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Symbolic Representation and Classification of $N = 1$ Supersymmetric Evolutionary Equations

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Abstract

We extend the symbolic representation to the ring of $N = 1$ supersymmetric differential polynomials, and demonstrate that operations on the ring, such as the super derivative, Fréchet derivative and super commutator, can be carried out in the symbolic way. Using the symbolic representation, we classify scalar λ -homogeneous $N = 1$ supersymmetric evolutionary equations with nonzero linear term when $\lambda > 0$ for arbitrary order and give a comprehensive description of all such integrable equations.

1 Introduction

Classification of all integrable equations for a given family of nonlinear equations is one of fundamental problems in the field of soliton theory, and can be tackled by many methods, among which the symmetry approach has been proved to be very efficient and powerful. The symmetry approach takes the existence of infinitely many higher order infinitesimal symmetries as the definition of integrability. It has been used to classify large classes of both nonlinear partial differential equations and difference-differential equations. We refer the reader to the recent review papers [1, 2] and references therein for details.

The ultimate goal is to obtain the classification result for integrable equations of any order, i.e. the global classification result [3]. So far, results of this kind have been obtained for scalar homogeneous evolutionary equations in both commutative [4] and non-commutative [5] cases, and a class of second order (in time) non-evolutionary equations of odd order in spatial variable [6]. They have been achieved mainly due to the symbolic representation of the ring of differential polynomials. The symbolic representation (an abbreviated form of the Fourier transformation), originally applied to the theory of integrable equations by Gel'fand and Dikii [7], enables us to translate the question of integrability into the problem of divisibility of certain multi-variable polynomials, which can be solved by results from number theory and algebraic geometry. It is a suitable tool to study integrability of non-commutative [5], non-evolutionary [6, 8], non-local (integro-differential) [9], multi-component [10] and multi-dimensional equations [11].

Supersymmetry was introduced into the soliton theory in the late of 1970s. Many classical integrable equations have been successfully embedded in their supersymmetric extensions, includ-

ing the supersymmetric sine-Gordon equation [12], the supersymmetric Korteweg-de Vries (KdV) equation [13, 14], the supersymmetric nonlinear Schrödinger equations [15], the supersymmetric two boson equation [16], the supersymmetric Harry Dym equations [17], etc. These supersymmetric integrable prototypes have been extensively studied in the past four decades, and shown to have various novel features, such as fermionic nonlocal conserved densities [18], non-unique roots of Lax operator [19] and odd Hamiltonian structures [20]. Recently, discrete integrable systems on Grassmann algebras [21, 22, 23, 24], as well as Grassmann extensions of Yang-Baxter maps [25], have been established due to the development of Darboux-Bäcklund transformations of super and/or supersymmetric integrable equations.

Though the list of supersymmetric integrable equations are long, it may be far from complete taking account of the fact that the (non-)existence of supersymmetric counterparts of some classical models, like the Ibragimov-Shabat equation [26] and the Kaup-Kupershmidt equation [27], is not proved. Some efforts have been made to enrich the garden of supersymmetric integrable equations. Fermionic extensions of Burgers and Boussinesq equations were constructed and shown to have infinitely many symmetries [28]. Wolf generated tables [29] of homogeneous supersymmetric systems with a certain higher order symmetry using the package SsTools [30]. A class of fifth order supersymmetric evolutionary equations (which are 3/2-homogeneous in our terminology) admitting seventh order symmetries were presented, and all of them are claimed to be integrable [31].

In this paper, we develop the symbolic representation for the ring of $N = 1$ supersymmetric differential polynomials so that we can globally classify scalar λ -homogeneous evolutionary equation of the form

$$\Phi_t = \Phi_n + K\left(\Phi, (\mathcal{D}\Phi), \Phi_1, (\mathcal{D}\Phi_1), \dots, \Phi_{n-1}, (\mathcal{D}\Phi_{n-1})\right), \quad (1)$$

where the super function $\Phi = \Phi(x, \theta, t)$ is fermionic (anti-commutative), Φ_k stands for the k -th order derivative of Φ with respect to x , and \mathcal{D} denotes the super derivative. (Detailed explanations about our notations will be given in Section 2.) We show that there are only eight hierarchies (*up to scaling transformations of both dependent and independent variables*) of nontrivial supersymmetric integrable equations, that is, the corresponding systems in Φ 's components are truly coupled (see Section 3 for the explicit explanation of trivial and nontrivial supersymmetric extensions). To the best of our knowledge, the hierarchy of the supersymmetric fifth order modified KdV equation (18) has not appeared in the literature. This classification result also enables us to give definite answers to the non-existence of the above equations.

The arrangement of this paper is as follows. In Section 2, we give a basic introduction on a super Lie bracket for the super differential ring of smooth functions in Φ and its (super) derivatives, and then define the concepts of infinitesimal symmetries and integrability for $N = 1$ supersymmetric evolutionary equations. In Section 3, we present the main result of this paper, i.e. a complete list of nontrivial λ -homogeneous ($\lambda > 0$) supersymmetric integrable equations. This result is global in the sense that if a λ -homogeneous $N = 1$ supersymmetric evolutionary equation of arbitrary order is nontrivial and integrable, then it must belong to one of symmetry hierarchies of eight equations we identified. The trivial supersymmetric integrable equations of such kind are listed in Appendix A. In Section 4, we develop the symbolic representation converting every $N = 1$ supersymmetric differential polynomial to a unique multi-variable polynomial (and vice versa), which is anti-symmetric with respect to symbols of fermionic variables and meanwhile symmetric with respect to symbols of bosonic variables. Inferred from this one-to-one correspondence, we show that operations of supersymmetric differential polynomials, such as partial derivatives, the super derivative, derivative with respect to x Fréchet derivative and super commutator, can be carried out in the symbolic way. Interestingly and

importantly, the super commutator of the linear term Φ_k with any supersymmetric differential polynomial has the same symbolic formulation as its counterpart in the classical case [4]. This remarkable fact guarantees that equation (1) can be globally classified via the same strategy. We explain it in Section 5 although some detailed proofs are skipped. Concluding remarks and discussions are given in the last section.

2 Symmetries of $N = 1$ supersymmetric evolutionary equations

In this section, we give a brief account on symmetries for $N = 1$ supersymmetric evolutionary equations. Algebraic backgrounds about evolutionary equations with fermionic variables can be found in [32, 33], while rudiments of super algebras in [34].

It is convenient to work with a super variable $\Phi = \Phi(x, \theta, t)$ where both x and t are usual commutative space-time coordinates, while θ is an anti-commutative independent variable such that $\theta^2 = 0$. The super variable Φ can be either fermionic (anti-commutative) or bosonic (commutative) in this section. The k -th order derivative of Φ with respect to x is denoted by $\Phi_k \equiv (\partial_x^k \Phi)(k \in \mathbb{N})^1$, (We often omit the subscript of Φ_0 , and simply write Φ .) and \mathcal{D} stands for the super derivative defined by $\mathcal{D} \equiv \partial_\theta + \theta \partial_x$ with the property $\mathcal{D}^2 = \partial_x$.

Let \mathfrak{C} denote the super (\mathbb{Z}_2 -graded) differential ring of smooth functions or simply polynomials in Φ and Φ 's (super) derivatives. As a matter of fact, every element in \mathfrak{C} is supersymmetric, namely invariant under the transformation

$$\begin{cases} x \mapsto x - \eta\theta \\ \theta \mapsto \theta + \eta \end{cases} \quad (\eta \text{ is a fermionic parameter}). \quad (2)$$

The multiplication on \mathfrak{C} is super commutative, that is, for $P, Q \in \mathfrak{C}$

$$PQ = (-1)^{|P||Q|}QP$$

where $||$ denotes the parity of a super object². Derivatives on \mathfrak{C} , such as the super derivative \mathcal{D} and partial derivatives with respect to Φ_k or $(\mathcal{D}\Phi_k)$, obey super Leibniz rules

$$\mathcal{D}(PQ) = (\mathcal{D}P)Q + (-1)^{|P|}P(\mathcal{D}Q), \quad \frac{\partial}{\partial \bullet}(PQ) = \frac{\partial P}{\partial \bullet}Q + (-1)^{|\bullet||P|}P\frac{\partial Q}{\partial \bullet}$$

where \bullet represents Φ_k or $(\mathcal{D}\Phi_k)$. For any element $P \in \mathfrak{C}$, there must be a least non-negative integer k such that

$$\frac{\partial P}{\partial \Phi_l} = \frac{\partial P}{\partial (\mathcal{D}\Phi_l)} = 0 \quad (l > k).$$

If, in addition, we have

$$\frac{\partial P}{\partial (\mathcal{D}\Phi_k)} = 0,$$

then P is said to be *of order* k . Otherwise, we say P is *of order* $k\frac{1}{2}$.

¹ \mathbb{N} denotes the set of non-negative integers, $\{0, 1, 2, 3, \dots\}$.

²Formulas involving parities are extended to all elements in \mathfrak{C} by linearity [34].

