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Fermionic quantization of Hopf solitons

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Abstract

In this paper we show how to quantize Hopf solitons using the Finkelstein-Rubinstein approach. Hopf solitons can be quantized as fermions if their Hopf charge is odd. Symmetries of classical minimal energy configurations induce loops in configuration space which give rise to constraints on the wave function. These constraints depend on whether the given loop is contractible. Our method is to exploit the relationship between the configuration spaces of the Faddeev-Hopf and Skyrme models provided by the Hopf fibration. We then use recent results in the Skyrme model to determine whether loops are contractible. We discuss possible quantum ground states up to Hopf charge $Q = 7$.

1 Introduction

The possibility of knot-like solitons in a nonlinear field theory was first proposed by Faddeev in 1975, [10]. In 1997, interest in the model was revived by an article by Faddeev and Niemi [11]: the advent of larger computer power and a better understanding of the initial conditions led to a series of papers. In [15] axially symmetric configurations were studied extensively. Papers by Battye and Sutcliffe showed that for higher Hopf charge twisted, knotted and linked configurations occur [7, 8]. The most recent results are due to Hietarinta and Salo [16, 17]. Stable and metastable static solutions have now been explored up to Hopf charge $Q = 8$.

Quantization of Hopf solitons was first discussed in [15]. More recently Su described a collective coordinate quantization in [27] which was motivated by the collective coordinate quantization of Skyrmions in [1]. However, collective coordinate quantizations can be potentially misleading unless the topology of configuration space is examined carefully [5].

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In this paper we describe the fermionic quantization of Hopf solitons following an old idea of Finkelstein and Rubinstein [12]. Solitons in scalar field theories can consistently be quantized as fermions provided the fundamental group of configuration space has a $\mathbb{Z}_2$ subgroup generated by a loop in which two identical solitons are exchanged. Loops in configuration space give rise to so-called Finkelstein-Rubinstein constraints which depend on whether the loop is contractible. The Skyrme model [28] was the main motivation for this approach; see [20] for further references. Symmetries of classical configurations induce loops in configuration space. After quantization these loops give rise to constraints on the wave function. Recently, a simple formula has been found to determine whether a loop in the configuration space of Skyrmions is contractible [20]. We shall exploit the fact that Skyrmions and Hopf solitons are related via the Hopf map to use Skyrmions as a tool to study Hopf solitons.

This paper is organized as follows. In section 2 we discuss the configuration space of Hopf solitons for general domains. The configuration space of Skyrmions can be related to Hopf solitons via the Hopf map which is a fibration. This mathematical structure enables us to prove that the Hopf map induces, in certain circumstances, an isomorphism between the fundamental groups of the Skyrme and Faddeev-Hopf configuration spaces. In section 3 we summarize some known facts about Hopf solitons. In section 4 we describe how to quantize a Hopf soliton as a fermion and calculate possible ground states in the Faddeev-Hopf model. In the following section, we discuss collective coordinate quantization in this context. We end with some concluding remarks.

2 The topology of configuration space

Let M be a compact, connected, oriented 3-manifold and $p_0 \in M$ be a marked point. The case of most interest is $M = S^3$, interpreted as the one point compactification of $\mathbb{R}^3$ with p_0 representing the boundary at infinity. The configuration space we seek to study is $(S^2)^M$, the space of based maps $M \to S^2$, that is continuous maps sending the chosen point p_0 to a chosen point in S^2, $(0, 0, 1)$ say. We also define the space $\mathrm{Free}(M, S^2)$ of unbased maps $M \to S^2$ and similarly $(S^3)^M$ and $\mathrm{Free}(M, S^3)$ where the chosen point is $(1, 0) \in S^3 \subset \mathbb{C}^2$, say. All such spaces are given the compact open topology (equivalent to the C^0 topology). Our goal in this section is to relate the topology of $(S^2)^M$, the Faddeev-Hopf configuration space, to that of $(S^3)^M$, the standard Skyrme configuration space.

The connected components of $(S^2)^M$ were enumerated and classified by Pontrjagin [25]. Let μ be a generating 2-cocycle for $H^2(S^2;\mathbb{Z}) = \mathbb{Z}$. Then given $\phi \in (S^2)^M$ one has an associated 2-cocycle $\phi^*\mu \in H^2(M;\mathbb{Z})$ by pullback. No two maps $M \to S^2$ having noncohomologous 2-cocycles can be homotopic, and every 2-cocycle on M is cohomologous to the pullback of μ by some map. Thus, the homotopy classes of maps $M \to S^2$ fall into disjoint families labelled by $H^2(M;\mathbb{Z})$. Within any such family, the classes are labelled by elements of $H^3(M;\mathbb{Z})/2[\phi^*\mu] \cup H^1(M;\mathbb{Z})$. Note that this group varies from family to family and that to compute it requires knowledge of the ring structure on $H^*(M;\mathbb{Z})$. The most important family is the one with $[\phi^*\mu] = 0$, the so-called algebraically inessential maps. Classes within this family are labelled by elements of $H^3(M;\mathbb{Z}) = \mathbb{Z}$, identified with the Hopf charge Q, which we would like to interpret as the soliton number of the configuration, that is, the excess of solitons over antisolitons. Let us denote the space of algebraically inessential maps by $(S^2)^M_* \subset (S^2)^M$. Note that these sets coincide if $H^2(M;\mathbb{Z}) = 0$, for example, when $M = S^3$. Configurations outside $(S^2)^M_*$ wrap some 2-cycle in M nontrivially around S^2. They are bound to some topological defect in physical space and so are arguably not localized topological solitons at all. We shall not consider their physics in this paper.

Our main tool will be the Hopf map $\pi : S^3 \to S^2$, most conveniently defined by identifying S^3 with the unit sphere in $\mathbb{C}^2$ and S^2 with $\mathbb{C}P^1$, for then

$$\pi : (z_1, z_2) \mapsto [z_1, z_2]. \tag{2.1}$$

Note that π sends the marked point $(1, 0) \in S^3$ to the marked point $[1, 0] \in S^2$, corresponding to the North pole, $(0, 0, 1)$. The map π is a fibration, that is, it has the homotopy lifting property with respect to all domains. A map $\phi : M \to S^2$ has a lift $\widetilde{\phi} : M \to S^3$ (where $\pi \circ \widetilde{\phi} = \phi$) if and only if $\phi^*\mu = 0$, that is, if and only if $\phi \in (S^2)^M_*$. The integer in $H^3(M;\mathbb{Z})$ labelling the class of ϕ is precisely the degree of $\widetilde{\phi} : M \to S^3$, that is, the baryon number of the Skyrme configuration $\widetilde{\phi}$. This was shown explicitly for $M = S^3$ in [24]. So, given a Skyrme configuration of degree Q, we may produce an algebraically inessential Hopf configuration of charge Q by composition with the Hopf map. In this way we produce a map $\pi_* : (S^3)^M \to (S^2)^M_*$. To what extent does the topology of $(S^3)^M$ determine that of $(S^2)^M_*$?

Theorem 1 *The map $\pi_* : (S^3)^M \to (S^2)^M_*$ induced by the Hopf fibration is a Serre fibration.*

Proof: We must prove that the map has the homotopy lifting property with respect to all *disks* D^k [23], that is, that the commutative diagram below left may be completed by a map $\widetilde{H}$ along the diagonal. Here H is a homotopy between two maps $f_0, f_1 : D^k \to (S^2)^M_*$ and $\widetilde{f}_0$ is a lift of f_0. Using the identification of $g : X \to Y^Z$ with $\hat{g} : Z \times X \to Y$, we produce the commutative diagram below right. Now the homotopy $\hat{H}$ certainly does lift to $\widetilde{\hat{H}}$ since π is a fibration. From $\widetilde{\hat{H}}$ we produce a map $\check{H} : D^k \times I \to \mathrm{Free}(M, S^3)$ by $(\check{H}(d,t))(p) = \widetilde{\hat{H}}(p, d, t)$. A priori, this is not necessarily the lifted homotopy we seek, however, since there is no reason why it should respect the basing condition.

$$\begin{tikzcd}
D^k \times \{0\} \arrow[r, "\widetilde{f}_0"] \arrow[d, "\iota"'] & (S^3)^M \arrow[d, "\pi_*"] \\
D^k \times [0,1] \arrow[r, "H"'] \arrow[ur, "\widetilde{H}"] & (S^2)^M_*
\end{tikzcd}
\qquad
\begin{tikzcd}
M \times D^k \times \{0\} \arrow[r, "\widetilde{\hat{f}}_0"] \arrow[d, "\iota"'] & S^3 \arrow[d, "\pi"] \\
M \times D^k \times [0,1] \arrow[r, "\hat{H}"'] \arrow[ur, "\widetilde{\hat{H}}"] & S^2
\end{tikzcd}$$

Let $U \subset S^2$ be a small closed ball centred on $(0, 0, 1)$ and choose a local trivialization of the Hopf bundle $S^1 \hookrightarrow S^3 \xrightarrow{\pi} S^2$ over U. Then by continuity of $\hat{H}$ and compactness of $D^k \times [0, 1]$, there exists a closed ball $B \subset M$ centred on p_0 so that the restriction $\widetilde{\hat{H}}| : B \times D^k \times I \to S^3$ takes values in $\pi^{-1}(U)$. We may write it, with respect to our local trivialization, as

$$\widetilde{\hat{H}}|(p, d, t) = (\hat{H}(p, d, t), \lambda(p, d, t))$$

where $\lambda : B \times D^k \times I \to S^1$. In this language, we are done if $\lambda| : \{p_0\} \times D^k \times I \to \{1\}$, for then the map $\check{H}$ *does* satisfy the basing criteria. Note that we are free to change λ to any continuous map λ_* we please, provided we do not change it on $\partial B \times D \times I$, since this just shifts $\widetilde{\hat{H}}$ along the fibres of S^3 which does not change $\pi \circ \widetilde{\hat{H}}$, so that the altered map is still a lift of $\hat{H}$, and is still continuous. Now since $\partial B \times D^k \times I$ deformation retracts to S^2 and $\pi_2(S^1) = 0$, $\lambda| : \partial B \times D^k \times I$ is nullhomotopic and we may construct the required $\lambda_* : B \times D \times I \to S^1$ by applying the null homotopy radially in B. □

Our main interest is to understand the fundamental group of each connected component of $(S^2)^M_*$. Given any map $\rho : X \to Y$, there is a natural homomorphism $\rho_* : \pi_1(X) \to \pi_1(Y)$ defined by composition of loops in X with ρ. The fact that π_*, which we will henceforth denote ρ, is a Serre fibration allows us to obtain a short exact sequence relating $\pi_1((S^3)^M)$ and $\pi_1((S^2)^M_*)$. In the case $M = S^3$ this reduces to the statement that the homomorphism ρ_* associated with ρ is actually an isomorphism. We can therefore determine the homotopy class of a loop in the Hopf configuration space by lifting it to a loop in the Skyrme configuration space and applying known results.

