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# On the nonemptiness of approximate cores of large games

Nizar Allouch<sup>1</sup> · Myrna Wooders<sup>2</sup>

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**Abstract** We provide a new proof of the nonemptiness of approximate cores of games with many players of a finite number of types. Earlier papers in the literature proceed by showing that, for games with many players, equal-treatment cores of their “balanced cover games,” which are nonempty, can be approximated by equal-treatment  $\varepsilon$ -cores of the games themselves. Our proof is novel in that we develop a limiting payoff possibilities set and rely on a fixed point theorem.

**Keywords** NTU games · Core · Approximate cores · Small group effectiveness · Coalition formation · Payoff-dependent balancedness

**JEL Classification** C71 · C78 · D71

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✉ Nizar Allouch  
n.allouch@qmul.ac.uk

Myrna Wooders  
myrna.wooders@vanderbilt.edu  
<http://www.myrnawooders.com>

<sup>1</sup> School of Economics and Finance, Queen Mary University of London, London, UK

<sup>2</sup> Department of Economics, Vanderbilt University, Nashville, TN, USA

## 1 Introduction

The core is an anchoring concept in game theory going back, in its origins, to Edgeworth's contract curve, and the contributions of [Debreu and Scarf \(1963\)](#) and [Aumann \(1964\)](#). The core remains a central concept in economics and most recently, in market design; see, for example, [Roth \(2002\)](#). Even in games with many, but finite numbers of players, however, the core may be empty. The addition of a single player to a large game with a nonempty core may result in a game with an empty core. The problem of the emptiness of the core is especially salient in economies with public goods subject to congestion and exclusion (local public goods) or in economies with clubs. Even in pure exchange economies, the nonemptiness of the core can depend on whether commodities are infinitely divisible. It is, however, a remarkable fact that, as established by [Wooders \(1983\)](#) and a number of subsequent papers, games with many players satisfying apparently mild conditions have nonempty approximate cores.

In this paper, inspired by the payoff-dependent balancedness notion<sup>1</sup> of Herings and [Predtetchinski and Herings \(2004\)](#) and [Bonnisseau and Iehlé \(2007\)](#),<sup>2</sup> we demonstrate nonemptiness of approximate cores for sequences of games with arbitrary distributions of players. Recall that much of the literature on approximate cores of NTU games, beginning with [Wooders \(1983\)](#) and most recently [Kovalenkov and Wooders \(2001\)](#), [Kovalenkov and Wooders \(2003\)](#) and [Wooders \(2008\)](#), establishes nonemptiness of approximate cores of large games by showing that payoffs in the cores of derived "balanced cover" games can be approximated by feasible payoffs of the original games. Quite surprisingly, a modification of a key construct from the literature on payoff-dependent balancedness, a correspondence from limiting feasible payoffs to distributions of players types<sup>3</sup> achieving them, enables us to establish that for large games limiting payoffs vary continuously with the distribution of player types. With such a correspondence in hand, we can bypass approximation of the original games by balanced cover games and simply appeal to a fixed point argument rather than to approximating balanced games. An especially interesting aspect of our proof is that we obtain a limiting payoff set that is the analogue of the limiting utility function of [Wooders \(1994\)](#) describing TU games with many players as market games. This paper lays a foundation for further investigation of many player nontransferable utility games as market games.

More specifically, for sequence of games with growing numbers of players of each of a finite number of types and arbitrary distributions of player types we introduce a set of limiting equal-treatment payoffs, denoted by  $\Gamma$ , and a correspondence from payoffs in  $\Gamma$  to distributions of players types able to achieve them. A limiting equal-

<sup>1</sup> Payoff-dependent balancedness generalizes the well-known notion of Scarf balancedness for NTU games.

<sup>2</sup> As they discuss, the intuition behind the [Bonnisseau and Iehlé \(2007\)](#) result comes from the existence of a general pricing rule equilibrium in an economy with a nonconvex production sectors, as in [Bonnisseau and Cornet \(1988\)](#), [Bonnisseau et al. \(1991\)](#) and [Bonnisseau \(1997\)](#), which show that core payoffs correspond to equilibrium allocations of a suitably constructed two-production-set auxiliary economy.

<sup>3</sup> In interpretation, a distribution of player types is simply a list of percentages of players of each type. For large games it also reflects a set of players *à la* [Aubin \(1979\)](#), where players have different participation rates (see also [Florenzano 1990](#); [Allouch and Florenzano 2004](#)).

treatment payoff is approximately feasible for some group, possibly large, described by the distribution of player types in the group. We require essentially four conditions for our results:

1. Superadditivity (SA): Any group  $N$  of players can realize at least the payoffs achievable by cooperation only within groups in a partition of  $N$ ;
2. Players of the same type are substitutes (PSTS): The payoff possibilities set of a group  $N$  depends on the types profile of the group and not on the names of its members;
3. Convexity (CONV): For each group  $N$  the payoff possibilities set is convex;
4. Small group effectiveness (SGE): All or almost all gains to group formation can be realized by groups uniformly bounded in size.

Our result extends that of [Wooders \(1983\)](#) in that our limiting construct is not restricted to games with a fixed distribution of player types; instead we consider all sequences of games with growing player sets converging to some given distribution of types. While we use SGE, [Wooders \(1983\)](#) uses the apparently milder condition of boundedness of per capita payoffs. We use SGE since it is easier to work with and closely related. Recall that [Wooders \(2008, Theorem 2\)](#) uses similar conditions as employed in this paper to demonstrate that, for games with a compact metric space of player types, given  $\varepsilon > 0$  there is an integer  $\eta_0(\varepsilon)$  such that all games with more than  $\eta_0(\varepsilon)$  players have nonempty equal-treatment  $\varepsilon$ -cores.

Although both [Predtetchinski \(2005\)](#) and [Allouch and Predtetchinski \(2008\)](#) use the notion of payoff-dependent balancedness, their approaches differ in many aspects from that of the current paper. First, the current paper deals with a sequence of games defined in characteristic form with possibly ever-increasing equal-treatment payoff sets. Our framework, as [Wooders \(1983\)](#) and subsequent papers on games with many players, can accommodate a general class of exchange economies including ones with (local) public goods and clubs. In contrast, [Predtetchinski \(2005\)](#) and [Allouch and Predtetchinski \(2008\)](#) treat a pure exchange economy, where equal-treatment payoff sets are identical under replications of the total player set. As a result, in our approach both feasible payoffs and core concepts are defined approximately for large finite games, in contrast to [Predtetchinski \(2005\)](#) and [Allouch and Predtetchinski \(2008\)](#) where feasible payoffs and core concepts are exactly defined. Moreover, the crucial argument in our paper, based on small group effectiveness, is to show that payoffs achieved in the limit by a distribution of player types vary continuously with the distribution of player types. However, in [Allouch and Predtetchinski \(2008\)](#) such a continuity argument is inferred directly from the upper semi-continuity of utility functions over feasible allocations. Finally, in our approach we seek a fixed point for an arbitrary limiting distribution of player types, (both rational and nonrational), unlike [Allouch and Predtetchinski \(2008\)](#) where the distribution of players type is fixed and rational.

The paper is organized as follows. In Sect. 2, we present the basic features of games with a finite number of types of players. In Sect. 3, we present our main result on the nonemptiness of approximate cores of a sequence of games with a finite number of types of players. Section 4 provides the proof of our main result, and we conclude in Sect. 5 with a comparison to the literature.

## 2 NTU games with a finite number of types of players

We investigate games with a fixed finite set of player types  $T = \{1, \dots, T\}$ . Let

$$\mathcal{N} = \{(t, q) \mid t = 1, \dots, T \text{ and } q \in \mathbb{Z}_+\},$$

where  $\mathbb{Z}_+$  is the set of nonnegative integers. Note that  $\mathcal{N}$  is a countably infinite set. A group of players is a finite subset of  $\mathcal{N}$ , with a typical group denoted by  $N \subset \mathcal{N}$ . The *profile* of  $N$ , denoted by  $\text{pro}(N)$ , is defined as follows:

$$\text{pro}(N) = (\text{pro}_1(N), \dots, \text{pro}_T(N)),$$

where  $\text{pro}_t(N)$  denotes the number of players of type  $t$  in  $N$ . Also, let  $|N| = \sum_{t \in T} \text{pro}_t(N)$  denote the number of players in  $N$ .