Definition 2.1. A partial differential operator of the form

$$\mathbf{P} = \sum_{k=0} \left(P_{2k} \frac{\partial}{\partial \Phi_k} + P_{2k+1} \frac{\partial}{\partial (\mathcal{D}\Phi_k)} \right) \quad (P_j \in \mathfrak{C}, \forall j \in \mathbb{N}) \quad (3)$$

is called a *vector field on \mathfrak{C}* .

Regarding the vector field \mathbf{P} defined by (3), if there exists $|\mathbf{P}| \in \mathbb{Z}_2$ such that

$$|\mathbf{P}| = (|P_j| - |\Phi| - j) \pmod{2},$$

where P_j is an arbitrary non-zero coefficient, then \mathbf{P} is of *parity* $|\mathbf{P}|$. All vector fields form a super vector space, denoted by \mathfrak{V} . In fact, \mathfrak{V} is a super Lie algebra with the bracket defined by

$$[\mathbf{P}, \mathbf{Q}] = \mathbf{P}\mathbf{Q} - (-1)^{|\mathbf{P}||\mathbf{Q}|} \mathbf{Q}\mathbf{P},$$

and the super Jacobi identity reads as

$$(-1)^{|\mathbf{P}||\mathbf{R}|} [\mathbf{P}, [\mathbf{Q}, \mathbf{R}]] + (-1)^{|\mathbf{Q}||\mathbf{P}|} [\mathbf{Q}, [\mathbf{R}, \mathbf{P}]] + (-1)^{|\mathbf{R}||\mathbf{Q}|} [\mathbf{R}, [\mathbf{P}, \mathbf{Q}]] = 0 \quad (\mathbf{P}, \mathbf{Q}, \mathbf{R} \in \mathfrak{V}).$$

Example 2.1. The prolongation of super derivative \mathcal{D} on \mathfrak{C} is a vector field of parity 1 given by

$$\sum_{k=0} \left((\mathcal{D}\Phi_k) \frac{\partial}{\partial \Phi_k} + \Phi_{k+1} \frac{\partial}{\partial (\mathcal{D}\Phi_k)} \right).$$

We shall still denote it by \mathcal{D} without confusion.

Given a vector field \mathbf{P} defined by (3) of parity $|\mathbf{P}|$, by straightforward calculation we obtain

$$[\mathbf{P}, \mathcal{D}] = \sum_{k=0} \left(\left(P_{2k+1} - (-1)^{|\mathbf{P}|} (\mathcal{D}P_{2k}) \right) \frac{\partial}{\partial \Phi_k} + \left(P_{2(k+1)} - (-1)^{|\mathbf{P}|} (\mathcal{D}P_{2k+1}) \right) \frac{\partial}{\partial (\mathcal{D}\Phi_k)} \right).$$

Hence, \mathbf{P} commutes with \mathcal{D} , i.e. $[\mathbf{P}, \mathcal{D}] = 0$ if and only if

$$P_j = (-1)^{|\mathbf{P}|} (\mathcal{D}P_{j-1}) = (-1)^{j|\mathbf{P}|} (\mathcal{D}^j P_0) \quad (j \in \mathbb{N}).$$

Definition 2.2. Given $P \in \mathfrak{C}$, a vector field

$$\sum_{k=0} \left((\partial_x^k P) \frac{\partial}{\partial \Phi_k} + (-1)^{(|\mathbf{P}|-|\Phi|)} (\mathcal{D}\partial_x^k P) \frac{\partial}{\partial (\mathcal{D}\Phi_k)} \right)$$

is called an *evolutionary* vector field, denoted by \mathbf{V}_P , whose parity is $(|\mathbf{P}| - |\Phi|) \pmod{2}$.

We call $P \in \mathfrak{C}$ the *characteristic* of \mathbf{V}_P . When we talk about evolutionary vector fields, we often simply refer to their characteristics. The super Lie bracket of two evolutionary vector fields is still evolutionary, and indeed, for $P, Q \in \mathfrak{C}$ we have

$$[\mathbf{V}_P, \mathbf{V}_Q] = \mathbf{V}_H, \quad H = \mathbf{V}_P(Q) - (-1)^{|\mathbf{V}_P||\mathbf{V}_Q|} \mathbf{V}_Q(P). \quad (4)$$

This defines a super Lie bracket and thus the set of evolutionary vector fields is a super Lie subalgebra of \mathfrak{V} . We are going to rewrite the Lie bracket in terms of Fréchet derivatives.

Definition 2.3. For any $Q \in \mathfrak{C}$, we define a linear differential operator

$$D_Q = \sum_{k=0} \left((-1)^{|\Phi|(|Q|-|\Phi|)} \frac{\partial Q}{\partial \Phi_k} \partial_x^k + (-1)^{(|\Phi|+1)(|Q|-|\Phi|+1)} \frac{\partial Q}{\partial (\mathcal{D}\Phi_k)} \mathcal{D}\partial_x^k \right), \quad (5)$$

and call it the *Fréchet derivative* of Q .

Note that the parity of the Fréchet derivative for Q is $(|Q| - |\Phi|) \pmod{2}$, which is the same as $|\mathbf{V}_Q|$. In fact, we have

$$\mathbf{V}_P(Q) = (-1)^{|\mathbf{V}_P||\mathbf{V}_Q|} \sum_{k=0} \left((-1)^{|\Phi|(|Q|-|\Phi|)} \frac{\partial Q}{\partial \Phi_k} (\partial_x^k P) + (-1)^{(|\Phi|+1)(|Q|-|\Phi|+1)} \frac{\partial Q}{\partial (\mathcal{D}\Phi_k)} (\mathcal{D}\partial_x^k P) \right).$$

Comparing to equation (5), it becomes to

$$\mathbf{V}_P(Q) = (-1)^{|\mathbf{V}_P||\mathbf{V}_Q|} D_Q(P).$$

Therefore, we endow \mathfrak{C} a super Lie bracket defined by

$$[P, Q] = D_P(Q) - (-1)^{|D_P||D_Q|} D_Q(P), \quad (6)$$

whose parity is $|[P, Q]| = (|P| + |Q| - |\Phi|) \pmod{2}$. This super Lie algebra is isomorphic to the super Lie subalgebra of evolutionary vector fields since we have

$$[\mathbf{V}_P, \mathbf{V}_Q] = -\mathbf{V}_{[P, Q]}.$$

The super Jacobi identity on \mathfrak{C} is given by

$$(-1)^{|D_P||D_R|} [P, [Q, R]] + (-1)^{|D_Q||D_P|} [Q, [R, P]] + (-1)^{|D_R||D_Q|} [R, [P, Q]] = 0 \quad (P, Q, R \in \mathfrak{C}). \quad (7)$$

Definition 2.4. For a $N = 1$ supersymmetric evolutionary equation³

$$\Phi_t = K, \quad (K \in \mathfrak{C} \text{ and } |K| = |\Phi|), \quad (8)$$

any $Q \in \mathfrak{C}$ such that $[K, Q] = 0$ is called an *infinitesimal symmetry*^A (abbreviated to symmetries) of equation (8).

Example 2.2. Equation (8) has at least two symmetries, namely Φ_1 and K , where K is obviously certified by a trivial identity $[K, K] = 0$, and Φ_1 is proved as follows.

The Fréchet derivatives of K , Φ_1 are respectively given by

$$D_K = \sum_{k=0} \left(\frac{\partial K}{\partial \Phi_k} \partial_x^k + (-1)^{(|\Phi|+1)} \frac{\partial K}{\partial (\mathcal{D}\Phi_k)} \mathcal{D}\partial_x^k \right) \quad \text{and} \quad D_{\Phi_1} = \partial_x.$$

Notice that $|D_K| = 0$, and according to the Lie bracket (6), we have $[K, Q] = D_K(Q) - D_Q(K)$. Hence, we easily conclude that Φ_1 is a symmetry of equation (8) since

$$D_K(\Phi_1) = \partial_x(K) = D_{\Phi_1}(K),$$

that is, $[K, \Phi_1] = 0$. In fact, Φ_1 indicates the invariance of equation (8) under x -translation, while K represents the invariance under t -translation.

³The temporal variable t is bosonic, so K must have the same parity as Φ .

⁴There is no constraints on parities of symmetries.

As an immediate corollary of the super Jacobi identity (7), if Q_1 and Q_2 are symmetries of equation (8), then so is $[Q_1, Q_2]$. Therefore, all symmetries of equations (8) constitute a super Lie subalgebra of \mathfrak{C} , which is the centralizer of K in \mathfrak{C} and denoted by $C_{\mathfrak{C}}(K)$.

Finally, we end the section with the definition of integrability used in this paper.

Definition 2.5. The supersymmetric equation (8) is said to be *integrable* if $C_{\mathfrak{C}}(K)$ is infinite-dimensional.