Theorem 2 *The map $\rho : (S^3)^M \to (S^2)^M_*$ obtained from the Hopf fibration induces a short exact sequence of groups*

$$0 \to \pi_1((S^3)^M) \xrightarrow{\rho_*} \pi_1((S^2)^M_*) \to H^1(M; \mathbb{Z}) \to 0.$$

Proof: Given any Serre fibration $F \hookrightarrow E \xrightarrow{\rho} B$, where F, E, B denote the fibre, total space and base, we have an induced long exact sequence of homotopy groups:

$$\ldots \to \pi_1(F) \to \pi_1(E) \xrightarrow{\rho_*} \pi_1(B) \to \pi_0(F) \to \pi_0(E) \xrightarrow{\rho_*} \pi_0(B) \to 0. \tag{2.2}$$

In the case at hand, $E = (S^3)^M$, $B = (S^2)^M_*$ and $F = (S^1)^M$. Using the identification $S^1 = U(1)$, we see that F is a topological group, so all its connected components are homeomorphic. The components of G^M for any Lie group G are enumerated in [3] while $\pi_1(G^M)$ is constructed in [4]. The relevant results here are $\pi_0(F) = H^1(M; \mathbb{Z})$ and $\pi_1(F) = 0$. Note also that $\pi_0(E) = \pi_0(B) = H^3(M; \mathbb{Z}) = \mathbb{Z}$ by the theorems of Hopf and Pontrjagin. Substituting in (2.2) gives

$$0 \to \pi_1(E) \xrightarrow{\rho_*} \pi_1(B) \to H^1(M; \mathbb{Z}) \to \mathbb{Z} \xrightarrow{\rho_*} \mathbb{Z} \to 0. \tag{2.3}$$

By exactness, the second ρ_* is surjective, and there are only two surjective homomorphisms $\mathbb{Z} \to \mathbb{Z}$ (namely $1 \mapsto 1$ and $1 \mapsto -1$), both of which are injective. So we see that the second ρ_* is an isomorphism. Since the second ρ_* has trivial kernel, the image of $H^1(M)$ in $\mathbb{Z}$ is 0 by exactness, and the sequence truncates as was claimed. □

We note in passing that this provides an algebraic proof that the Hopf map takes degree Q Skyrme configurations to Hopf charge Q (or $-Q$ if the orientation on M or S^3 is swapped) Faddeev-Hopf configurations, since this is precisely the statement that $\rho_* : \pi_0((S^3)^M) \to \pi_0((S^2)^M_*)$ is an isomorphism. By identifying the Hopf degree of $\phi \in (S^2)^M_*$ with the degree of its lift $\widetilde{\phi} \in (S^3)^M$, we adopt the standard convention that the Hopf map $\pi \in (S^2)^{S^3}$ itself has Hopf degree $+1$.

The short exact sequence does not tell us precisely what $\pi_1((S^2)^M_*)$ is in general. One useful class of domains (which includes $M = S^3$) where we do know the answer is those with finite fundamental group.

Corollary 3 *If $\pi_1(M)$ is finite then $\rho_* : \pi_1((S^3)^M) \to \pi_1((S^2)^M_*)$ induced by the Hopf map is an isomorphism.*

Proof: The result follows once we show that $H^1(M;\mathbb{Z}) = 0$. By the Universal Coefficient Theorem, $H^1(M;\mathbb{Z})$ is isomorphic to the free part of $H_1(M;\mathbb{Z})$, since $H_0(M;\mathbb{Z}) = \mathbb{Z}$ has no torsion. But $H_1(M;\mathbb{Z})$ is isomorphic to the abelianization of $\pi_1(M)$ which, being finite, can have no free part. □

These results are useful because a lot is known about the topology of $(S^3)^M$ since it can be identified with the topological group G^M where $G = SU(2)$. The canonical identification is given by

$$S^3 \to SU(2) : (z_1, z_2) \mapsto U = \begin{pmatrix} z_1 & -\bar{z}_2 \\ z_2 & \bar{z}_1 \end{pmatrix}. \tag{2.4}$$

This map is well-defined because $U^\dagger U = UU^\dagger = \mathbb{I}_2$ and $|z_1|^2 + |z_2|^2 = 1$ implies that $\det U = 1$. Also note that $(1, 0) \mapsto \mathbb{I}_2$. Since $(SU(2))^M$ is a topological group all connected components of $(S^3)^M$ are homeomorphic, and the fundamental group is abelian. A loop in the identity component of $SU(2)^M$ based at the constant map $M \to \{\mathbb{I}_2\}$ may be thought of as a map from $S^1 \wedge M$ to $SU(2)$, where $\wedge$ denotes smash product. If $M = S^3$ then $S^1 \wedge M = S^4$ and $\pi_4(SU(2)) = \mathbb{Z}_2$, so we have that $\pi_1((S^2)^{S^3}_*) = \pi_1(SU(2)^{S^3}) = \mathbb{Z}_2$ for all components. Using a similar argument for the *vacuum sector* $(S^2)^M_0$ of the Faddeev-Hopf model, we could very easily have shown that, for $M = S^3$, $\pi_1((S^2)^M_0) = \pi_4(S^2) = \mathbb{Z}_2$. Note that we have actually proved much more than this, however: the fundamental group of *every* connected component of the Faddeev-Hopf configuration space is $\mathbb{Z}_2$, and crucially, that the map from the Skyrme configuration space induced by Hopf fibration is an isomorphism.

The above results will suffice for our purposes. In fact, one can say much more about the algebraic topology of $(S^2)^M$, with M a general compact oriented 3-manifold. It turns out that all components of $(S^2)^M_*$ are homeomorphic, though the same fails to be true for the full space $(S^2)^M$. Furthermore, it is possible to compute both the fundamental group and the whole real cohomology ring (including its cup product structure) of any component of $(S^2)^M$. These results are obtained [4] by exploiting a somewhat less obvious relationship between $(S^2)^M$ and the vacuum (degree 0) sector of $SU(2)^M$. Essentially, all Faddeev-Hopf configurations in a given sector may be obtained from a fixed map in that sector by acting on the codomain with some degree 0 Skyrme configuration. This gives natural maps from the vacuum sector of the Skyrme model to each sector of the Faddeev-Hopf model, which can be shown to have many topologically natural properties. The topological results we present here are not so powerful as those of [4], but they are also less technical and may be visualized rather concretely. Most importantly, they are particularly well-suited to the study of Finkelstein-Rubinstein quantization in the Faddeev-Hopf model.

3 The Faddeev-Hopf model

From now on we consider only the case $M = S^3$, interpreted as the one point compactification of $\mathbb{R}^3$ with the point p_0 representing the boundary at infinity. The most extensively studied model of this kind is due to Faddeev [10] who suggested the following Lagrangian density

$$\mathcal{L} = \frac{1}{2}\partial_\mu \mathbf{n} \cdot \partial^\mu \mathbf{n} - \frac{\lambda}{4}\left(\partial_\mu \mathbf{n} \times \partial_\nu \mathbf{n}\right) \cdot \left(\partial^\mu \mathbf{n} \times \partial^\nu \mathbf{n}\right) \tag{3.1}$$

where the field $\mathbf{n} = (n_1, n_2, n_3)$ takes values on the 2-sphere, that is $|\mathbf{n}|^2 = 1$, λ is a coupling constant, and the boundary condition is $\mathbf{n}(\infty) = (0, 0, 1)$. We have changed notation from ϕ to $\mathbf{n}$ for the field so as to fit in with the existing literature on the model. Note that the second term in (3.1) stabilizes the solitons against radial rescaling. As discussed in section 2 the Hopf charge Q can be identified with the degree of any lift of $\mathbf{n}$ to $\widetilde{\mathbf{n}} : \mathbb{R}^3 \to S^3$. The energy E of a static configuration of Hopf charge Q is bounded below by

$$E \geq c|Q|^{\frac{3}{4}} \tag{3.2}$$

where c is a constant. For more details see [29, 30].

