derived by varying the following energy [17]:

$$E = \int \left(-\tfrac{1}{2}\mathrm{Tr}\left(R_i R_i\right) - \tfrac{1}{16}\mathrm{Tr}\left([R_i, R_j][R_i, R_j]\right)\right) \mathrm{d}^3\mathbf{x}, \tag{1}$$

where $R_i = (\partial_i U)U^\dagger$ is a right invariant $\mathfrak{su}(2)$ valued current. A static solution of the variational equations could be a saddle point. Only solutions which minimise the energy are called Skyrmions. In order to have finite energy the Skyrme fields have to take a constant value, $U(|\mathbf{x}| = \infty) = 1$, at infinity.[1] Therefore, the space of Skyrme configurations consists of all maps $U : \mathbb{R}^3 \to S^3$ with boundary condition $U(|\mathbf{x}| = \infty) = 1$ which effectively compactifies $\mathbb{R}^3$ to S^3. Such maps can be characterised by their degree which is an element of the third homotopy group $\pi_3(S^3) \cong \mathbb{Z}$. We will call the space of Skyrme configurations Q^* in order to emphasise that the maps are based, *i.e.* the point "∞" is mapped to 1. The configuration space Q^* is split into connected components Q^*_B which are the homotopy classes of $\pi_3(S^3)$. The fact that Q^*_B is connected follows from a famous theorem of Hopf (*e.g.* [19]). Furthermore, the energy of configurations in Q^*_B is bounded below by the topological charge: $E \geq 12\pi^2 B$, [20].

2.1 Finkelstein-Rubinstein Constraints

In the following we describe an idea of Finkelstein and Rubinstein, how to quantize a scalar field theory and obtain fermions. Quantization usually implies replacing the classical configuration space by (wave) functions on configuration

[1] The value of U at infinity is fixed by assuming there is a (infinitesimal) pion mass term.

space. Finkelstein and Rubinstein argued that if the configuration space Q^* is not simply connected, then the wave functions have to be defined not on configuration space Q^*, but on the covering space of configuration space CQ^*. To simplify matters, we assume that baryons are conserved in our theory, that is baryons cannot decay. This means that the wave function has to be non vanishing only on one component, otherwise there could be transitions between different sectors. Therefore, we impose the (superselection) rule that the wave functions are defined on CQ^*_B.

In order to have fermionic quantization, a rotation of a wave function ψ by 2π has to result in $-\psi$. However, the $SO(3)$ action is not well defined on CQ^*_B. In order to define this group action one has to keep track of the component of the covering space and this is the origin of the FR constraints.

Finkelstein showed in Ref. [21] that the fundamental group $\pi_1(Q^*_0)$ is isomorphic to $\pi_4(S^3)$. It is a standard result of algebraic topology that $\pi_4(S^3) \cong \mathbb{Z}_2$, *e.g.* [19]. Furthermore, Whitehead proved the following theorem in Ref. [22].

Theorem 2.1 (Whitehead) *Let $F^p(X, x_0)$ be based maps $f : S^p \to X$ such that $f(1) = x_0$. Then the connected components of $F^p(X, x_0)$ are homotopy equivalent.* □

This implies in particular that the fundamental groups $\pi_1(Q^*_{B_1})$ and $\pi_1(Q^*_{B_2})$ for different components are isomorphic. It follows that[2]

$$\pi_1(Q^*_B) \cong \pi_4(S^3) \cong \mathbb{Z}_2. \tag{2}$$

The covering space CQ^*_B can now be defined as the space of all paths starting at a fixed point $q_0 \in Q^*_B$ modulo homotopy. Elements in CQ^*_B will be denoted by $[q]$. Let $[a]$ be the generator of the fundamental group of Q^*_B. Then we can define the path $[q \cdot a]$, where $[a]$ is a loop, starting at q. This is well defined because Q^*_B is connected so that the loop is independent of the base point q_0 (up to isomorphism). $[q \cdot a]$ and $[q]$ are different points in CQ^*_B which project to the same point $q \in Q^*_B$. As $\pi_1(Q^*_B) = \mathbb{Z}_2$, $[a^2]$ is a trivial loop, therefore,

$$[q \cdot a \cdot a] = [q]. \tag{3}$$

Define a wave function

$$\psi : CQ^* \times \mathbb{R} \to \mathbb{C} : ([q], t) \mapsto \psi([q], t) \tag{4}$$

as an element of a formal Hilbert space $\mathcal{H}$, such that ψ is square integrable and normalised:

$$\int |\psi([q], t)|^2 \mathrm{d}[q] = 1. \tag{5}$$

[2]An alternative proof was given in Ref. [5] which relies on the fact that the target space of the Skyrme model is a group, namely $SU(2)$.

Define $L_a\psi([q],t) = \psi([q\cdot a],t)$. Conventionally, a wave function is determined by configuration space, modulo a phase. Then

$$L_a\psi([q],t) = \mathrm{e}^{i\phi}\psi([q],t), \tag{6}$$

and equation (3) implies that $\mathrm{e}^{2i\phi} = 1$. If the phase is trivial then we can define the function on Q_B^*. In physicists' language, this corresponds to a theory which only contains bosons. Therefore, we impose a nontrivial phase

$$L_a\psi([q],t) = -\psi([q],t). \tag{7}$$

We now impose rotational and isorotational symmetry on the wave function as follows. We insist that the wave function ψ vanishes on the path $[q]$ unless the corresponding configuration q is invariant under a rotation through angle α around axis $\mathbf{n}$ followed by an isorotation through angle β around axis $\mathbf{N}$. Then equation (7) implies the following constraint on the wave function:

$$\mathrm{e}^{-i\alpha\mathbf{n}\cdot\mathbf{J}}\mathrm{e}^{-i\beta\mathbf{N}\cdot\mathbf{I}}\psi([q],t) = \begin{cases} \psi([q],t) & \text{if the induced loop is contractible,} \\ -\psi([q],t) & \text{otherwise.} \end{cases} \tag{8}$$

Here $\mathbf{J}$ and $\mathbf{I}$ are the spin and isospin operators respectively, with quantum numbers J and I. This imposes a constraint on the allowed quantum states, namely equation (8) can only be satisfied for suitable values of I and J. Particularly important are rotations or isorotations by 2π because they are always a symmetry of the system. When a 2π rotation gives rise to a nontrivial loop then this corresponds to half integer spin. It is worth noting that Finkelstein and Rubinstein proved a connection between spin and statistics in this setting. Namely, Q_B^* admits half integer spin if and only if it admits odd exchange statistics.

2.2 Zero Mode Quantization

The Skyrme model is not renormalizable as a field theory. The usual approach is to quantize it semi-classically. The key idea is the following. The classical dynamics of (slow-moving) solitons can often be described by geodesic motion on the moduli space of static solutions [23]. This has been shown for monopoles in [24], where this idea was also used to quantize monopoles. For an overview of exact results about the geodesic approximation for vortices, see *e.g.* [25] and references therein. Contrary to monopoles and vortices the minimum energy configuration of a given sector is found to be unique up to the action of the symmetry group, namely translations in space and rotations in space and target space. For $B = 1$ the Skyrmion has spherical symmetry so that the symmetry orbit is 6 dimensional. For $B = 2$ there is axial symmetry and the orbit is 8 dimensional, whereas for higher baryon number there are only discrete symmetries and the orbit is 9 dimensional. Since the formal Hilbert space $\mathcal{H}$ of the previous section is difficult to handle in practice, a simple approximation is often used in the literature, namely that the wave function is only non-vanishing on the static minimal energy solution of a given sector, *i.e.* a Skyrmion, and its symmetry orbit. This quantization is known as zero mode quantization.[3]

The Skyrme Lagrangian is invariant under the Poincaré group of (3+1) dimensional space, $SO(3)$ rotations in target space and some discrete parity transformations, which will not be considered here. Similarly, the space of static solutions, that is configurations which minimise the energy (1), is invariant under the Euclidean group $\mathbb{E}_3$, isorotations and parity transformations. By acting with the latter symmetry group on a static Skyrmion U_0 we generate a set of new static solutions

$$U(\mathbf{x}) = AU_0\left(D(A')\left(\mathbf{x}-\mathbf{X}\right)\right)A^\dagger, \tag{9}$$

where A and A' are SU(2) matrices, $D(A')$ is the associated SO(3) rotation and $\mathbf{X}$ is a vector. In the zero mode approximation the matrices A and A' and the vector $\mathbf{X}$ are considered to be time dependent. This leads to the following reduced Lagrangian

$$L = -M + \tfrac{1}{2}M\dot{X}^2 + \tfrac{1}{2}a_iU_{ij}a_j + \tfrac{1}{2}b_iV_{ij}b_j - a_iW_{ij}b_j, \tag{10}$$

where M is the classical mass of the Skyrmion, and

$$a_k = -i\mathrm{Tr}\left(\tau_k A^\dagger \dot{A}\right) \quad \text{and} \quad b_k = -i\mathrm{Tr}\left(\tau_k \dot{A}' A'^\dagger\right). \tag{11}$$

