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Clarkson, Peter and Law, Chun-Kong and Lin, Chia-Hua (2016) An algebraic proof for the Umemura polynomials for the third Painlevé equation. Technical report. arxiv.org

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An algebraic proof for the Umemura polynomials for the third Painlevé equation

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September 5, 2016

arXiv:1609.00495v1 [math.CA] 2 Sep 2016

Abstract

We are concerned with the Umemura polynomials associated with the third Painlevé equation. We extend Taneda's method, which was developed for the Yablonskii–Vorob'ev polynomials associated with the second Painlevé equation, to give an algebraic proof that the rational functions generated by the nonlinear recurrence relation satisfied by Umemura polynomials are indeed polynomials.

1 Introduction

The third Painlevé equation ($\mathrm{P_{III}}$)

$$\frac{d^2w}{dz^2} = \frac{1}{w}\left(\frac{dw}{dz}\right)^2 - \frac{1}{z}\frac{dw}{dz} + \frac{\alpha w^2 + \beta}{z} + \gamma w^3 + \frac{\delta}{w}, \tag{1.1}$$

where $' = d/dz$ and α, β, γ and δ are arbitrary parameters. We are concerned with the generic case when $\gamma\delta \neq 0$, so we set $\gamma = 1$ and $\delta = -1$, without loss of generality (by rescaling w and z if necessary), and so consider

$$\frac{d^2w}{dz^2} = \frac{1}{w}\left(\frac{dw}{dz}\right)^2 - \frac{1}{z}\frac{dw}{dz} + \frac{\alpha w^2 + \beta}{z} + w^3 - \frac{1}{w}. \tag{1.2}$$

The six Painlevé equations ($\mathrm{P_I}$–$\mathrm{P_{VI}}$), were discovered by Painlevé, Gambier and their colleagues whilst studying second order ordinary differential equations of the form

$$\frac{d^2w}{dz^2} = F\left(z, w, \frac{dw}{dz}\right), \tag{1.3}$$

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AMS Subject Classification (2010) : 34A55, 34B24, 47A75.

Keywords: Umemura polynomials, third Painlevé equation, recurrence relation

where F is rational in dw/dz and w and analytic in z. The Painlevé equations can be thought of as nonlinear analogues of the classical special functions. Indeed Iwasaki, Kimura, Shimomura and Yoshida [35] characterize the six Painlevé equations as "the most important nonlinear ordinary differential equations" and state that "many specialists believe that during the twenty-first century the Painlevé functions will become new members of the community of special functions". Subsequently this has happened as the Painlevé equations are a chapter in the NIST *Digital Library of Mathematical Functions* [60, §32].

The general solutions of the Painlevé equations are transcendental in the sense that they cannot be expressed in terms of known elementary functions and so require the introduction of a new transcendental function to describe their solution. However, it is well known that $\mathrm{P_{II}}$–$\mathrm{P_{VI}}$ possess rational solutions and solutions expressed in terms of the classical special functions — Airy, Bessel, parabolic cylinder, Kummer and hypergeometric functions, respectively — for special values of the parameters, see, e.g. [15, 23, 33] and the references therein. These hierarchies are usually generated from "seed solutions" using the associated Bäcklund transformations and frequently can be expressed in the form of determinants.

Vorob'ev [73] and Yablonskii [75] expressed the rational solutions of $\mathrm{P_{II}}$

$$\frac{d^2w}{dz^2} = 2w^3 + zw + \alpha, \tag{1.4}$$

with α an arbitrary constant, in terms of special polynomials, now known as the *Yablonskii–Vorob'ev polynomials*, which are defined through the recurrence relation (a second-order, bilinear differential-difference equation)

$$Q_{n+1}Q_{n-1} = zQ_n^2 - 4\left[Q_n\frac{d^2Q_n}{dz^2} - \left(\frac{dQ_n}{dz}\right)^2\right], \tag{1.5}$$

with $Q_0(z) = 1$ and $Q_1(z) = z$. It is clear from the recurrence relation (1.5) that the Q_n are rational functions, though it is not obvious that they are polynomials since one is dividing by Q_{n-1} at every iteration. In fact it is somewhat remarkable that the Q_n are polynomials. Taneda [68], see also [26], used an algebraic method to prove that the functions Q_n defined by (1.5) are indeed polynomials. Clarkson and Mansfield [20] investigated the locations of the roots of the Yablonskii–Vorob'ev polynomials in the complex plane and showed that these roots have a very regular, approximately triangular structure; the term "approximate" is used since the patterns are not exact triangles as the roots lie on arcs rather than straight lines. An earlier study of the distribution of the roots of the Yablonskii–Vorob'ev polynomials is given in [41]; see also [35, p. 255, 339]. Recently Bertola and Bothner [3] and Buckingham and Miller [8, 9] have studied the Yablonskii–Vorob'ev polynomials $Q_n(z)$ in the limit as $n \to \infty$ and shown that the roots lie in a "triangular region" with elliptic sides which meet with interior angle $\frac{2}{5}\pi$, suggesting a limit to a solution of $\mathrm{P_I}$. Indeed Buckingham & Miller [8, 9] show that in the limit as $n \to \infty$, the rational solution of $\mathrm{P_{II}}$ tends to the *tritronquée solution* of $\mathrm{P_I}$.

Okamoto [57] obtained special polynomials, analogous to the Yablonskii–Vorob'ev polynomials, which are associated with some of the rational solutions of $\mathrm{P_{IV}}$. Noumi and Yamada [54] generalized Okamoto's results and expressed all rational solutions of $\mathrm{P_{IV}}$ in terms of special polynomials, now known as the *generalized Hermite polynomials* and *generalized Okamoto polynomials*. The structure of the roots of these polynomials is studied in [12], where it is shown that the roots of the generalized Hermite polynomials have an approximate rectangular structure and the roots of the generalized Okamoto polynomials have a combination of approximate rectangular and triangular structures. As for P_{II}, "approximate" is used since the patterns are not exact triangles and rectangles as the roots lie on arcs rather than straight lines.

Umemura [71] derived special polynomials with certain rational and algebraic solutions of P_{III}, P_V and P_{VI} which have similar properties to the Yablonskii–Vorob'ev polynomials and the Okamoto polynomials. Recently there have been further studies of the special polynomials associated with rational solutions of P_{II} [3, 8, 9, 26, 37, 39, 42, 68], rational and algebraic solutions of P_{III} [1, 11, 38, 51, 52, 59, 56, 72], rational solutions of P_{IV} [14, 22, 26, 54, 40, 55], rational and algebraic solutions of P_V [13, 49, 53, 58, 74], algebraic solutions of P_{VI} [43, 48, 67, 69]; a comprehensive review is given in [16]. Several of these papers are concerned with the combinatorial structure and determinant representation of the polynomials, often related to the Hamiltonian structure and affine Weyl symmetries of the Painlevé equations. Additionally the coefficients of these special polynomials have some interesting, indeed somewhat mysterious, combinatorial properties [70, 71].