We take as given a correspondence  $\mathcal{V}$  mapping each group  $N$  into a subset of  $\mathbb{R}^N$ . For each group of players  $N$ , the correspondence has the following properties:

- $\mathcal{V}(N)$  is a closed subset of  $\mathbb{R}^N$ ;
- $0 \in \text{int}\mathcal{V}(N)$ ;
- $\mathcal{V}(N)$  is comprehensive from below (that is, if  $x \in \mathcal{V}(N)$  and if  $y \in \mathbb{R}^N, y \leq x$  then  $y \in \mathcal{V}(N)$ );
- $\mathcal{V}(N) \cap \mathbb{R}_+^N$  is bounded above.

We also assume that correspondence  $\mathcal{V}$  satisfies the following properties.

### 2.1 Superadditivity (SA)

For any group of players  $N$  and any partition  $\mathcal{P}(N) = (N_k)_{k=1}^K$  of  $N$  into groups with the property that  $v \in \mathcal{V}(N_k) \times \mathbb{R}^{N \setminus N_k}$  for each  $k$ , it holds that  $v \in \mathcal{V}(N)$ , that is,

$$\bigcap_{k=1}^K \left( \mathcal{V}(N_k) \times \mathbb{R}^{N \setminus N_k} \right) \subset \mathcal{V}(N).$$

Superadditivity implies that any payoff vector that can be realized by groups in a partition of a group of players is feasible for the entire group of players.

The following notion of substitute players in NTU games was introduced in Wooders (1983). For NTU games, to capture the notion of substitutes it is necessary to require not only that substitute players make the same contribution to any group they may join but also that they are interchangeable when they are both in the same group.

### 2.2 Players of the same type are substitutes (PSTS)

For any group of players  $N$  and any two players  $(t, q)$  and  $(t, q')$  (a) if  $(t, q) \notin N$  and  $(t, q') \notin N$ , given any  $v \in \mathcal{V}(N \cup \{(t, q)\})$  it holds that  $v' \in \mathcal{V}(N \cup \{(t, q')\})$ , where  $v'$  is defined by  $v'_{tq'} = v_{tq}$  and  $v'_\ell = v_\ell$  for all  $\ell \in N, \ell \neq (t, q), (t, q')$  and (b)

if  $(t, q), (t, q') \in N$ , given any  $v \in \mathcal{V}(N)$  it holds that  $v' \in \mathcal{V}(N)$  where  $v'_{tq'} = v_{tq}$ ,  $v'_{tq} = v_{tq}$  and  $v'_\ell = v_\ell$  for all  $\ell \in N$ ,  $\ell \neq (t, q), (t, q')$ .<sup>4</sup>

Given  $v = (v_1, \dots, v_T) \in \mathbb{R}^T$ , for any group of players  $N$  define  $v_N \in \mathbb{R}^N$  such that for each  $(t, q) \in N$  it holds that  $(v_N)_{tq} = v_t$ . When  $v_N \in \mathcal{V}(N)$  we say that  $v$  represents an equal-treatment payoff in  $\mathcal{V}(N)$ . Let

$$\mathcal{V}^{\text{etp}}(N) \stackrel{\text{def}}{=} \{v \in \mathbb{R}^T \mid v_N \in \mathcal{V}(N)\},$$

denote the subset of payoff vectors that represent equal-treatment payoffs in  $\mathcal{V}(N)$ . Note that  $\mathcal{V}^{\text{etp}}(N)$  is nonempty since it always contains the 0 payoff and is unconstrained for player types that do not appear in  $N$ . Moreover, in view of PSTS, it holds that the equal-treatment payoff set  $\mathcal{V}^{\text{etp}}(N)$  of a group of players  $N$  depends only on the profile  $\text{pro}(N)$ .<sup>5</sup>

### 2.3 Convexity (CONV)

For each group  $N \subset \mathcal{N}$  the set  $\mathcal{V}(N)$  is convex.

Convexity of payoff sets is often used in studies of NTU games and is satisfied for the special case of games with transferable utility. For our purposes in this paper, convexity is used to ensure that the average of any finite set of feasible payoffs is feasible.

### 2.4 Small group effectiveness (SGE)

For every  $\varepsilon > 0$ , there is a positive integer  $\tau(\varepsilon)$  such that each group  $N \subset \mathcal{N}$  has a partition  $\mathcal{P}(N) = (N_k)_{k=1}^K$  with the properties that  $|N_k| \leq \tau(\varepsilon)$  for each  $k$ , and

$$\mathcal{V}^{\text{etp}}(N) \subset \bigcap_{k=1}^K \mathcal{V}^{\text{etp}}(N_k) + \varepsilon \mathbf{1},$$

where  $\mathbf{1} = (1, \dots, 1) \in \mathbb{R}^T$ .

Small group effectiveness ensures that, given arbitrarily small  $\varepsilon$ , almost all (within  $\varepsilon$ ) gains to group formation can be realized by a partition of the group of players into groups uniformly bounded in size.<sup>6</sup>

<sup>4</sup> For the reader familiar with Wooders (2008), we note with this requirement, that players of the same type are substitutes, the correspondence  $\mathcal{V}$  determines a “pregame” with a finite number of types.

<sup>5</sup> To see this, let  $N$  and  $N'$  be groups of players with the same profiles. Let  $\tau$  be a type-preserving one-to-one mapping from  $N$  onto  $N'$  (that is, if  $(t, q) \in N$  then  $\tau((t, q)) = (t, q')$  for some  $q'$  such that  $(t, q') \in N'$ ). Given a set  $Y \subset \mathbb{R}^N$ , let  $\sigma_\tau(Y) \subset \mathbb{R}^{N'}$  denote the set formed from  $Y$  by substituting the values of the coordinates according to the mapping  $\tau$ . In view of PSTS, it holds that  $\mathcal{V}(N) = \sigma_\tau^{-1}(\mathcal{V}(N'))$ , which implies that  $\mathcal{V}^{\text{etp}}(N) = \mathcal{V}^{\text{etp}}(N')$ . Hence, PSTS implies that the equal-treatment payoff set  $\mathcal{V}^{\text{etp}}(N)$  of a group of players  $N$  depends only on the profile  $\text{pro}(N)$ .

<sup>6</sup> While other related conditions appear in the literature, such as boundedness of the set of equal-treatment payoffs and strict small group effectiveness, the condition of SGE, which precisely limits the power of

### 3 The limiting utility possibilities set for NTU games

Following Scarf’s (1967) definition of an NTU game, with every group  $N \subset \mathcal{N}$  we associate an NTU game  $(N, V)$  defined by the property that for each nonempty subset  $S$  of  $N$  it holds that  $V(S) = \mathcal{V}(S) \times \mathbb{R}^{N \setminus S}$ . Thus, given the player set  $N$ , the correspondence  $\mathcal{V}$  determines the NTU game (in coalitional function form)  $(N, V)$ , where  $N$  is a finite set (the set of *players*) and  $V$  is a set-valued function that assigns to each nonempty subset  $S$  of  $N$  (a *group* or *coalition*) a nonempty subset  $V(S)$  of  $\mathbb{R}^N$ , called a *payoff possibilities set* or simply a *payoff set*.

A *payoff vector for the game*  $(N, V)$  is a vector  $x$  in  $\mathbb{R}^N$ . A payoff vector  $x$  is *feasible* for  $N$  if  $x \in V(N)$ . A payoff vector  $x$  is in the  $\varepsilon$ -core of the game  $(N, V)$  if it is feasible for  $N$  and if, for every subset  $S$  of  $N$ ,  $x + \varepsilon \mathbf{1}_N \notin \text{int } V(S)$ . Informally, a feasible payoff vector  $x$  is in the  $\varepsilon$ -core if no group of players can improve upon  $x$  by more than  $\varepsilon$  for each player in the group.<sup>7</sup>

During the proof of the following theorem, we will use the following notation: Denote by  $\|\cdot\|$  the sum-metric in  $\mathbb{R}^T$ ; that is, for  $s \in \mathbb{R}^T$  we have  $\|s\| = \sum_{t=1}^T |s_t|$ . For each point  $s \in \mathbb{R}^T$  let  $\text{supp}(s)$  denote the set  $\{t \in T \mid s_t > 0\}$ , called the *support of  $s$* . Let  $\Delta$  denote the simplex in  $\mathbb{R}^T : \Delta = \{s \in \mathbb{R}_+^T \mid \|s\| = 1\}$  and let  $\text{int } \Delta$  denote its (relative) interior.