The Lagrangian of the model has $E(3) \times O(3)$ symmetry. Since spatial translations are rather trivial we will not discuss them any further. The target space $O(3)$ symmetry is broken to $O(2)$ symmetry by the boundary condition. Kundu and Rybakov showed in [21] that topologically nontrivial configurations admit at most an axial (one-parameter) symmetry. General configurations with axial symmetry are discussed in [15]. Special configurations with axial symmetry have been studied recently in [17] and can be described in the following way. Introduce toroidal coordinates (η, ξ, ϕ) on $\mathbb{R}^3$ defined by

$$x = a\frac{\sinh\eta\cos\phi}{\cosh\eta - \cos\xi}, \;\; y = a\frac{\sinh\eta\sin\phi}{\cosh\eta - \cos\xi}, \;\; z = a\frac{\sin\xi}{\cosh\eta - \cos\xi}. \tag{3.3}$$

These coordinates form a canonically oriented orthogonal system covering all of $\mathbb{R}^3$ except the circle $C = \{x^2 + y^2 = a^2, z = 0\}$ and the z-axis. Surfaces of constant $\eta \in (0, \infty)$ are tori of revolution about the z-axis, but with non-circular generating curves. As $\eta \to \infty$ these tori collapse to the circle C and as $\eta \to 0$ they collapse to the z-axis. Each torus of constant η is parametrized by the angular coordinates (ϕ, ξ); ϕ is the angle around the z axis, ξ is an angular coordinate around the not quite circular generating curve of the torus. The maps of interest are most easily written in terms of a complex stereographic coordinate W on S^2. Projecting from $(0, 0, 1)$, so that $W = (n_1 + in_2)/(1 - n_3)$, they take the form[1]

$$W = f(\eta)\mathrm{e}^{i(m\xi - n\phi)}, \tag{3.4}$$

where $f(\eta)$ satisfies the boundary conditions $f(0) = \infty$ and $f(\infty) = 0$. Inverting the stereographic projection yields

$$\mathbf{n} = \left(\frac{2f}{f^2+1}\cos(m\xi - n\phi), \frac{2f}{f^2+1}\sin(m\xi - n\phi), \frac{f^2-1}{f^2+1}\right). \tag{3.5}$$

This ansatz will be referred to as the toroidal ansatz. Here the word "ansatz" is used rather loosely, for an approximation which is a good initial guess for the numerically calculated static

[1] Note that we have changed the sign of n in [17].

solution. It is worth mentioning that the toroidal ansatz gives rise to exact solutions for the Lagrangian density $\mathcal{L} = (H_{\mu\nu}H^{\mu\nu})^{\frac{3}{4}}$ where $H_{\mu\nu} = \mathbf{n}\cdot(\partial_\mu \mathbf{n} \times \partial_\nu \mathbf{n})$, [2].

Under rotation by α around the z axis the toroidal coordinates change to $(\eta, \xi, \phi+\alpha)$ which rotates the vector $\mathbf{n}$ by $-n\alpha$ around the third axis in target space. Obviously, this rotation can be undone by a rotation around the third axis in target space.

There is an obvious lift of any map $\mathbb{R}^3 \to S^2$ within this ansatz to a Skyrme configuration $\mathbb{R}^3 \to S^3$, obtained as follows. For given f, m and n, let

$$\widetilde{\mathbf{n}} : (\eta, \phi, \xi) \mapsto (z_1, z_2) \in \mathbb{C}^2, \qquad \text{where} \qquad z_1 = \frac{f}{\sqrt{f^2+1}}\mathrm{e}^{im\xi}, \quad z_2 = \frac{1}{\sqrt{f^2+1}}\mathrm{e}^{in\phi}. \tag{3.6}$$

Then $|z_1|^2+|z_2|^2 = 1$ so that $\widetilde{\mathbf{n}}$ is actually S^3-valued, and the composition of this map with the Hopf map is clearly $\mathbf{n}$, since the stereographic coordinate W coincides with the inhomogeneous coordinate $W = z_1/z_2$ under the identification $S^2 \equiv \mathbb{C}P^1$. It is now straightforward to compute the degree of $\widetilde{\mathbf{n}}$, and hence the Hopf degree of $\mathbf{n}$. Since the degree of $\widetilde{\mathbf{n}}$ is a homotopy invariant, we may deform f to any convenient function satisfying the boundary conditions, for example, $f(\eta) = \eta^{-1}$. In this case, $(2^{-\frac{1}{2}}, 2^{-\frac{1}{2}}) \in S^3$ is a regular value of $\widetilde{\mathbf{n}}$ with precisely $|mn|$ preimages, namely the points with $\eta = \exp(im\xi) = \exp(in\phi) = 1$. At each of these preimages, the image of the canonically oriented coordinate frame under $d\widetilde{\mathbf{n}}$ is

$$d\widetilde{\mathbf{n}} : [\partial_\eta, \partial_\xi, \partial_\phi] \mapsto [(-2^{-\frac{3}{2}}, 0, 2^{-\frac{3}{2}}, 0), (0, \frac{m}{\sqrt{2}}, 0, 0), (0, 0, 0, \frac{n}{\sqrt{2}})]$$

where we have identified $\mathbb{C}^2 \cong \mathbb{R}^4$. The orientation of the image frame is given by the sign of the determinant

$$\det[d\widetilde{\mathbf{n}}\partial_\eta,\ d\widetilde{\mathbf{n}}\partial_\xi,\ d\widetilde{\mathbf{n}}\partial_\phi,\ \widetilde{\mathbf{n}}] = \frac{mn}{8}.$$

Hence each of the $|mn|$ preimages has multiplicity $+1$ if $mn > 0$ and -1 if $mn < 0$, so the Hopf charge of $\mathbf{n}$ is mn, in agreement with the calculation in [15].

Numerical evidence suggests that the energy minimals for $Q = 1, 2$ and 4 have axial symmetry. In general, minimals are more complicated, having knotted or linked structures with at most discrete symmetries. In principle any cyclic group C_q is a possible discrete symmetry. However, in practice only the simplest nontrivial symmetry — the twofold symmetry C_2 — seems to occur. Clearly any nonconstant smooth field configuration cannot be symmetric under a rotation in target space without a compensating spatial rotation. It is possible, however, for a configuration to be invariant under a spatial rotation without a compensating rotation in target space. For example, the axial configuration in (3.5) with even n has a C_2 symmetry generated by spatial rotation by π about the z-axis. We will discuss symmetries further in the next section when we calculate the constraints they impose on the wave function.

4 Finkelstein-Rubinstein constraints

In this section we describe how to use ideas of Finkelstein and Rubinstein [12] to quantize a scalar field theory and obtain fermions. Quantization usually implies replacing the classical configuration space by wave functions on configuration space. However, if the configuration space is not simply connected it is possible to define wave functions on the universal cover of configuration space. As shown in section 2, the fundamental group of each connected component of our configuration space $\mathcal{Q} = (S^2)^{S^3}_*$ is $\mathbb{Z}_2$. So the universal cover $\widetilde{\mathcal{Q}}$ is a twofold cover. We will also assume that the topological charge is conserved in the quantum theory, as it is in the classical theory, so the wave functions are defined on the covering space of a component of configuration space $\mathcal{Q}_Q$ with fixed Hopf charge Q. We shall formally think of the quantum state of the model as being specified by a wave function $\Psi \in L^2(\widetilde{\mathcal{Q}}_Q)$ with respect to some measure on $\widetilde{\mathcal{Q}}_Q$. Let $\Pi : \widetilde{\mathcal{Q}}_Q \to \widetilde{\mathcal{Q}}_Q$ be the deck transformation, that is, the map which takes p to the unique point in $\widetilde{\mathcal{Q}}_Q$ which differs from p but projects to the same point in $\mathcal{Q}_Q$. This induces a linear map $\Pi^* : L^2 \to L^2$ by pullback: $(\Pi^*\Psi)(p) := \Psi(\Pi(p))$. Since the states $\Pi^*\Psi$ and Ψ are physically indistinguishable, we must have $\Psi(p) = e^{i\theta(p)}\Pi^*\Psi(p)$ for all $p \in \widetilde{\mathcal{Q}}_Q$ and all Ψ. But $\Pi^*\Pi^* = 1$, so the only possibilities are $\Pi^*\Psi = \Psi$ or $\Pi^*\Psi = -\Psi$. In order to allow for fermionic solitons, we must consistently choose the latter possibility: our wavefunctions must always be *odd* under Π.

Spinoriality then arises as follows. Consider the loop in $\mathcal{Q}_Q$ defined by spatial rotation about a fixed axis through 2π of a fixed base configuration $\mathbf{n}$. Since $\pi_1(\mathcal{Q}_Q) = \mathbb{Z}_2$, this may not be contractible, and its contractibility is independent of the basepoint $\mathbf{n}$ chosen. If it is noncontractible, both lifts of the loop to $\widetilde{\mathcal{Q}}_Q$ fail to close, but are rather paths connecting a Π-related pair of points (both of which project to $\mathbf{n}$). Having insisted on Π-oddness, therefore, we see that every allowable state in this sector aquires a minus sign under spatial rotation by 2π, the hallmark of spinoriality. That this is equivalent to fermionicity (that is, odd exchange statistics) was proved by Finkelstein and Rubinstein in [12].

The question of whether Hopf solitons can be consistently quantized as fermions thus reduces to the question of whether 2π spatial rotation loops in $\mathcal{Q}_Q$ are noncontractible when Q is odd and contractible when Q is even. To answer this, we only need to determine the contractibility for a representative of each sector. Consider the loop $\gamma : [0,1] \to (S^3)^{S^3}$ defined by $\hat{\gamma}(\eta, \xi, \phi, t) = \widetilde{\mathbf{n}}(\eta, \xi, \phi + 2\pi t)$, where $\widetilde{\mathbf{n}} : S^3 \to S^3$ is defined in (3.6), and we once again use the natural identification of $g : X \to Y^Z$ with $\hat{g} : Z \times X \to Y$. This is a 2π spatial rotation loop (about the z axis) of the degree $Q = mn$ Skyrme configuration $\widetilde{\mathbf{n}}$. Note that $\pi \circ \gamma : [0,1] \to \mathcal{Q}_Q$ is also a 2π rotation loop, but in the Faddeev-Hopf configuration space. Corollary 3 states that $\pi \circ \gamma$ is contractible if and only if γ is contractible, which is true if and only if the degree Q is odd, by work of Giulini [14]. Hence imposing Π-oddness on our quantum states Ψ does indeed produce a consistent fermionic quantization of Hopf solitons.