Further these special polynomials associated with rational solutions of the Painlevé equations arise in several applications:

- the Yablonskii–Vorob'ev polynomials arise in the transition behaviour for the semi-classical sine-Gordon equation [7] and in moving boundary problems [65, 66];
- the Umemura polynomials associated with rational solutions of P_{III} and P_V arise as multivortex solutions of the complex sine-Gordon equation [2, 4, 5, 61];
- the generalized Hermite polynomials associated with rational solutions of P_{IV} arise as multiple integrals in random matrix theory [6, 24], in the description of vortex dynamics with quadrupole background flow [18], and as coefficients of recurrence relations for semi-classical orthogonal polynomials [10, 19];
- the generalized Okamoto polynomials associated with rational solutions of P_{IV} generate previously unknown rational-oscillatory solutions of the de-focusing nonlinear Schrödinger equation [17];
- these special polynomials associated with rational solutions of Painlevé equations are examples of exceptional orthogonal polynomials [27, 28, 29, 30, 31, 32, 45, 46, 47], for which there is much current interest.

We emphasize that the fact that the nonlinear recurrence relation (1.5) generates polynomials also follows from the τ-function theory associated with the theory of Painlevé equations. The τ-functions are in general entire functions. It can be shown that for P_{II} with $\alpha = m$, the associated τ-function is

$$\tau_m(z) = Q_m(z) \exp\left(-\frac{z^3}{24}\right),$$

Consequently the rational function $Q_m(z)$ has to be a polynomial. Taneda [68] and Fukutani, Okamoto and Umemura [26] independently gave a direct algebraic proof, which is one of the first studies of nonlinear recurrence relations for polynomials. In particular, Taneda [68] defined a Hirota-like operator

$$\ell(f) = f\frac{d^2 f}{dz^2} - \left(\frac{df}{dz}\right)^2,$$

and showed that if $f(z)$ is a polynomial in z, and $g = zf^2 - 4\,\ell(f)$, then f divides $2z\,g^2 - 4\,\ell(g)$. Hence if $f(z) = Q_{m-1}(z)$, then $g(z) = Q_m(z)Q_{m-2}(z)$ and

$$2z\,g^2 - 4\,\ell(g) = Q_m^2 Q_{m-3} Q_{m-1} + Q_{m-2}^2 (zQ_m^2 - 4\,\ell(Q_m)),$$

so that Q_m divides $zQ_m^2 - 4\,\ell(Q_m)$, implying that Q_{m+1} is a polynomial. This is based on the assumption that each Q_m has simple zeros (implying that Q_m and Q_{m-1} have no common zeros), which in turn can be proved using another identity derived from $\mathrm{P_{II}}$,

$$\frac{dQ_{m+1}}{dz} Q_{m-1} - Q_{m+1} \frac{dQ_{m-1}}{dz} = (2m+1)Q_m^2, \tag{1.6}$$

which is proved in [26, 68] (see also [42]).

Now for $\mathrm{P_{III}}$ (1.2), the recurrence relation becomes

$$S_{n+1}S_{n-1} = -z\left[S_n \frac{d^2 S_n}{dz^2} - \left(\frac{dS_n}{dz}\right)^2\right] - S_n \frac{dS_n}{dz} + (z+\mu)S_n^2, \tag{1.7}$$

where μ is a complex parameter. Here there is one more term $S_n \dfrac{dS_n}{dz}$, and z in the main term implies that the root $z = 0$ of S_n, if exists, will accumulate. We continue to employ Taneda's trick, by defining another Hirota-like operator

$$\mathcal{L}_z(f) = f\frac{d^2 f}{dz^2} - \left(\frac{df}{dz}\right)^2 - \frac{f}{z}\frac{df}{dz}.$$

Also we need another identity. We find that it is suitable to use the fourth order differential equation in S_n (4.6) found in [11]. This higher order equation in fact comes from the σ-equation equivalent to $\mathrm{P_{III}}$

$$\left(z\frac{d^2\sigma}{dz^2} - \frac{d\sigma}{dz}\right)^2 + 4\left(\frac{d\sigma}{dz}\right)^2\left(z\frac{d\sigma}{dz} - 2\sigma\right) + 4z\lambda_1\frac{d\sigma}{dz} - z^2\left(z\frac{d\sigma}{dz} - 2\sigma + 2\lambda_0\right) = 0, \tag{1.8}$$

with $\lambda_0 = \frac{1}{8}(\alpha^2 + (\beta-2)^2)$ and $\lambda_1 = -\frac{1}{4}\alpha(\beta-2)$; or equivalently

$$z^2\frac{d^3\sigma}{dz^3} - z\frac{d^2\sigma}{dz^2} + 6z\left(\frac{d\sigma}{dz}\right)^2 - 8\sigma\frac{d\sigma}{dz} + \frac{d\sigma}{dz} - \tfrac{1}{2}z^3 - 2z\lambda_1 = 0, \tag{1.9}$$

which is obtained by differentiating (1.8). The higher order equation

$$\begin{aligned} z^2\left[S_n\frac{d^4 S_n}{dz^4} - 4\frac{dS_n}{dz}\frac{d^3 S_n}{dz^3} + 3\left(\frac{d^2 S_n}{dz^2}\right)^2\right] + 2z\left(S_n\frac{d^3 S_n}{dz^3} - \frac{dS_n}{dz}\frac{d^2 S_n}{dz^2}\right) \\ - 4z(z+\mu)\left[S_n\frac{d^2 S_n}{dz^2} - \left(\frac{dS_n}{dz}\right)^2\right] - 2S_n\frac{d^2 S_n}{dz^2} + 4\mu S_n\frac{dS_n}{dz} = 2n(n+1)S_n^2, \end{aligned}$$

which is derived from (1.9), is also instrumental in the analysis of the case when $z = 0$ is a root of S_n (see §4 below).

In §2 we describe rational solutions of equation (1.2). In §§3 and 4 we extend Taneda's algebraic proof for equation (1.5) to equation (2.3) and in §5 we discuss our results.

2 Rational solutions of $\mathrm{P_{III}}$

The location of rational solutions of equation (1.2), which is $\mathrm{P_{III}}$ with $\gamma = 1$ and $\delta = -1$, are given in the following theorem.

Theorem 2.1. *Equation* (1.2) *has a rational solution if and only if* $\alpha \pm \beta = 4n$, $n \in \mathbb{Z}$.