The matrices U_{ij}, V_{ij} and W_{ij} are dependent on the classical solution U_0 and are given by [11]

$$\begin{aligned}
U_{ij} &= -\tfrac{1}{8}\int d^3x\ \mathrm{Tr}\left(T_iT_j + \tfrac{1}{4}\Big[R_k,T_i\Big]\Big[R_k,T_j\Big]\right), && (12)\\
W_{ij} &= \tfrac{1}{8}\int d^3x\ \epsilon_{jlm}x_l\ \mathrm{Tr}\left(T_iR_m + \tfrac{1}{4}\Big[R_k,T_i\Big]\Big[R_k,R_m\Big]\right), && (13)\\
V_{ij} &= -\tfrac{1}{8}\int d^3x\ \epsilon_{ilm}\epsilon_{jno}x_lx_n\ \mathrm{Tr}\left(R_mR_o + \tfrac{1}{4}\Big[R_k,R_m\Big]\Big[R_k,R_o\Big]\right), && (14)
\end{aligned}$$

where $R_k = (\partial_k U_0)U_0^\dagger$ is the right invariant $\mathfrak{su}(2)$ current which has been defined above, and

$$T_i = i\left[\frac{\tau_i}{2}, U_0\right]U_0^\dagger \tag{15}$$

is also an $\mathfrak{su}(2)$ current. Lagrangian (10) is no longer Lorentz invariant which is consistent with the fact that the moduli approximation only works for small velocities. For a covariant treatment for $B=1$ see for example [26].

[3]Here, the word "zero mode" is used rather loosely to refer to symmetry transformations that leave the energy (1) invariant.

Lagrangian (10) can now be canonically quantized. The momenta conjugate to a_i and b_i become the body-fixed spin and isospin angular momenta K_i and L_i satisfying the $SU(2)$ commutation relations, see [11] for details. The usual space-fixed spin and isospin angular momenta J_i and I_i are related to the body-fixed operators by

$$J_i = -D_{ij}(A')L_j, \qquad I_i = -D_{ij}(A)K_j. \tag{16}$$

The remaining nonvanishing commutation relations are

$$\begin{aligned} \left[L_i, A'\right] &= -\tfrac{\tau_i}{2}A', \qquad \left[J_i, A'\right] = \tfrac{\tau_i}{2}A', \\ \left[I_i, A\right] &= -\tfrac{\tau_i}{2}A, \qquad \left[K_i, A\right] = \tfrac{\tau_i}{2}A. \end{aligned} \tag{17}$$

Therefore, $\mathbf{L}^2 = \mathbf{J}^2$ and $\mathbf{I}^2 = \mathbf{K}^2$. A basis for this Hilbert space is given by

$$|J, J_3, L_3\rangle \otimes |I, I_3, K_3\rangle \tag{18}$$

with $-J \le J_3, L_3 \le J$ and $-I \le I_3, K_3 \le I$. In this approximation, the ground states are the states with the lowest values of I and J that are compatible with the FR constraints arising from the symmetries of the given Skyrmion.

The values of the integrals U_{ij}, V_{ij}, and W_{ij} strongly depend on the symmetries of the Skyrmion. For tetrahedral symmetry T, octahedral symmetry O, and icosahedral symmetry Y, the matrices often only have one eigenvalue, see [16] for a detailed discussion. In this case, the Hamiltonian is that of a spherical top. If there is an axis of symmetry of higher than second order then the matrices have at most two eigenvalues. If these eigenvalues are distinct then the Hamiltonian is that of a symmetric top. In both cases, the states in (18) will be energy eigenstates as well. If the states only have dihedral D_2 symmetry, then the Hamiltonian is that of an asymmetric top and the states (18) are no longer energy eigenstates, see *e.g.* [27].

The integrals U_{ij}, V_{ij}, and W_{ij} determine the values of the moment of inertia for J and I. For $B = 1$ these moments of inertia are equal, however, in general the moments of inertia are larger for rotations than for isorotations, which implies that for $I = J$ isorotations contribute more to the energy.

3 The Rational Map Ansatz

Equation (1) can only by solved numerically. However, there is a good approximation, called the rational map ansatz [17]. In this ansatz Skyrme fields as maps from $S^3 \to S^3$ are given in terms of rational maps which are holomorphic maps from $S^2 \to S^2$. This ansatz not only gives good approximations for the Skyrme configurations and in particular their symmetries. It also gives a good approximation to the topology of configuration space Q^*_B. We will prove the following theorem.

Theorem 3.1 *The rational map ansatz induces a surjective homomorphism from the fundamental group of based rational maps $\pi_1(Rat_B^*)$ onto the fundamental group of Skyrme configurations $\pi_1(Q_B^*)$.*

3.1 Description of the Rational Map Ansatz

The rational map ansatz is best derived in the geometric approach to the Skyrme model [28]. For more details see [17, 29]. Here, we only state the main results. First note that the angular coordinates (θ, ϕ) can be related to the complex plane z by the stereographic projection $z = \mathrm{e}^{i\phi}\tan\frac{\theta}{2}$. Then the rational map ansatz is given by

$$U(\mathbf{x}) = \exp(if(r)\ \hat{\mathbf{n}}_R(z)\cdot\boldsymbol{\tau}). \tag{19}$$

Here $f(r)$ is a shape function with boundary condition $f(0) = \pi$ and $f(\infty) = 0$, and τ_i are the Pauli matrices. The angular behaviour is determined by the function $R(z)$ which is also the stereographic projection of an S^2 in target space S^3. The unit vector $\hat{\mathbf{n}}_R(z)$ is

$$\hat{\mathbf{n}}_R = \frac{1}{1+|R|^2}\left(2\Re(R), 2\Im(R), 1-|R|^2\right). \tag{20}$$

There is one further restriction on the map $R(z)$ in that it is a holomorphic map of degree B. This implies that $R(z)$ can be written as a ratio of two polynomials,

$$R(z) = \frac{p(z)}{q(z)}, \tag{21}$$

where $p(z)$ and $q(z)$ are polynomials with maximal polynomial degree B which have no common factor. Moreover, at least one of the polynomials has polynomial degree B, see *e.g.* [30]. We will call the class of such maps Rat_B and will show in Sect. 3.2 that such maps give rise to configurations with baryon number B.

In practice, the rational map ansatz is used in the following way. Inserting (19) into (1) gives

$$E = 4\pi\int\left(f'^2r^2 + 2B(f'^2+1)\sin^2 f + \mathcal{I}\frac{\sin^4 f}{r^2}\right)\mathrm{d}r, \tag{22}$$

where[4]

$$\mathcal{I} = \frac{1}{4\pi}\int\left(\frac{1+|z|^2}{1+|R|^2}\left|\frac{\mathrm{d}R}{\mathrm{d}z}\right|\right)^4\frac{2i\mathrm{d}z\mathrm{d}\bar{z}}{(1+|z|^2)^2}. \tag{23}$$

Now, the minimum ansatz field is found by choosing polynomials $p(z)$ and $q(z)$ which minimise $\mathcal{I}$ and then calculating the shape function $f(r)$ numerically. These minimum energy ansatz fields have been calculated for all the known

[4] $\mathcal{I}$ is also known as the harmonic 4-energy of holomorphic maps $R: S^2 \to S^2$.

Skyrmions and are found to have energies that exceed the true, numerically determined minima by less than 3%. Moreover, in almost all the cases the rational map ansatz gives the correct symmetries of the Skyrme fields [10]. In the following, we will describe the symmetries of Skyrmions, and their rational maps, in more detail.

Skyrmions can be rotated by an $SU(2)$ matrix A' and isorotated by A as in equation (9). The matrices A and A' induce the following transformations of the rational map $R(z)$.

$$R(z) \to \tilde{R}(z) = M_A\left(R\left(M_{A'}(z)\right)\right). \tag{24}$$

Configurations are invariant under a subgroup of these transformations if $\tilde{R}(z) = R(z)$. Since the corresponding Möbius transformations M_A and $M_{A'}$ do not agree with the canonical $SU(2)$ action, we derive the transformations explicitly.

In 3-dimensional space a rotation by θ around the unit vector $\boldsymbol{\omega}$ is given by $\exp(-i\theta\omega_i J_i)$ where the (space fixed) angular momentum operator is $(J_k)_{lm} = -i\epsilon_{klm}$. In order to describe rotations for rational maps we relate the unit vector $\mathbf{n}$ to a complex vector

$$V = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \in \mathbb{C}^2 \tag{25}$$

via the formula

$$n_i = \frac{V^\dagger \tau_i V}{V^\dagger V}, \tag{26}$$

where τ_i are the standard Pauli matrices. If $V^\dagger V = 1$ then this map is the famous Hopf map. Note also that because of the denominator in equation (26) this map is well defined for $V \in \mathbb{CP}^1$ where $[v_1, v_2] \cong [\lambda v_1, \lambda v_2]$ for $\lambda \in \mathbb{C}^\times$. In this case the map is one to one.