3 A complete list of scalar λ -homogeneous ($\lambda > 0$) $N = 1$ supersymmetric integrable evolutionary equations

In this section, we give a global classification result for scalar λ -homogeneous $N = 1$ supersymmetric evolutionary equations like equation (1) when $\lambda > 0$. It should be reminded *the super variable Φ is assumed to be fermionic* as usual. The completeness is in the sense that every other such integrable equation is contained in the symmetry hierarchy of one of these equations presented here. A n th order supersymmetric evolutionary equation (1) is said to be λ -homogeneous if it is invariant under the scaling transformation

$$(x, \theta, t, \Phi) \mapsto (e^{-\epsilon}x, e^{-\epsilon/2}\theta, e^{-n\epsilon}t, e^{\lambda\epsilon}\Phi), \quad (\epsilon \in \mathbb{R}).$$

For example, the supersymmetric KdV equation [13, 14]

$$\Phi_t = \Phi_3 + 3\Phi_1(\mathcal{D}\Phi) + 3\Phi(\mathcal{D}\Phi_1)$$

is a 3/2-homogeneous integrable equation.

By letting $\Phi = \xi(x, t) + \theta v(x, t)$, where ξ and v are often referred to as components of the super variable Φ , any scalar $N = 1$ supersymmetric equation in Φ can be rewritten as a system in ξ and v . When ξ vanishes, the reduced equation in v is called the bosonic limit of that supersymmetric model. A supersymmetric equation is said to be a *nontrivial* extension of its bosonic limit if the corresponding system in ξ and v are truly coupled, otherwise it is categorized as a *trivial* extension since it is essentially equivalent to a certain classical equation under simple changes of variables, and trivially inherits integrability from the latter. However, it is noticed that trivial extensions happen to be involved with supersymmetric extensions of matrix models, or conformal field theories coupled to gravity [35], so they are worthy of more investigations as well. In the rest of this section, we list the nontrivial cases as well as their component forms, for which we use the notations $v_k \equiv (\partial_x^k v)$ and $\xi_k \equiv (\partial_x^k \xi)$ ($k \in \mathbb{N}$). For the trivial cases, we list them in appendix A.

There are no second order nontrivial λ -homogeneous $N = 1$ supersymmetric integrable equations of the form (1).

3.1 Third order nontrivial integrable equations

(i). Supersymmetric KdV equation [13, 14] ($\lambda = \frac{3}{2}$)

$$\Phi_t = \Phi_3 + 3\Phi_1(\mathcal{D}\Phi) + 3\Phi(\mathcal{D}\Phi_1). \quad (9)$$

We rewrite it in components:

$$\begin{cases} v_t = v_3 + 6v_1v + 3\xi_2\xi \\ \xi_t = \xi_3 + 3\xi_1v + 3\xi v_1. \end{cases}$$

As a reduction of the supersymmetric KP hierarchy introduced by Manin and Radul [13], the supersymmetric KdV equation (9) can be encoded in the Lax equation $L_t = [4(L^{3/2})_+, L]$ where $L = \partial_x^2 + \Phi \mathcal{D}$ and the subscript $_+$ means taking the differential part of a super pseudo-differential operator. Through r -matrix approach, its bi-Hamiltonian structure was established by Oevel and Popowicz [19], and its recursion operator is consequently given by

$$\mathfrak{R} = \left(\partial_x^2 \mathcal{D} + 2\Phi \partial_x + 2\partial_x \Phi + \mathcal{D}\Phi \mathcal{D} \right) \partial_x^{-1} \left(\partial_x \mathcal{D} + \Phi \right) \partial_x^{-1}. \quad (10)$$

(ii). Supersymmetric potential KdV equation ($\lambda = \frac{1}{2}$)

$$\Phi_t = \Phi_3 + 3\Phi_1(\mathcal{D}\Phi_1) \quad (11)$$

and it is rewritten in components as

$$\begin{cases} v_t = v_3 + 3v_1^2 + 3\xi_2\xi_1 \\ \xi_t = \xi_3 + 3\xi_1v_1. \end{cases}$$

If equation (11) holds, then Φ_1 solves the supersymmetric KdV equation (9). The recursion operator of equation (11) is

$$\mathfrak{R} = \partial_x^{-1} \left(\partial_x^2 \mathcal{D} + 2\Phi_1 \partial_x + 2\partial_x \Phi_1 + \mathcal{D}\Phi_1 \mathcal{D} \right) \partial_x^{-1} \left(\partial_x \mathcal{D} + \Phi_1 \right).$$

(iii). Supersymmetric modified KdV equation [14] ($\lambda = \frac{1}{2}$)

$$\Phi_t = \Phi_3 - 3\Phi_1(\mathcal{D}\Phi)^2 - 3\Phi(\mathcal{D}\Phi_1)(\mathcal{D}\Phi) \quad (12)$$

and its component form is

$$\begin{cases} v_t = v_3 - 6v_1v^2 - 3\xi_2\xi v - 3\xi_1\xi v_1 \\ \xi_t = \xi_3 - 3\xi_1v^2 - 3\xi v_1v. \end{cases}$$

If Φ is a solution to equation (12), then $\Phi_1 - \Phi(\mathcal{D}\Phi)$, which is referred to as the super Miura transformation [14], solves the supersymmetric KdV equation (9). Substituting the super Miura transformation into the recursion operator (10) and factorizing it produces the recursion operator [37] of equation (12)

$$\mathfrak{R} = 4\partial_x \Phi \mathcal{D}^{-1} \Phi \mathcal{D} - (\partial_x - \mathcal{D}\Phi \mathcal{D}^{-1} \Phi - 2\partial_x \Phi \partial_x^{-1} \Phi) \mathcal{D} (\mathcal{D} - \Phi \mathcal{D}^{-1} \Phi - 2\Phi \partial_x^{-1} \Phi \mathcal{D}).$$

(iv). Supersymmetric third order Burgers equation [31] ($\lambda = \frac{1}{2}$)

$$\Phi_t = \Phi_3 + 3\Phi_1(\mathcal{D}\Phi_1) + 3\Phi(\mathcal{D}\Phi_2) + 3\Phi(\mathcal{D}\Phi_1)(\mathcal{D}\Phi) \quad (13)$$

and in components it can be written as

$$\begin{cases} v_t = v_3 + 3v_2v + 3v_1^2 + 3v_1v^2 + 3\xi_3\xi + 3\xi_2\xi_1 + 3\xi_2\xi v + 3\xi_1\xi v_1 \\ \xi_t = \xi_3 + 3\xi_1v_1 + 3\xi v_2 + 3\xi v_1v. \end{cases}$$

Equation (13) does not commute with the trivial supersymmetric Burgers equation (48) in Appendix A, in other words, its right hand side is not a symmetry of equation (48). In fact, the third order symmetry of equation (48) should be

$$\Phi_3 + 3\Phi_2(\mathcal{D}\Phi) + 3\Phi_1(\mathcal{D}\Phi_1) + 3\Phi_1(\mathcal{D}\Phi)^2.$$

Until now, we have neither a recursion operator nor a master symmetry for equation (13).

3.2 Fifth order nontrivial integrable equations

(i). Supersymmetric Sawada-Kotera equation [36] ($\lambda = \frac{3}{2}$)

$$\Phi_t = \Phi_5 + 5\Phi_3(\mathcal{D}\Phi) + 5\Phi_2(\mathcal{D}\Phi_1) + 5\Phi_1(\mathcal{D}\Phi)^2 \quad (14)$$

and in components it is of the form

$$\begin{cases} v_t = v_5 + 5v_3v + 5v_2v_1 + 5v_1v^2 - 5\xi_3\xi_1 \\ \xi_t = \xi_5 + 5\xi_3v + 5\xi_2v_1 + 5\xi_1v^2. \end{cases}$$

The recursion operator of the supersymmetric Sawada-Kotera equation (14) was deduced from its Lax equation [36], and later factorized by Popowicz [20] into two ‘‘odd’’ Hamiltonian operators, which allow us to rewrite the recursion operator in a neater form as

$$\mathfrak{R} = \left(\partial_x^2 \mathcal{D} + 2\Phi \partial_x + 2\partial_x \Phi + \mathcal{D}\Phi \mathcal{D} \right) \partial_x^{-1} \left(\partial_x^2 \mathcal{D} + 2\Phi \partial_x + 2\partial_x \Phi + \mathcal{D}\Phi \mathcal{D} \right) \mathcal{D}^{-1} (\partial_x \mathcal{D} + \Phi)^2 \mathcal{D}^{-1}.$$

(ii). Supersymmetric fifth order KdV equation [31] ($\lambda = \frac{3}{2}$)

$$\Phi_t = \Phi_5 + 10\Phi_3(\mathcal{D}\Phi) + 15\Phi_2(\mathcal{D}\Phi_1) + 5\Phi_1(\mathcal{D}\Phi_2) + 15\Phi_1(\mathcal{D}\Phi)^2 + 15\Phi(\mathcal{D}\Phi_1)(\mathcal{D}\Phi) \quad (15)$$

and in components it can be written as

$$\begin{cases} v_t = v_5 + 10v_3v + 20v_2v_1 + 30v_1v^2 - 5\xi_3\xi_1 + 15\xi_2\xi v + 15\xi_1\xi v_1 \\ \xi_t = \xi_5 + 10\xi_3v + 15\xi_2v_1 + 5\xi_1v_2 + 15\xi_1v^2 + 15\xi v_1v. \end{cases}$$

Applying the recursion operator (10) to the supersymmetric KdV equation (9), its fifth order symmetry is obtained

$$\Phi_5 + 5\Phi_3(\mathcal{D}\Phi) + 10\Phi_2(\mathcal{D}\Phi_1) + 10\Phi_1(\mathcal{D}\Phi_2) + 5\Phi(\mathcal{D}\Phi_3) + 10\Phi_1(\mathcal{D}\Phi)^2 + 20\Phi(\mathcal{D}\Phi_1)(\mathcal{D}\Phi). \quad (16)$$

This fact implies equation (15) does not belong to the Manin-Radul’s supersymmetric KdV hierarchy leading by equation (9). In fact, equation (15) has a sixth order recursion operator [31], rewritten as

$$\mathfrak{R} = \mathcal{A} \partial_x^{-1} \left(\partial_x^3 \mathcal{D} + 6\Phi \partial_x^2 + 4(\mathcal{D}\Phi) \partial_x \mathcal{D} + 8\Phi_1 \partial_x + 2(\mathcal{D}\Phi_1) \mathcal{D} + 3\Phi_2 + 9\Phi(\mathcal{D}\Phi) \right) \partial_x^{-1} \mathcal{A} \mathcal{D}^{-1},$$

where $\mathcal{A} \equiv \partial_x^2 \mathcal{D} + 3\Phi \partial_x + (\mathcal{D}\Phi) \mathcal{D} + 2\Phi_1$.