Proof. See Gromak, Laine and Shimomura [33, p. 174]; also [51, 52]. □

Umemura [71] derived special polynomials associated with rational solutions of $\mathrm{P_{III}}$ (1.2), which are defined in Theorem 2.2, and states that these polynomials are the analogues of the Yablonskii–Vorob'ev polynomials associated with rational solutions of $\mathrm{P_{II}}$ [73, 75] and the Okamoto polynomials associated with rational solutions of $\mathrm{P_{IV}}$ [57].

Theorem 2.2. *Suppose that* $T_n(z;\mu)$ *satisfies the recurrence relation*

$$zT_{n+1}T_{n-1} = -z\left[T_n\frac{d^2T_n}{dz^2} - \left(\frac{dT_n}{dz}\right)^2\right] - T_n\frac{dT_n}{dz} + (z+\mu)T_n^2, \tag{2.1}$$

with $T_{-1}(z;\mu) = 1$ *and* $T_0(z;\mu) = 1$. *Then*

$$w_n(z;\mu) \equiv w(z;\alpha_n,\beta_n) = \frac{T_n(z;\mu-1)\,T_{n-1}(z;\mu)}{T_n(z;\mu)\,T_{n-1}(z;\mu-1)} \equiv 1 + \frac{d}{dz}\ln\frac{T_n(z;\mu-1)}{z^n\,T_n(z;\mu)}, \tag{2.2}$$

satisfies $\mathrm{P_{III}}$ (1.2), *with* $\alpha_n = 2n+2\mu-1$ *and* $\beta_n = 2n-2\mu+1$.

Proof. See Umemura [71]; also [11, 38]. □

The "polynomials" $T_n(z;\mu)$ are rather unsatisfactory since they are actually polynomials in $\xi = 1/z$ rather than polynomials in z, which would be more natural. However it is straightforward to determine a recurrence relation which generates functions $S_n(z;\mu)$ which are polynomials in z. These are given in the following theorem which generalizes the work of Kajiwara and Masuda [38].

Theorem 2.3. *Suppose that* $S_n(z;\mu)$ *satisfies the recurrence relation*

$$S_{n+1}S_{n-1} = -z\left[S_n\frac{d^2S_n}{dz^2} - \left(\frac{dS_n}{dz}\right)^2\right] - S_n\frac{dS_n}{dz} + (z+\mu)S_n^2, \tag{2.3}$$

with $S_{-1}(z;\mu) = S_0(z;\mu) = 1$. *Then*

$$w_n = w(z;\alpha_n,\beta_n) = \frac{S_n(z;\mu-1)\,S_{n-1}(z;\mu)}{S_n(z;\mu)\,S_{n-1}(z;\mu-1)} \equiv 1 + \frac{d}{dz}\ln\frac{S_{n-1}(z;\mu-1)}{S_n(z;\mu)}, \tag{2.4}$$

satisfies $\mathrm{P_{III}}$ (1.2) *with* $\alpha_n = 2n+2\mu-1$ *and* $\beta_n = 2n-2\mu+1$.

Proof. See Clarkson [11] and Kajiwara [36]. □

Remarks 2.4.

1. The polynomials $S_n(z;\mu)$ and $T_n(z;\mu)$, defined by (2.3) and (2.1), respectively, are related through $S_n(z;\mu) = z^{n(n+1)/2}T_n(z;\mu)$. The polynomials $S_n(z;\mu)$ also have the symmetry property $S_n(z;\mu) = S_n(-z;-\mu)$.

2. The rational solutions of $\mathrm{P_{III}}$ (1.2) lie on the lines $\alpha+\epsilon\beta=4n$, with $\epsilon=\pm1$, in the α-β plane. Further, letting $w(z)=u(\zeta)/\sqrt{\zeta}$, with $\zeta=\frac{1}{4}z^2$, in $\mathrm{P_{III}}$ (1.2) yields

$$\frac{d^2u}{d\zeta^2}=\frac{1}{u}\left(\frac{du}{d\zeta}\right)^2-\frac{1}{\zeta}\frac{du}{d\zeta}+\frac{\alpha u^2}{2\zeta^2}+\frac{\beta}{2\zeta}+\frac{u^3}{\zeta^2}-\frac{1}{u}, \tag{2.5}$$

which is known as $\mathrm{P_{III'}}$ (cf. Okamoto [59]) and is frequently used to determine properties of solutions of $\mathrm{P_{III}}$. However (2.5) has algebraic solutions rather than rational solutions [51, 52].

3. For any $n\in\mathbb{N}\cup\{0\}$, if $\alpha_n=2n+2\mu-1$ and $\beta_n=2n-2\mu+1$, with $\mu\in\mathbb{C}$, then $\alpha_n+\beta_n=4n$.

Kajiwara and Masuda [38] derived representations of rational solutions for $\mathrm{P_{III}}$ (1.2) in the form of determinants, which are described in the following theorem.

Theorem 2.5. *Let $p_k(z;\mu)$ be the polynomial defined by*

$$\sum_{j=0}^{\infty}p_j(z;\mu)\lambda^j=(1+\lambda)^{\mu}\exp\left(z\lambda\right), \tag{2.6}$$

with $p_j(z;\mu)=0$ for $j<0$, and $\tau_n(z)$, for $n\geq1$, be the $n\times n$ determinant

$$\tau_n(z;\mu)=\mathcal{W}\left(p_1(z;\mu),p_3(z;\mu),\ldots,p_{2n-1}(z;\mu)\right), \tag{2.7}$$

where $\mathcal{W}(\phi_1,\phi_2,\ldots,\phi_n)$ is the Wronskian. Then

$$w_n=w(z;\alpha_n,\beta_n,1,-1)=1+\frac{d}{dz}\ln\frac{\tau_{n-1}(z;\mu-1)}{\tau_n(z;\mu)}, \tag{2.8}$$

for $n\geq1$, satisfies $\mathrm{P_{III}}$ (1.2) *with $\alpha_n=2n+2\mu-1$ and $\beta_n=2n-2\mu+1$.*

Proof. See Kajiwara and Masuda [38]. □

Remarks 2.6.

1. We note that $p_k(z;\mu)=L_k^{(\mu-k)}(-z)$, where $L_k^{(m)}(\zeta)$ is the *associated Laguerre polynomial*, cf. [60, §18].