It is important to realize that, having imposed Π-oddness, *every* noncontractible loop in $\mathcal{Q}_Q$ must be associated with a sign flip in $\Psi : \widetilde{\mathcal{Q}}_Q \to \mathbb{C}$, regardless of whether the loop is generated by a spatial rotation. Let $\mathbf{n}$ be a Hopf degree $Q \neq 0$ energy minimal of the Faddeev-Hopf model which is invariant under a simultaneous spatial rotation by α about some axis $\mathbf{e}$ and rotation by β around the third axis in target space (the only axis compatible with the boundary conditions). Since for $Q \neq 0$ the maximal symmetry of a configuration is $O(2) \times O(2)$, only one spatial rotation axis $\mathbf{e}$ is possible for a given $\mathbf{n}$, and we may choose it, without loss of generality, to lie along the z axis. Let us call such a combined transformation an (α, β)-rotation. Then we may construct a loop $L(\alpha, \beta)_{\mathbf{n}}$ in $\mathcal{Q}_Q$ based at $\mathbf{n}$ which consists of rotation by $2t\alpha$ around the z-axis for time $t \in [0, \frac{1}{2}]$, followed by rotation by $(2t-1)\beta$ around the third axis in target space for $t \in (\frac{1}{2}, 1]$. In this language, the fact that $\mathbf{n}$ has the specified symmetry is precisely the statement that $L(\alpha,\beta)_{\mathbf{n}}$ is a loop, i.e. closed. There are two points $p, \Pi(p) \in \widetilde{\mathcal{Q}}_Q$ corresponding to $\mathbf{n}$, and any physical state must have $\Psi(\Pi(p)) = -\Psi(p)$. Now if $L(\alpha,\beta)_{\mathbf{n}}$ is noncontractible then p and $\Pi(p)$ are connected by the lifts of $L(\alpha,\beta)_{\mathbf{n}}$, starting at p and $\Pi(p)$, respectively. Hence, evaluated at the specific point p (or $\Pi(p)$) we must have

$$(\mathrm{e}^{-i\alpha\hat{L}_3}\mathrm{e}^{-i\beta\hat{K}_3}\Psi)(p) = -\Psi(p), \tag{4.1}$$

for any allowed state, where $\hat{L}_3$ is the third component of the spin operator $\hat{\mathbf{L}}$ and $\hat{K}_3$ is the third (and only) component of the spin operator in target space (henceforth called isospin).

If $L(\alpha,\beta)_{\mathbf{n}}$ were contractible, however, it would lift to a pair of closed loops in $\widetilde{\mathcal{Q}}_Q$ based at p and $\Pi(p)$, so that

$$(\mathrm{e}^{-i\alpha\hat{L}_3}\mathrm{e}^{-i\beta\hat{K}_3}\Psi)(p) = \Psi(p), \tag{4.2}$$

simply by continuity of Ψ. In the spirit of semiclassical quantization we assume that, at least for low lying states, the symmetry of the classical energy minimal is not broken by quantum effects. Thus we seek quantum states Ψ which are also invariant under (α,β)-rotations, so that

$$(\mathrm{e}^{-i\alpha\hat{L}_3}\mathrm{e}^{-i\beta\hat{K}_3}\Psi)(x) = e^{i\theta(x)}\Psi(x), \tag{4.3}$$

for all $x \in \widetilde{\mathcal{Q}}_Q$. But, assuming the (α,β)-rotation generates a finite group, there must exist an integer q such that $(\mathrm{e}^{-i\alpha\hat{J}_3}\mathrm{e}^{-i\beta\hat{I}_3})^q\Psi \equiv \Psi$, which implies, by continuity, that $\theta(x)$ must in fact be constant. But then $\theta(x) = \theta(p) = \pi$ if $L(\alpha,\beta)_{\mathbf{n}}$ is noncontractible by (4.1), or $\theta(x) \equiv 0$ if $L(\alpha,\beta)_{\mathbf{n}}$ is contractible, by (4.2). Hence, we obtain the so-called Finkelstein-Rubinstein constraints on symmetric quantum states:

$$\mathrm{e}^{-i\alpha\hat{L}_3}\mathrm{e}^{-i\beta\hat{K}_3}\psi = \begin{cases} \psi & \text{if the induced loop is contractible,} \\ -\psi & \text{otherwise.} \end{cases} \tag{4.4}$$

Equation (4.4) imposes constraints on the spin and isospin quantum numbers L, L_3 and K_3.

It is worth pausing here to discuss the relationship between body-fixed and space-fixed angular momentum. The Lagrangian of the Hopf model is invariant under a $SO(3) \times SO(3)$ symmetry group consisting of rotations in space and target space. For these symmetries we can define left and right actions which are generated by the space-fixed and body-fixed angular momenta $\mathbf{J}$ and $\mathbf{L}$ acting on space and by space-fixed and body-fixed angular momenta $\mathbf{I}$ and $\mathbf{K}$ acting on target space. The body-fixed and space-fixed angular momentum operators are related by rotations which implies that $\mathbf{J}^2 = \mathbf{L}^2$. For rotations in target space only rotations around the third axis are compatible with the boundary conditions. This implies $I_3^2 = K_3^2$. When the model is quantized the angular momentum operators $\hat{\mathbf{J}}^2 = \hat{\mathbf{L}}^2$, $\hat{J}_3$, $\hat{L}_3$, $\hat{I}_3$ and $\hat{K}_3$ form a set of commuting observables. The quantum wave function ψ can then be labelled by the usual spin quantum number as follows $\psi = |L, L_3, J_3, K_3, I_3\rangle$. Since the Finkelstein-Rubinstein constraints do not impose any restrictions on the values of J_3 and I_3, these values will often be suppressed and the wave function is given as $\psi = |L, L_3, K_3\rangle$. In order to make predictions, we are interested in states with given J and I_3. Therefore, we have to consider states with quantum numbers $L = J$ and $K_3 = \pm I_3$. Then the Finkelstein-Rubinstein constraints have the following effect. By restricting the allowed quantum states for given J and I_3 the degeneracy of the states is changed. In the extreme case that the degeneracy is zero, certain combinations of J and I_3 get excluded.

We now return to our discussion of loops in configuration space and Finkelstein-Rubinstein constraints. Just as for 2π spatial rotation loops, we can use the isomorphism $\pi_1((S^2)_*^{S^3}) \rightarrow \pi_1((S^3)^{S^3})$ induced by the Hopf fibration to calculate whether a given loop $L(\alpha,\beta)_{\mathbf{n}}$ is contractible. For every configuration $\mathbf{n}$ we can choose a configuration $\tilde{\mathbf{n}}$ in the configuration space $(S^3)^{S^3}$ of Skyrmions. Then $L(\alpha,\beta)_{\tilde{\mathbf{n}}}$ is a loop in $(S^3)^{S^3}$ which projects to the loop $L(\alpha,\beta)_{\mathbf{n}}$ in $(S^2)_*^{S^3}$ under π. The action of $SO(3)$ on the target space of $\widetilde{\mathbf{n}}$, that is S^3, is now identified with the adjoint action of $SU(2)$ on itself. Once again, Corollary 3 shows that $L(\alpha,\beta)_{\mathbf{n}}$ is contractible if and only if $L(\alpha,\beta)_{\tilde{\mathbf{n}}}$ is contractible. Contractibility of the latter loop can be determined by means of an explicit formula recently derived for Skyrmions with discrete symmetries, [20]. This states that the loop $L(\alpha,\beta)_{\tilde{\mathbf{n}}}$ is contractible if and only if

$$N = \frac{Q}{2\pi}\,(Q\alpha - \beta) \tag{4.5}$$

is even. Note that there is a slight subtlety with the choice of the sign of β.

We can immediately recover our earlier result that the Π-odd quantization is consistently fermionic from formula (4.5). To see this, note that every configuration is symmetric under $(2\pi, 0)$-rotation, and substituting $\alpha = 2\pi$, $\beta = 0$ into (4.5) shows that N is odd if and only if Q is odd. Hence the spin quantum numbers L and J are half integer if and only if Q is odd. Similarly, considering the case $\alpha = 0$, $\beta = 2\pi$ (pure isorotation by 2π) shows that the isospin quantum numbers K_3 and I_3 are also half integer if and only if Q is odd.

New constraints on low-lying quantum states Ψ are obtained if we assume that they are invariant under the symmetry groups of the corresponding classical energy minimals. The Faddeev-Hopf model has received much less numerical attention than the Skyrme model, so our understanding of these minimals and their symmetries is comparatively limited. For this reason, we will discuss the Finkelstein-Rubinstein constraints for general symmetries first, then apply the analysis to those symmetries which have been observed in numerical experiments. Since we are interested in symmetries which can be generated by loops in configuration space we disregard reflections and look only at subgroups of $T^2 = SO(2) \times SO(2)$. Note that T^2, and hence every subgroup of T^2, is abelian. This severely limits the symmetry groups possible, and accounts in part for the numerical observation that Hopf degree Q minimals tend to possess far less symmetry than degree Q Skyrmions. The symmetry group $G_{\mathbf{n}} < T^2$ of a configuration $\mathbf{n}$ is either continuous, in which case $G_{\mathbf{n}} \cong SO(2)$ corresponding to axial symmetry, or discrete, hence finite (T^2 is compact). Every finite abelian group is isomorphic to a product of finite cyclic groups of coprime order, so it suffices to understand the Finkelstein-Rubinstein constraints for q-fold cyclic symmetry C_q.