**Theorem** *Assume that  $\mathcal{V}$  satisfies SA, PSTS, SGE, and CONV. Let  $\{(N^n, V)\}_n$  be a sequence of games such that  $|N^n| \rightarrow \infty$  and*

$$\lim_{n \rightarrow +\infty} \frac{\text{pro}(N^n)}{|N^n|} = s^* \in \text{int } \Delta.$$

*Then there exists  $v^* \in \mathbb{R}^T$  satisfying the property: for every  $\varepsilon > 0$  there is an integer  $r_\varepsilon$  such that for each  $n \geq r_\varepsilon$ ,  $(v^* - \varepsilon \mathbf{1})_{N^n}$  is in the  $\varepsilon$ -core of  $(N^n, V)$ .*

Our novel proof is contained in the next section. In the remainder of this section, we introduce some notation used in the proof and indicate how the result is obtained.

Define a subset  $\Gamma$  of  $\mathbb{R}^T$  as follows:

$$\Gamma \stackrel{\text{def}}{=} \left\{ v \in \mathbb{R}^T \left| \begin{array}{l} \text{There exists } s \in \Delta \cap \mathbb{Q}^T \text{ such that,} \\ \text{for each } \varepsilon > 0, \text{ there exists a group } N_\varepsilon \text{ satisfying} \\ \frac{\text{pro}(N_\varepsilon)}{|N_\varepsilon|} = s \text{ and } (v - \varepsilon \mathbf{1}) \in \mathcal{V}^{\text{etp}}(N_\varepsilon) \end{array} \right. \right\}.$$

The set  $\Gamma$  represents equal-treatment payoffs that are feasible or approximately feasible for some group, possibly large, described by the fixed distribution of player types in the group. When  $(v - \varepsilon \mathbf{1}) \in \mathcal{V}^{\text{etp}}(N_\varepsilon)$  we say that  $N_\varepsilon$  *approximately achieves*  $v$ .

Footnote 6 continued

small groups, was introduced in Wooders (2008) for NTU games and in Wooders (1992) for TU games. Our notion here of SGE is slightly more restrictive than that of Wooders (2008), but the difference is simply for convenience.

<sup>7</sup> This notion of  $\varepsilon$ -core is sometimes called the *uniform  $\varepsilon$ -core*.

Note that given  $v \in \Gamma$  it may be that there does not exist a group  $N$  that can fully achieve  $v$ , that is, there need not exist a group  $N$  such that  $v \in \mathcal{V}^{\text{etp}}(N)$ ; there will exist such a group only if *all* gains to group formation can be exhausted by groups bounded in size.<sup>8</sup> Note also that, by SA, if  $(v - \varepsilon \mathbf{1}) \in \mathcal{V}^{\text{etp}}(N_\varepsilon)$  then  $(v - \varepsilon \mathbf{1}) \in \mathcal{V}^{\text{etp}}(N'_\varepsilon)$  for every group  $N'_\varepsilon$  containing a positive integer multiple of players of each type as  $N_\varepsilon$ , that is, for every group  $N$  such that  $\text{pro}(N'_\varepsilon) = k\text{pro}(N_\varepsilon)$  for any positive integer  $k$ .

Given  $v \in \Gamma$ , there are multiple groups with different distributions that can all approximately achieve  $v$ . Thus, we define the correspondence  $\Pi : \Gamma \rightrightarrows \Delta$  as follows:

$$\Pi(v) \stackrel{\text{def}}{=} \left\{ s \in \Delta \cap \mathbb{Q}^T \mid \begin{array}{l} \text{For each } \varepsilon > 0 \\ \text{there exists a group } N_\varepsilon \text{ satisfying} \\ \frac{\text{pro}(N_\varepsilon)}{|N_\varepsilon|} = s \text{ and } (v - \varepsilon \mathbf{1}) \in \mathcal{V}^{\text{etp}}(N_\varepsilon) \end{array} \right\}.$$

The set  $\Pi(v)$  consists of those distributions  $s$  of player types for which  $v$  is approximately feasible, that is, those distributions of player types that, in the definition of  $\Gamma$  are required to exist. Note that the groups  $N_\varepsilon$  may need to become arbitrarily large as  $\varepsilon$  becomes small. Note also that nonemptiness of the set  $\Pi(v)$  follows immediately from the definitions of  $\Gamma$  and  $\Pi$ .

The graph of the correspondence  $\Pi$  is denoted by  $G(\Pi)$  and defined by

$$G(\Pi) = \{(v, s) \in \Gamma \times (\Delta \cap \mathbb{Q}^T) \mid s \in \Pi(v)\}.$$

Define the correspondence  $\tilde{\Pi} : \text{cl}(\Gamma) \rightrightarrows \Delta$  as follows: for each  $v \in \text{cl}(\Gamma)$

$$\tilde{\Pi}(v) \stackrel{\text{def}}{=} \{s \in \Delta \mid \exists \{(v^n, s^n)\}_n \text{ in } G(\Pi) \text{ converging to } (v, s)\}.$$

We will show that, for each  $v \in \text{cl}(\Gamma)$ , the set  $\tilde{\Pi}(v)$  is nonempty and convex.

Let  $G(\tilde{\Pi})$  denote the graph of the correspondence  $\tilde{\Pi}$ . We will show that  $G(\tilde{\Pi})$  is the closure of  $G(\Pi)$  with respect to  $\mathbb{R}^T \times \Delta$ .

Our proof proceeds by showing that limiting payoffs vary continuously with the distribution of player types. That is, we show that  $\tilde{\Pi}$  is a continuous correspondence from distributions of player types to limiting equal-treatment payoffs achievable, or almost achievable, by large games with close distributions of players types. We can then appeal to a fixed point theorem to obtain the result that there is a point  $v^* \in \partial(\text{cl}(\Gamma))$  such that  $s^* \in \tilde{\Pi}(v)$ , which turns out to be sufficient to prove our main theorem.

### 4 The proof of the main result

**Proposition 1** *Let  $(v, s) \in \mathbb{R}^T \times \Delta$ . Let  $\{(v^n, s^n)\}_n$  be a sequence in  $G(\Pi)$  converging to  $(v, s)$  :*

- (1) *If  $s \in \Delta \cap \mathbb{Q}^T$  then  $(v, s) \in G(\Pi)$ .*

<sup>8</sup> That is, unless a form of *strict small group effectiveness* is satisfied.



(2) For any sequence of groups  $\{N^n\}_n$  satisfying  $\lim_{n \rightarrow +\infty} \frac{\text{pro}(N^n)}{|N^n|} = s \in \text{int}\Delta$  (with possibly  $\frac{\text{pro}(N^n)}{|N^n|} \neq s^n$ ) and  $|N^n| \rightarrow \infty$ , for every  $\varepsilon > 0$  there exists  $r_\varepsilon$  such that for each  $n \geq r_\varepsilon$ , it holds that  $(v - \varepsilon \mathbf{1}) \in \mathcal{V}^{\text{etp}}(N^n)$ .

*Proof of Proposition 1* (1) The proof of (1) has two parts. In Part (a), given the sequence  $\{(v^n, s^n)\}_n$  converging to  $(v, s)$ , using SGE, we determine a finite collection of distributions of player types such that for any distribution  $s'$  in the collection it holds that  $(v - \frac{2\varepsilon}{3} \mathbf{1}) \in \mathcal{V}^{\text{etp}}(S')$  for any group  $S'$  with distribution of types equal to  $s'$ . Let  $\mathcal{M}^*$  denote this collection of profiles. We use this result in Part (b) where it is shown that we can restrict attention to distributions in the collection  $\mathcal{M}^*$ .