2. The relationship between the polynomial $S_n(z,\mu)$ and the Wronskian $\tau_n(z;\mu)$ is

$$S_n(z,\mu)=c_n\tau_n(z;\mu),\qquad c_n=\prod_{j=1}^{n}(2j+1)^{n-j}.$$

3. In the special case when $\mu=1$, then

$$S_n(z,1)=z^{n(n-1)/2}\theta_n(z),$$

where $\theta_n(z)$ is the *Bessel polynomial* (sometimes known as the *reverse Bessel polynomial*, cf. [34]) given by

$$\theta_n(z)=\sqrt{\frac{2}{\pi}}\,z^{n+1/2}e^zK_{n+1/2}(z)\equiv\frac{n!}{(-2)^n}L_n^{(-2n-1)}(2x),$$

with $K_\nu(z)$ the *modified Bessel function*, which recently arose in the description of point vortex equilibria [62].

$$
\begin{aligned}
S_1(z;\mu) &= z+\mu,\\
S_2(z;\mu) &= \xi^3-\mu,\\
S_3(z;\mu) &= \xi^6-5\mu\xi^3+9\mu\xi-5\mu^2,\\
S_4(z;\mu) &= \xi^{10}-15\mu\xi^7+63\mu\xi^5-225\mu\xi^3+315\mu^2\xi^2-175\mu^3\xi+36\mu^2,\\
S_5(z;\mu) &= \xi^{15}-35\mu\xi^{12}+252\mu\xi^{10}+175\mu^2\xi^9-2025\mu\xi^8+945\mu^2\xi^7\\
&\qquad -1225\mu(\mu^2-9)\xi^6-26082\mu^2\xi^5+33075\mu^3\xi^4-350\mu^2(35\mu^2+36)\xi^3\\
&\qquad +11340\mu^3\xi^2-225\mu^2(49\mu^2-36)\xi+7\mu^3(875\mu^2-828).
\end{aligned}
$$

Table 2.1: The first few Umemura polynomials $S_n(z;\mu)$, with $\xi = z+\mu$.

$$
\begin{aligned}
\mathrm{Dis}(S_2(z;\mu)) &= -3^3\mu^2,\\
\mathrm{Dis}(S_3(z;\mu)) &= 3^{12}5^5\mu^6(\mu^2-1)^2,\\
\mathrm{Dis}(S_4(z;\mu)) &= 3^{27}5^{20}7^7\mu^{14}(\mu^2-1)^6(\mu^2-4)^2,\\
\mathrm{Dis}(S_5(z;\mu)) &= 3^{66}5^{45}7^{28}\mu^{26}(\mu^2-1)^{14}(\mu^2-4)^6(\mu^2-9)^2,\\
\mathrm{Dis}(S_6(z;\mu)) &= -3^{147}5^{80}7^{63}11^{11}\mu^{44}(\mu^2-1)^{26}(\mu^2-4)^{14}(\mu^2-9)^6(\mu^2-16)^2.
\end{aligned}
$$

Table 2.2: The discriminants of the Umemura polynomials $S_n(z;\mu)$.

The recurrence relation (2.3) is nonlinear, so in general, there is no guarantee that the rational function S_{n+1} thus derived is indeed a polynomial (since one is dividing by S_{n-1}), as was the case for the recurrence relation (1.5). However, the Painlevé theory guarantees that this is the case through an analysis of the τ-function. A few of these Umemura polynomials $S_n(z;\mu)$, with μ an arbitrary complex parameter, are given in Table 2.1.

Plots of the roots of the polynomials $S_n(z;\mu)$ for various μ are given in [11]. Initially for μ sufficiently large and negative, the $\frac{1}{2}n(n+1)$ roots form an approximate triangle with n roots on each side. Then as μ increases, the roots in turn coalesce and eventually for μ sufficiently large and positive they form another approximate triangle, similar to the original triangle, though with its orientation reversed. It is straightforward to determine when the roots of $S_n(z;\mu)$ coalesce using discriminants of polynomials. Suppose that

$$f(z) = z^m + a_{m-1}z^{m-1} + \ldots + a_1 z + a_0,$$

is a monic polynomial of degree m with roots $\alpha_1, \alpha_2, \ldots, \alpha_m$, so $f(z) = \prod_{j=1}^m (z-\alpha_j)$. Then the *discriminant* of $f(z)$ is

$$\mathrm{Dis}(f) = \prod_{1\le j<k\le m} (\alpha_j-\alpha_k)^2.$$

Hence the polynomial f has a multiple root when $\mathrm{Dis}(f)=0$.

The discriminants of the first few Umemura polynomials $S_n(z;\mu)$ are given in Table 2.2. Thus $S_2(z;\mu)$ has multiple roots when $\mu=0$, $S_3(z;\mu)$ has multiple roots when $\mu = 0, \pm1$, $S_4(z;\mu)$ has multiple roots when $\mu = 0,\pm1,\pm2$, $S_5(z;\mu)$ has multiple roots when $\mu = 0,\pm1,\pm2,\pm3$, and $S_6(z;\mu)$ has multiple roots when $\mu = 0,\pm1,\pm2,\pm3,\pm4$. Further the multiple roots occur at $z=0$. This leads to the following theorem.

Theorem 2.7. *The discriminant of the polynomial $S_n(z;\mu)$ is given by*

$$|\mathrm{Dis}(S_n)| = \prod_{j=1}^{n}(2j+1)^{(2j+1)(n-j)^2} \prod_{k=-n}^{n}(\mu-k)^{c_{n-|k|}},$$

where $c_n = \frac{1}{6}n^3 + \frac{1}{4}n^2 - \frac{1}{6}n - \frac{1}{8}[1-(-1)^n]$ and $\mathrm{Dis}(S_n) < 0$ if and only $n = 2 \bmod 4$. Further the polynomial $S_n(z;\mu)$ has multiple roots at $z = 0$ when $\mu = 0, \pm1, \pm2, \ldots, \pm(n-2)$.

Proof. See Amdeberhan [1]. □

Using the Hamiltonian formalism for $\mathrm{P_{III}}$, it is shown in [11] that the polynomials $S_n(z;\mu)$ satisfy an fourth order bilinear equation and a sixth order, hexa-linear (homogeneous of degree six) difference equation.

3 Application of Taneda's method

In this section, we use the algebraic method due to Taneda [68] to prove that the rational functions S_n satisfying (2.3) are indeed polynomials, assuming all the functions S_n's have simple zeros.

We define an operator $\mathcal{L}_z$ as follows

$$\mathcal{L}_z(f) = f\frac{d^2f}{dz^2} - \left(\frac{df}{dz}\right)^2 + \frac{f}{z}\frac{df}{dz}. \tag{3.1}$$

Lemma 3.1. *Let $f(z)$ and $g(z)$ be arbitrary polynomials. Then*

(a) $\mathcal{L}_z(kf) = k^2\,\mathcal{L}_z(f)$, with k a constant;

(b) $\mathcal{L}_z(f\,g) = f^2\mathcal{L}_z(g) + g^2\mathcal{L}_z(f)$;

(c) If $h = -z\mathcal{L}_z(f) + k(z+\mu)f^2$, with k and μ constants, then

$$f \mid z\mathcal{L}_z(h) - 2k(z+\mu)h^2, \tag{3.2}$$

where the symbol $\mid$ means that the right hand side is divisible by the left hand side.