First, we deal with axial symmetry. Consider the axial configurations (3.4) with Hopf charge $Q = mn$. These are invariant under $(\alpha, n\alpha)$-rotations for all $\alpha \in \mathbb{R}$. Since the loop $L(\alpha, n\alpha)_{\mathbf{n}}$ exists for all $\alpha \in \mathbb{R}$ it is homotopic to the constant loop ($\alpha = 0$). So $L(\alpha, n\alpha)_{\mathbf{n}}$ is contractible and gives rise to the following constraint on wave functions:

$$\mathrm{e}^{-i\alpha\hat{L}_3}\mathrm{e}^{-in\alpha\hat{K}_3}\Psi = \Psi. \tag{4.6}$$

Since formula (4.6) is valid for all α we can expand the equation in α. The first order term gives rise to the following constraint for the spin operators:

$$(\hat{L}_3 + n\hat{K}_3)\Psi = 0. \tag{4.7}$$

Equation (4.7) implies for the spin quantum numbers that $L_3 = -nK_3$.

If the axial symmetry is broken then the symmetry group must be isomorphic to a product of finite cyclic groups. Not every cyclic subgroup of T^2 is possible for a given Q, however, since the generator (α, β) of $C_q < T^2$ must satisfy equation (4.5), that is, N must be an *integer*. There are precisely q different C_q subgroups of T^2 which are candidates for symmetry groups, generated by $(2\pi/q, 2k\pi/q)$ where $k = 0, 1, \ldots, q-1$, since pure isorotation can never leave a nonconstant configuration invariant. Let us denote these groups C_q^k. To illustrate, let us assume that q is prime so that C_q is a finite field. Then formula (4.5) applied to the generator of C_q^k implies that $Q(Q-k) = 0 \mod q$ and hence $Q = 0 \mod q$ or $Q = k \mod q$ by the field property. Hence, unless Q is a multiple of q, formula (4.5) rules out all possible C_q symmetries except $C_q^{Q \bmod q}$. Similar criteria can be derived for q not prime, but they are not so neat. Of particular interest given the current state of numerics is the case $q = 2$. The argument above shows that, for odd Q, only C_2^1 symmetry is possible, not C_2^0.

Given a candidate symmetry group C_q^k, formula (4.5) gives us a one-dimensional (hence irreducible) representation of $C_{\bar{q}}$, where $\bar{q} = q$ if $Q(k+1)$ is even and $\bar{q} = 2q$ if $Q(k+1)$ is odd, by mapping the generator $(2\pi/q, 2k\pi/q)$ to $(-1)^N$. This representation may also be thought of as a homomorphism $C_{\bar{q}} \to \mathbb{Z}_2 = \{1, -1\}$ and is thus necessarily trivial if q is odd and $Q(k+1)$ is even. We call this the Finkelstein-Rubinstein representation of $C_{\bar{q}}$. There is also a natural representation of $C_{\bar{q}}$ on the spin-isospin L, K_3 quantum state space, defined by the inclusion $C_q^k < SO(3) \times SO(2)$. A state Ψ with quantum numbers L, K_3 is thus compatible with C_q^k symmetry if and only if the decomposition of the spin-isospin L, K_3 representation of $C_{\bar{q}}$ into irreducible representations contains a copy of the Finkelstein-Rubinstein representation. Given that we consider only cyclic groups, in practice we need only check compatibility on the generator $(\alpha, \beta) = (2\pi/q, 2\pi k/q)$. Thus L_3, K_3 must satisfy

$$e^{-2\pi i(L_3 + kK_3)/q} = (-1)^N = e^{i\pi Q(Q-k)/q} \tag{4.8}$$

$$\Leftrightarrow \quad L_3 + kK_3 = -\frac{1}{2}Q(Q-k) + \ell q \tag{4.9}$$

where ℓ is an integer.

A good candidate for the ground state in the charge Q sector is the state with the lowest values of L and $|K_3|$ (and hence J and $|I_3|$) compatible in this way with the symmetries of the classical minimal.

To illustrate this symmetry analysis, we compute the quantum ground state for stable and metastable Hopf solitons of degrees $Q = 1, \ldots 7$, using the classical solutions obtained numerically by Hietarinta et al [17]. Only axial and C_2 symmetries ever arise for these solutions. In the C_2 case for *even* Q, we distinguish between the two possible groups C_2^0 and C_2^1 using the colour coding information in [17]. The results are presented in table 1. The first entry is the Hopf number Q. A star indicates that the state is metastable, that is, the classical solution is not a global minimal. The next entry is the energy E_Q which has been calculated in [17] and corresponds to $\lambda = 1/4$. The following entry gives the shape of the Hopf configuration. The entry "symmetry" shows which symmetry has been used to calculate the Finkelstein-Rubinstein constraints. Here (n, m) corresponds to the axial symmetry of the corresponding toroidal ansatz (3.4). C_2^0 is generated by π rotation in space whereas C_2^1 is generated by rotation by π in space followed by rotation by π in target space. As a word of caution, while axial symmetry has been checked numerically, the C_2 symmetry is obtained by inspection from the figures in [17] and [8]. For low Q the symmetries are apparent. However, for higher Hopf charge, $Q > 4$, the symmetries are difficult to guess, if indeed they exist at all. Where no entry is given, the classical solution has no obvious symmetry and the only constraint applicable is that of consistent fermionicity.

$\|Q\|$	E_Q	shape	symmetry	FR	ground state	excited state (1)	excited state (2)
1	135.2	unknot	$(1,1)$	1	$\|\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\rangle$	$\|\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}\rangle$	$\|\frac{3}{2}, -\frac{3}{2}, \frac{3}{2}\rangle$
2	220.6	unknot	$(2,1)$	1	$\|0,0,0\rangle$	$\|1,0,0\rangle$	$\|2,-2,1\rangle$
2^*	249.6	unknot	$(1,2)$	1	$\|0,0,0\rangle$	$\|1,0,0\rangle$	$\|1,-1,1\rangle$
3	308.9	unknot	C_2^1	-1	$\|\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\rangle$	$\|\frac{3}{2}, \frac{1}{2}, \frac{1}{2}\rangle$	$\|\frac{1}{2}, -\frac{1}{2}, \frac{3}{2}\rangle$
3^*	311.3	unknot	$(3,1)$	1	$\|\frac{3}{2}, -\frac{3}{2}, \frac{1}{2}\rangle$	$\|\frac{5}{2}, -\frac{3}{2}, \frac{1}{2}\rangle$	$\|\frac{9}{2}, -\frac{9}{2}, \frac{3}{2}\rangle$
4	385.5	unknot	$(2,2)$	1	$\|0,0,0\rangle$	$\|1,0,0\rangle$	$\|2,-2,1\rangle$
4^*	392.7	unknot	C_2^0	1	$\|0,0,0\rangle$	$\|1,0,0\rangle$	$\|0,0,1\rangle$
4^*	405.0	unknot	$(4,1)$	1	$\|0,0,0\rangle$	$\|1,0,0\rangle$	$\|4,-4,1\rangle$
5	459.8	link	—	—	$\|\frac{1}{2}, \pm\frac{1}{2}, \frac{1}{2}\rangle$	$\|\frac{3}{2}, \pm\frac{1}{2}, \frac{1}{2}\rangle$	$\|\frac{1}{2}, \pm\frac{1}{2}, \frac{3}{2}\rangle$
5^*	479.2	unknot	C_2^1	1	$\|\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\rangle$	$\|\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}\rangle$	$\|\frac{1}{2}, \frac{1}{2}, \frac{3}{2}\rangle$
6	521.0	link	—	—	$\|0,0,0\rangle$	$\|1,0,0\rangle$	$\|0,0,1\rangle$
6^*	536.2	link	—	—	$\|0,0,0\rangle$	$\|1,0,0\rangle$	$\|0,0,1\rangle$
7	589.0	knot	—	—	$\|\frac{1}{2}, \pm\frac{1}{2}, \frac{1}{2}\rangle$	$\|\frac{3}{2}, \pm\frac{1}{2}, \frac{1}{2}\rangle$	$\|\frac{1}{2}, \pm\frac{1}{2}, \frac{3}{2}\rangle$

Table 1: Ground states and excited states for $Q = 1, \ldots, 7$.

"FR" gives the Finkelstein-Rubinstein constraints $(-1)^N$ where N is calculated with equation (4.5) for the generator of the discrete symmetries. Note that axial symmetry implies FR = 1. Then ground states are calculated as explained above. They are given in the form $|LL_3K_3\rangle$. The quantum numbers J_3 and I_3 are suppressed. Recall that $J = L$ and $|I_3| = |K_3|$. We have also included two excited states. "Excited state (1)", is obtained from the ground state by increasing L by 1 and finding the lowest K_3 such that all constraints are satisfied. Similarly, "excited state (2)" is obtained by increasing K_3 by 1. Note that changing the sign of $\hat{L}_3$ and $\hat{K}_3$ in the constraints (4.4) given by a loop $L(\alpha, \beta)_{\mathbf{n}}$ can be interpreted as constraints for the loop $L(-\alpha, -\beta)_{\mathbf{n}}$. Since the fundamental group is $\mathbb{Z}_2$ the loop $L(\alpha, \beta)_{\mathbf{n}}$ is contractible if and only if $L(-\alpha, -\beta)_{\mathbf{n}}$ is contractible. Therefore, whenever $|L, L_3, K_3\rangle$ satisfies the constraints imposed by a symmetry, so does $|L, -L_3, -K_3\rangle$. In table 1, we only display states with $K_3 \geq 0$.