Part (a): Let us fix  $\varepsilon > 0$ . Since  $s^n \in \Pi(v^n)$  and thus has rational coefficients, there exists a group  $N_\varepsilon^n$  such that

$$\frac{\text{pro}(N_\varepsilon^n)}{|N_\varepsilon^n|} = s^n \text{ and } \left(v^n - \frac{\varepsilon}{3} \mathbf{1}\right) \in \mathcal{V}^{\text{etp}}(N_\varepsilon^n).$$

Since  $\mathcal{V}$  satisfies SGE, there is an integer  $\tau(\frac{\varepsilon}{3})$  and a partition  $\mathcal{P}(N_\varepsilon^n) = (N_{\varepsilon,k}^n)_{k=1}^K$  of  $N_\varepsilon^n$  such that

$$\mathcal{V}^{\text{etp}}(N_\varepsilon^n) \subset \bigcap_{k=1}^K \mathcal{V}^{\text{etp}}(N_{\varepsilon,k}^n) + \frac{\varepsilon}{3} \mathbf{1},$$

with the property  $|N_{\varepsilon,k}^n| \leq \tau(\frac{\varepsilon}{3})$  for each  $N_{\varepsilon,k}^n \in \mathcal{P}(N_\varepsilon^n)$ . Since there is only a finite number of profiles for groups  $N$  satisfying  $|N| \leq \tau(\frac{\varepsilon}{3})$ , we can denote their number by a finite integer  $M$  and let  $p_1, \dots, p_m, \dots, p_M$  be a list of these profiles. Thus, we can write

$$\text{pro}(N_\varepsilon^n) = \sum_{m=1}^M \beta_m^n p_m,$$

where  $\beta_m^n$  is the number of subsets  $N_{\varepsilon,k}^n \in \mathcal{P}(N_\varepsilon^n)$  such that  $\text{pro}(N_{\varepsilon,k}^n) = p_m$ . Since  $0 \leq \beta_m^n \leq |N_\varepsilon^n|$  we can assume, without loss of generality, that for each  $m = 1, \dots, M$ , the sequence  $(\frac{\beta_m^n}{|N_\varepsilon^n|})$  converges to a real number  $\beta_m^*$ . Let

$$\mathcal{M}^* = \{m \mid \beta_m^* > 0\}.$$

Then, it holds that

$$s = \sum_{m \in \mathcal{M}^*} \beta_m^* p_m. \tag{1}$$

Thus, it holds that

$$\text{supp}(s) = \cup_{m \in \mathcal{M}^*} \text{supp}(p_m). \tag{2}$$

Moreover, by PSTS, for every group  $N$  such that  $\text{pro}(N) = p_m$  for some  $m \in \mathcal{M}^*$ , for all  $n$  sufficiently large, it holds that

$$\left(v^n - \frac{\varepsilon}{3} \mathbf{1}\right) \in \mathcal{V}^{\text{etp}}(N) + \frac{\varepsilon}{3} \mathbf{1}.$$

Rearranging terms, it follows that

$$\left( v^n - \frac{2\varepsilon}{3} \mathbf{1} \right) \in \mathcal{V}^{\text{etp}}(N).$$

Since  $\mathcal{V}^{\text{etp}}(N)$  is a closed set it holds that

$$\left( v - \frac{2\varepsilon}{3} \mathbf{1} \right) \in \mathcal{V}^{\text{etp}}(N). \tag{3}$$

Part (b): We next show that we can restrict attention to groups with profiles in the set  $\mathcal{M}^*$ . Since  $s \in \Delta \cap \mathbb{Q}^T$ , there is an integer  $d$  such that  $ds$  has integer components. In view of (2), for each  $n$ , the set

$$\mathcal{A}^n = \left\{ l \in \mathbb{Z}_+ \mid lds \geq \sum_{m \in \mathcal{M}^*} \beta_m^n p_m \right\}$$

is nonempty. Let  $l^n = \inf \mathcal{A}^n$ . We claim that

$$\lim_{n \rightarrow \infty} \frac{\sum_{m \in \mathcal{M}^*} \beta_m^n p_m}{\|l^n ds\|} = s.$$

Suppose not. Then, passing to a subsequence if necessary, we may assume that

$$\lim_{n \rightarrow \infty} \frac{\sum_{m \in \mathcal{M}^*} \beta_m^n p_m}{\|l^n ds\|} = \lim_{n \rightarrow \infty} \frac{\|\sum_{m \in \mathcal{M}^*} \beta_m^n p_m\|}{\|l^n ds\|} \frac{\sum_{m \in \mathcal{M}^*} \beta_m^n p_m}{\|\sum_{m \in \mathcal{M}^*} \beta_m^n p_m\|} = \lambda^* s$$

where  $\lambda^* \in [0, 1[$ . Thus, since  $l^n \rightarrow \infty$ , it also holds that,

$$\lim_{n \rightarrow \infty} \frac{(l^n - 1)ds - \sum_{m \in \mathcal{M}^*} \beta_m^n p_m}{\|l^n ds\|} = (1 - \lambda^*)s$$

Hence, in view of (2), given  $\delta \in ]0, 1 - \lambda^*[$ , for all  $n$  sufficiently large, it holds that

$$\frac{(l^n - 1)ds - \sum_{m \in \mathcal{M}^*} \beta_m^n p_m}{\|l^n ds\|} \geq \delta s,$$

which implies that

$$(l^n - 1)ds - \sum_{m \in \mathcal{M}^*} \beta_m^n p_m \geq 0.$$

This implies that  $(l^n - 1) \in \mathcal{A}^n$ , which is a contradiction to  $l^n$  being the infimum of  $\mathcal{A}^n$ .

Let  $N_\varepsilon^{n,*} \subset N_\varepsilon^n$  be a group satisfying  $\text{pro}(N_\varepsilon^{n,*}) = \sum_{m \in \mathcal{M}^*} \beta_m^n p_m$ . Let also  $E_\varepsilon^{n,s}$  be a group satisfying  $N_\varepsilon^{n,*} \subset E_\varepsilon^{n,s}$  and  $\text{pro}(E_\varepsilon^{n,s}) = l^n ds$ . Given the construction of

these groups, we have

$$\lim_{n \rightarrow \infty} \frac{|E_\varepsilon^{n,s} \setminus N_\varepsilon^{n,*}|}{|N_\varepsilon^{n,s}|} = \lim_{n \rightarrow \infty} \frac{\|l^n ds - \sum_{m \in \mathcal{M}^*} \beta_m^n p_m\|}{\|l^n ds\|} = 0.$$

Moreover, by PSTS, SA and (3), it holds that

$$\left(v - \frac{2\varepsilon}{3}\mathbf{1}\right) \in \mathcal{V}^{\text{etp}}(N_\varepsilon^{n,*}).$$

The equal-treatment payoff  $(v - \frac{2\varepsilon}{3}\mathbf{1})$  is feasible for all groups  $D$  in  $E_\varepsilon^{n,s}$  with  $\text{pro}(D) = \text{pro}(N_\varepsilon^{n,*})$ . Consider the payoff vector

$$\left(\left(v - \frac{2\varepsilon}{3}\mathbf{1}\right)_D, 0_{E_\varepsilon^{n,s} \setminus D}\right) \in \mathcal{V}(E_\varepsilon^{n,s}).$$

Take the average payoff vector over all such groups  $D$ , which yields an equal-treatment payoff vector that belongs to  $\mathcal{V}(E_\varepsilon^{n,s})$  since CONV.<sup>9</sup> For sufficiently large  $n$ , this average payoff vector will be greater than  $(v - \varepsilon\mathbf{1})_{E_\varepsilon^{n,s}}$ . Hence, it holds that

$$(v - \varepsilon\mathbf{1}) \in \mathcal{V}^{\text{etp}}(E_\varepsilon^{n,s}),$$

which implies that  $s \in \Pi(v)$ .

(2) In the following, we will show that for any sequence of groups  $\{N^n\}_n$  with a limiting distribution of player types given by  $s$ , for all  $n$  sufficiently large,  $N^n$  can be approximated (in terms of numbers of players of each type) by groups with profiles in the collection  $(p_m)_{m \in \mathcal{M}^*}$ , which, in turn, implies that  $N^n$  can approximately achieve the payoff  $v$ .

We require the following theorem about projections on a convex set.

### 4.1 Projection theorem

Let  $C$  be a nonempty closed convex set of  $\mathbb{R}^T$ .

- (i) For any  $x \in \mathbb{R}^T$  there exists a unique vector  $P_C(x) = \arg \min_{z \in C} \|z - x\|_2$  called the projection of  $x$  on  $C$ .<sup>10</sup>
- (ii) The vector  $P_C(x)$  can be defined as the only vector with the property

$$(y - P_C(x)) \cdot (x - P_C(x)) \leq 0, \quad \forall y \in C.$$

<sup>9</sup> This approach to the “leftovers problem” was initiated in Wooders (1983). Details of this sort of argument are contained therein.