Let

$$R_{ij} = \exp\left(-i\theta\omega_k J_k\right)_{ij} \tag{27}$$

be a rotation by θ around the unit vector ω_i. Then the rotation of $\mathbf{n}$ corresponds to a $SU(2)$ rotation A' of V:

$$n_i' = R_{ij} n_j = \frac{V^\dagger A'^\dagger \tau_i A' V}{V^\dagger A'^\dagger A' V} \tag{28}$$

A straight forward calculation shows that A' is given by

$$A' = \pm\exp\left(-i\frac{\theta}{2}\omega_k\tau_k\right). \tag{29}$$

The complex number z is related to $\mathbf{n}$ by the stereographic projection

$$z = \frac{n_1 + i n_2}{1 + n_3} = \frac{v_2}{v_1}, \tag{30}$$

where the second equality follows from the definition (26) by direct calculation. Therefore, we obtain the transformation law

$$\tilde{z} = \frac{\big(\cos\frac{\theta}{2} + i\omega_3 \sin\frac{\theta}{2}\big)z + \big(\omega_2 - i\omega_1\big)\sin\frac{\theta}{2}}{\big(-\omega_2 - i\omega_1\big)\sin\frac{\theta}{2}z + \big(\cos\frac{\theta}{2} - i\omega_3\sin\frac{\theta}{2}\big)}. \tag{31}$$

Now, let $A = \exp(-i\theta\omega_i I_i)$ be the isospin rotations by θ around the unit vector ω_i and $I_i = \frac{1}{2}\tau_i$. Then A induces a rotation R_{ij} on the unit vector $\mathbf{n}_R$ in equation (20). We repeat the derivation above and obtain

$$\tilde{R} = \frac{\big(\cos\frac{\theta}{2} + i\omega_3 \sin\frac{\theta}{2}\big)R + \big(\omega_2 - i\omega_1\big)\sin\frac{\theta}{2}}{\big(-\omega_2 - i\omega_1\big)\sin\frac{\theta}{2}R + \big(\cos\frac{\theta}{2} - i\omega_3\sin\frac{\theta}{2}\big)}. \tag{32}$$

Note that rational maps can also be defined as maps $\mathbb{CP}^1 \to \mathbb{CP}^1$ using homogeneous coordinates $[v_1, v_2] \to [p, q]$, where square brackets imply the equivalence relation $[u, v] = [\lambda u, \lambda v]$ for $\lambda \in \mathbb{C}^\times$. Then p and q can be considered as homogeneous polynomials in v_1 and v_2, see [17, 31] for details. The advantage of these coordinates is that both the rotation group and the isospin rotation group act linearly.

3.2 The Rational Map Ansatz as a Suspension

A construction like the rational map ansatz is very common in algebraic topology and is known as a suspension. In fact the suspension $\mathcal{S}M$ of a given manifold M is constructed by considering the product space $I \times M$, where I is the interval $I = [0, 1]$, and taking the quotient with respect to the two end points of I:

$$\mathcal{S}M = (I \times M)/(\{0\} \times M \cup \{1\} \times M). \tag{33}$$

In other words, there is an equivalence relation $\sim$ which identifies all the points $\{0\} \times M$ to one point and also identifies all the points $\{1\} \times M$ to one point. The quotient space $(I \times M)/\sim$ is again a manifold. One reason why this construction is so important is its natural compatibility with spheres: $\mathcal{S}S^n = S^{n+1}$. However, not only spaces, but also maps can be suspended. Let $f : M \to N$ then the suspension $\mathcal{S}f$ of f is given by $\mathcal{S}f : \mathcal{S}M \to \mathcal{S}N$. This map is induced by $Id \otimes f : I \times M \to I \times N$ where $Id : I \to I$ is the identity map. The suspension is very convenient for studying homotopy group as the following theorem suggests, see [19, Corollary 4.24].

Theorem 3.2 (Freudenthal suspension theorem) *The suspension map $\pi_i(S^n) \to \pi_{i+1}(S^{n+1})$ is an isomorphism for $i < 2n-1$ and a surjection for $i = 2n-1$.* □

It is easy to verify that the rational map ansatz is a suspension. Both domain and codomain are given by Cartesian products of an interval and a two-sphere. The fact that the radial coordinate $r \in [0, \infty)$ can be considered as a closed interval relies on the boundary conditions. Moreover, at each of the end points the map collapses to a point. So domain and codomain are suspensions of S^2 and the rational map ansatz suspends holomorphic maps from $S^2 \to S^2$.

An immediate consequence of theorem 3.2 is that rational maps of degree B induce Skyrme configurations of degree B which proves our claim in Sect. 3.1. In the following we prove theorem 3.1. This theorem can be used to calculate the FR phase directly from rational maps. Denote the space of based rational maps Rat_B^* by

$$Rat_B^* = \{R \in Rat_B : R(\infty) = 1\}. \tag{34}$$

In order to apply the Freudenthal suspension theorem 3.2 we need to consider more general maps between two-spheres. Let M_B denote continuous maps from $S^2 \to S^2$ of degree B, and M_B^* the space of based maps in M_B. The following theorem by Segal, [32], implies that for our purposes, rational maps are sufficiently general.

Theorem 3.3 (Segal) *M_B^* and Rat_B^* are homotopy equivalent up to B.* □

Proof of theorem 3.1:

First we show that the suspension $\mathcal{S}$ induces an surjective homomorphism $\mathcal{S}_* : \pi_1(M_B^*) \to \pi_1(Q_B^*)$. We only need to prove this for $B = 0$ because Whitehead's theorem 2.1 implies that all the fundamental groups $\pi_1(M_B^*)$ are isomorphic and similarly for $\pi_1(Q_B^*)$. It can be shown by the same argument as in [21] that

$$\pi_1(M_0^*) \cong \pi_3(S^2) \cong \mathbb{Z}. \tag{35}$$

Recall that $\pi_1(Q_0^*) \cong \pi_4(S^3)$. Then the Freudenthal suspension theorem 3.2 implies that $\mathcal{S}_*$ is a surjective homomorphism.

It follows from Segal's theorem that this result also holds for rational maps as long as $B > 0$. Namely, we have the following homomorphisms

$$\pi_1(Rat_B^*) \to \pi_1(M_B^*) \to \pi_1(Q_B^*), \tag{36}$$

where the first map is an isomorphism because of theorem 3.3 and the second map is a surjective homomorphism. □

3.3 The Fundamental Group of Rational Maps

In this section we show how to calculate the homotopy class of a loop that is generated by a given symmetry. Segal has proven the following theorem in [32, Proposition 6.4]:

Theorem 3.4 (Segal) *$\pi_1(Rat_B^*) = \mathbb{Z}$ and is generated by the loop which moves one zero of a rational function once (clockwise) around one pole.* □

Note that the inclusion induces a surjective homomorphism of fundamental groups between based and unbased rational maps

$$\pi_1(Rat_B^*) \to \pi_1(Rat_B) \tag{37}$$

such that the generator of $\pi_1(Rat_B^*)$ is mapped to the generator of $\pi_1(Rat_B)$. The fundamental group of unbased maps is $\pi_1(Rat_B) = \mathbb{Z}_{2B}$ [33, 32].

Let z_i be the zeros, and p_j the poles of the rational map R. For a given loop L in Rat_B^* the zeros and poles move around in the complex plane as a function of a parameter $\phi \in [0, \Phi]$ such that $z_i(0) = z_i(\Phi) = z_i$ and $p_j(0) = p_j(\Phi) = p_j$. Define the integral

$$N(L) = \frac{i}{2\pi} \sum_{i,j=1}^{B} \int_0^{\Phi} \frac{(z_i'(\phi) - p_j'(\phi))\,\mathrm{d}\phi}{(z_i(\phi) - p_j(\phi))}. \tag{38}$$

where $'$ denotes differentiation with respect to ϕ. Note that the denominators in the integrals do not vanish because poles and zeros cannot coalesce.

Lemma 3.5 *$N(L)$ is a homotopy invariant and counts the number of times zeros move around poles. Therefore, $N(L)$ provides an isomorphism $\pi_1(Rat_B^*) \to \mathbb{Z}$.*

Proof of lemma 3.5: For a given loop, L, zeros and poles move around in the complex plane such that zeros move into zeros and poles move into poles. Therefore, a loop induces a permutation of poles and zeros. These permutations split up into cycles. Changing variables via $z(\phi) = z_i(\phi) - p_j(\phi)$ we obtain

$$N(L) = \frac{i}{2\pi} \sum_k \oint_{C_k(L)} \frac{\mathrm{d}z}{z}, \tag{39}$$

where the second sum runs over all the cycles and the integral is written as a contour integral. The loop L in the space of rational maps induces one or more contour integrals over the contours $C_k(L)$. The value of the integral is given according to Cauchy's theorem by the residue of the simple pole at $z = 0$ and the number of times the $C_k(L)$ enclose this pole.

Consider now a loop $\tilde{L}$ which is homotopic to L. This induces a set of contours $\tilde{C}_j$ where the number of cycles could have changed. Then $N(\tilde{L}) - N(L)$ is a difference of contour integrals. By adding and subtracting additional paths we can express $N(\tilde{L}) - N(L)$ as a sum of closed contours $K_i(L, \tilde{L})$. However, because $\tilde{L}$ and L are homotopic, there can be no residue inside $K_i(L, \tilde{L})$ as this would correspond to zeros coalescing with poles. Therefore, the integral over $K_i(L, \tilde{L})$ vanishes by Cauchy's theorem and we have shown that $N(\tilde{L}) = N(L)$, *i.e.* $N(L)$ is a homotopy invariant.