Proof.

(a) This follows directly from the definition.

(b) We observe that

$$\begin{aligned}\mathcal{L}_z(fg) &= fg\frac{d^2}{dz^2}(fg) - \left[\frac{d}{dz}(fg)\right]^2 + \frac{fg}{z}\frac{d}{dz}(fg)\\ &= f^2\left[g\frac{d^2g}{dz^2} - \left(\frac{dg}{dz}\right)^2 + \frac{g}{z}\frac{dg}{dz}\right] + g^2\left[f\frac{d^2f}{dz^2} - \left(\frac{df}{dz}\right)^2 + \frac{f}{z}\frac{df}{dz}\right]\\ &= f^2\mathcal{L}_z(g) + g^2\mathcal{L}_z(f),\end{aligned}$$

so the result is valid.

(c) Finally, by definition

$$
\begin{aligned}
h &= -z\left[f\frac{d^2f}{dz^2} - \left(\frac{df}{dz}\right)^2 + \frac{f}{z}\frac{df}{dz}\right] + k(z+\mu)f^2 \\
&= z\left(\frac{df}{dz}\right)^2 + f \times (a\ polynomial) \\
\frac{dh}{dz} &= -2f\frac{d^2f}{dz^2} - z\left(f\frac{d^3f}{dz^3} - \frac{df}{dz}\frac{d^2f}{dz^2}\right) + kf^2 + 2k(z+\mu)f\frac{df}{dz} \\
&= z\frac{df}{dz}\frac{d^2f}{dz^2} + f \times (a\ polynomial) \\
\frac{d^2h}{dz^2} &= -\frac{df}{dz}\frac{d^2f}{dz^2} - 3f\frac{d^3f}{dz^3} - z\left[f\frac{d^4f}{dz^4} - \left(\frac{d^2f}{dz^2}\right)^2\right] + 4kf\frac{df}{dz} + 2k(z+\mu)\left[f\frac{d^2f}{dz^2} + \left(\frac{df}{dz}\right)^2\right] \\
&= -\frac{df}{dz}\frac{d^2f}{dz^2} + z\left(\frac{d^2f}{dz^2}\right)^2 + 2k(z+\mu)\left(\frac{df}{dz}\right)^2 + f \times (a\ polynomial).
\end{aligned}
$$

Then we can see

$$
\begin{aligned}
\mathcal{L}_z(h) &= h\frac{d^2h}{dz^2} - \left(\frac{dh}{dz}\right)^2 + \frac{h}{z}\frac{dh}{dz} \\
&= 2z(z+\mu)\left(\frac{df}{dz}\right)^4 - f\frac{df}{dz}\left(\frac{d^2f}{dz^2}\right)^2 + \frac{2k(z+\mu)f}{z}\left(\frac{df}{dz}\right)^3 + f \times (a\ polynomial) \\
&= 2z(z+\mu)\left(\frac{df}{dz}\right)^4 + \frac{2k\mu f}{z}\left(\frac{df}{dz}\right)^3 + f \times (a\ polynomial).
\end{aligned}
$$

Since $z\mathcal{L}_z(h) - 2k(z+\mu)h^2 = f \times (a\ polynomial)$, then

$$f \mid z\mathcal{L}_z(h) - 2k(z+\mu)h^2, \tag{3.3}$$

as required.

□

Theorem 3.2. *Suppose $\{S_n\}$ is a sequence of rational functions with simple nonzero roots, satisfying* (2.3)*. For all $N \in \mathbb{N} \cup \{0\}$, if $z = 0$ is **not** a root of any $S_n(z)$ for $0 \le n \le N$, then:*

(a) $S_{N+1}(z)$ is a polynomial of degree $\frac{1}{2}(N+1)(N+2)$;

(b) $S_{N+1}(z)$ and $S_N(z)$ do not have a common root.

Proof. We shall prove these using induction. We have $S_{-1}(z) = S_0(z) = 1$, then $S_1(z) = z+\mu$ and $S_2(z) = (z+\mu)^3 - \mu$. Clearly, (a) and (b) hold for $n = 0, 1, 2$, when $\mu \ne 0, \pm 1$. We assume that (a) and (b) hold for $n = N-2, N-1, N$, with $N \ge 2$. Then we will prove that the statements also hold for $n = N+1$.

We let f be S_{N-1}. Then $n = N-1$ and $h = S_N S_{N-2}$ in Lemma 3.1. Then (3.3) becomes

$$S_{N-1} \mid z\mathcal{L}_z(S_N S_{N-2}) - 2k(z+\mu)(S_N S_{N-2})^2. \tag{3.4}$$

Hence

$$\begin{aligned}
&\left[\mathcal{L}_z(S_N S_{N-2}) - \frac{2(z+\mu)}{z}(S_N S_{N-2})^2\right](-z) \\
&\quad = -z\left[S_{N-2}^2\mathcal{L}_z(S_N) + S_N^2\mathcal{L}_z(S_{N-2})\right] + 2(z+\mu)(S_N S_{N-2})^2 \\
&\quad = S_{N-2}^2[-z\mathcal{L}_z(S_N) + (z+\mu)S_N^2] + S_N^2[-z\mathcal{L}_z(S_{N-2}) + (z+\mu)S_{N-2}^2] \\
&\quad = S_{N-2}^2[-z\mathcal{L}_z(S_N) + (z+\mu)S_N^2] + S_N^2 S_{N-1}S_{N-3}.
\end{aligned}$$

Then by (3.3) and (b) with $n = N-1$, we have

$$S_{N-1} \mid -z\mathcal{L}_z(S_N) + (z+\mu)S_N^2 = -z\left[S_N\frac{d^2S_N}{dz^2} - \left(\frac{dS_N}{dz}\right)^2\right] - S_N\frac{dS_N}{dz} + (z+\mu)S_N^2$$

So, according to (2.3), S_{N+1} is a polynomial by induction.