(iii). Supersymmetric Fordy-Gibbons equation [37] ($\lambda = \frac{1}{2}$)

$$\begin{aligned} \Phi_t = & \Phi_5 - 5\Phi_2(\mathcal{D}\Phi_2) - 5\Phi_1(\mathcal{D}\Phi_3) - 5\Phi_3(\mathcal{D}\Phi)^2 - 10\Phi_3\Phi_1\Phi - 5\Phi_2(\mathcal{D}\Phi_1)(\mathcal{D}\Phi) \\ & - 10\Phi(\mathcal{D}\Phi_2)(\mathcal{D}\Phi_1) - 5\Phi_1(\mathcal{D}\Phi_2)(\mathcal{D}\Phi) - 5\Phi_1(\mathcal{D}\Phi_1)^2 - 10\Phi_2\Phi_1\Phi(\mathcal{D}\Phi) \\ & + 10\Phi_1(\mathcal{D}\Phi_1)(\mathcal{D}\Phi)^2 - 10\Phi(\mathcal{D}\Phi_1)^2(\mathcal{D}\Phi) + 5\Phi_1(\mathcal{D}\Phi)^4 \end{aligned} \quad (17)$$

and in components it is of the form

$$\begin{cases} v_t = v_5 - 5v_3v_1 - 5v_3v^2 - 5v_2^2 - 20v_2v_1v - 5v_1^3 + 5v_1v^4 - 5\xi_4\xi_1 - 5\xi_3\xi_2 \\ \quad - 5\xi_3\xi_1v - 5\xi_2\xi_1v_1 - 10\xi_2\xi(v_2 + v_1v) - 10\xi_1\xi(v_3 + v_2v + v_1^2) \\ \xi_t = \xi_5 - 5\xi_3v^2 - 10\xi_3\xi_1\xi - 5\xi_2(v_2 + v_1v) - 10\xi_2\xi_1\xi v - 5\xi_1(v_3 + v_2v) \\ \quad - 5\xi_1(v_1^2 - 2v_1v^2 - v^4) - 10\xi(v_2v_1 + v_1^2v). \end{cases}$$

If Φ is a solution to equation (17), then $\Phi_1 - \Phi(\mathcal{D}\Phi)$ solves the supersymmetric Sawada-Kotera equation (14). With the super Miura transformation, the bi-Hamiltonian structure of equation (17) was established [37] from that of the supersymmetric Sawada-Kotera equation (14). The recursion operator of equation (17) is given by

$$\mathfrak{R} = \mathcal{DB}\partial_x^{-1}\mathcal{B}^\dagger\mathcal{DB}\mathcal{D}^{-1}(\mathcal{D} - \Phi)\mathcal{D}(\mathcal{D} + \Phi)(\mathcal{D} - \Phi)\mathcal{D}(\mathcal{D} + \Phi)\mathcal{D}^{-1}\mathcal{B}^\dagger,$$

where $\mathcal{B} \equiv \partial_x - \Phi\mathcal{D} + 2(\mathcal{D}\Phi)$ and \mathcal{B}^\dagger is the formal adjoint of \mathcal{B} , i.e. $\mathcal{B}^\dagger = -\partial_x + \Phi\mathcal{D} + (\mathcal{D}\Phi)$.

(iv). Supersymmetric fifth order modified KdV equation ($\lambda = \frac{1}{2}$)

$$\begin{aligned} \Phi_t = & \Phi_5 + 5\Phi_3(\mathcal{D}\Phi_1) - 5\Phi_1(\mathcal{D}\Phi_3) - 10\Phi_3(\mathcal{D}\Phi)^2 - 15\Phi_3\Phi_1\Phi - 15\Phi_2(\mathcal{D}\Phi_1)(\mathcal{D}\Phi) \\ & - 10\Phi_1(\mathcal{D}\Phi_2)(\mathcal{D}\Phi) - 10\Phi_1(\mathcal{D}\Phi_1)^2 - 15\Phi(\mathcal{D}\Phi_2)(\mathcal{D}\Phi_1) - 15\Phi_2\Phi_1\Phi(\mathcal{D}\Phi) \\ & + 15\Phi_1(\mathcal{D}\Phi_1)(\mathcal{D}\Phi)^2 - 15\Phi(\mathcal{D}\Phi_1)^2(\mathcal{D}\Phi) + 15\Phi_1(\mathcal{D}\Phi)^4 + 15\Phi(\mathcal{D}\Phi_1)(\mathcal{D}\Phi)^3 \end{aligned} \quad (18)$$

and in components it is of the form

$$\begin{cases} v_t = v_5 - 10v_3v^2 - 40v_2v_1v - 10v_1^3 + 30v_1v^4 - 5\xi_4\xi_1 - 5\xi_3\xi_2 - 5\xi_3\xi_1v \\ \quad - 5\xi_2\xi_1v_2 - 15\xi_2\xi(v_2 + v_1v - v^3) - 15\xi_1\xi(v_3 + v_2v + v_1^2 - 3v_1v^2) \\ \xi_t = \xi_5 + 5\xi_3(v_1 - 2v^2) - 15\xi_3\xi_1\xi - 15\xi_2v_1v - 15\xi_2\xi_1\xi v - 5\xi_1(v_3 + 2v_2v) \\ \quad - 5\xi_1(2v_1^2 - 3v_1v^2 - 3v^4) - 15\xi(v_2v_1 + v_1^2v - v_1v^3). \end{cases}$$

Equation (18) is not a symmetry for the supersymmetric modified KdV equation (12), whose fifth order symmetry is given by

$$\begin{aligned} & \Phi_5 - 5\Phi_3(\mathcal{D}\Phi)^2 - 15\Phi_2(\mathcal{D}\Phi_1)(\mathcal{D}\Phi) - 15\Phi_1(\mathcal{D}\Phi_2)(\mathcal{D}\Phi) - 10\Phi_1(\mathcal{D}\Phi_1)^2 \\ & - 5\Phi(\mathcal{D}\Phi_3)(\mathcal{D}\Phi) - 10\Phi(\mathcal{D}\Phi_2)(\mathcal{D}\Phi_1) + 10\Phi_1(\mathcal{D}\Phi)^4 + 20\Phi(\mathcal{D}\Phi_1)(\mathcal{D}\Phi)^3. \end{aligned}$$

Equation (18) is related to equation (15) by a Muira transformation: *supposing Φ is a solution to equation (15), then $\Phi_1 - \Phi(\mathcal{D}\Phi)$ solves equation (15)*. Thus its integrability immediately follows from that of equation (15) due to this relation.

The main result of this paper is to show that the above list is complete for positive λ .

Theorem 1. *A scalar λ -homogeneous ($\lambda > 0$) $N = 1$ supersymmetric equation (1) is nontrivial integrable if and only if $\lambda = \frac{1}{2}$ or $\frac{3}{2}$ and the equation belongs to one the preceding eight symmetry hierarchies up to scaling transformations of t and Φ .*

In particular, unlike the classical scalar case [4], some equations such as the second order Burgers equation, the Ibragimov-Shabat equation and the Kaup-Kupershmidt equation have no nontrivial supersymmetric integrable analogues for scalar λ -homogeneous ($\lambda > 0$) supersymmetric equations of form (1). On the other hand, there are new supersymmetric third order Burgers, fifth order KdV and modified KdV equations. Supersymmetrization both decreases and increases the number of integrable equations in nontrivial cases.

4 $N = 1$ Supersymmetric differential polynomials and symbolic representation

In this section, we extend symbolic representation to the ring of supersymmetric differential polynomials, that is, the elements in \mathfrak{C} are polynomials in Φ and its (super) derivatives. As in the classical case [4], we concentrate ourselves on supersymmetric differential monomials to develop symbolic representation, and all results are straightforwardly generalized to supersymmetric differential polynomials by linearity.