It is obvious from the properties of integration that $N(L_1 + L_2) = N(L_1) + N(L_2)$. Therefore, N is a homomorphism. Furthermore, it is easy to check by direct calculation that moving a zero once clockwise around a pole gives rise to $N = 1$. Therefore, N provides an isomorphism between the fundamental group of based rational maps and the integers. $\square$

Corollary 3.6 *The rational map ansatz induces an isomorphism* $N \mod 2 \rightarrow \pi_1(Q_B^*)$.

Proof of corollary 3.6: Due to theorem 3.1 there is a surjective homomorphism $\pi_1(Rat_B^*) \rightarrow \pi_1(Q_B^*)$. According to lemma 3.5, N provides an isomorphism $\pi_1(Rat_B^*) \rightarrow \mathbb{Z}$. Since $\pi_1(Q_B^*) \cong \mathbb{Z}_2$ there is an isomorphism between $N \mod 2$ and $\pi_1(Q_B^*)$. $\square$

3.4 Axially-Symmetric Skyrmions

In the next two sections, we show how to calculate N for a loop given by a combined rotation and isorotation. In general, the isospin action gives rise to a complicated movement of zeros and poles. However, for axially symmetric maps the isospin action can be calculated explicitly. Therefore, we consider axially symmetric maps first and then generalise the results to maps which are symmetric under discrete symmetries.

There are many possible $U(1) \times U(1)$ actions on Rat_B^*. However, for degree B rational maps there is only a one parameter family of axially symmetric maps.

Lemma 3.7 *The most general axially symmetric map in* Rat_B^* *for* $B \neq 0$ *is given by*

$$R(z) = \frac{z^B - b}{z^B + b}, \tag{40}$$

for $b \neq 0$.

Proof of lemma 3.7: It is easy to check that the only rotation that leaves the boundary condition $R(z = \infty) = 1$ invariant is a rotation by an angle α around the x_3-axis,

$$z \rightarrow \mathrm{e}^{i\alpha} z. \tag{41}$$

Similarly, the only isorotation that leaves the boundary conditions invariant is an isorotation by β around the X_1-axis,[5]

$$R \rightarrow \frac{\cos(\beta/2)R - i\sin(\beta/2)}{-i\sin(\beta/2)R + \cos(\beta/2)}. \tag{42}$$

[5]Here, x_i denotes a Cartesian coordinate system of the domain, and X_i is a local Cartesian coordinate system of the codomain. R and z are related to X_i and x_i, respectively, by stereographic projection.

In the following, when no rotation or isorotation axis is mentioned we implicitly assume (41) and (42), respectively.

Under an infinitesimal rotation by α followed by an infinitesimal isorotation by β a rational map

$$R(z) = \frac{p(z)}{q(z)} \tag{43}$$

of degree $B \neq 0$ transforms into

$$\tilde{R}(z) = R(z) + \frac{i}{q(z)^2}\left(\frac{\beta}{2}\left(p(z)^2 - q(z)^2\right) + \alpha z\left(p'(z)q(z) - p(z)q'(z)\right)\right) + O(\alpha, \beta), \tag{44}$$

where $'$ denotes differentiation with respect to z. The map is invariant under a continuous symmetry generated by α and β if and only if the term in big brackets vanishes. Assuming that $\beta = 0$ leads to

$$\alpha z R'(z) = 0 \tag{45}$$

which implies that either $\alpha = 0$ so that there is no continuous symmetry or $R(z)$ is constant almost everywhere in contradiction to $B \neq 0$. Therefore, β does not vanish.

As $R(z)$ has degree B we can write the polynomials $p(z)$ and $q(z)$ as

$$p(z) = \sum_{k=0}^{B} a_k z^k, \tag{46}$$

$$q(z) = \sum_{k=0}^{B} b_k z^k, \tag{47}$$

and because of the boundary conditions we can set $a_B = b_B = 1$. The rational map is invariant, if the first order term in equation (44) vanishes, *i.e.* all the coefficients of the polynomials are equal to zero. The coefficient of z^n can be written as

$$\sum_k a_k a_{n-k} - b_k b_{n-k} + \gamma(2k - n)(a_k b_{n-k} - a_{n-k} b_k), \tag{48}$$

where $\gamma = \alpha/\beta$ and the sum runs from $k = 0$ to n if $n \leq B$ and from $k = n - B$ to B if $n > B$. For $n = 0$ we obtain $a_0^2 - b_0^2 = 0$.

Case: $a_0 = b_0$

By induction let $a_i = b_i$ for $0 \leq i < n$. Using the coefficients with $n \leq B$ we obtain

$$(a_n - b_n)(1 + n\gamma) = 0. \tag{49}$$

For $1 + n\gamma \neq 0$ this implies by induction that $p(z) = q(z)$, *i.e.* $R(z) = 1$ in contradiction to the assumption that the degree of $R(z)$ is $B \neq 0$.

We can also use the coefficients for $n \geq B$ to perform an induction for decreasing i by $a_i = b_i$ for $B \geq i > n - B$, noting that $a_B = b_B = 1$. This leads to

$$(a_{n-B} - b_{n-B})(1 + \gamma(n - 2B)) = 0. \tag{50}$$

Setting $\gamma = -1/m$ and $m = n - B$ in (50) we obtain

$$B(a_m - b_m) = 0, \tag{51}$$

which implies by induction and equation (49) that $a_i = b_i$ for all $0 \leq i \leq B$, *i.e.* $p(z) = q(z)$. This is again a contradiction to $R(z) \neq 1$. Therefore, $a_0 = b_0$ does not give any solutions.

Case: $a_0 = -b_0$

By induction let $a_i = -b_i$ for $0 \leq i < n$. Using the coefficients with $n \leq B$ we obtain

$$(a_n + b_n)(1 - n\gamma) = 0. \tag{52}$$

This leads to a contradiction for $n = B$ if $\gamma \neq 1/B$. Therefore, we have to set $\gamma = 1/B$. Now, we can perform the same inductions as in the previous case. From equation (50) we obtain

$$(a_{n-B} - b_{n-B})(1 - (2B - n)/B) = 0. \tag{53}$$

This implies that

$$a_i = -b_i \quad \text{and} \quad a_i = b_i, \tag{54}$$

for $i = 1, \ldots, B - 1$, so that all such a_i and b_i vanish. Furthermore, $a_n = b_n = 1$ and $a_0 = -b_0 = -b$. This is a well-defined rational map of degree B unless $b = 0$. $\square$

The zeros and poles of the rational map (40) are distributed equidistantly on the unit circle in the z plane. Under an isorotation (42) by 2π a pole moves into a neighbouring pole and a zero moves into a neighbouring zero. However, under a rotation by 2π all zeros and poles move round once around the unit circle and end at their respective starting position. A combination of a rotation by α and an isorotation by β is a symmetry of the axial map (40) if and only if $B\alpha - \beta = 2\pi k$ for $k \in \mathbb{Z}$. We will denote a path generated by a rotation by α and an isorotation by β by $L(\alpha, \beta)$.

Lemma 3.8 *Given the axially symmetric map* (40) *of degree* $B \neq 0$ *and a loop generated by rotations by* α *and isorotations by* β *such that* $B\alpha - \beta = 2\pi k$ *for* $k \in \mathbb{Z}$. *Then* $N(L(\alpha, \beta))$ *is given by*

$$N(L(\alpha, \beta)) = \frac{B}{2\pi}(B\alpha - \beta). \tag{55}$$

Proof: Rotations around the x_3-axis are particularly easy to handle. Let z_i and p_j be the zeros and poles of a rational map R. Then $z \to \mathrm{e}^{i\phi} z$ changes the zeros and poles to $\tilde{z}_i = \mathrm{e}^{-i\phi} z_i$ and $\tilde{p}_i = \mathrm{e}^{-i\phi} p_i$. A rotation by α then induces a path $L(\alpha, 0)$. With equation (38) we obtain

$$N(L(\alpha,0)) = \frac{\alpha}{2\pi} B^2. \tag{56}$$

Note that this is true for any rational map $R \in Rat_B^*$.