Since no constraints with FR $= -1$ occur for even Hopf charge Q all the ground states are given by $|0, 0, 0\rangle$ and "excited states (1)" are $|1, 0, 0\rangle$. The influence of the Finkelstein-

Rubinstein constraints can only be seen for "excited state (2)". For odd Q the Finkelstein-Rubinstein constraints influence the ground states and all the excited states.

One might ask why the first and second excited states are expected to have spin and isospin one unit higher than the ground state, respectively. One reason is that this is consistent with the collective coordinate quantization of Hopf solitons, to which we turn in the next section.

5 Collective coordinate quantization

The simplest non-trivial quantitative application of our results is the collective coordinate quantization, [27]. In this case the wave function is only non-vanishing on the space of minimal energy configurations in a given sector, also called the moduli space. The effective Lagrangian L_{eff} in this approximation is obtained by restricting the full Lagrangian to fields which, at each fixed time, lie in the moduli space.[2] From L_{eff} one can construct an effective Hamiltonian and canonically quantize the system in the standard manner. For Hopf charge $Q = 1$ the reduced Hamiltonian is given in [27] using "$SU(2)$ notation".

The Lagrangian L (3.1) can be split up into kinetic energy T and potential energy V, namely $L = T - V$ where

$$T = \int_{\mathbb{R}^3} \frac{1}{2}|\partial_t \mathbf{n}|^2 + \frac{\lambda}{2}\sum_i |\partial_t \mathbf{n} \times \partial_i \mathbf{n}|^2, \tag{5.1}$$

$$V = \int_{\mathbb{R}^3} \frac{1}{2}\sum_i |\partial_i \mathbf{n}|^2 + \frac{\lambda}{4}\sum_{i,j} |\partial_i \mathbf{n} \times \partial_j \mathbf{n}|^2. \tag{5.2}$$

Now let $\mathsf{M} \subset \mathcal{Q}_Q$ be the moduli space of charge Q energy minimizers, and $\mathbf{n}(t)$ be a trajectory in M. Since $\mathbf{n}(t)$ is a critical point of V for all t, V must remain constant, $V[\mathbf{n}(t)] = M_0$ say, interpreted as the classical mass of the Hopf soliton. It follows that the effective Lagrangian is $L_{\text{eff}} = T|_{\mathsf{M}} - M_0$, so the reduced dynamics is determined purely by the kinetic energy restricted to M. This has a natural geometric interpretation: being quadratic in first time derivatives, T defines a positive quadratic form and hence a unique Riemannian metric γ on M, and the classical dynamics descending from L_{eff} is nothing other than geodesic motion in (M, γ). Since the Faddeev-Hopf model is not of Bogomol'nyi type, M is just the orbit of any energy minimizer under the symmetry group of the model, that is, all zero modes arise due to symmetry. The centre of mass motion decouples, so we may, without loss of generality, assume that the centre of mass is fixed at the origin, so that M is the orbit of some minimizer $\mathbf{n}_0$ under $G = SO(3) \times SO(2)$, acting as described in section 3. So (M, γ) is a homogeneous space, diffeomorphic to G/K where $K < G$ is the isotropy group of $\mathbf{n}_0$. It follows that γ is uniquely determined by its value on $T_{\mathbf{n}_0}\mathsf{M}$.

Generically, as we have described, K is discrete, so M has dimension 4, and γ is specified by 6 constants, which may be interpreted as the components of the Hopf soliton's inertia tensor. However, we shall concentrate on the case where $\mathbf{n}$ has axial symmetry. Then

$$K = \{k(\alpha) = ([\text{diag}(e^{i\alpha/2}, e^{-i\alpha/2})], e^{in\alpha}) : \alpha \in \mathbb{R}\} \tag{5.3}$$

[2] As has been discussed in the Skyrme model, [6, 9], this approximation breaks down if centrifugal effects are taken into account. This problem can be avoided by introducing a (sufficiently large) mass term for the vector $\mathbf{n}$ so that the fields decay fast enough at infinity.

for some divisor n of Q, where we have used the standard isomorphisms $SO(3) \equiv PU(2)$ and $SO(2) \equiv U(1)$ to identify $SO(3)$ matrices with projective equivalence classes of $U(2)$ matrices, and $SO(2)$ matrices with complex phases. Let $\theta_1, \theta_2, \theta_3$ be the usual basis of left invariant vector fields on $SO(3)$ and $\theta_4 = \partial_\xi$ on $SO(2) \equiv \{e^{i\xi} : \xi \in \mathbb{R}\}$. Let $\langle \cdots \rangle$ denote linear span. Then the Lie algebra of G, is $\mathfrak{g} = \langle \theta_1, \ldots, \theta_4 \rangle$, and the Lie algebra of K is $\mathfrak{k} = \langle \theta_3 + n\theta_4 \rangle$. We may identify $T_{\mathbf{n}_0}\mathsf{M}$ with the complementary space $\mathfrak{p} = \langle \theta_1, \theta_2, \theta_3 \rangle$. Note $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ since $n \neq 0$. So γ is equivalent to a positive symmetric bilinear form $\bar{\gamma} : \mathfrak{p} \oplus \mathfrak{p} \to \mathbb{R}$, and this must be invariant under the adjoint action of K on $\mathfrak{p}$. Relative to the basis $\{\theta_1, \theta_2, \theta_3\}$ this is

$$Ad_{k(\alpha)} = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{5.4}$$

Let $\mathfrak{p}^*$ denote the dual space to $\mathfrak{p}$, so that $\bar{\gamma} \in \mathfrak{p}^* \odot \mathfrak{p}^*$, where $\odot$ denotes the symmetric tensor product. The induced action of K on $\mathfrak{p}^* \odot \mathfrak{p}^*$ may be decomposed into irreducible representations, whence one finds that the dimension of the space of invariant symmetric bilinear forms on $\mathfrak{p} \oplus \mathfrak{p}$ is [19]

$$\frac{1}{2\pi}\int_0^{2\pi} d\alpha \, \frac{1}{2}[(\mathrm{tr}\, Ad_{k(\alpha)})^2 + \mathrm{tr}\,(Ad_{k(\alpha)}^2)] = 2. \tag{5.5}$$

Hence there exist positive constants a, b such that

$$\bar{\gamma} = a(\sigma_1^2 + \sigma_2^2) + b\sigma_3^2 \tag{5.6}$$

where $\{\sigma_i\}$ are the one forms dual to $\{\theta_i\}$. Thus the metric γ on M is determined by just two constants.

The static solution $\mathbf{n}_0$, and hence its classical mass M_0 and moments of inertia a, b, all depend parametrically on the coupling λ. In fact, this dependence is quite simple, as we shall now show. Let us temporarily denote all λ dependence explicitly, so that T_λ, V_λ are the kinetic and potential energy functionals at coupling λ, $\mathbf{n}_\lambda$ is the static solution, $M_0(\lambda)$ is its mass, and $a(\lambda)$, $b(\lambda)$ its moments of inertia. A simple rescaling of the integration variables in (5.2) shows that, for any fixed map $\mathbf{n} : \mathbb{R}^3 \to S^2$,

$$V_\lambda[\mathbf{n}(\mathbf{x})] \equiv \sqrt{\lambda}\, V_1[\mathbf{n}(\sqrt{\lambda}\,\mathbf{x})]. \tag{5.7}$$

Hence, given an extremal $\mathbf{n}_*$ of V_1 (here and henceforth, the subscript $*$ will indicate that a quantity refers to the $\lambda = 1$ model), $\mathbf{n}_\lambda(\mathbf{x}) = \mathbf{n}_*(\lambda^{-\frac{1}{2}}\mathbf{x})$ is an extremal of V_λ, and furthermore its energy is

$$M_0(\lambda) = V_\lambda[\mathbf{n}_\lambda] = \sqrt{\lambda}\, V_1[\mathbf{n}_*] = \sqrt{\lambda}\, M_*. \tag{5.8}$$

So the classical soliton masses scale as $\lambda^{\frac{1}{2}}$. A similar argument works for the moments of inertia too. The coefficients $a(\lambda)$, $b(\lambda)$ are, by definition, twice the kinetic energies of the time-dependent fields, $\mathbf{n}_\lambda^{(i)}(\mathbf{x}, t)$ say, obtained from $\mathbf{n}_\lambda$ by subjecting it to spatial rotation at unit angular velocity about the x_i-axis with $i = 1, 3$ respectively. Let $R_i(t)$ denote rotation through angle t about the x_i-axis. Then

$$\mathbf{n}_\lambda^{(i)}(\mathbf{x}, t) = \mathbf{n}_\lambda(R_i(t)\mathbf{x}) = \mathbf{n}_*(\lambda^{-\frac{1}{2}} R_i(t)\mathbf{x}) = \mathbf{n}_*(R_i(t)\lambda^{-\frac{1}{2}}\mathbf{x}) = \mathbf{n}_*^{(i)}(\lambda^{-\frac{1}{2}}\mathbf{x}, t), \tag{5.9}$$

by linearity of R_i. Rescaling the integration variables in (5.1) as before, one sees that $T_\lambda[\mathbf{n}_\lambda^{(i)}] = \lambda^{\frac{3}{2}} T_1[\mathbf{n}_*^{(i)}]$, and so the moments of inertia scale as $\lambda^{\frac{3}{2}}$:

$$a(\lambda) = \lambda^{\frac{3}{2}}\, a_*, \qquad b(\lambda) = \lambda^{\frac{3}{2}}\, b_*. \tag{5.10}$$

Note that neither of these arguments appealed to axial symmetry, so the same scaling behaviour applies to solitons with only discrete (for example, trivial) symmetry groups, also. This includes the scaling behaviour of the moment of inertia associated with isorotation (where this no longer coincides with spatial rotation) because

$$\mathbf{n}_\lambda^{(\mathrm{iso})}(\mathbf{x}, t) = R_3(t)\mathbf{n}_\lambda(\mathbf{x}) = R_3(t)\mathbf{n}_*(\lambda^{-\frac{1}{2}}\mathbf{x}) = \mathbf{n}_*^{(\mathrm{iso})}(\lambda^{-\frac{1}{2}}\mathbf{x}, t). \tag{5.11}$$

From now on, we will no longer denote the λ dependence explicitly, but will retain the $*$ subscript for quantities associated with the $\lambda = 1$ model.