<sup>10</sup> As usual,  $\|\cdot\|_2$  denote the Euclidian norm.

Let  $\{N^n\}_n$  be an arbitrary sequence of groups satisfying  $\lim_{n \rightarrow +\infty} \frac{\text{pro}(N^n)}{|N^n|} = s$  (with possibly  $\frac{\text{pro}(N^n)}{|N^n|} \neq s^n$ ) and  $|N^n| \rightarrow \infty$ . Then, taking  $\varepsilon$  (and other definitions) as given in the proof of Part (a) in (1) above, let  $c(N^n)$  denote the projection of  $\text{pro}(N^n)$  on the convex cone  $\mathcal{C}$  spanned by  $(p_m)_{m \in \mathcal{M}^*}$ . First, note that since  $\mathcal{C}$  is a convex cone spanned by  $(p_m)_{m \in \mathcal{M}^*}$  it follows from (1) that  $s \in \mathcal{C}$ . We claim that

$$\lim_{n \rightarrow +\infty} \frac{c(N^n)}{|N^n|} = s.$$

Suppose not. Then, passing to a subsequence if necessary, we may assume that

$$\lim_{n \rightarrow +\infty} \frac{c(N^n)}{|N^n|} = s' \neq s.$$

Since  $s \in \mathcal{C}$  and  $\mathcal{C}$  is a convex cone, it holds that for each  $n$

$$(|N^n|s - c(N^n)) \cdot (\text{pro}(N^n) - c(N^n)) \leq 0,$$

or equivalently,

$$\left(s - \frac{c(N^n)}{|N^n|}\right) \cdot \left(\frac{\text{pro}(N^n)}{|N^n|} - \frac{c(N^n)}{|N^n|}\right) \leq 0.$$

Taking the limit it holds that  $\|s - s'\|_2^2 \leq 0$ , which is a contradiction.

For each  $n$ , let  $\mathcal{F}^n = \{\theta \in [0, 1] \mid \text{pro}(N^n) - \theta c(N^n) \geq 0\}$ , which is nonempty since it contains 0. Let  $\theta^n = \max \mathcal{F}^n$ . We claim that  $\lim_{n \rightarrow +\infty} \theta^n = 1$ . Suppose not. Then passing to a subsequence if necessary, we may assume that  $\lim_{n \rightarrow +\infty} \theta^n = \theta^*$  for some  $\theta^* \in [0, 1[$ . Hence,

$$\lim_{n \rightarrow +\infty} \frac{\text{pro}(N^n) - \frac{(\theta^n+1)}{2} c(N^n)}{|N^n|} = \frac{(1 - \theta^*)}{2} s.$$

Then, since (2) and  $s \in \text{int}\Delta$ , for some  $\delta > 0$ , for all  $n$  sufficiently large, it holds that

$$\frac{\text{pro}(N^n) - \frac{(\theta^n+1)}{2} c(N^n)}{|N^n|} \geq \delta s,$$

which implies that  $\frac{(\theta^n+1)}{2} \in \mathcal{F}^n$ . This is a contradiction since, for all  $n$  sufficiently large, it holds that  $\theta^n < \frac{(\theta^n+1)}{2}$ .

Note that we can write  $c(N^n) = \sum_{m \in \mathcal{M}^*} \xi_m^n p_m$ , for some real numbers  $\xi_m^n \in \mathbb{R}_+$ . Let

$$\text{Integer}(\theta^n c(N^n)) = \sum_{m \in \mathcal{M}^*} [\theta^n \xi_m^n] p_m,$$

where, for any  $u \in \mathbb{R}_+$ ,  $[u]$  denotes the integer part of  $u$ . Let  $\widehat{N}^n \subset N^n$  be a group satisfying  $\text{pro}(\widehat{N}^n) = \text{Integer}(\theta^n c(N^n))$ . Note that

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{|N^n \setminus \widehat{N}^n|}{|N^n|} &= \lim_{n \rightarrow \infty} \frac{\|\text{pro}(N^n) - \theta^n c(N^n)\| + \|\theta^n c(N^n) - \text{Integer}(\theta^n c(N^n))\|}{|N^n|} \\ &\leq 0 + \lim_{n \rightarrow \infty} \frac{\|\sum_{m \in \mathcal{M}^*} p_m\|}{|N^n|} = 0. \end{aligned}$$

Moreover, from SA and (3) it follows that for a large enough  $n$

$$\left(v - \frac{2\varepsilon}{3} \mathbf{1}\right) \in \mathcal{V}^{\text{etp}}(\widehat{N}^n).$$

Therefore, by CONV, one could subsidize the left-overs  $(N^n \setminus \widehat{N}^n)$  so that

$$(v - \varepsilon \mathbf{1}) \in \mathcal{V}^{\text{etp}}(N^n).$$

□

**Proposition 2** *There is a bound  $B \in \mathbb{R}_+$  such that for all  $(v, s) \in G(\Pi)$ , for each  $t \in \text{supp}(s)$  it holds that  $v_t < B$ .*

*Proof of Proposition 2* Taking again  $\varepsilon$  (and other definitions) as given in the proof of Part (a) in (1) above, for each  $m \in \{1, \dots, M\}$ , let  $N_m$  denote an arbitrary group satisfying

$$\begin{aligned} \text{pro}(N_m) &= p_m, \\ B_m &= \max_{\substack{t \in \text{supp}(p_m) \\ v \in \mathcal{V}^{\text{etp}}(N_m)}} v_t, \end{aligned}$$

and

$$B = \max_{m=1, \dots, M} B_m + \frac{2}{3} \varepsilon.$$

Clearly,  $B$  is well defined and finite. Let  $t \in \text{supp}(s)$ . From (2) it follows that  $t \in \text{supp}(p_m)$  for some  $m \in \mathcal{M}^*$ , which in view of PSTS and (3), implies that  $v_t \leq B$ . □

Recall that

$$G(\Pi) = \{(v, s) \in \Gamma \times (\Delta \cap \mathbb{Q}^T) \mid s \in \Pi(v)\}.$$

Obviously, given that the domain of  $G(\Pi)$  is  $\Gamma \times (\Delta \cap \mathbb{Q}^T)$ , there are some converging sequences  $\{(v^n, s^n)\}_n$  with each element in the sequence contained in the graph but the limits of the sequences are not. Given that having a closed graph is crucial to be able to use a fixed point argument, we get around this difficulty by constructing an auxiliary correspondence with a closed graph.

Define the correspondence  $\widetilde{\Pi} : \text{cl}(\Gamma) \rightrightarrows \Delta$  as follows: for each  $v \in \text{cl}(\Gamma)$

$$\widetilde{\Pi}(v) \stackrel{\text{def}}{=} \{s \in \Delta \mid \exists \{(v^n, s^n)\}_n \text{ in } G(\Pi) \text{ converging to } (v, s)\}.$$

Let  $G(\tilde{\Pi})$  denote the graph of the correspondence  $\tilde{\Pi}$ . The following proposition shows that  $G(\tilde{\Pi})$  is the closure of  $G(\Pi)$  with respect to  $\mathbb{R}^T \times \Delta$ .

**Proposition 3**  $G(\tilde{\Pi}) = \text{cl}(G(\Pi))$ .

*Proof of Proposition 3* Indeed,  $G(\tilde{\Pi}) = \text{cl}(G(\Pi))$  holds from the definition of  $\tilde{\Pi}$ . □

**Proposition 4** For each  $v \in \text{cl}(\Gamma)$  the set  $\tilde{\Pi}(v)$  is nonempty and convex.

*Proof of Proposition 4* Let  $v \in \text{cl}(\Gamma)$ . Then, there exists a sequence  $\{v^n\}_n \in \Gamma$  that converges to  $v$ . This implies that there exists a sequence  $\{(v^n, s^n)\}_n$  satisfying  $s^n \in \Delta$  and  $s^n \in \Pi(v^n)$ . Since  $\Delta$  is compact, passing to a subsequence if necessary, we may assume that  $\lim_{n \rightarrow +\infty} s^n = s \in \Delta$ . Hence,  $s \in \tilde{\Pi}(v)$ . Hence,  $\tilde{\Pi}(v)$  is nonempty.