A supersymmetric differential monomial of the form

$$\Phi_{k_1} \Phi_{k_2} \cdots \Phi_{k_m} (\mathcal{D}\Phi_{l_1})(\mathcal{D}\Phi_{l_2}) \cdots (\mathcal{D}\Phi_{l_n})$$

is said to be of *degree* (m, n) explicitly indicating its dependences on variables of different parities: m fermionic variables and n bosonic variables, and sometimes we say that its *total degree* is $r = m + n > 0$ to emphasize the whole multiplicity in Φ and its (super) derivatives.

Let $\mathfrak{R}^{(m,n)}$ be the space linearly spanned by all supersymmetric differential monomials of degree (m, n) with coefficients in \mathbb{C} , then for a fixed positive integer r ,

$$\mathfrak{R}^r = \bigoplus_{m+n=r} \mathfrak{R}^{(m,n)} \quad (m, n \in \mathbb{N})$$

is the space containing all monomials of total degree r and their linear combinations, and hence

$$\mathfrak{R} = \bigoplus_{r>0} \mathfrak{R}^r = \bigoplus_{m,n} \mathfrak{R}^{(m,n)} \quad (m, n \in \mathbb{N}, m + n > 0)$$

is the super commutative algebra of all supersymmetric differential polynomials in Φ and its (super) derivatives. It is noticed that $\mathbb{C} \not\subset \mathfrak{R}$.

With the super Lie bracket (6), \mathfrak{R} is a graded super Lie algebra since

$$[\mathfrak{R}^r, \mathfrak{R}^{r'}] \subset \mathfrak{R}^{r+r'-1} \quad (r, r' > 0).$$

Moreover, we have

$$[\mathfrak{R}^{(m,n)}, \mathfrak{R}^{(p,q)}] \subset \mathfrak{R}^{(m+p-1, n+q)} \bigoplus \mathfrak{R}^{(m+p+1, n+q-2)}, \quad (19)$$

where n or q are positive integers. In particular, $[\mathfrak{R}^{(1,0)}, \mathfrak{R}^{(p,q)}] \subset \mathfrak{R}^{(p,q)}$.

To describe supersymmetric differential polynomials in symbolic language, the basic idea is to convert the fermionic variable Φ_k to $\phi\zeta^k$, and the bosonic one $(\mathcal{D}\Phi_k)$ to $u z^k$, where ϕ or u are used to count degrees, and Φ 's subscript k indicating the order of derivatives is replaced by powers of symbols. As results of this replacement, the differentiation with respect to x is transformed to multiplications of symbols. Variables in the same monomials of higher (total) degrees are assigned with different symbols, for instance, the quadratic term $\Phi_j \Phi_k$ is converted to $\phi^2 \zeta_1^j \zeta_2^k$, while $(\mathcal{D}\Phi_j)(\mathcal{D}\Phi_k)$ is replaced by $u^2 z_1^j z_2^k$. With these two simple examples, it is noticed that there are some confusions aroused by the super commutativity of multiplication. Specifically, we have

$$\Phi_j \Phi_k = -\Phi_k \Phi_j \quad \text{and} \quad (\mathcal{D}\Phi_j)(\mathcal{D}\Phi_k) = (\mathcal{D}\Phi_k)(\mathcal{D}\Phi_j).$$

Following above discussion, $-\Phi_k \Phi_j$ is replaced by $-\phi^2 \zeta_1^k \zeta_2^j$, while $(\mathcal{D}\Phi_k)(\mathcal{D}\Phi_j)$ by $u^2 z_1^k z_2^j$. From equivalent products, different symbolic expressions are obtained. To overcome these obstacles, the

unique symbolic representation is defined to be the average of all possible symbolic expressions. Hence, symbolic forms of $\Phi_j\Phi_k$ and $(\mathcal{D}\Phi_j)(\mathcal{D}\Phi_k)$ are respectively given by

$$\frac{1}{2}\phi^2(\zeta_1^j\zeta_2^k - \zeta_2^j\zeta_1^k) \quad \text{and} \quad \frac{1}{2}u^2(z_1^jz_2^k + z_2^jz_1^k).$$

Definition 4.1. The symbolic representation of a monomial in $\mathfrak{R}^{(m,n)}$ is defined by

$$\Phi_{k_1}\Phi_{k_2}\cdots\Phi_{k_m}(\mathcal{D}\Phi_{l_1})(\mathcal{D}\Phi_{l_2})\cdots(\mathcal{D}\Phi_{l_n}) \mapsto \phi^m u^n \langle \zeta_1^{k_1}\zeta_2^{k_2}\cdots\zeta_m^{k_m} \rangle_{\mathcal{S}_m^\zeta} \langle z_1^{l_1}z_2^{l_2}\cdots z_n^{l_n} \rangle_{\mathcal{S}_n^z}$$

where $\langle \rangle_{\mathcal{S}_m^\zeta}$ stands for anti-symmetrization of the product $\zeta_1^{k_1}\zeta_2^{k_2}\cdots\zeta_m^{k_m}$, i.e.

$$\langle \zeta_1^{k_1}\zeta_2^{k_2}\cdots\zeta_m^{k_m} \rangle_{\mathcal{S}_m^\zeta} = \frac{1}{m!} \sum_{\sigma \in \mathcal{S}_m} \text{sgn}(\sigma) \zeta_{\sigma(1)}^{k_1} \zeta_{\sigma(2)}^{k_2} \cdots \zeta_{\sigma(m)}^{k_m},$$

and $\langle \rangle_{\mathcal{S}_n^z}$ means to symmetrise the product $z_1^{l_1}z_2^{l_2}\cdots z_n^{l_n}$, namely

$$\langle z_1^{l_1}z_2^{l_2}\cdots z_n^{l_n} \rangle_{\mathcal{S}_n^z} = \frac{1}{n!} \sum_{\sigma \in \mathcal{S}_n} z_{\sigma(1)}^{l_1} z_{\sigma(2)}^{l_2} \cdots z_{\sigma(n)}^{l_n}.$$

A differential polynomial is a finite linear combination of monomials. The symbolic representation of a polynomial $P \in \mathfrak{R}^{(m,n)}$ denoted by \widehat{P} is in general of the form

$$\widehat{P} = \phi^m u^n F(\zeta_1, \dots, \zeta_m; z_1, \dots, z_n),$$

where the multi-variable polynomial F is anti-symmetric with respect to its first m arguments, but symmetric with respect to the others. Suppose a polynomial $Q \in \mathfrak{R}^{(p,q)}$ has the symbolic representation

$$\widehat{Q} = \phi^p u^q H(\zeta_1, \dots, \zeta_p; z_1, \dots, z_q),$$

then the product PQ has the following symbolic representation:

$$\widehat{PQ} = \phi^{m+p} u^{n+q} \langle F(\zeta_1, \dots, \zeta_m; z_1, \dots, z_n) H(\zeta_{m+1}, \dots, \zeta_{m+p}; z_{n+1}, \dots, z_{n+q}) \rangle_{\mathcal{S}_{m+p}^\zeta, \mathcal{S}_{n+q}^z}, \quad (20)$$

where $\langle \rangle_{\mathcal{S}_{m+p}^\zeta, \mathcal{S}_{n+q}^z}$ means to anti-symmetrize ζ_j 's, and meanwhile symmetrize z_i 's.

This defines the corresponding multiplication for the symbols. Formula (20) is consistent with Definition 4.1. Symbolic representations of super differential monomials can be deduced from $\Phi_k \mapsto \phi\zeta_1^k$, $\mathcal{D}\Phi_k \mapsto uz_1^k$ and the multiplication rule (20).

All operations for the ring of supersymmetric differential polynomials can be carried out with symbols in much simpler way. Partial derivatives of P with respect to Φ_k or $(\mathcal{D}\Phi_k)$ can be calculated via its symbolic representation as follows:

$$\frac{\partial \widehat{P}}{\partial \Phi_k} = m\phi^{m-1}u^n \frac{1}{k!} \frac{\partial^k F}{\partial \zeta_m^k}(\zeta_1, \dots, \zeta_{m-1}, 0; z_1, \dots, z_n); \quad (21)$$

$$\frac{\partial \widehat{P}}{\partial (\mathcal{D}\Phi_k)} = n\phi^m u^{n-1} \frac{1}{k!} \frac{\partial^k F}{\partial z_n^k}(\zeta_1, \dots, \zeta_m; z_1, \dots, z_{n-1}, 0). \quad (22)$$

According to the definition of the super derivative \mathcal{D} and using the above two formulas, it follows that

$$\widehat{(\mathcal{D}P)} = (-1)^{m-1} m\phi^{m-1}u^{n+1} \langle F(\zeta_1, \dots, \zeta_{m-1}, z_{n+1}; z_1, \dots, z_n) \rangle_{\mathcal{S}_{n+1}^z}$$

$$+ (-1)^m n \phi^{m+1} u^{n-1} \langle F(\zeta_1, \dots, \zeta_m; z_1, \dots, z_{n-1}, \zeta_{m+1}) \zeta_{m+1} \rangle_{\mathcal{S}_{m+1}^\zeta}. \quad (23)$$

$$\begin{aligned} &= (-1)^{m-1} \frac{m}{n+1} \phi^{m-1} u^{n+1} \sum_{i=1}^{n+1} F(\zeta_1, \dots, \zeta_{m-1}, z_i; z_1, \dots, \bar{z}_i, \dots, z_{n+1}) \\ &+ \frac{n}{m+1} \phi^{m+1} u^{n-1} \sum_{j=1}^{m+1} (-1)^{j-1} F(\zeta_1, \dots, \bar{\zeta}_j, \dots, \zeta_{m+1}; z_1, \dots, z_{n-1}, \zeta_j) \zeta_j, \end{aligned} \quad (24)$$

where \bar{z}_i means that z_i is omitted and $\bar{\zeta}_j$ is defined in the same manner.