We wish to quantize geodesic motion on M, which may be formulated as a Hamiltonian flow on $T^*\mathsf{M}$, within the framework of Finkelstein and Rubinstein. As it stands, there is a problem with this, however. As shown above, the fundamental group of $\mathcal{Q}_Q$, the topological sector containing M, is $\mathbb{Z}_2$, whereas $\pi_1(\mathsf{M}) = \mathbb{Z}_{2n}$, where n is the divisor of Q appearing in (5.3). A proof of this is presented in the appendix. So $\pi_1(\mathsf{M}) \neq \pi_1(\mathcal{Q}_Q)$ unless $n = 1$, and this type of axial symmetry occurs only for $Q = 1$ and the metastable $Q = 2$ state, according to Hieterinta et al [17]. Nevertheless, a fermionic collective coordinate approximation is still possible, the key point being that in all cases the 2π spatial rotation loop has order 2 in $\pi_1(\mathsf{M})$. It is slightly unfortunate that this is true independent of Q, that is, whether Q is odd or even. For consistency we must thus *choose* bosonic quantization for Q even, it is not imposed on us by the topology of M. This illustrates that collective coordinate quantization can be quite treacherous in the absence of a good understanding of the topology of the full configuration space.

To construct the collective coordinate quantization it is convenient to exploit the n-fold covering map $\varrho : SO(3) \to G/K$ which maps $g \in SO(3)$ to the coset $(g, 1)K$, that is, the left coset of K containing $(g, 1) \in G$. Note that ϱ commutes with the natural $SO(3)$ left actions on $SO(3)$ and M. Geodesics in (M, γ) are the images of geodesics in $(SO(3), \varrho^*\gamma)$, where the lifted metric $\varrho^*\gamma$ is precisely (5.6), but with σ_i now interpreted as (global) left invariant one forms on $SO(3)$, rather than basis vectors in $\mathfrak{p}^*$. The Hamiltonian generating geodesic flow in $(SO(3), \varrho^*\gamma)$ is

$$H = \frac{1}{2a}(L_1^2 + L_2^2) + \frac{1}{2b}L_3^2 = \frac{1}{2a}|\mathbf{L}|^2 + \left(\frac{1}{2b} - \frac{1}{2a}\right) L_3^2 \tag{5.12}$$

where $L_i : T^*SO(3) \to \mathbb{R}$ are the angular momenta corresponding to the vector fields θ_i (the components of the moment map for the Hamiltonian action of $SO(3)$ on $T^*SO(3)$). Their Poisson bracket algebra is well known: $\{L_1, L_2\} = L_3$ and cyclic permutations. We may now quantize in the usual way, replacing classical angular momenta by $\hat{L}_i$, self-adjoint linear operators on $L^2(SO(3))$ and Poisson brackets by commutators. Note that $\{\hat{H}, \hat{\mathbf{L}}^2, \hat{L}_3\}$ is a compatible set of observables. In this set-up, we are thinking of the wavefunction as defined on the covering space, $\psi : SO(3) \to \mathbb{C}$; it is important to note that for Q odd (even) only those functions which are double-valued (single-valued) under the projection ϱ make physical sense. The deck transformation group for ϱ is generated by $\exp(2\pi\theta_3/n)$, so we find that the eigenvalues of $\hat{L}_3$ must be integer multiples of $n/2$. This conclusion may be reached another way. Note that $\theta_3+n\theta_4 \in \mathfrak{k}$ vanishes on M, so the corresponding classical momenta are linearly dependent: $L_3+nK_3=0$. Hence the quantum operators must satisfy $(\hat{L}_3+n\hat{K}_3)\psi=0$ on any physical state, and the conclusion follows because $\hat{K}_3$ has half-integer spectrum. Of course, this is nothing other than the FR constraint for axial symmetry (4.7). We may use the linear dependence of the third components of spin and isospin to rewrite $\hat{H}$ in terms of $\hat{K}_3$, or both $\hat{L}_3$ and $\hat{K}_3$ if we wish. A convenient way to write the quantum hamiltonian is

$$\hat{H} = M_0 + \frac{1}{2a}\hat{\mathbf{L}}^2 + \left(\frac{1}{2b} - \frac{1}{2a}\right)\hat{L}_3^2. \tag{5.13}$$

It is now trivial to express the quantum energy spectrum in terms of the quantum numbers L^2 and K_3:

$$E = \sqrt{\lambda}\, M_* + \frac{\hbar^2}{\lambda^{\frac{3}{2}}}\left[\frac{L(L+1)}{2a_*} + \left(\frac{1}{2b_*} - \frac{1}{2a_*}\right)n^2K_3^2\right] \tag{5.14}$$

where we have used the constraint $L_3 = -nK_3$ to eliminate L_3, and the scaling behaviour obtained in (5.8),(5.10) to render all λ dependence explicit. Recall that $*$-subscript quantities refer to the $\lambda=1$ soliton. As discussed in the previous section, the body-fixed and space-fixed angular momenta satisfy $\hat{\mathbf{J}}^2 = \hat{\mathbf{L}}^2$ and $\hat{I}_3^2 = \hat{K}_3^2$. Therefore, we can also express the energy in terms of the space-fixed angular momentum quantum numbers, which are the quantities measured in a physical experiment, by replacing $L(L+1)$ by $J(J+1)$ and K_3^2 by I_3^2 in formula (5.14).

We would like to order these states by increasing energy. Clearly, this order depends on n and the relative size of the constants a_* and b_*. As discussed above, to determine these constants, one must compute the kinetic energy of time dependent fields $\mathbf{n}(t) = (\exp(t\theta_1),1)\cdot\mathbf{n}_0$ and $\mathbf{n}(t) = (\exp(t\theta_3),1)\cdot\mathbf{n}_0$ respectively, where $\cdot$ denotes the action of G on M. This is computationally very expensive if one uses for $\mathbf{n}_0$ the genuine axially symmetric energy minimizers found in [17], since even to construct $\mathbf{n}_0$ requires one to solve nonlinear PDEs. Instead, we shall again exploit the Hopf fibration and assume that $\mathbf{n}_0$ is well approximated by the image under the Hopf map ρ of a Skyrme configuration $U:\mathbb{R}^3\to SU(2)$ within the rational map ansatz of Houghton, Manton and Sutcliffe [18]. This idea was introduced in [8].[3] The rational map ansatz may be described as follows. Using $\exp:\mathfrak{su}(2)\to SU(2)$, one may identify $SU(2)$ with the closed ball of radius π in $\mathfrak{su}(2)\equiv\mathbb{R}^3$. The entire boundary of this ball gets mapped to $-\mathbb{I}_2$. Partition physical space $\mathbb{R}^3$ into concentric 2-spheres of radius $r\in[0,\infty)$. Choose a fixed holomorphic map $\mathbf{R}:S^2\to S^2\subset\mathbb{R}^3$ of degree Q and a smooth decreasing surjection $f:[0,\infty)\to(0,\pi]$ (the profile function). Then the corresponding degree Q Skyrme configuration is

$$U(r,x_1,x_2) = \exp\left(f(r)\mathbf{R}(x_1,x_2)\right) \tag{5.15}$$

where x_1, x_2 is any coordinate system on S^2. With respect to stereographic coordinates z, R on its domain and codomain, $\mathbf{R}$ is the eponymous rational map $R(z)$. We may then write

[3] Su has also discussed the rational map ansatz for Hopf solitons, using a different notation, [26].

Q	M_*	M_*^H	M_*^G	M_*^B	a_*	b_*
1	275.0	270.4	278.6	252.5	418.8	369.7
2	462.9	441.2	446.9	418.0	1265.0	1309.4
3^*	665.5	622.6	—	590.5	3272.7	3556.1

Table 2: Classical energy M_* and moments of inertia a_*, b_* of various axially symmetric solitons, at $\lambda = 1$, within the rational map ansatz. For comparison, we also quote the classical energies of the corresponding numerical solutions found in the literature (M_*^H: Hietarinta and Salo [17], M_*^G: Gladikowski and Hellmund [15], M_*^B: Battye and Sutcliffe [7]). Note that M_*^H and M_*^G have been inferred using the scaling rule (5.8).

$U(r, z)$ more explicitly as

$$U(r,z) = \frac{1}{1+|R|^2} \begin{pmatrix} e^{-if} + |R|^2 e^{if} & 2i\bar{R}\sin f \\ 2iR\sin f & e^{if} + |R|^2 e^{-if} \end{pmatrix}. \tag{5.16}$$

The corresponding Faddeev-Hopf configuration $\pi \circ U$ can easily be calculated with equations (2.1) and (2.4),

$$W(r,z) = \frac{|R(z)|^2 e^{if(r)} + e^{-if(r)}}{2iR(z)\sin f(r)}, \tag{5.17}$$

where again we choose stereographic coordinates on S^2. The idea is to approximate the true energy minimizer $\mathbf{n}_0$ by a configuration of this form and minimize over all possible R and f. In fact, to obtain axial symmetry, we must assume $R(z) = z^Q$ (note this assumes the divisor n of Q is simply $n = Q$, so our results apply only to $Q = 1, 2$ and the metastable $Q = 3^*$, 4^* solitons). We then minimize the potential energy V over all possible profile functions f. This yields a nonlinear second order ODE for $f(r)$ which is easily solved numerically. We may, without loss of generality, set λ to unity.