Now, let z_n and p_n be the zeros and poles of R in (40). Then the zeros and poles induced by the isorotation are given by

$$\tilde{z}_n = \mathrm{e}^{i\beta/B} z_n, \tag{57}$$
$$\tilde{p}_n = \mathrm{e}^{i\beta/B} p_n. \tag{58}$$

This induces a path $L(0, \beta)$. Then $N(L(0, \beta))$ is given by

$$N(L(0,\beta)) = -\frac{\beta}{2\pi} B. \tag{59}$$

For a given symmetry both $N(L(\alpha, 0))$ and $N(L(0, \beta))$ need not be integers. However, since $L(\alpha, \beta) = L(\alpha, 0) + L(0, \beta)$ is a loop, the homotopy invariant $N(L(\alpha, \beta))$ is an integer and is given by equation (55). This completes the proof of lemma 3.8. □

3.5 General Formula for $N(L)$

In this section we generalise the formula of lemma 3.8 to rational maps in Rat_B. First we discuss how to relate symmetric rational maps $R \in Rat_B$ to symmetric maps $\tilde{R} \in Rat_B^*$. Let $R \in Rat_B$ have the following symmetry

$$R(z) = \tilde{M}\left(R\left(M(z)\right)\right), \tag{60}$$

where M corresponds to a rotation by α around $\mathbf{n}$ and $\tilde{M}$ is an isorotation by β around $\mathbf{N}$. For $\alpha \neq 2\pi k$ for $k \in \mathbb{Z}$, M only leaves the points

$$z_{\mathbf{n}} = \frac{n_1 + i n_2}{1 + n_3} \tag{61}$$

and

$$z_{-\mathbf{n}} = -\frac{n_1 + i n_2}{1 - n_3} \tag{62}$$

fixed. Similarly, $\tilde{M}$ only leaves $R_{\pm\mathbf{N}}$ fixed, where $R_{\pm\mathbf{N}}$ is defined as in (61) and (62), respectively. Therefore, equation (60) implies that $R(z_{-\mathbf{n}}) = R_{\pm\mathbf{N}}$. By redefining $\mathbf{N}$ and β if necessary we set

$$R(z_{-\mathbf{n}}) = R_{\mathbf{N}}. \tag{63}$$

Lemma 3.9 *There are transformations $\tilde{M}_{\tilde{T}}$ and M_T such that the map*

$$\tilde{R}(\tilde{z}) = \tilde{M}_{\tilde{T}}(R(M_T^{-1}(\tilde{z}))) \tag{64}$$

is invariant under

$$\tilde{R}(\tilde{z}) = \tilde{M}_1(\tilde{R}(M_3(\tilde{z}))), \tag{65}$$

where $\tilde{M}_1$ is an isorotation by β around the X_1-axis and M_3 is a rotation by α around the x_3-axis. Furthermore, $\tilde{R}(\infty) = 1$. $\tilde{R}$ is unique up to rotations around x_3 and isorotations around X_1.

Proof: M_T is induced by the $SO(3)$ rotation

$$T : \mathbf{n} \mapsto \mathbf{n}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \tag{66}$$

We can choose orthonormal vectors $\mathbf{n}$, $\mathbf{u}$, and $\mathbf{v}$ such that

$$T^{-1} = T^t = (\mathbf{u}, \mathbf{v}, \mathbf{n}), \tag{67}$$

and $\det T = 1$. Similarly, $\tilde{M}_{\tilde{T}}$ is induced by

$$\tilde{T} : \mathbf{N} \mapsto \mathbf{N}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}. \tag{68}$$

Again, we choose orthonormal vectors $\mathbf{N}$, $\mathbf{U}$, and $\mathbf{V}$, such that

$$\tilde{T}^{-1} = (\mathbf{N}, \mathbf{U}, \mathbf{V}), \tag{69}$$

and $\det \tilde{T} = 1$. There is a $U(1) \times U(1)$ family of choices how to define $\mathbf{u}$ and $\mathbf{v}$, and $\mathbf{U}$ and $\mathbf{V}$. This is generated by rotations and isorotations around the symmetry axis $\mathbf{n}$ and $\mathbf{N}$. Now, we can express the rotations and isorotations as

$$M(z) = M_T^{-1}(M_3(M_T(z))) \tag{70}$$

and

$$\tilde{M}(R) = \tilde{M}_{\tilde{T}}^{-1}(\tilde{M}_1(\tilde{M}_{\tilde{T}}(R))), \tag{71}$$

where M_3 is a rotation by α around the x_3-axis and $\tilde{M}_1$ is an isorotation by β around the X_1-axis. By inserting equations (70) and (71) into equation (60) and acting with $\tilde{M}_{\tilde{T}}$, we obtain

$$\tilde{M}_{\tilde{T}}(R(M_T^{-1}(M_T(z)))) = \tilde{M}_1(\tilde{M}_{\tilde{T}}(R(M_T^{-1}(M_3(M_T(z)))))). \tag{72}$$

Setting $\tilde{z} = M_T(z)$ and $\tilde{R}(\tilde{z}) = \tilde{M}_{\tilde{T}}(R(M_T^{-1}\tilde{z}))$ we have shown equation (65). Furthermore,

$$\tilde{R}(\infty) = \tilde{M}_{\tilde{T}}(R(M_T^{-1}(\infty)) = \tilde{M}_{\tilde{R}}(R(z_{-\mathbf{n}})) = \tilde{M}_{\tilde{T}}(R_{\mathbf{N}}) = 1. \tag{73}$$

$\tilde{R}(z)$ is unique up to the above mentioned $U(1) \times U(1)$ family of choices which corresponds to rotations around x_3 and isorotations around X_1. $\square$

In order to uniquely define the rotation angle α we make the following choice. A rotation is expressed as $\exp(-i\alpha \mathbf{n} \cdot \mathbf{J})$ where $\alpha \in [-2\pi, 2\pi]$ and the sign of $\mathbf{n}$ is chosen such that $n_3 > 0$ or $n_3 = 0$ and $n_2 > 0$ or $n_1 = 1$. Similarly, $\beta \in [-2\pi, 2\pi]$ and $\mathbf{N}$ is given by $R(z_{-\mathbf{n}})$. With the above lemmata we prove the following theorem.

Theorem 3.10 *The value of N for a given symmetry of a rational map $R \in Rat_B$ only depends on the rotation angle α and the isorotation angle β, where the angles are defined such that $R(z_{-\mathbf{n}}) = R_{\mathbf{N}}$. It is given by $N = \frac{B}{2\pi}(B\alpha - \beta)$.*

Proof: Let R_0 be a based rational map that is symmetric under a rotation by α followed by an isorotation by β. If this generates axial symmetry then we can apply lemma 3.8. We will now consider the case that the symmetry is finite.[6] Then α and β can be written as $\alpha = 2\pi m/n$ and $\beta = 2\pi k/l$ where $k, m \in \mathbb{Z}$ and $n, l \in \mathbb{N}$.

In order to compute $N(L)$ we use the fact that $N(L)$ is an isomorphism. Rather than calculating $N(L)$ for α and β we calculate $N(\tilde{L})$ for $\tilde{\alpha} = nl\alpha$ and $\tilde{\beta} = nl\beta$, *i.e.* nl times the original loop L. Then $N(L)$ is given by $N(\tilde{L})/(nl)$. The advantage of the new loop $\tilde{L}$ is that it contains complete $U(1) \times U(1)$ orbits of rotations and isorotations. Such orbits always generate closed loops.

Let R_0 be a based rational map. Since Rat_B^* is connected there is a path R_t from the original map R_0 to a map with axial symmetry denoted by R_1. Since rotations and isorotations preserve the degree and rotations M_3 around x_3 and isorotations $\tilde{M}_1$ around X_1 preserve the base point, M_3 and $\tilde{M}_1$ give rise to a homotopy of loops $\tilde{L}_t$ starting at R_t and generated by $\tilde{\alpha}$ and $\tilde{\beta}$. All the loops $\tilde{L}_t$ are well defined and closed, and $\tilde{L}_0 = \tilde{L}$ and $\tilde{L}_1$ is the loop for an axial symmetric map. We can apply the formula of lemma 3.8 to $\tilde{L}_1$ and obtain

$$\begin{aligned} N(L) &= \frac{N(\tilde{L})}{nl} = \frac{1}{nl}\frac{B}{2\pi}(B\tilde{\alpha} - \tilde{\beta}) \\ &= \frac{B}{2\pi}(B\alpha - \beta). \end{aligned}$$

Now, let R be an arbitrary rational map of degree B, *i.e.* $R \in Rat_B$. Let its symmetry be a rotation by α around $\mathbf{n}$ and an isorotation by β around $\mathbf{N}$ such that $R(z_{-\mathbf{n}}) = R_{\mathbf{N}}$. Then we can use lemma 3.9 to transform $R \in Rat_B$ into a map R_0 and calculate N as above. This transformation is unique up to a $U(1) \times U(1)$ family of choices which does not change the value of N. This completes the proof of theorem 3.10. $\square$

[6] A finite dimensional compact Lie group cannot have discrete subgroups of infinite order.

Formula (38) can also be used to calculate N for a given symmetry numerically. We performed these calculations for the symmetries of Skyrmions up to $B = 10$ and confirmed the results of theorem 3.10.

We will now explore a few simple consequences of theorem 3.10 and corollary 3.6. First note that the loop generated by a rotation (or isorotation) is homotopic to an element of the fundamental group of Rat_B, namely a^k where a is the generator of Rat_B and k is an integer. We know that $a^{2k} = 1$, therefore, either $k \equiv B \mod 2B$ or $k \equiv 0 \mod 2B$.

Consider first a 2π rotation, *i.e.* $N = B^2$. If B is even then

$$B^2 \equiv 0 \mod 2B \tag{74}$$

and the loop is contractible (even in Rat_B). If B is odd then

$$B \equiv 1 \mod 2 \text{ implies that } B^2 \equiv B \mod 2B \tag{75}$$

so that the loop is non-contractible and is homotopic to a^B. Notice that for 2π isorotations we obtain $N = B$. Therefore, rotations by 2π and isorotations by 2π are homotopic in Rat_B if and only if B is odd.