Now, let  $s_1, s_2 \in \tilde{\Pi}(v)$  and  $\alpha \in [0, 1]$ . Then for each  $\varepsilon > 0$ , there exists two sequences  $\{(v_1^n, s_1^n)\}_n$  and  $\{(v_2^n, s_2^n)\}_n$  converging to, respectively,  $(v, s_1)$  and  $(v, s_2)$  such that  $s_1^n \in \Pi(v_1^n)$  and  $s_2^n \in \Pi(v_2^n)$ , for each  $n$ . This implies that  $v_1^n, v_2^n \in \Gamma$ . Then, there exists two groups  $N_\varepsilon^1$  and  $N_\varepsilon^2$  satisfying

$$\frac{\text{pro}(N_\varepsilon^1)}{|N_\varepsilon^1|} = s_1^n, \quad \frac{\text{pro}(N_\varepsilon^2)}{|N_\varepsilon^2|} = s_2^n, \quad (v_1^n - \varepsilon \mathbf{1}) \in \mathcal{V}^{\text{ep}}(N_\varepsilon^1), \quad \text{and} \quad (v_2^n - \varepsilon \mathbf{1}) \in \mathcal{V}^{\text{ep}}(N_\varepsilon^2).$$

Let  $v_{1,2}^n = (\inf\{(v_1^n)_1, (v_2^n)_1\}, \dots, \inf\{(v_1^n)_T, (v_2^n)_T\})$ . Note that  $v_{1,2}^n$  also converges to  $v$ . Moreover, given that  $\mathcal{V}^{\text{ep}}(N_\varepsilon^1)$  and  $\mathcal{V}^{\text{ep}}(N_\varepsilon^2)$  are comprehensive from below it holds that

$$(v_{1,2}^n - \varepsilon \mathbf{1}) \in \mathcal{V}^{\text{ep}}(N_\varepsilon^1) \cap \mathcal{V}^{\text{ep}}(N_\varepsilon^2). \tag{4}$$

Let  $\alpha^n = \frac{a^n}{b^n}$ , where  $a^n, b^n \in \mathbb{Z}_+$  and  $a^n < b^n$ , be a sequence of rationals converging to  $\alpha$ . Let  $\widehat{N}_\varepsilon^1, \widehat{N}_\varepsilon^2 \subset \mathcal{N}$  such that  $\widehat{N}_\varepsilon^1 \cap \widehat{N}_\varepsilon^2 = \emptyset$ ,

$$\text{pro}(\widehat{N}_\varepsilon^1) = a^n |N_\varepsilon^2| \text{pro}(N_\varepsilon^1), \quad \text{and} \quad \text{pro}(\widehat{N}_\varepsilon^2) = (b^n - a^n) |N_\varepsilon^1| \text{pro}(N_\varepsilon^2).$$

By SA and (4), it holds that

$$(v_{1,2}^n - \varepsilon \mathbf{1})_{\widehat{N}_\varepsilon^1 \cup \widehat{N}_\varepsilon^2} \in \left( \left( \mathcal{V}(\widehat{N}_\varepsilon^1) \times \mathbb{R}^{\widehat{N}_\varepsilon^2} \right) \cap \left( \mathbb{R}^{\widehat{N}_\varepsilon^1} \times \mathcal{V}(\widehat{N}_\varepsilon^2) \right) \right) \subset \mathcal{V}(\widehat{N}_\varepsilon^1 \cup \widehat{N}_\varepsilon^2),$$

which implies

$$(v_{1,2}^n - \varepsilon \mathbf{1}) \in \mathcal{V}^{\text{ep}}(\widehat{N}_\varepsilon^1 \cup \widehat{N}_\varepsilon^2).$$

Moreover, note that

$$\frac{\text{pro}(\widehat{N}_\varepsilon^1 \cup \widehat{N}_\varepsilon^2)}{|\widehat{N}_\varepsilon^1 \cup \widehat{N}_\varepsilon^2|} = \frac{a^n |N_\varepsilon^2| \text{pro}(N_\varepsilon^1) + (b^n - a^n) |N_\varepsilon^1| \text{pro}(N_\varepsilon^2)}{b^n |N_\varepsilon^1| |N_\varepsilon^2|} = \alpha^n s_1^n + (1 - \alpha^n) s_2^n.$$

Hence

$$(v_{1,2}^n, \alpha^n s_1^n + (1 - \alpha^n) s_2^n) \in G(\Pi),$$

which, by taking the limit, implies that

$$(v, \alpha s_1 + (1 - \alpha)s_2) \in G(\tilde{\Gamma}).$$

Thus,  $\tilde{\Gamma}(v)$  is convex. □

The set  $\text{cl}(\Gamma)$  is a nonempty, closed, and comprehensive from below subset of  $\mathbb{R}^T$ . Note that the set  $\text{cl}(\Gamma)$  is a proper set of  $\mathbb{R}^T$ . Define  $W$  as the set

$$W \stackrel{\text{def}}{=} \text{cl}(\Gamma) \cap [-\infty, B + 1]^T,$$

where  $B$  is defined in Proposition 2. A point  $v \in W$  belongs to the boundary of  $W$  if and only if either  $v \in \partial(\text{cl}(\Gamma))$  or has  $v_t = B + 1$  for some  $t = 1, \dots, T$ .

**Proposition 5** *There is a homeomorphism  $h$  from the space  $\Delta$  to the space  $\partial W \cap [-1, +\infty]^T$  such that  $h(s)_t = -1$  whenever  $s \in \Delta$  and  $t \in T \setminus \text{supp}(s)$ .*

*Proof of Proposition 5* Let  $s \in \Delta$  be given. Let  $R$  be the ray emanating from  $-\mathbf{1} = (-1, \dots, -1)$  in the direction of  $s$ . Thus, every point  $\rho$  of  $R$  is of the form  $\rho = -\mathbf{1} + \sigma s$  for some nonnegative real number  $\sigma$ . It is clear that, since  $W$  is closed, comprehensive from below, and bounded from above,  $R$  intersects the boundary of  $W$  at exactly one point.

To see that  $R$  does intersect  $\partial W$ , observe that  $-\mathbf{1}$  belongs to both the set  $W$  and the ray  $R$ . Thus, the set  $R_1 = R \cap W$  is nonempty. Furthermore, there is a  $\rho \in R$  such that  $\rho_t \geq B + 1$  for some  $t \in \text{supp}(s)$ , so that  $\rho$  lies outside the interior of  $W$ . Therefore, the set  $R_2 = R \cap (\mathbb{R}^T \setminus \text{int}W)$  is nonempty. Thus,  $R_1$  and  $R_2$  are nonempty closed subsets of  $R$  whose union is  $R$ . By connectedness of  $R$ , the set  $R_1 \cap R_2 = R \cap \partial W$  is nonempty.

To show that the intersection of  $R$  and  $W$  is a singleton, suppose that the set  $R \cap \partial W$  contains two distinct points  $\underline{v}$  and  $\bar{v}$ . Thus,  $\underline{v} = -\mathbf{1} + \underline{\sigma}s$  and  $\bar{v} = -\mathbf{1} + \bar{\sigma}s$  for some nonnegative reals  $\underline{\sigma}$  and  $\bar{\sigma}$ . Without loss of generality, we can assume that  $\underline{\sigma} < \bar{\sigma}$ . For each  $t \in \text{supp}(s)$ , we have  $\underline{v}_t < \bar{v}_t < B + 1$ . For each  $t \in T \setminus \text{supp}(s)$  (possibly empty), we then have  $\underline{v}_t = \bar{v}_t = -1$ . Note that since  $\bar{v} \in \text{cl}(\Gamma)$ , it is easy to check that

$$\hat{v}_t = \begin{cases} \bar{v}_t & \text{if } t \in \text{supp}(s) \\ 0 & \text{otherwise} \end{cases}$$

also belongs  $\text{cl}(\Gamma)$ . Since  $\underline{v}_t < \hat{v}_t$  for each  $t \in T$ , it follows that  $\underline{v}$  is in the interior of  $\text{cl}(\Gamma)$ . Moreover, since  $\underline{v}_t < B + 1$  for each  $t \in T$ , it follows from comprehensiveness that  $\underline{v}$  is in the interior of  $W$ , which is a contradiction.