To prove (b), we use a proof by contradiction. If S_N and S_{N-1} have the same root $z_0 \neq 0$, then by (2.3), z_0 is also a root of

$$S_N\frac{d^2S_n}{dz^2} - \left(\frac{dS_N}{dz}\right)^2,$$

and hence $\frac{dS_N}{dz}$. This implies z_0 is at least a double root of S_N, which contradicts our assumption about S_N. □

4 Further analysis on $S_n(0,\mu)$

Theorem 4.1. *Let $\phi_n = S_n(0,\mu)$, and*

$$\phi_n' := \frac{\partial S_n}{\partial z}(0,\mu), \qquad \phi_n'' := \frac{\partial^2 S_n}{\partial z^2}(0,\mu),$$

etc. Then for all $n \geq 3$,

$$\phi_{n+1} = \frac{\phi_n\phi_{n-1}}{\phi_{n-2}}\left(2\mu^2 - 2n^2 + 2n - 1 - \frac{\phi_n\phi_{n-3}}{\phi_{n-1}\phi_{n-2}}\right); \tag{4.1}$$

$$\phi_{n+1}' = -\frac{\phi_n\phi_{n+2}}{\phi_{n+1}} + \mu\phi_{n+1}. \tag{4.2}$$

Proof. Differentiating (2.3) with respect to z gives

$$\begin{aligned}
\frac{dS_{n+1}}{dz} = \frac{1}{S_{n-1}}\Bigg\{&S_n^2 + 2(z+\mu)S_n\frac{dS_n}{dz} - 2S_n\frac{d^2S_n}{dz^2} \\
&+ z\left(\frac{dS_n}{dz}\frac{d^2S_n}{dz^2} - S_n\frac{d^3S_n}{dz^3}\right) - S_{n+1}\frac{dS_{n-1}}{dz}\Bigg\}.
\end{aligned} \tag{4.3}$$

Substitute $z = 0$ into (2.3) and (4.3). We obtain

$$\phi_{n+1} = \frac{\phi_n}{\phi_{n-1}}\left(\mu\phi_n - \phi_n'\right), \tag{4.4}$$

and

$$\phi'_{n+1} = \frac{\phi_n}{\phi_{n-1}} \left(\phi_n + 2\mu\phi'_n - 2\phi''_n - \frac{\phi'_{n-1}\phi_{n+1}}{\phi_n} \right). \tag{4.5}$$

Now (4.4) implies that (4.2) is valid. Furthermore, in [11, p. 9519],

$$z^2 \left[S_n \frac{d^4 S_n}{dz^4} - 4 \frac{dS_n}{dz}\frac{d^3 S_n}{dz^3} + 3 \left(\frac{d^2 S_n}{dz^2} \right)^2 \right] + 2z \left(S_n \frac{d^3 S_n}{dz^3} - \frac{dS_n}{dz}\frac{d^2 S_n}{dz^2} \right)$$
$$- 4z(z+\mu) \left[S_n \frac{d^2 S_n}{dz^2} - \left(\frac{dS_n}{dz} \right)^2 \right] - 2S_n \frac{d^2 S_n}{dz^2} + 4\mu S_n \frac{dS_n}{dz} = 2n(n+1)S_n^2. \tag{4.6}$$

This implies, as ϕ_n is not identically zero, that

$$2\mu\phi'_n - \phi''_n = n(n+1)\phi_n. \tag{4.7}$$

Hence by (4.2),

$$\phi''_n = 2\mu\phi'_n - n(n+1)\phi_n = [2\mu^2 - n(n+1)]\phi_n - \frac{2\mu\phi_{n-1}\phi_{n+1}}{\phi_n}.$$

Now substitute this equation and (4.2) into (4.5) to obtain, after simplification,

$$-\frac{\phi_n\phi_{n+2}}{\phi_{n+1}} = \frac{\phi_n^2}{\phi_{n-1}} (2n^2 + 2n + 3 - 2\mu^2) + \frac{\phi_{n+1}\phi_n\phi_{n-2}}{\phi_{n-1}^2}.$$

Therefore we have

$$\phi_{n+2} = \frac{\phi_n\phi_{n+1}}{\phi_{n-1}} \left(2\mu^2 - 2n^2 - 2n - 3 - \frac{\phi_{n+1}\phi_{n-2}}{\phi_n\phi_{n-1}} \right).$$

Thus (4.1) is also valid. □

Corollary 4.2.

(a) For all $n \in \mathbb{N}$,

$$\phi_n(\mu) = \mu^{\gamma_0^n} (\mu^2 - 1)^{\gamma_1^n} (\mu^2 - 4)^{\gamma_2^n} \dots \left[\mu^2 - (n-1)^2\right]^{\gamma_{n-1}^n},$$

where for $0 \le j < k$,

$$\gamma_{2j}^n = \lceil \frac{n}{2} \rceil - j = k - j \quad \text{if } n = 2k \text{ or } 2k-1;$$
$$\gamma_{2j+1}^n = \lfloor \frac{n}{2} \rfloor - j = k - j \quad \text{if } n = 2k \text{ or } 2k+1.$$

(b) $\phi'_n(\mu) = \phi_{n-1}(\mu)g(\mu)$, where $g(\mu)$ is a polynomial of degree n or $n-2$ according as n is even or odd.

Proof. If $n = 2k$ is even, then by induction hypothesis,

$$\phi_{2k} = \mu^k(\mu^2-1)^k(\mu^2-4)^{k-1}(\mu^2-9)^{k-1}\dots\left[\mu^2-(2k-2)^2\right]\left[\mu^2-(2k-1)^2\right].$$
$$= \mu^k(\mu^2-1)^k \prod_{j=1}^{k-1} \left[\mu^2-(2j)^2\right]^{k-j} \left[\mu^2-(2j+1)^2\right]^{k-j}.$$

while ϕ_{n-1} and ϕ_{n-2} have similar expressions. Thus

$$\frac{\phi_n\phi_{n-1}}{\phi_{n-2}} = \mu^{k+1}(\mu^2-1)^k(\mu^2-4)^k(\mu^2-9)^{k-1}\dots$$
$$\dots\left[\mu^2-(2k-2)^2\right]^2\left[\mu^2-(2k-1)^2\right].$$

Hence it follows from Theorem 4.1 that

$$\phi_{n+2} = \mu^{k+1}(\mu^2-1)^k(\mu^2-4)^k(\mu^2-9)^{k-1}\dots$$
$$\dots\left[\mu^2-(2k-2)^2\right]^2\left[\mu^2-(2k-1)^2\right]\left[\mu^2-(2k)^2\right].$$

The situation is similar for odd n.

We also consider odd n's and even n's in part (b). When $n=2k$, we note that

$$\phi_{2k} = \phi_{2k-1}\prod_{j=1}^{k}\left[\mu^2-(2j-1)^2\right].$$

Hence by Theorem 4.1,

$$\phi'_{2k} = \mu\phi_{2k-1}\left\{\prod_{j=1}^{k}\left[\mu^2-(2j-1)^2\right]-\prod_{j=1}^{k-1}\left[\mu^2-(2j)^2\right]\right\}.$$

When $n=2k+1$, then

$$\phi_{2k+1} = \phi_{2k}\prod_{j=0}^{k}\left[\mu^2-(2j)^2\right],$$

and

$$\phi'_{2k+1} = \phi_{2k}\left\{\prod_{j=0}^{k}\left[\mu^2-(2j)^2\right]-\prod_{j=0}^{k}\left[\mu^2-(2j+1)^2\right]\right\}.$$