With the aid of corollary 3.6 we recover the result of Giulini that rotations by 2π give rise to nontrivial loops if and only if B is odd, [5]. The same is true for isorotations by 2π. In the following section, we will consider nontrivial loops due to symmetries of rational maps.

4 Results

In this section we use the results of the previous sections to construct the ground states and some excited states of the Skyrme model, which are compatible with the FR constraints. First we describe the construction, then we present our results in a table and compare them to the literature and experimental data.

4.1 Construction of Ground States

Battye and Sutcliffe have calculated the minimal energy rational maps for B up to 22, and these maps are unique up to arbitrary rotations and isorotations. Since only $B = 1$ and $B = 2$ have continuous symmetries and these cases are already extensively discussed in the literature, we restrict our attention to $B > 2$. Only the following discrete symmetries occur empirically: Dihedral symmetry D_n for $2 \leq n \leq 6$, tetrahedral symmetry T, octahedral symmetry O and icosahedral symmetry Y. All these symmetry groups can be generated by two generators. In order to calculate the ground state for a given baryon number B we perform the following steps:

1. Choose a representative R of the minimal energy rational map.

2. Choose two generators of the symmetry group.

3. For each generator determine $\mathbf{n}$ and α, and $\mathbf{N}$ and β such that $R(z_{-\mathbf{n}}) = R_{\mathbf{N}}$.

4. Calculate $N \pmod 2$ for each generator. These are the FR constraints.

5. Decompose rotations and isorotations into irreducible representations.

6. Starting with the $J = 0$ and $I = 0$ (or $J = \frac{1}{2}$ and $I = \frac{1}{2}$) state calculate which states are allowed, that is are consistent with the imposed symmetry, using group theory.

In the following we describe how to calculate irreducible representations, see *e.g.* [34]. Let $D_{ij}(g)$ be a matrix representation of a finite group G of order $|G|$. Then $\chi(g) = \mathrm{Tr}\ (D_{ij}(g))$ is its character. Any representation of G can be decomposed into irreducible representations with character χ_i, and the following orthogonality relation holds:

$$\frac{1}{|G|} \sum_{g \in G} \bar{\chi}_i(g) \chi_j(g) = \delta_{ij}. \tag{76}$$

The irreducible representations have been calculated for all the relevant groups, *e.g.* in the table in [35], also see [36] for the icosahedral group. We use the notation of [34] whereby one dimensional representations are labelled with an A and two dimensional representations by E. In this context a $'$ means that it is the representation of the double group, that is a 2π rotation gives rise to minus the identity.

There is a simple formula for characters of $SU(2)$ representations of dimension D. For a rotation by an angle θ around a unit vector $\mathbf{n}$ the character is given by

$$\chi\big(\exp\left(-i\theta\ \mathbf{n} \cdot \mathbf{J}\right)\big) = \frac{\sin \frac{D\theta}{2}}{\sin \frac{\theta}{2}}. \tag{77}$$

There is a significant difference between rotations and isorotations. For even baryon numbers B rotations by 2π are contractible. We can define an $SU(2)$ action on the homogeneous coordinates $[p, q]$. If B is odd, then a rotation by 2π gives rise to $[-p, -q]$, whereas if B is even, it gives $[p, q]$. Therefore, the $SU(2)$ representation of the symmetry group can always be mapped to an $SO(3)$ representation if B is even. This implies that we can choose the two dimensional irreducible representation E_1' of the double group to impose the symmetry.

For isorotations, the situation is different because 2π isorotations are always noncontractible in the space of rational maps, so a 2π isorotation always gives $[-p, -q]$. If B is odd, the isospin transformation corresponds to a double group representation E_k'. However, if B is even, it is given by a representation which is not in the double group.

Let g be a rotation by α around $\mathbf{n}$ followed by an isorotation by β around $\mathbf{N}$. Then we can calculate the FR constraints for the symmetry transformation, and we define

$$\chi_{FR}(g) = \begin{cases} 1 & \text{if contractible,} \\ -1 & \text{otherwise.} \end{cases} \tag{78}$$

$\chi_{FR}(g)$ forms a one dimensional representation of the symmetry group, which is identical to its character.

Let the wave function ψ transform under a tensor product of rotations and isorotations, namely the $2J+1$ dimensional representation J and the $2I+1$ dimensional representation I. As shown in the previous section, I and J are integers if B is even and half-integers if B is odd. If a Skyrmion is invariant under a symmetry group then equation (8) imposes the following additional constraint on the wave function:

$$\exp\left(-i\alpha\ \mathbf{n}\cdot\mathbf{J}\right)\exp\left(-i\beta\ \mathbf{N}\cdot\mathbf{I}\right)\psi = \chi_{FR}(g)\psi. \tag{79}$$

The character of a tensor product is given by the product of the characters. Therefore, the number n of representations for given quantum numbers J and I that are compatible with the FR constraints is

$$n = \frac{1}{|G|}\sum_{g\in G}\bar{\chi}_{rot,J}(g)\bar{\chi}_{iso,I}(g)\chi_{FR}(g). \tag{80}$$

In the following we describe table 1. The first column is the baryon number B. In the second column, we display the symmetry of the rational map, and also the corresponding irreducible representations of rotations and isorotations. In the following two columns we show the value of N for the two generators of the symmetry. We have chosen to display N rather than $N \mod 2$ because it contains some more information about how the symmetries are imposed. Yet, it is $N \mod 2$ that implies the FR representation given in the last column.

For $B=1$ the FR constraints are trivial because an isorotation is equivalent to a rotation. Similarly, for $B=2$ a rotation by α corresponds to an isorotation by 2α, so that N vanishes for axial symmetry. Yet, the C_2 rotation gives a non-trival value for N. For some values of B there are additional "excited" states labelled by B^*. These are the particularly symmetric maps for $B=5$ and 11 in Ref. [17], and also maps that are very close to minimal energy rational maps as mentioned in [18]. These excited states B^* give an indication of how important the symmetry of a configuration is for determining its ground state.

4.2 Ground States for $B = 1,\ldots,22$

Table 2 shows the ground states that have been calculated with our methods and compares our results to the experimental data in the table of isotopes [37]. It is

B	Symmetry	N_{g_1}	N_{g_2}	FR-Rep
1	$SO(3)$	—	—	A_1
2	D_∞	$N_{C_\infty} = 0$	$N_{C_2} = 1$	A_2
3	$T(E_1', E_1')$	$N_{C_3} = 2$	$N_{C_2} = 6$	A_1
4	$O(E_1', E_1)$	$N_{C_4} = 2$	$N_{C_3} = 8$	A_1
5	$D_2(E_1', E_1')$	$N_{C_2} = 10$	$N_{C_2'} = 10$	A_1
5*	$O(E_1', E_1')$	$N_{C_4} = 5$	$N_{C_3} = 10$	A_2
6	$D_4(E_1', E_1)$	$N_{C_4} = 12$	$N_{C_2'} = 15$	A_2
7	$Y(E_1', E_2')$	$N_{C_5} = 14$	$N_{C_3} = 14$	A_1
8	$D_6(E_1', E_1)$	$N_{C_6} = 8$	$N_{C_2'} = 28$	A_1
9	$D_4(E_1', E_1')$	$N_{C_4} = 18$	$N_{C_2'} = 36$	A_1
9*	$T(E_1', E_1')$	$N_{C_3} = 24$	$N_{C_2} = 36$	A_1
10	$D_4(E_1', E_1)$	$N_{C_4} = 20$	$N_{C_2'} = 45$	A_2
10*	$D_3(E_1', E_1)$	$N_{C_3} = 40$	$N_{C_2} = 45$	A_2
11	$D_3(E_1', E_1')$	$N_{C_3} = 44$	$N_{C_2'} = 55$	A_2
11*	$Y(E_1', E_1')$	$N_{C_5} = 22$	$N_{C_3} = 44$	A_1
12	$T(E_1', A_2 \oplus A_3)$	$N_{C_3} = 40$	$N_{C_2} = 72$	A_1
13	$O(E_1', E_1')$	$N_{C_4} = 39$	$N_{C_3} = 52$	A_2
13*	$D_4(E_1', E_1')$	$N_{C_4} = 39$	$N_{C_2'} = 78$	A_3
14	$D_2(E_1', A_1 \oplus A_3)$	$N_{C_2} = 98$	$N_{C_2'} = 91$	A_3
15	$T(E_1', E_1')$	$N_{C_3} = 80$	$N_{C_2} = 120$	A_1
16	$D_2(E_1', A_1 \oplus A_3)$	$N_{C_2} = 128$	$N_{C_2'} = 120$	A_1
16*	$D_3(E_1', E_1)$	$N_{C_3} = 96$	$N_{C_2} = 120$	A_1
17	$Y(E_1', E_2')$	$N_{C_5} = 68$	$N_{C_3} = 102$	A_1
17*	$O(E_1', E_1')$	$N_{C_4} = 68$	$N_{C_3} = 102$	A_1
18	$D_2(E_1', A_1 \oplus A_3)$	$N_{C_2} = 162$	$N_{C_2'} = 153$	A_3
19	$D_3(E_1', E_1')$	$N_{C_3} = 114$	$N_{C_2} = 190$	A_1
19*	$T(E_1', E_1')$	$N_{C_3} = 114$	$N_{C_2} = 190$	A_1
20	$D_6(E_1', E_1)$	$N_{C_6} = 60$	$N_{C_2} = 190$	A_1
21	$T(E_1', E_1')$	$N_{C_3} = 154$	$N_{C_2} = 210$	A_1
22	$D_5(E_1', E_1)$	$N_{C_5} = 88$	$N_{C_2} = 231$	A_2
22*	$D_3(E_1', E_1)$	$N_{C_3} = 176$	$N_{C_2} = 231$	A_2

Table 1: This table shows the Finkelstein-Rubinstein constraints for $B = 1$ to 22. For more details see text.

worth emphasising that nuclei with the same value of I and J are degenerate in the Skyrme model, since it only models the strong interaction. In other words, only the number of nucleons matters, not whether they are protons or neutrons.