Example 4.1. Given $P = \Phi_3 \Phi_2 \Phi_1$, we calculate $\partial P / \partial \Phi_2$ using symbolic representation. The symbolic representation for P is

$$\widehat{P} = \phi^3 F(\zeta_1, \zeta_2, \zeta_3) = \frac{1}{6} \phi^3 (\zeta_1^3 \zeta_2^2 \zeta_3^1 + \zeta_2^3 \zeta_3^2 \zeta_1^1 + \zeta_3^3 \zeta_1^2 \zeta_2^1 - \zeta_1^3 \zeta_3^2 \zeta_2^1 - \zeta_2^3 \zeta_1^2 \zeta_3^1 - \zeta_3^3 \zeta_2^2 \zeta_1^1).$$

According to formula (21), we have

$$\frac{\widehat{\partial P}}{\partial \Phi_2} = 3\phi^2 \frac{1}{2!} \frac{\partial^2 F}{\partial \zeta_3^2}(\zeta_1, \zeta_2, 0) = -\frac{1}{2} \phi^2 (\zeta_1^3 \zeta_2^1 - \zeta_2^3 \zeta_1^1),$$

which conversely implies $\partial P / \partial \Phi_2 = -\Phi_3 \Phi_1$, the expected result.

Example 4.2. For $P = \Phi_1 \Phi$, let's calculate $(\mathcal{D}P)$ and $(\partial_x P)$ through symbolic representation. Because

$$\widehat{P} = \phi^2 F(\zeta_1, \zeta_2) = \frac{1}{2} \phi^2 (\zeta_1^1 \zeta_2^0 - \zeta_2^1 \zeta_1^0),$$

it follows from formula (23) that

$$\widehat{(\mathcal{D}P)} = (-1)2\phi u F(\zeta_1, z_1) = -\phi u (\zeta_1^1 z_1^0 - z_1^1 \zeta_1^0),$$

which means $(\mathcal{D}P) = -\Phi_1(\mathcal{D}\Phi) + \Phi(\mathcal{D}\Phi_1)$. Applying formula (23) to $\widehat{(\mathcal{D}P)}$ yields

$$\widehat{(\partial_x P)} = -u^2 \langle z_2^1 z_1^0 \rangle_{\mathcal{S}_2^z} + u^2 \langle z_1^1 z_2^0 \rangle_{\mathcal{S}_2^z} + \phi^2 \langle \zeta_1^1 \zeta_2^1 \rangle_{\mathcal{S}_2^\zeta} - \phi^2 \langle \zeta_2^2 \zeta_1^0 \rangle_{\mathcal{S}_2^\zeta} = -\frac{1}{2} \phi^2 (\zeta_2^2 \zeta_1^0 - \zeta_1^2 \zeta_2^0),$$

which give us $(\partial_x P) = \Phi_2 \Phi$ as expected.

In the above example, we notice that $\widehat{(\partial_x P)} = (\zeta_1 + \zeta_2) \widehat{P}$ for $P = \Phi_1 \Phi$. Here we give the general formula for it.

Proposition 1. Let $P \in \mathfrak{R}^{(m,n)}$ with symbolic representation $\widehat{P} = \phi^m u^n F(\zeta_1, \dots, \zeta_m; z_1, \dots, z_n)$. Then

$$\widehat{(\partial_x P)} = \widehat{P} \left(\sum_{j=1}^m \zeta_j + \sum_{i=1}^n z_i \right).$$

Furthermore,

$$\widehat{(\partial_x^k P)} = \widehat{P} \left(\sum_{j=1}^m \zeta_j + \sum_{i=1}^n z_i \right)^k \quad (k \in \mathbb{N}).$$

Proof. Since $(\partial_x P) = \mathcal{D}(\mathcal{D}P)$, the statement for $(\partial_x P)$ can be proved by applying the symbolic rule of super derivative (23) or (24) to $(\widehat{\mathcal{D}\Phi})$. We rewrite $\widehat{\mathcal{D}P} = \widehat{Q}_1 + \widehat{Q}_2$, where

$$\begin{aligned}\widehat{Q}_1 &= (-1)^{m-1} \frac{m}{n+1} \phi^{m-1} u^{n+1} \sum_{i=1}^{n+1} F(\zeta_1, \dots, \zeta_{m-1}, z_i; z_1, \dots, \bar{z}_i, \dots, z_{n+1}), \\ \widehat{Q}_2 &= \frac{n}{m+1} \phi^{m+1} u^{n-1} \sum_{j=1}^{m+1} (-1)^{j-1} F(\zeta_1, \dots, \bar{\zeta}_j, \dots, \zeta_{m+1}; z_1, \dots, z_{n-1}, \zeta_j) \zeta_j\end{aligned}$$

and $Q_1 \in \mathfrak{R}^{(m-1, n+1)}$ while $Q_2 \in \mathfrak{R}^{(m+1, n-1)}$. We apply formula (23) again to \widehat{Q}_1 and \widehat{Q}_2 separately, and carry out the symmetrization and anti-symmetrization accordingly. This leads to the proof of the statement. The calculation is straightforward and we won't include all details. \square

Note that formulas in Proposition 1 are essentially the same as the symbolic rule for derivatives with respect to x in classical case [4].

We introduce the symbol for the super derivative \mathcal{D} as η , and consequently η^2 stands for ∂_x . According to Definition 2.3 and using formulas (21) and (22), we obtain the following statement for Fréchet derivatives.

Proposition 2. *Let $P \in \mathfrak{R}^{(m, n)}$ with symbolic representation $\widehat{P} = \phi^m u^n F(\zeta_1, \dots, \zeta_m; z_1, \dots, z_n)$. The Fréchet derivative of P is represented in the following expression:*

$$\widehat{D}_P = m\phi^{m-1} u^n F(\zeta_1, \dots, \zeta_{m-1}, \eta^2; z_1, \dots, z_n) + n\phi^m u^{n-1} F(\zeta_1, \dots, \zeta_m; z_1, \dots, z_{n-1}, \eta^2) \eta, \quad (25)$$

where η is the symbol for \mathcal{D} .

Example 4.3. Let $P = \Phi_2 \Phi(\mathcal{D}\Phi_1)(\mathcal{D}\Phi)$. It has the symbolic representation

$$\widehat{P} = \phi^2 u^2 F(\zeta_1, \zeta_2; z_1, z_2) = \frac{1}{4} \phi^2 u^2 (\zeta_1^2 \zeta_2^0 - \zeta_2^2 \zeta_1^0) (z_1^1 z_2^0 + z_2^1 z_1^0).$$

According to Proposition 2, we have

$$\begin{aligned}\widehat{D}_P &= 2\phi u^2 F(\zeta_1, \eta^2; z_1, z_2) + 2\phi^2 u F(\zeta_1, \zeta_2; z_1, \eta^2) \eta \\ &= \frac{1}{2} \phi u^2 \zeta_1^2 (z_1^1 z_2^0 + z_2^1 z_1^0) \eta^0 - \frac{1}{2} \phi u^2 \zeta_1^0 (z_1^1 z_2^0 + z_2^1 z_1^0) \eta^4 + \frac{1}{2} \phi^2 u (\zeta_1^2 \zeta_2^0 - \zeta_2^2 \zeta_1^0) z_1^1 \eta \\ &\quad + \frac{1}{2} \phi^2 u (\zeta_1^2 \zeta_2^0 - \zeta_2^2 \zeta_1^0) z_1^0 \eta^3,\end{aligned}$$

which implies that the Fréchet derivative of P is given by

$$D_P = \Phi_2(\mathcal{D}\Phi_1)(\mathcal{D}\Phi) - \Phi(\mathcal{D}\Phi_1)(\mathcal{D}\Phi) \partial_x^2 + \Phi_2 \Phi(\mathcal{D}\Phi_1) \mathcal{D} + \Phi_2 \Phi(\mathcal{D}\Phi) \mathcal{D} \partial_x.$$

Having derived the symbolic formulas for the super derivative \mathcal{D} , the usual derivative ∂_x and Fréchet derivatives, we are in the place to write down the symbolic expression for $D_P(Q)$ for $P \in \mathfrak{R}^{(m, n)}$ and $Q \in \mathfrak{R}^{(p, a)}$ as in classical case [4]. However, the formula is rather long. We only give two special and simple cases, which we are going to use later.