Having constructed our approximate energy minimizer, $W(r, z)$, we must compute the kinetic energy at $t = 0$ of

$$W(t,r,z) = \frac{|R(\tilde{z}(t,z))|^2 e^{if(r)} + e^{-if(r)}}{2iR(\tilde{z}(t,z))\sin f(r)} \tag{5.18}$$

where

$$\tilde{z}(t,z) = \frac{z\cos t/2 + i\sin t/2}{iz\sin t/2 + \cos t/2}, \quad \text{and} \quad \tilde{z}(t,z) = ze^{it}, \tag{5.19}$$

yielding $a_*/2$ and $b_*/2$ respectively. The calculations are elementary, but lengthy, and all reduce to radial integrals of expressions involving $f(r)$ and $f'(r)$. The results for $Q = 1$, 2 and

Q	groundstate	E_0	excited state (1)	E_1	excited state (2)	E_2
1	$\vert\frac{1}{2},-\frac{1}{2},\frac{1}{2}\rangle$	6.63 TeV	$\vert\frac{3}{2},-\frac{1}{2},\frac{1}{2}\rangle$	9.67 TeV	$\vert\frac{3}{2},-\frac{3}{2},\frac{3}{2}\rangle$	9.93 TeV
2	$\vert 0,0,0\rangle$	9.82 TeV	$\vert 1,0,0\rangle$	10.49 TeV	$\vert 2,-2,1\rangle$	11.79 TeV
3^*	$\vert\frac{3}{2},-\frac{3}{2},\frac{1}{2}\rangle$	14.58 TeV	$\vert\frac{5}{2},-\frac{3}{2},\frac{1}{2}\rangle$	15.23 TeV	$\vert\frac{9}{2},-\frac{9}{2},\frac{3}{2}\rangle$	17.12 TeV

Table 3: Groundstates and first excited states, and their energies, of super heavy smoke ring solitons in the collective coordinate approximation, using the rational map ansatz.

the metastable $Q = 3^*$ are summarized in table 2. These data, along with formula (5.14) give the complete quantum energy spectrum for these solitons, at arbitrary coupling.

To illustrate our approach we shall interpret the Hopf solitons as super heavy fermion states in the strongly coupled pure Higgs sector of the standard model, as advocated by Gipson and Tze [13]. To make contact with their work, we must take the unit of energy to be $e_0 = 300$ GeV, $\hbar = 1$, and the coupling constant to be $\lambda = \ln(m_H/e_0)/24\pi^2$, where m_H is the Higgs mass. In this model, the Higgs sector is strongly coupled, so the Higgs mass assumes the rather large value $m_H \approx 1$ TeV, so that $\lambda \approx 0.005$. The unit of length is the Compton wavelength of a particle of rest energy e_0, namely $d_0 = \hbar c/e_0 \approx 0.66\ 10^{-3}$ fm. Then the $Q = 1$ ground state represents what Gipson and Tze call a "smoke ring soliton" of energy 6.63 TeV which is compatible with the lower bound of 5.5 TeV given in [13]. A sensible measure for the size of the Hopf soliton is the value of the radius in the rational map ansatz at which the profile function takes the value $\pi/2$. We find that our Hopf soliton has a radius of $0.08\ 10^{-3}$ fm which is comparable with the lower bound of $0.2\ 10^{-3}$ fm in [13] where the radius is defined in a slightly different way. We display the groundstates and the first two excited states in the collective coordinate approximation in table 3. The energies of the states are dominated by the classical contribution. As anticipated in table 1, the groundstate has the lowest energy followed by excited state (1) and excited state (2). The energy of the states increases with the Hopf charge Q. The size of the Hopf solitons also increases with the charge; $0.08\ 10^{-3}$ fm for $Q = 1$, $0.09\ 10^{-3}$ fm for $Q = 2$ and $0.13\ 10^{-3}$ fm for $Q = 3$.

Clearly, the relative size of the quantum excitation energy of an excited state to the ground state energy depends on the coupling λ. If λ is small, as in the application above, the quantum corrections become significant. In an application where the solitons are taken to model real physical structures, whose energies and sizes are known experimentally (rather than hypothetical exotic matter states as in the current case), one would tune the energy and length scales independently so as to fit some reference data as well as possible. This amounts to tuning both λ and the value of $\hbar$, which is why we retained explicit $\hbar$ dependence in equation (5.14). In the case of the Skyrme system as a model of nucleons, for example, one finds that

$\hbar \approx 46.8$ in natural units [22]. Even if λ is large, therefore, quantum corrections may still be significant, provided $\hbar/\lambda$ remains large. So the relative importance of quantum corrections depends strongly on the physical interpretation of the model under consideration.

6 Conclusion

We have described how to quantize Hopf solitons using the Finkelstein-Rubinstein construction and thereby demonstrated that Hopf solitons can be quantized as fermions when their Hopf charge Q is odd. An important ingredient of the proof is the fact that the Hopf map $S^3 \to S^2$ induces a Serre fibration $(S^3)^M \to (S^2)^M_*$. Using this fibration we could show that the fundamental group of Skyrmions is isomorphic to the fundamental group of Hopf solitons, when physical space has finite fundamental group, and this isomorphism is induced by the Hopf map. This enabled us to use results which have been derived for the Skyrme model.

In a semiclassical quantization we expect that classical symmetries are not broken by quantum effects. Then the symmetries of the classical configurations induce non-trivial constraints on the wave function. We calculated possible ground states of Hopf solitons for $Q = 1, \ldots, 7$ from the minimal energy configurations given in [17]. Since Hopf solitons do not have many symmetries, the constraints on the wave functions are quite weak. Often, only the degeneracy of a state changes, rather than the state being excluded completely. Excited states have been included to better illustrate the influence of the Finkelstein-Rubinstein constraints.

In order to get quantitative predictions of the quantum energy spectrum of Hopf solitons, we resorted to a collective coordinate approximation. In general, naive collective coordinate quantization can give spurious results if the topology of the moduli space is incompatible with that of the full configuration space. We concentrated on the case where the moduli space consists of axially symmetric configurations, which provides a good example of this difficulty. As discussed in the previous section, such a moduli space allows for fermionic quantization for both odd and even Hopf charge. In order to describe the physics correctly, we have to impose bosonic quantization for even Q and fermionic quantization for odd Q. In other words, we must impose some of the Finkelstein-Rubinstein constraints arising from the topology of the full configuration space "by hand" on the wave function on the moduli space. They do not arise from the topology of the moduli space itself.

The Faddeev-Hopf model contains a single coupling constant λ. By simple rescaling arguments, we derived the scaling behaviour of the classical energy and moments of inertia of a soliton as λ varies. This allowed us to find a formula for the quantum energy spectrum of axially symmetric solitons, within the collective coordinate approximation, with all λ dependence explicit. The numerical constants M_*, a_* and b_* in this formula were approximated, for three such axially symmetric solitons, by constructing approximate energy minimizers within the rational map ansatz. Our aim in this paper was to illustrate the general approach of fermionic soliton quantization within the Faddeev-Hopf model. This can now be applied to a variety of physical models that admit Hopf solitons.

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Appendix: The fundamental group of the moduli space

We wish to compute the fundamental group of M, the orbit of a configuration $\mathbf{n} : \mathbb{R}^3 \to S^2$ under $G = SO(3) \times SO(2)$, when $\mathbf{n}$ is invariant under the axial symmetry group $K = \{(R_3(\alpha), e^{in\alpha}) : \alpha \in \mathbb{R}\} < G$, where $R_3(\alpha)$ denotes rotation through α about the x_3 axis. Since $\mathsf{M} \cong G/K$ and $p : G \to G/K$ is a fibration, we have the associated homotopy exact sequence

$$\begin{array}{ccccccc} & K & \stackrel{\iota}{\hookrightarrow} & G & \stackrel{p}{\to} & G/K & & \\ \Rightarrow & \pi_1(K) & \stackrel{\iota_*}{\to} & \pi_1(G) & \stackrel{p_*}{\to} & \pi_1(\mathsf{M}) & \to & \pi_0(K) \\ & \mathbb{Z} & \stackrel{\iota_*}{\to} & \mathbb{Z}_2 \oplus \mathbb{Z} & \stackrel{p_*}{\to} & \pi_1(\mathsf{M}) & \to & 0. \end{array}$$

Hence p_* surjects, so $\pi_1(\mathsf{M}) \equiv \pi_1(G)/\ker p_*$ by the Isomorphism Theorem. But $\ker p_*$ is, by exactness, the image of $\pi_1(K)$ under inclusion, clearly the infinite cyclic group generated by $1 \oplus n \in \pi_1(G)$. This group has precisely $2n$ cosets in $\pi_1(G)$, labelled by the elements

$$0 \oplus 0,\ 0 \oplus 1,\ \ldots,\ 0 \oplus (2n-1),$$

for example. Let us denote the coset $g + \ker p_*$ by $[g]$. It follows immediately that the quotient group $\pi_1(G)/\ker p_*$ is cyclic of order $2n$, generated by $[0 \oplus 1]$. Note also that the 2π spatial rotation loop lies in $1 \oplus 0 \in \pi_1(G)$, which projects to $[0 \oplus n] = n[0 \oplus 1]$ in $\pi_1(G)/\ker p_*$, since $1 \oplus 0 = 0 \oplus n - 1 \oplus n$. Hence the 2π spatial rotation loop in M is noncontractible of order 2, independent of n (and Q).

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