□

Remark 4.3. Part (a) of Theorem 4.2 above means that $z=0$ is a root of S_n if and only if $\mu=0,\pm1,\pm2,\dots,\pm(n-1)$. In particular, the first few $\phi_n(\mu)$'s are

$$\begin{aligned}
\phi_1(\mu) &= \mu,\\
\phi_2(\mu) &= \mu(\mu^2-1),\\
\phi_3(\mu) &= \mu^2(\mu^2-1)(\mu^2-4),\\
\phi_4(\mu) &= \mu^2(\mu^2-1)^2(\mu^2-4)(\mu^2-9),\\
\phi_5(\mu) &= \mu^3(\mu^2-1)^2(\mu^2-4)^2(\mu^2-9)(\mu^2-16).
\end{aligned}$$

Theorem 4.4. *Fix $n\in\mathbb{N}$. If $\mu=\pm n$, then for the recurrence relation* (2.3) *and starting polynomials $S_{-1}=S_0=1$, we have:*

(a) all the zeros of rational functions S_k are simple, when $k=0,1,\dots,n$;

(b) each S_k is a monic polynomial in z of degree $\frac{1}{2}k(k+1)$, for $k=0,1,\dots,n$.

Proof. We shall make use of the identity (4.6) again. Suppose $z_0 \neq 0$ is a root of S_n. Then from (4.6),

$$3\left[\frac{d^2S_n}{dz^2}(z_0)\right]^2 = \frac{dS_n}{dz}(z_0)\left[4\frac{d^3S_n}{dz^3}(z_0) + 2\frac{d^2S_n}{dz^2}(z_0) + 4(z_0+\mu)\frac{dS_n}{dz}(z_0)\right].$$

Hence if z_0 is a root of $\frac{dS_n}{dz}$, then it also has to be a root of $\frac{d^2S_n}{dz^2}$. That is, if z_0 is not a simple root of S_n, then its order ≥ 3. Let k be the order of the root z_0. Analyzing on the identity (4.6), the term

$$S_n\frac{d^3S_n}{dz^3} - 4\frac{dS_n}{dz}\frac{d^3S_n}{dz^3} + 3\left(\frac{d^2S_n}{dz^2}\right)^2,$$

has the zero z_0 with order $2k-4$, while the other terms has order greater than or equal to $2k+3$. Therefore let $S_n(z) = (z-z_0)^k g(z)$, where $g(z)$ is a polynomial and $\frac{dg}{dz}(z_0) \neq 0$. The there exists a polynomial $h(z)$ such that

$$S_n\frac{d^4S_n}{dz^4} - 4\frac{dS_n}{dz}\frac{d^3S_n}{dz^3} + 3\left(\frac{d^2S_n}{dz^2}\right)^2 = (z-z_0)^{2k-4}\left[(z-z_0)h(z) + 6k(k-1)g^2(z)\right].$$

But it must have z_0 as a root, which gives a contradiction. We see that every z_0 is at most a simple root. □

Theorem 4.5. *Let $\mu \in \mathbb{Z}\setminus\{0\}$. Suppose that $S_n(z) = z^\sigma g(z)$, where $g(z) = \sum_{j=0}^k a_j z^j$, with $a_0 \neq 0$. Then*

(a) $a_1 = \mu a_0$; and

(b) if $\sigma = \frac{1}{2}(n-|\mu|)(n-|\mu|+1)$, then

$$a_2 = \tfrac{1}{2}\left(\mu^2 - \frac{|\mu|}{2m+1}\right)a_0.$$

Proof. We use the auxilliary identity (4.6) for the proof. First

$$\frac{dS_n}{dz} = z^\sigma\frac{dg}{dz} + \sigma z^{\sigma-1}g, \tag{4.8a}$$

and

$$\frac{d^2S_n}{dz^2} = z^\sigma\frac{d^2g}{dz^2} + 2\sigma z^{\sigma-1}\frac{dg}{dz} + \sigma(\sigma-1)z^{\sigma-2}g, \tag{4.8b}$$

$$\frac{d^3S_n}{dz^3} = z^\sigma\frac{d^3g}{dz^3} + 3\sigma z^{\sigma-1}\frac{d^2g}{dz^2} + 3\sigma(\sigma-1)z^{\sigma-2}\frac{dg}{dz} + \sigma(\sigma-1)(\sigma-2)z^{\sigma-3}g, \tag{4.8c}$$

$$\begin{aligned}\frac{d^4S_n}{dz^4} = z^\sigma\frac{d^4g}{dz^4} &+ 4\sigma z^{\sigma-1}\frac{d^3g}{dz^3} + 6\sigma(\sigma-1)z^{\sigma-2}\frac{d^2g}{dz^2} + 4\sigma(\sigma-1)(\sigma-2)z^{\sigma-3}\frac{dg}{dz}\\ &+ \sigma(\sigma-1)(\sigma-2)(\sigma-3)z^{\sigma-4}g.\end{aligned} \tag{4.8d}$$

Express (4.6) as

$$
\begin{aligned}
2n(n+1)S_n^2 + 4z^2 \left[S_n \frac{d^2 S_n}{dz^2} - \left(\frac{dS_n}{dz} \right)^2 \right] \\
= z^2 \left[S_n \frac{d^4 S_n}{dz^4} - 4 \frac{dS_n}{dz} \frac{d^3 S_n}{dz^3} + 3 \left(\frac{d^2 S_n}{dz^2} \right)^2 \right] + 2z \left(S_n \frac{d^3 S_n}{dz^3} - \frac{dS_n}{dz} \frac{d^2 S_n}{dz^2} \right) \\
- 4\mu z \left[S_n \frac{d^2 S_n}{dz^2} - \left(\frac{dS_n}{dz} \right)^2 \right] - 2S_n \frac{d^2 S_n}{dz^2} + 4\mu S_n \frac{dS_n}{dz}. \qquad (4.9)
\end{aligned}
$$

Then we substitute (4.8) into (4.9) and obtain, after simplification,

$$
\begin{aligned}
&[2n(n+1) - 4\sigma] z^{2\sigma} g^2 + \ldots \\
&\quad = -8\sigma z^{2\sigma-1} g \frac{dg}{dz} + 8\mu\sigma z^{2\sigma-1} g^2 - (8\sigma+2) z^{2\sigma} g \frac{d^2 g}{dz^2} + 8\sigma z^{2\sigma} \left(\frac{dg}{dz} \right)^2 + 4\mu z^{2\sigma} g \frac{dg}{dz} + \ldots
\end{aligned}
$$

Comparing coefficients of $z^{2\sigma-1}$ in the resulting polynomials, we obtain

$$
8\sigma a_0 a_1 - 8\mu\sigma a_0^2 = 0.
$$

This implies part (a). Next we compare coefficients of $z^{2\sigma}$ to get

$$
[2n(n+1) - 4\sigma] a_0^2 = -8\sigma(a_1^2 + 2a_0 a_2) + (16\sigma+4)\sigma a_0 a_1 - (16\sigma+4) a_0 a_2 + 8\sigma a_1^2.
$$