Proposition 3. *Let $P \in \mathfrak{R}^{(m, n)}$ and $P \mapsto \widehat{P} = \phi^m u^n F(\zeta_1, \dots, \zeta_m; z_1, \dots, z_n)$. Then*

$$\widehat{D}_P(\widehat{\Phi}_k) = \widehat{P} \left(\sum_{j=1}^m \zeta_j^k + \sum_{i=1}^n z_i^k \right). \quad (26)$$

Proof. Notice that $\widehat{\Phi}_k = \phi \zeta_1^k$, $\eta \widehat{\Phi}_k = u z_1^k$ and $\eta^2 \widehat{\Phi}_k = \phi \zeta_1^{k+1}$. Using Proposition 2, we have

$$\widehat{D}_P(\widehat{\Phi}_k) = m \phi^m u^n \langle F(\zeta_1, \dots, \zeta_{m-1}, \zeta_m; z_1, \dots, z_n) \zeta_m^k \rangle_{S_m^\zeta} + n \phi^m u^n \langle F(\zeta_1, \dots, \zeta_m; z_1, \dots, z_{n-1}, z_n) z_n^k \rangle_{S_n^z}.$$

After symmetrisation and anti-symmetrisation, we obtain the required formula. \square

Combining the result in this proposition and Proposition 1, we have the following formula for a super commutator (cf. equation (6)), which plays an important role in later classification.

Corollary 1. *Let $P \in \mathfrak{R}^{(m,n)}$ and $P \mapsto \widehat{P} = \phi^m u^n F(\zeta_1, \dots, \zeta_m; z_1, \dots, z_n)$. The super commutator $[\Phi_k, P]$ is symbolically formulated as*

$$[\widehat{\Phi}_k, P] = \widehat{P} \left\{ \left(\sum_{j=1}^m \zeta_j + \sum_{i=1}^n z_i \right)^k - \left(\sum_{j=1}^m \zeta_j^k + \sum_{i=1}^n z_i^k \right) \right\}. \quad (27)$$

Proposition 4. *Let $P, Q \in \mathfrak{R}^{(1,1)}$ and $P \mapsto \widehat{P} = \phi u F(\zeta_1, z_1)$, $Q \mapsto \widehat{Q} = \phi u H(\zeta_1, z_1)$. Then*

$$\begin{aligned} \widehat{D}_P(\widehat{Q}) &= \phi u^2 \langle F(\zeta_1 + z_2, z_1) H(\zeta_1, z_2) + F(\zeta_1; z_1 + z_2) H(z_1, z_2) \rangle_{S_z^2} \\ &\quad - \phi^3 \langle F(\zeta_1, \zeta_2 + \zeta_3) H(\zeta_2, \zeta_3) \zeta_3 \rangle_{S_\zeta^3} \end{aligned}$$

Proof. The statement can be proved by directly applying Proposition 2 and Proposition 1. \square

In the end of this section, we calculate a fifth order symmetry of the supersymmetric KdV equation (9) using the symbolic method to illustrate its mechanism. We know that the supersymmetric KdV equation is of the form

$$\Phi_t = \Phi_3 + K_2 \quad \text{with} \quad K_2 = 3\Phi_1(\mathcal{D}\Phi) + 3\Phi(\mathcal{D}\Phi_1) \in \mathfrak{R}^{(1,1)}.$$

We compute its fifth order symmetry

$$Q = \Phi_5 + Q_2 + Q_3 + \dots \quad \text{with} \quad Q_i \in \mathfrak{R}^i \quad (i = 2, 3, \dots)$$

term by term according to its degree. Firstly the quadratic term $Q_2 \in \mathfrak{R}^2$ is determined by

$$[\Phi_3, Q_2] + [K_2, \Phi_5] = 0.$$

Following Corollary 1, we convert it to symbolics

$$\widehat{Q}_2 \left((\zeta_1 + z_1)^3 - \zeta_1^3 - z_1^3 \right) = \widehat{K}_2 \left((\zeta_1 + z_1)^5 - \zeta_1^5 - z_1^5 \right),$$

where $\widehat{K}_2 = 3\phi u (\zeta_1^1 z_1^0 + \zeta_1^0 z_1^1)$. Hence, we obtain

$$\widehat{Q}_2 = \frac{(\zeta_1 + z_1)^5 - \zeta_1^5 - z_1^5}{(\zeta_1 + z_1)^3 - \zeta_1^3 - z_1^3} \widehat{K}_2 = 5\phi u (\zeta_1^3 z_1^0 + \zeta_1^0 z_1^3) + 10\phi u (\zeta_1^1 z_1^2 + \zeta_1^2 z_1^1),$$

which implies $Q_2 = 5\Phi_3(\mathcal{D}\Phi) + 5\Phi(\mathcal{D}\Phi_3) + 10\Phi_1(\mathcal{D}\Phi_2) + 10\Phi_2(\mathcal{D}\Phi_1)$, which is completely determined by the quadratic terms of the equation.

Secondly, the cubic terms $Q_3 \in \mathfrak{R}^3$ is determined by

$$[\Phi_3, Q_3] + [K_2, Q_2] = 0. \quad (28)$$

According to Proposition 4, for $K_2, Q_2 \in \mathfrak{R}^{(1,1)}$ we have

$$\begin{aligned} [\widehat{Q_2, K_2}] = & 30\phi u^2 (\zeta_1^3 z_1^1 z_2^0 + \zeta_1^3 z_2^1 z_1^0 + 2\zeta_1^2 z_1^2 z_2^0 + 2\zeta_1^2 z_2^2 z_1^0 + 4\zeta_1^2 z_1^1 z_2^1 + \zeta_1^1 z_1^3 z_2^0 + \zeta_1^1 z_2^3 z_1^0) \\ & + 30\phi u^2 (4\zeta_1^1 z_1^2 z_2^1 + 4\zeta_1^1 z_2^2 z_1^1 + \zeta_1^0 z_1^3 z_2^1 + \zeta_1^0 z_2^3 z_1^1 + 2\zeta_1^1 z_1^2 z_2^2). \end{aligned}$$

Thus $[Q_2, K_2] \in \mathfrak{R}^{(1,2)}$. According to equation (19), for any $P \in \mathfrak{R}^{(m,n)}$ we have $[\Phi_k, P] \in \mathfrak{R}^{(m,n)}$. So equation (28) implies $Q_3 \in \mathfrak{R}^{(1,2)}$. From the symbolic representation of equation (28), we deduce

$$\widehat{Q_3} = \frac{[\widehat{Q_2, K_2}]}{(\zeta_1 + z_1 + z_2)^3 - \zeta_1^3 - z_1^3 - z_2^3} = 10\phi u^2 (\zeta_1^1 z_1^0 z_2^0 + \zeta_1^0 z_1^1 z_2^0 + \zeta_1^0 z_2^1 z_1^0),$$

which gives us $Q_3 = 10\Phi_1(\mathcal{D}\Phi)^2 + 20\Phi(\mathcal{D}\Phi_1)(\mathcal{D}\Phi)$.

It is straightforward to check $[K_2, Q_3] = 0$. Therefore, a fifth order symmetry of the supersymmetric KdV equation (9) starting with Φ_5 is the one given by (16) generated by its recursion operator (10).

5 Classification of λ -homogeneous equations

In this section, we use the symbolic representation to prove our main classification Theorem 1. The computations are remarkably similar to the classical commutative [4] and non-commutative [5] cases. The key differences are

- the polynomials arising in the symbolic computations include two sets of variables: one set symmetrised and the other anti-symmetrised under permutations;
- while the bounds on the orders of the equation and its symmetries happen to be the same as in the classical cases, the symbolic computation relies on whether or not the variables commute and this leads to rather different classification results from the classical ones.

In [4], we gave extensive results about the mutual divisibility of certain particular multivariate polynomials, called “ G -functions”, which play a crucial role in proving the (non-)existence of symmetries. In Corollary 1, we demonstrated that the same G -functions appear in the computation. Thus all the results discussed in [4, Section 5] are immediately applicable.

Definition 5.1. The G -functions are the (commutative) polynomials

$$G_m^{(l)}(y_1, \dots, y_l) = (y_1 + \dots + y_l)^m - y_1^m - \dots - y_l^m \quad (l \geq 2).$$

Formula (27) can be rewritten in terms of G -function as

$$[\widehat{\Phi_k, Q}] = G_k^{(2l+n+1)}(\zeta_1, \dots, \zeta_{2l+1}, z_1, \dots, z_n) \widehat{Q}, \quad Q \in \mathfrak{R}^{(2l+1, n)}. \quad (29)$$

An immediate consequence is the following known result:

Proposition 5. Consider the linear evolutionary equation $\Phi_t = \sum_{k=1}^n \lambda_k \Phi_k$, where λ_k 's are constants and $\lambda_n \neq 0$. The space of its symmetries is

- \mathfrak{R} if and only if $n = 1$;
- \mathfrak{R}^1 if and only if $n > 1$.