B	$\|J\rangle\|I\rangle_0$	$\|J\rangle\|I\rangle_1$	$\|J\rangle\|I\rangle_2$	Experiment	$\|J\rangle\|I\rangle_{Exp.}$	Match
1	$\|\frac{1}{2}\rangle\|\frac{1}{2}\rangle$	$\|\frac{3}{2}\rangle\|\frac{1}{2}\rangle$	$\|\frac{1}{2}\rangle\|\frac{3}{2}\rangle$	${}^{1}_{1}$H	$\|\frac{1}{2}\rangle\|\frac{1}{2}\rangle$	■
2	$\|1\rangle\|0\rangle$	$\|3\rangle\|0\rangle$	$\|0\rangle\|1\rangle$	${}^{2}_{1}$H	$\|1\rangle\|0\rangle$	■
3	$\|\frac{1}{2}\rangle\|\frac{1}{2}\rangle$	$\|\frac{5}{2}\rangle\|\frac{1}{2}\rangle$	$\|\frac{3}{2}\rangle\|\frac{3}{2}\rangle$	${}^{3}_{2}$He	$\|\frac{1}{2}\rangle\|\frac{1}{2}\rangle$	■
4	$\|0\rangle\|0\rangle$	$\|4\rangle\|0\rangle$	$\|0\rangle\|1\rangle$	${}^{4}_{2}$He	$\|0\rangle\|0\rangle$	■
5	$\|\frac{1}{2}\rangle\|\frac{1}{2}\rangle$	$\|\frac{3}{2}\rangle\|\frac{1}{2}\rangle$	$\|\frac{1}{2}\rangle\|\frac{3}{2}\rangle$	(${}^{5}_{2}$He)	$\|\frac{3}{2}\rangle\|\frac{1}{2}\rangle$	(${}^{5}_{2}$He*)
5^*	$\|\frac{5}{2}\rangle\|\frac{1}{2}\rangle$	$\|\frac{7}{2}\rangle\|\frac{1}{2}\rangle$	$\|\frac{3}{2}\rangle\|\frac{3}{2}\rangle$	(${}^{5}_{2}$He)	$\|\frac{3}{2}\rangle\|\frac{1}{2}\rangle$	(${}^{5}_{2}$He*)
6	$\|1\rangle\|0\rangle$	$\|3\rangle\|0\rangle$	$\|1\rangle\|1\rangle$	${}^{6}_{3}$Li	$\|1\rangle\|0\rangle$	■
7	$\|\frac{7}{2}\rangle\|\frac{1}{2}\rangle$	$\|\frac{13}{2}\rangle\|\frac{1}{2}\rangle$	$\|\frac{3}{2}\rangle\|\frac{3}{2}\rangle$	${}^{7}_{3}$Li	$\|\frac{3}{2}\rangle\|\frac{1}{2}\rangle$	(${}^{7}_{3}$Li**)
8	$\|0\rangle\|0\rangle$	$\|2\rangle\|0\rangle$	$\|0\rangle\|1\rangle$	(${}^{8}_{4}$Be)	$\|0\rangle\|0\rangle$	■
9	$\|\frac{1}{2}\rangle\|\frac{1}{2}\rangle$	$\|\frac{3}{2}\rangle\|\frac{1}{2}\rangle$	$\|\frac{1}{2}\rangle\|\frac{3}{2}\rangle$	${}^{9}_{4}$Be	$\|\frac{3}{2}\rangle\|\frac{1}{2}\rangle$	(${}^{9}_{4}$Be*)
9^*	$\|\frac{1}{2}\rangle\|\frac{1}{2}\rangle$	$\|\frac{5}{2}\rangle\|\frac{1}{2}\rangle$	$\|\frac{3}{2}\rangle\|\frac{3}{2}\rangle$	${}^{9}_{4}$Be	$\|\frac{3}{2}\rangle\|\frac{1}{2}\rangle$	(${}^{9}_{4}$Be*)
10	$\|1\rangle\|0\rangle$	$\|3\rangle\|0\rangle$	$\|1\rangle\|1\rangle$	${}^{10}_{5}$B	$\|3\rangle\|0\rangle$	(${}^{10}_{5}$B*)
10^*	$\|1\rangle\|0\rangle$	$\|3\rangle\|0\rangle$	$\|1\rangle\|1\rangle$	${}^{10}_{5}$B	$\|3\rangle\|0\rangle$	(${}^{10}_{5}$B*)
11	$\|\frac{1}{2}\rangle\|\frac{1}{2}\rangle$	$\|\frac{3}{2}\rangle\|\frac{1}{2}\rangle$	$\|\frac{1}{2}\rangle\|\frac{3}{2}\rangle$	${}^{11}_{5}$B	$\|\frac{3}{2}\rangle\|\frac{1}{2}\rangle$	(${}^{11}_{5}$B*)
11^*	$\|\frac{1}{2}\rangle\|\frac{1}{2}\rangle$	$\|\frac{11}{2}\rangle\|\frac{1}{2}\rangle$	$\|\frac{3}{2}\rangle\|\frac{3}{2}\rangle$	${}^{11}_{5}$B	$\|\frac{3}{2}\rangle\|\frac{1}{2}\rangle$	(${}^{11}_{5}$B*)
12	$\|0\rangle\|0\rangle$	$\|3\rangle\|0\rangle$	$\|0\rangle\|1\rangle$	${}^{12}_{6}$C	$\|0\rangle\|0\rangle$	■
13	$\|\frac{5}{2}\rangle\|\frac{1}{2}\rangle$	$\|\frac{7}{2}\rangle\|\frac{1}{2}\rangle$	$\|\frac{3}{2}\rangle\|\frac{3}{2}\rangle$	${}^{13}_{6}$C	$\|\frac{1}{2}\rangle\|\frac{1}{2}\rangle$	(${}^{13}_{6}$C***)
13^*	$\|\frac{3}{2}\rangle\|\frac{1}{2}\rangle$	$\|\frac{5}{2}\rangle\|\frac{1}{2}\rangle$	$\|\frac{1}{2}\rangle\|\frac{3}{2}\rangle$	${}^{13}_{6}$C	$\|\frac{1}{2}\rangle\|\frac{1}{2}\rangle$	(${}^{13}_{6}$C**)
14	$\|1\rangle\|0\rangle$	$\|2\rangle\|0\rangle$	$\|0\rangle\|1\rangle$	${}^{14}_{7}$N	$\|1\rangle\|0\rangle$	■
15	$\|\frac{1}{2}\rangle\|\frac{1}{2}\rangle$	$\|\frac{5}{2}\rangle\|\frac{1}{2}\rangle$	$\|\frac{3}{2}\rangle\|\frac{3}{2}\rangle$	${}^{15}_{7}$N	$\|\frac{1}{2}\rangle\|\frac{1}{2}\rangle$	■
16	$\|0\rangle\|0\rangle$	$\|2\rangle\|0\rangle$	$\|0\rangle\|1\rangle$	${}^{16}_{8}$O	$\|0\rangle\|0\rangle$	■
16^*	$\|0\rangle\|0\rangle$	$\|2\rangle\|0\rangle$	$\|0\rangle\|1\rangle$	${}^{16}_{8}$O	$\|0\rangle\|0\rangle$	■
17	$\|\frac{7}{2}\rangle\|\frac{1}{2}\rangle$	$\|\frac{13}{2}\rangle\|\frac{1}{2}\rangle$	$\|\frac{3}{2}\rangle\|\frac{3}{2}\rangle$	${}^{17}_{8}$O	$\|\frac{5}{2}\rangle\|\frac{1}{2}\rangle$	(${}^{17}_{8}$O$^{*(8)}$)
17^*	$\|\frac{1}{2}\rangle\|\frac{1}{2}\rangle$	$\|\frac{7}{2}\rangle\|\frac{1}{2}\rangle$	$\|\frac{3}{2}\rangle\|\frac{3}{2}\rangle$	${}^{17}_{8}$O	$\|\frac{5}{2}\rangle\|\frac{1}{2}\rangle$	(${}^{17}_{8}$O*)
18	$\|1\rangle\|0\rangle$	$\|2\rangle\|0\rangle$	$\|0\rangle\|1\rangle$	${}^{18}_{8}$O	$\|0\rangle\|1\rangle$	(${}^{18}_{9}$F)
19	$\|\frac{1}{2}\rangle\|\frac{1}{2}\rangle$	$\|\frac{3}{2}\rangle\|\frac{1}{2}\rangle$	$\|\frac{1}{2}\rangle\|\frac{3}{2}\rangle$	${}^{19}_{9}$F	$\|\frac{1}{2}\rangle\|\frac{1}{2}\rangle$	■
19^*	$\|\frac{1}{2}\rangle\|\frac{1}{2}\rangle$	$\|\frac{5}{2}\rangle\|\frac{1}{2}\rangle$	$\|\frac{3}{2}\rangle\|\frac{3}{2}\rangle$	${}^{19}_{9}$F	$\|\frac{1}{2}\rangle\|\frac{1}{2}\rangle$	■
20	$\|0\rangle\|0\rangle$	$\|2\rangle\|0\rangle$	$\|0\rangle\|1\rangle$	${}^{20}_{10}$Ne	$\|0\rangle\|0\rangle$	■
21	$\|\frac{1}{2}\rangle\|\frac{1}{2}\rangle$	$\|\frac{5}{2}\rangle\|\frac{1}{2}\rangle$	$\|\frac{3}{2}\rangle\|\frac{3}{2}\rangle$	${}^{21}_{10}$Ne	$\|\frac{3}{2}\rangle\|\frac{1}{2}\rangle$	(${}^{21}_{10}$Ne*)
22	$\|1\rangle\|0\rangle$	$\|3\rangle\|0\rangle$	$\|1\rangle\|1\rangle$	${}^{22}_{10}$Ne	$\|0\rangle\|1\rangle$	(${}^{22}_{11}$Na*)
22^*	$\|1\rangle\|0\rangle$	$\|3\rangle\|0\rangle$	$\|1\rangle\|1\rangle$	${}^{22}_{10}$Ne	$\|0\rangle\|1\rangle$	(${}^{22}_{11}$Na*)