Define the map  $h$  from  $\Delta$  to  $\partial W \cap [-1, +\infty]^T$  by letting  $h(s)$  be the unique point in the intersection of the ray  $R$  and the set  $\partial W$ . We now demonstrate that  $h$  has an inverse. Let  $g$  denote the map from  $\partial W \cap [-1, +\infty]^T$  to  $\Delta$  given by the equation

$$g(v) = \frac{v + \mathbf{1}}{\sum_{t \in T} (v_t + 1)}.$$

The map  $g$  is well defined since the point  $-\mathbf{1}$  lies in the interior of  $W$ . It is easy to see that  $g$  is indeed the inverse of  $h$ , that is,  $h \circ g$  and  $g \circ h$  are equal to the respective identity maps. Clearly,  $g$  is a continuous map. Furthermore, because its domain is compact and the codomain is Hausdorff, it carries closed sets to closed sets. Therefore,  $h$  is also a continuous map. This proves that  $h$  is a homeomorphism.  $\square$

The rest of the proof relies on the [Talman and Yang \(2009\)](#) version of the well-known Fan’s coincidence theorem, as stated below. That theorem addresses a domain consisting of a nonempty and convex subset  $Y$  of  $\mathbb{R}^N$ . For ease of translation of the theorem to our context, we restrict the domain to be the simplex in  $\mathbb{R}^T$ . Let  $N(\Delta, s) = \{z \in \mathbb{R}^N \mid (s - s')^\top z \geq 0 \text{ for each } s' \in \Delta\}$  denote the normal cone of the set  $\Delta$  at the point  $s$ . A zero point of a correspondence  $\Phi : \Delta \rightrightarrows \mathbb{R}^T$  is a point  $s$  of  $\Delta$  such that  $\Phi(s)$  contains the zero point.

For ease in reading, we state the following result using the simplex rather than an arbitrary compact convex set.

**Theorem** (Talman and Yang) *Let  $\Phi : \Delta \rightrightarrows \mathbb{R}^T$  be a correspondence with nonempty convex values having a compact graph. Suppose that for each  $s \in \Delta$  and for each  $z \in N(\Delta, s)$  there exists a  $\phi \in \Phi(s)$  such that  $z^\top \phi \leq 0$ . Then,  $\Phi$  has a zero point.*

**Proposition 6** *There exists  $v^* \in \partial(\text{cl}(\Gamma))$  such that  $s^* \in \tilde{\Pi}(v^*)$ .*

*Proof of Proposition 6* Define the correspondence  $\Phi : \Delta \rightrightarrows \mathbb{R}^T$  by letting  $\Phi(s) = \tilde{\Pi}(h(s)) - \{s^*\}$  for each  $s \in \Delta$ . Clearly, the correspondence  $\Phi$  has nonempty and convex values. Its graph is closed, because  $h$  is continuous and the graph of  $\tilde{\Pi}$  is closed. Since  $\Phi$  maps a compact set  $\Delta$  into a compact set  $\Delta - \{s^*\}$ , its graph is, in fact, a compact set.

We now need to verify that the conditions of the above-fixed point theorem are satisfied. Let  $s \in \Delta$  be given and let  $v$  denote the vector  $h(s)$ . Then, the normal cone of  $\Delta$  at  $s$  is the set

$$N(\Delta, s) = \left\{ z \in \mathbb{R}^T \mid z = a\mathbf{1} + \sum_{t \in T \setminus \text{supp}(s)} l_t \mathbf{e}_t, a \in \mathbb{R}, l_t \leq 0 \right\},$$

where  $(\mathbf{e}_1, \dots, \mathbf{e}_T)$  is the standard canonical basis of  $\mathbb{R}^T$ . Let  $z \in N(\Delta, s)$  be given. If  $\text{supp}(s) = T$ , then every  $z \in N(\Delta, s)$  is proportional to the vector  $\mathbf{1}$ . In this case, since  $\Phi(s) \subset \Delta - \{s^*\}$ , the equality  $z^\top \phi = 0$  holds for each  $\phi \in \Phi(s)$ . If  $T \setminus \text{supp}(s)$  is nonempty, then,  $v_t = h(s)_t = -1$ , for each  $t \in T \setminus \text{supp}(s)$ . Let  $\hat{s} \in \Delta$  defined as follows

$$\hat{s} = \begin{cases} \hat{s}_t = s_t^* + \frac{1}{|T \setminus \text{supp}(s)|} \sum_{j \in \text{supp}(s)} s_j^* & \text{if } t \in T \setminus \text{supp}(s), \\ 0 & \text{otherwise} \end{cases}$$

and  $\{(\hat{s}^n)\}_n$  be a sequence in  $\Delta \cap \mathbb{Q}^T$  such that  $\lim_{n \rightarrow +\infty} \hat{s}^n = \hat{s}$  and  $\hat{s}_t^n = 0$ , for each  $t \in \text{supp}(s)$ . Let  $N^n$  be a group satisfying

$$\frac{\text{pro}(N^n)}{|N^n|} = \hat{s}^n.$$



Since  $v_{N^n} = -\mathbf{1}_{N^n} \in \text{int}\mathcal{V}(N^n)$  it holds that  $\hat{s}^n \in \Pi(v)$  for each  $n$ , which implies  $\hat{s} \in \tilde{\Pi}(v) = \tilde{\Pi}(h(\hat{s}))$ . The vector  $\phi = \hat{s} - s^*$  is therefore an element of  $\Phi(s)$ . Since  $0 \leq \phi_t$  for each  $t \in T \setminus \text{supp}(s)$ , the inequality  $z^\top \phi \leq 0$  holds for each  $z \in N(\Delta, s)$ . By Fan’s coincidence theorem, the correspondence  $\Phi$  has a zero point, say  $\tilde{s}$ . Letting  $v^*$  be equal to  $h(\tilde{s})$ , we see that  $v^* \in \partial W$  and  $s^* \in \tilde{\Pi}(v^*)$ . Since  $s^* \in \text{int}\Delta$  it follows from (taking the limit in) Proposition 2 that  $v_t^* < B + 1$  for each  $t$  and thus  $v^* \in \partial(\text{cl}(\Gamma))$ . □

Finally, Proposition 6, together with (2) in Proposition 1, implies that for every  $\varepsilon > 0$  there exists  $r_\varepsilon$  such that for each  $n \geq r_\varepsilon$ ,  $(v^* - \varepsilon\mathbf{1})_{N^n}$  is in the  $\varepsilon$ -core of  $(N^n, V)$ . □

### 5 Some relationships to the literature

To relate our work to the literature, we begin with Wooders (1983). We note that the results of that paper have played a role in much subsequent work on cooperative games with many players (see, for example, Wooders (1994) and Kovalenkov and Wooders (2001, 2003) and on economies with clubs or local public goods (see, for example, Wooders 1997; Conley and Wooders 2001; Allouch and Wooders 2008).

Before proceeding we require another definition:

#### 5.1 Per capita boundedness (PCB)

The correspondence  $\mathcal{V}$  satisfies per capita boundedness if there is a constant  $K$  such that if  $v \in \mathcal{V}^{\text{cp}}(N)$  then  $v_t \leq K$  for each  $t$  such that  $(t, q) \in N$  for some  $q$ .

Note that PCB is less restrictive than SGE, but when there are many players of each type then the two concepts are closely related. Following is a restatement of the main theorem of Wooders (1983) using the notation of this paper and Lemmas in Wooders’s proof.

**Theorem (Wooders 1983)** *Assume  $\mathcal{V}$  satisfies SA, PSTS, PCB, and CONV. Let  $s \in \text{int}\Delta \cap \mathbb{Q}^T$  and let  $\{(N^n, V)\}_n$  be a sequence of games such that  $|N^n| \rightarrow \infty$  and, for each  $n$ ,*

$$\frac{\text{pro}(N^n)}{|N^n|} = s.$$

*Then, there exists  $v^* \in \mathbb{R}^T$  satisfying the property: for every  $\varepsilon > 0$  there is an integer  $r_\varepsilon$  such that for each  $n \geq r_\varepsilon$ ,  $(v^* - \varepsilon\mathbf{1})_{N^n}$  is in the  $\varepsilon$ -core of  $(N^n, V)$ .*

Note that  $s$  can be any vector in  $\text{int}\Delta \cap \mathbb{Q}^T$ . Thus, for sequences of games where all games in the sequence have the same percentage of players of each type, the main theorem of Wooders (1983) implies the result of this paper. The proof, however, is quite different. In some sense, Wooders (1983) approaches the limiting payoff  $v^*$  “from below” while the current paper starts with defining a limiting set for every possible player profile. Essentially, Wooders’s proof shows that the limit of the set of equal-treatment payoff vectors achievable by a sequence of games with a fixed distribution

of player types is the same as the limit of those equal-treatment games achievable by the associated sequence of balanced cover games. In contrast, we look directly at the limiting sets. We also show that the sets of equal-treatment payoffs vary continuously with as the distribution of player types varies. Our result also extends that of [Wooders \(1983\)](#) since she treats a fixed distribution of player types while we treat all sequences of growing games converging to the same limiting distribution of player types; this difference is reflected in the part of our proof using the projection theorem.