Since $a_1 = \mu a_0$, we have

$$
(32\sigma+4) a_0 a_2 = [\mu^2(16\sigma+4) - 2n(n+1) + 4\sigma] a_0^2,
$$

so that

$$
a_2 = \frac{\mu^2(16\sigma+4) - 2n(n+1) + 4\sigma}{32\sigma+4}\, a_0.
$$

Now let $m = n - |\mu|$ and $\sigma = \frac{1}{2} m(m+1)$,

$$
2n(n+1) = 2[\mu^2 + (2m+1)|\mu| + m(m+1)],
$$

while

$$
8\sigma + 1 = (2m+1)^2.
$$

Therefore part (b) is valid. □

Theorem 4.6. *Let $\mu \in \mathbb{Z}$. Then for all $n > |\mu|$:*

(a) $z = 0$ is a root of order $\frac{1}{2}(n - |\mu|)(n - |\mu| + 1)$ for S_n;

(b) S_n is a monic polynomial of degree $\frac{1}{2} n(n+1)$;

(c) all other roots of S_n are simple.

Proof. Part (c) follows from the proof of Theorem 4.4. For parts (a) and (b), the case when $\mu = 0$ is simple. For then, $S_n(z) = z^{\sigma_n}$. In general, let $\mu \in \mathbb{Z} \setminus \{0\}$. Let $k = |\mu|$, by Corollary 4.2, $z = 0$ is a root of S_{k+1}. Let $S_{k+1} = zg_{1,\mu}(z)$. By (4.3) and (4.7),

$$\frac{dS_{k+1}}{dz}(0;k) = \frac{\phi_k(k)\left[\phi_k(k) + 2\mu\phi_k'(k) - \phi_k''(k)\right]}{\phi_{k-1}(k)} = \frac{\phi_k^2(k)}{\phi_{k-1}(k)}(k^2+k+1) \neq 0.$$

Thus z is a simple root of S_{k+1}. So $S_{k+1} = zg_{1,\mu}(z)$ where $g_{1,\mu} = \sum a_j^{1,\mu} x^j$ is a degree $\sigma_{k+1} - 1$ polynomial with nonzero roots.

Let $n = k+1$. By (2.3),

$$S_k S_{k+2} = z^3 \left[\left(\frac{dg_{1,\mu}}{dz}\right)^2 - g_{1,\mu}\frac{d^2 g_{1,\mu}}{dz^2} + g_{1,\mu}^2\right] + z^2\left(\mu g_{1,\mu}^2 - g_{1,\mu}\frac{dg_{1,\mu}}{dz}\right).$$

For the right hand side, the coefficient of z^3 is given by

$$a_0^{1,\mu}\left(a_0^{1,\mu} + 2\mu a_1^{1,\mu} - 4a_2^{1,\mu}\right) = \left(a_0^{1,\mu}\right)^2 (1 + \tfrac{2}{3}\mu) \neq 0,$$

by Lemma 4.5. Hence $S_{k+2} = z^3 g_{2,\mu}(z)$ where $g_{2,\mu}$ had nonzero constant term. Use induction on $m = n - |\mu| > 1$. Let

$$S_{n-1} = z^{\sigma_{m-1}} g_{m-1,\mu}, \qquad S_n = z^{\sigma_m} g_{m,\mu},$$

where $g_{m-1,\mu}$ and $g_{m,\mu}$ are polynomials of degrees σ_{m-1} and σ_m respectively with nonzero roots. Then by (2.3),

$$z^{\sigma_{m-1}} g_{m-1,\mu} S_{m-1} = z^{2\sigma_m}\left(\mu g_{m,\mu}^2 - g_{m,\mu}\frac{dg_{m,\mu}}{dz}\right) + z^{2\sigma_m+1}\left[\left(\frac{dg_{m,\mu}}{dz}\right)^2 - g_{m,\mu}\frac{d^2 g_{m,\mu}}{dz^2} + g_{m,\mu}^2\right].$$

Let $a_j^{m,\mu}$ be the coefficients of $g_{m,\mu}$. By Lemma 4.5, $a_1^{m,\mu} = \mu a_0^{m,\mu}$. So we may write $S_{n+1} = z^{\sigma_{m+1}} g_{m+1,\mu}$, where

$$\begin{aligned}
a_0^{m+1,\mu} a_0^{m-1,\mu} &= 2\mu a_0^{m,\mu} a_1^{m,\mu} - 4a_0^{m\mu} a_2^{m,\mu} + (a_0^{m,\mu})^2 \\
&= a_0^{m,\mu}\left(2\mu a_1^{m,\mu} + a_0^{m,\mu} - 4a_2^{m,\mu}\right) \\
&= (a_0^{m,\mu})^2\left(1 + \frac{2|\mu|}{2m+1}\right).
\end{aligned}$$

So $a_0^{m+1,\mu}$ is nonzero, and the rational function $g_{m+1,\mu}$ does not have $z = 0$ as a root.

Next we want to show that $g_{m+1,\mu}$ is a polynomial. From the proof of Theorem 4.4 and Theorem 3.2(b), we know that all nonzero roots of S_n and S_{n-1} are simple and not common. Furthermore, we still have

$$S_{n-1} \,|\, S_{n-2}^2\left[-z\mathcal{L}_z(S_n) + (z+\mu)S_n^2\right],$$

where

$$\Delta := -z\mathcal{L}_z(S_n) + (z+\mu)S_n^2 = z\left[\left(\frac{dS_n}{dz}\right)^2 - S_n\frac{d^2 S_n}{dz^2}\right] - S_n\frac{dS_n}{dz} + (z+\mu)S_n^2.$$

We conclude that $g_{m-1,\mu}$ divides Δ, which means that $g_{m+1,\mu}$ is also a polynomial. Consequently parts (a) and (b) follow by induction. □

5 Conclusions

We have given a direct algebraic proof that the nonlinear recurrence relation (2.3) generates polynomials S_n, rather than rational functions without direct resort to the τ-function theory of Painlevé equations. However we critically needed a higher order equation derived from the corresponding σ-equation, which seems to be inevitable in the nonlinear scenario. We believe that the method can be developed to apply to the fifth Painlevé equation (P_V) as well, though we shall not pursue this further here.

Acknowledgements

PAC thanks the Department of Applied Mathematics, National Sun Yat-sen University, Kaohsiung, Taiwan and the Department of Mathematics, National Taiwan University, Taipei, Taiwan, for their hospitality during his visit where some of this work was done. CKL is partially supported by Ministry of Science and Technology, Taiwan, under contract number MOST104-2115-M-110-008.

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