Table 2: These are the ground states and two excited states for $B = 1$ to 22. For further details see text.

Table 2 is organised as follows. The first column gives the baryon number B corresponding to the number of nucleons in the atomic nuclei. In the following three columns we display the ground state and two excited states which have been obtained from the zero mode quantization of the Skyrme model. The ground state is the state with the lowest value of I, *i.e.* either 0 or $\frac{1}{2}$, and the lowest value of J which is compatible with the constraints. One excited state is given by the second lowest value of J. The other excited state is given by the second lowest value of I, *i.e.* 1 or $\frac{3}{2}$, and the lowest value of J that is compatible with the constraints. This calculation makes assumptions about the relative magnitudes of U_{ij}, V_{ij} and W_{ij} in equation (10), see Sect. 2.2. These integrals have to be evaluated explicitly using the numerical solutions, and this might change the relative order of the states. Such calculations will be left for a future publication.

In the column "Experiment" we cite the ground state for a given baryon number and the following column gives the value for J and I. For $B = 21$ and $B = 22$ the value of I is not given in [37]. We make the reasonable assumption that $I = \frac{1}{2}$ and $I = 1$, respectively. The last column labelled "Match" has a black box when the Skyrme ground state agrees with experiment and otherwise shows the lowest nuclear state with the given values of I and J. Here $*$ is short for first excited state, $**$ is the second excited state, etc. Nuclei in brackets are unstable. For some excited state the value of I is missing in [37]. Here we assume that it agrees with the value for the ground state. The values for $B = 1$ and $B = 2$ are taken from the literature, [2, 11]. Also note that the results agree with Carson [14] for $B = 3$ and also with Irwin [16] who considered $B = 4$ to 9.[7]

The results for even baryon number are promising. Our calculations of the ground state agree with experiment for all but three cases, namely $B = 10$, 18 and 22. For $B = 10$ calculations predict the state $|1\rangle|0\rangle$ rather than $|3\rangle|0\rangle$. It is difficult to see in our scheme why the $|1\rangle|0\rangle$ state should have higher energy than the $|3\rangle|0\rangle$ state. For $B = 18$ the experimental ground state is $|0\rangle|1\rangle$ rather than $|1\rangle|0\rangle$. This deserves further investigation, because the ordering of the states makes assumptions about the integrals (12) – (14). Finally for $B = 22$ our calculations predict a $|1\rangle|0\rangle$ state whereas the experimental ground state is probably a $|0\rangle|1\rangle$ state, which is incompatible with our results. Note that the value of the binding energy per nucleon has local peaks at $B = 4$, 8, 12, 16 and 20. All these ground states are predicted correctly.

The results for odd baryon numbers are not as good. Our calculations agree with experiment for $B = 1$, 3, 15, and 19 but the disagree for $B = 5$, 7, 9, 11, 13, 17, and 21. It is worth mentioning that for $B = 5$ there is no stable nucleus, and in general atomic nuclei are more stable for even B than for odd B. However, there are reasons to believe that the zero mode approximation is less reliable for odd B. For $B = 1$ it has been argued by many authors that the correct energy to minimise is not the classical mass M (given by E in (1)) but the rotationally improved energy

$$\tilde{E} = M + \frac{1}{2\Theta} I(I+1) + E_{\boldsymbol{\pi}}, \tag{81}$$

[7] The only exception is that for $B = 4$ we found an excited state $|0\rangle|1\rangle$ which is lower than the $|2\rangle|1\rangle$ state predicted in [16].

where Θ is the moment of inertia and $E_{\boldsymbol{\pi}}$ is the additional energy for including a pion mass term, see [38]. A similar construction has to be implemented for $B > 1$. This could deform the Skymion, change its symmetry and thereby change its ground state. For even B this does not play such a big role as $I = 0$ in the ground state so that the additional term does not contribute, and as a first approximation, the J terms are assumed to be a smaller perturbation. For odd B, however, $I = \frac{1}{2}$ so that the term always has to be taken into account.[8]

5 Conclusion

In this paper we showed that it is possible to calculate the FR constraints directly from the rational map ansatz. The key idea is to think of the rational map ansatz as a suspension. We proved that a loop in configuration space Q_B^* is contractible if and only if it is homotopic to a suspension of a loop in Rat_B which is an element of even order in $\pi_1(Rat_B)$. Even though the fundamental group of rational maps is more complicated than the fundamental group of configuration space it is nevertheless possible to derive a formula to calculate the homotopy class for loops generated by rotations and isorotations. It is worth emphasising that this formula is mathematically rigorous and there is no approximation involved. Therefore, the rational map ansatz is not only a good approximation to the minimal energy configurations, but it also captures important topology of the configuration space.

In order to quantize Skyrmions as described in Sect. 2.2 it is important to have an approximate moduli space, and it is here that approximations come into play. In this paper we chose the zero mode approximation which only takes the rotational and isorotational degrees of freedom into account, because this is the simplest nontrivial application of our results about FR constraints. In Sect. 4 we calculated the ground states of the Skyrme model for baryon numbers up to $B = 22$. We found agreement with Irwin [16] who calculated the FR constraints using an analogy with monopoles. Our results agree with experiments for all even baryon numbers apart from $B = 10$, 18 and 22. It also appears that the Skyrme model works best for stable nuclei. In the odd baryon sector, the results are not as promising. Yet, there are reasons why the zero mode approximation does not work as well for odd B, as discussed at the end of the previous section.

In the following we will comment on future work. In this paper we only relied on group theory making assumptions about the relative magnitude of integrals which involve the minimal energy configurations. With the numerical solutions in [18] it is possible to refine our results, verify the ground states, and calculate quantities such as mass, charge radii, and magnetic moments. By considering the reflection symmetries of Skyrmions the parity of the ground states can be calculated, [16]. Finally, it is vital to go beyond the zero mode approximation.

[8]For a path integral derivation of the rotationally improved Skyrmion see Ref. [39].

The problem of imposing FR constraints was solved in this paper, so the difficult question is how to find a suitable approximation to the Skyrmion moduli space.

The first step is to consider rotationally improved Skyrmions. These could change the symmetries of the configurations and therefore, give rise to improved ground states. These changes will be particularly significant for odd B because in this case the contribution of isorotations to the energy cannot vanish.

Another possibility is to calculate the lowest vibrational modes of a Skyrmion and their frequencies [40, 41, 42, 43, 44]. Then the excited states, and maybe even some ground states, are combinations of rotational and vibrational states. Another approach would be to construct a better approximation to the Skyrmion moduli space. For $B = 2$ Manton has constructed a 12 dimensional "unstable" manifold [45] which describes the configuration space of low energy configurations and can be thought of as a moduli space with a potential. Leese *et al.* found a 10 dimensional submanifold which corresponds to Skyrmions in the attractive channel. In [13] the $B = 2$ Skyrmion was quantized in this attractive channel approximation and the result was significantly better than in the zero mode approximation. Note that the moduli space of monopoles is conjectured to be related to the moduli space of attractive channel Skyrmions. Therefore, it might be possible to use monopole fields for quantizing Skyrmions.

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