Our work in this paper also relates to [Wooders \(2008\)](#) which considers a compact metric space of player types and demonstrates nonemptiness of approximate cores for *all* sufficiently large (but finite) games derived from the structure. That paper uses somewhat different conditions than the current paper and has a different purpose—namely to relax the finite-type assumption, so we will not discuss the relationships in any detail.

## 6 Conclusions

We present an extension of the main result of [Wooders \(1983\)](#). In work in progress, we use our approach in this paper to relate the Aumann-core (with coalitions of positive measure) and the  $f$ -core of [Kaneko and Wooders \(1986\)](#) and [Kaneko and Wooders \(1996\)](#)<sup>11</sup> and show their equivalence for games with many players. We also establish the equivalence of core outcomes for other models allowing “large” players (or positive measure).

In conclusion, we note that the theory of cooperative games satisfying either per capita boundedness or small group effectiveness has found application in club theory and the theory of local public goods (c.f., [Allouch and Wooders \(2008\)](#) and references therein), in the study of market games ([Wooders 1994](#)), and recently, in security exchanges ([Faias and Luque 2016](#)). All these papers advance the notion that large cooperative games, with a compact metric space of player types, are market games. The general theories of approximate cores of games with many players can also be used to obtain a special case of a non-emptiness result due to [Gersbach et al. \(2015\)](#) on household formation. While we require convexity of payoff sets, if we had instead allowed a “left-over” set of players as in [Shubik and Wooders \(1983\)](#) or [Kovalenkova and Wooders \(2003\)](#), then a result analogous to that of this paper could be immediately applied to matching markets, as in the celebrated volume of [Roth and Sotomayer \(1990\)](#) and in multiple extensions of matching markets such as [Legros and Newman \(2002\)](#). Another application, in progress, uses SGE in the study of economies with incomplete information (see [Kamishiro et al. 2016](#)). In view of the multiplicity of applications already in the literature, we anticipate that our work in this paper will help in the development of yet further applications.

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<sup>11</sup> See also [Hammond et al. \(1989\)](#).

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## References

- Allouch, N., Florenzano, M.: Edgeworth and Walrasian equilibrium of an arbitrage-free exchange economy. *Econ. Theory* **23**, 353–371 (2004)
- Allouch, N., Predtetchinski, A.: On the non-emptiness of the fuzzy core. *Int. J. Game Theory* **37**, 203–210 (2008)
- Allouch, N., Wooders, M.: Price taking equilibrium in club economies with multiple memberships and unbounded club sizes. *J. Econ. Theory* **141**, 246–278 (2008)
- Aubin, J.P.: *Mathematical Methods of Games and Economic Theory*. North Holland, Amsterdam (1979)
- Aumann, R.J.: Markets with a continuum of traders. *Econometrica* **32**, 39–50 (1964)
- Bonnisseau, J.-M.: Existence of equilibria in economies with externalities and nonconvexities. *Set Valued Anal.* **5**, 209–226 (1997)
- Bonnisseau, J.-M., Cornet, B.: Existence of equilibria when firms follow bounded losses pricing rules. *J. Math. Econ.* **17**, 119–147 (1988)
- Bonnisseau, J.-M., Cornet, B.: General equilibrium with increasing returns the existence problem. In: Barnett, W.A., et al. (eds.) *Equilibrium Theory and Applications. Proceedings of Sixth International Symposium in Economic Theory Econometrics*, pp. 65–82. Cambridge University Press, Cambridge (1991)
- Bonnisseau, J.-M., Iehlé, V.: Payoff-dependent balancedness and cores. *Games Econ. Behav.* **61**, 1–26 (2007)
- Conley, J.P., Wooders, M.: Tiebout economies with differential genetic types and endogenously chosen crowding characteristics. *J. Econ. Theory* **98**(261), 294 (2001)
- Debreu, G., Scarf, H.: A limit theorem on the core of an economy. *Int. Econ. Rev.* **4**, 235–246 (1963)
- Faias, M., Luque, J.: Endogenous formation of security exchanges. *Econ. Theory* (2016). doi:[10.1007/s00199-016-0989-9](https://doi.org/10.1007/s00199-016-0989-9)
- Florenzano, M.: Edgeworth equilibria, fuzzy core and equilibria of a production economy without ordered preferences. *J. Math. Anal. Appl.* **153**, 18–36 (1990)
- Gersbach, H., Haller, H., Konishi, H.: Household formation and equilibrium. *Econ. Theory* **59**, 461 (2015). doi:[10.1007/s00199-015-0859-x](https://doi.org/10.1007/s00199-015-0859-x)
- Hammond, P., Kaneko, M., Wooders, M.: Continuum economies with finite coalitions: core, equilibria, and widespread externalities. *J. Econ. Theory* **49**, 113–134 (1989)
- Kamishiro, Y., Serrano, R., Wooders, M.: Information and small group effectiveness in large quasilinear economies. SSRN (2016). [https://papers.ssrn.com/sol3/papers.cfm?abstract\\_id=2705542](https://papers.ssrn.com/sol3/papers.cfm?abstract_id=2705542). Accessed 20 Oct 2016
- Kaneko, M., Wooders, M.: The core of a game with a continuum of players and finite coalitions: the model and some results. *Math. Soc. Sci.* **12**, 105–137 (1986)
- Kaneko, M., Wooders, M.: The nonemptiness of the f-core of a game without side payments. *Int. J. Game Theory* **25**, 245–258 (1996)
- Kovalenkov, A., Wooders, M.: Epsilon cores of games with limited side payments: nonemptiness and equal treatment. *Games Econ. Behav.* **36**, 193–218 (2001)
- Kovalenkov, A., Wooders, M.: Approximate cores of games and economies with clubs. *J. Econ. Theory* **110**, 87–120 (2003)
- Legros, P., Newman, A.: Monotone matching in perfect and imperfect worlds. *Rev Econ Stud* **69**(4), 925–942 (2002)
- Predtetchinski, A., Herings, P.J.J.: A necessary and sufficient condition for the non-emptiness of the core of a non-transferable utility game. *J. Econ. Theory* **116**, 84–92 (2004)
- Predtetchinski, A.: The fuzzy core and the  $(\Pi, \beta)$ -balanced core. *Econ. Theory* **26**, 717–724 (2005)
- Roth, A., Sotomayer, M.: *Two-sided Matching*. Cambridge University Press, Cambridge, A Study in Game-theoretic Modeling and Analysis (1990)
- Roth, A.E.: The economist as engineer: game theory, experimentation, and computation as tools for design economics. *Econometrica* **70**, 1341–1378 (2002)
- Scarf, H.: The Core of an n-person Game. *Econometrica* **35**, 50–69 (1967)

- Shubik, M., Wooders, M.H.: Approximate cores of replica games and economies: Part II. set-up costs and firm formation in coalition production economies. *Math. Soc. Sci.* **6**, 285–306 (1983)
- Talman, A.J.J., Yang, Z.: A constructive proof of Ky Fan's coincidence theorem. *Math. Program.* **118**, 317–325 (2009)
- Wooders, M.: The epsilon core of a large replica game. *J. Math. Econ.* **11**, 277–300 (1983)
- Wooders, M.: Inessentiality of large groups and the approximate core property: an equivalence theorem. *Econ. Theory* **2**, 129–147 (1992)
- Wooders, M.: Equivalence of games and markets. *Econometrica* **62**, 1141–1160 (1994)
- Wooders, M.: Equivalence of Lindahl Equilibria with participation prices and the core. *Econ. Theory* **9**, 115–128 (1997)
- Wooders, M.: Small group effectiveness, per capita boundedness and nonemptiness of approximate cores. *J. Math. Econ.* **44**, 888–906 (2008)