Proof. Let $Q = \sum_{l,p} Q_{(l,p)}$, where $Q_{(l,p)} \in \mathfrak{R}^{(l,p)}$. Because \mathfrak{R} is a graded Lie algebra, and particularly $[\mathfrak{R}^{(1,0)}, \mathfrak{R}^{(l,p)}] \subset \mathfrak{R}^{(l,p)}$, Q is a symmetry of this equation if and only if $[\sum_{k=1}^n \lambda_k \Phi_k, Q] = 0$. From formula (29) it follows that

$$\sum_{k=1}^n \lambda_k G_k^{l+p}(\zeta_1, \dots, \zeta_l, z_1, \dots, z_p) \widehat{Q_{(l,p)}} = 0.$$

Under the assumption, this holds if and only if either $n = 1$ or $n > 1$ and $l + p = 1$. □

We recall the divisibility properties of the G -functions proved in [38, 4].

Theorem 2. *The symmetric polynomials $G_m^{(l)}$ ($l \geq 2$) can be factorized as*

$$G_m^{(l)} = t_m^{(l)} H_m^{(l)},$$

where $(H_m^{(l)}, H_n^{(l)}) = 1$ for all $n > m$, and $t_m^{(l)}$ is one of the following polynomials.

- $l = 2$:
 - ▷ $m = 0 \pmod{2}$: $y_1 y_2$
 - ▷ $m = 3 \pmod{6}$: $y_1 y_2 (y_1 + y_2)$
 - ▷ $m = 5 \pmod{6}$: $y_1 y_2 (y_1 + y_2) (y_1^2 + y_1 y_2 + y_2^2)$
 - ▷ $m = 1 \pmod{6}$: $y_1 y_2 (y_1 + y_2) (y_1^2 + y_1 y_2 + y_2^2)^2$
- $l = 3$:
 - ▷ $m = 0 \pmod{2}$: 1
 - ▷ $m = 1 \pmod{2}$: $(y_1 + y_2)(y_1 + y_3)(y_2 + y_3)$
- $l \geq 4$: 1

We now consider λ -homogeneous ($\lambda > 0$) $N = 1$ supersymmetric equations of the form

$$\Phi_t = K = K_1 + K_2 + K_3 + \dots, \tag{30}$$

where $K_1 = \Phi_n$ ($n \geq 2$) and

$$K_i \in \bigoplus_{l=0}^{\lfloor \frac{i-1}{2} \rfloor} \mathfrak{R}^{(2l+1, i-2l-1)} \subset \mathfrak{R}^i.$$

For each K_i , the degree of its symbolic representation \widehat{K}_i is determine by

$$d_i = \begin{cases} n - (i-1)\lambda - \frac{1}{2} & (i \text{ is even.}) \\ n - (i-1)\lambda & (i \text{ is odd.}) \end{cases}$$

Note that if d_i is not in \mathbb{N} then $K_i = 0$. This constraint restricts equation (30) to finite terms, and also reduces the number of relevant λ to be finite.

Due to Proposition 5, a nontrivial symmetry $Q \in \mathfrak{R}$ of equation (30) is supposed to be

$$Q = Q_1 + Q_2 + Q_3 + \cdots, \quad (Q_i \in \mathfrak{R}^i) \quad (31)$$

where the leading term Q_1 is either Φ_m or $(D\Phi_m)$. For the latter, according to equation (6) we obtain the super commutator of Q with itself, given by

$$[Q, Q] = 2\Phi_{2m+1} + \cdots$$

which is a new symmetry of equation (30). Hence, without loss of generality, we only assume Q is an m -th order nontrivial symmetry, which starts with $Q_1 = \Phi_m$ ($2 \leq m \neq n$).

To solve the symmetry condition $[K, Q] = 0$, we break it up into

$$\sum_{i=1}^r [K_i, Q_{r+1-i}] = 0, \quad (32)$$

which holds for all $r \geq 1$. Clearly we have $[K_1, Q_1] = 0$ when $r = 1$. The next equation to be solved is for $r = 2$, that is,

$$[K_1, Q_2] + [K_2, Q_1] = 0, \quad (33)$$

which is trivially satisfied if equation (30) has no quadratic terms, i.e. $K_2 = 0$. Let us concentrate on the case $K_2 \neq 0$. We first rewrite equation (33) in symbolic form using Corollary 1 or formula (29). Because $K_2 \in \mathfrak{R}^{(1,1)}$, we assume that $\widehat{K}_2 = \phi u F(\zeta_1, z_1)$. Then \widehat{Q}_2 is solved and formulated as

$$\widehat{Q}_2 = \phi u \frac{F(\zeta_1, z_1) G_m^{(2)}(\zeta_1, z_1)}{G_n^{(2)}(\zeta_1, z_1)} = \phi u \frac{\Lambda(\zeta_1, z_1)}{\zeta_1 z_1 (\zeta_1 + z_1)} G_m^{(2)}(\zeta_1, z_1), \quad (34)$$

where $\lim_{\zeta_1+z_1 \rightarrow 0} \Lambda(\zeta_1, z_1)$ exists.

Taking $r = 3$ in equation (32), we have

$$[K_1, Q_3] + [K_2, Q_2] + [K_3, Q_1] = 0. \quad (35)$$

To find Q_3 , we have to compute $[K_2, Q_2]$, whose symbolic representation can be obtained by using Proposition 4. Notice that there is a unique decomposition

$$[K_2, Q_2] = P_{(1,2)} + P_{(3,0)} \quad \text{where } P_{(1,2)} \in \mathfrak{R}^{(1,2)} \text{ and } P_{(3,0)} \in \mathfrak{R}^{(3,0)}.$$

It follows from Theorem 2 that the nontrivial common factor occurs when both n and m are odd. Thus we need to treat this case specially. In the same way as we did for classification of scalar homogeneous both commutative and non-commutative evolutionary equations, we are able to show that $\widehat{P}_{(1,2)}$ is exactly divided by $(\zeta_1 + z_1)(\zeta_1 + z_2)(z_1 + z_2)$, while $\widehat{P}_{(3,0)}$ by $(\zeta_1 + \zeta_2)(\zeta_1 + \zeta_3)(\zeta_2 + \zeta_3)$ when both m and n are odd and $(\zeta_1 + z_1) | \widehat{K}_2$ or $\zeta_1 z_1 | \widehat{K}_2$. Therefore, in this case, the existence of a symmetry is uniquely determined by the existence of its quadratic term. The proof is the same as we did in classical cases [4, 5]. For completeness, we only present the statement.

Theorem 3. Suppose equation (30) has a nonzero symmetry Q of the form (31) with $m \geq 2$. Let

$$S = \Phi_{m'} + S_2 + \cdots \quad S_i \in \mathfrak{R}^i. \quad (36)$$

If a nonzero quadratic differential polynomial $S_2 \in \mathfrak{R}^2$ satisfies $[\Phi_n, S_2] + [K_2, \Phi_{m'}] = 0$ (cf. equation (33)) when n and m' are both odd, then it uniquely determines a symmetry of the form (36) for equation (30). Moreover, these symmetries commute with each other, i.e. $[Q, S] = 0$.

This theorem shows that the existence of one symmetry implies the existence of infinitely many symmetries as long as we know the existence of either quadratic or cubic terms of the symmetries for equation (30). We are now ready to prove an important classification result.

Theorem 4. A nontrivial symmetry of a λ -homogeneous supersymmetric evolutionary equation with $\lambda > 0$ is part of a hierarchy starting at order 2, 3, 5 or 7.

Proof. It follows from Theorem 2 that the symbolic expression of \widehat{Q}_2 (cf. equation (34)) can be reformulated as

$$\frac{\widehat{K}_2 (\zeta_1^2 + \zeta_1 z_1 + z_1^2)^{s-s'} H_m^{(2)}(\zeta_1, z_1)}{H_n^{(2)}(\zeta_1, z_1)}, \quad (37)$$

where $s = \frac{m+3}{2} \pmod{3}$ and $s' = \frac{n+3}{2} \pmod{3}$, when n and m are both odd. The quadratic term Q_2 is only well-defined if expression (37) is a polynomial, and further uniquely determines a symmetry $Q = \Phi_n + Q_2 + \cdots$. As a necessary condition for this requirement, we deduce that $\lambda \leq \frac{5}{2} + 2 \min(s, s')$. The evolutionary equations defined by Q has the same symmetries as equation (30). So instead of equation (30) we may consider the equation given by Q . The lowest possible m is $2s+3$ for $s = 0, 1, 2$. Therefore we only need to consider λ -homogeneous equations with $\lambda \leq 6\frac{1}{2}$ of orders not greater than 7.

A similar observation can be made for even $n > 2$. Suppose we have found a nontrivial symmetry with quadratic term

$$\phi u \frac{F(\zeta_1, z_1) G_m^{(2)}}{\zeta_1 z_1 H_n^{(2)}},$$

which immediately implies $\lambda \leq \frac{3}{2}$. Then the quadratic term corresponding to $2\phi u \frac{F(\zeta_1, z_1)}{H_n^{(2)}}$ defines a symmetry starting with Φ_2 . Therefore, we only need to find the symmetries of equations of order 2 to get the complete classification of symmetries of scalar λ -homogeneous equations (with $\lambda \leq \frac{3}{2}$) starting with an even linear term.

Finally, we analyze the case when equation (30) has no quadratic terms. If $K_i = 0$ for $i = 2, \dots, j-1$, then we translate $[K_1, Q_j] + [K_j, Q_1] = 0$ to symbolics, i.e.

$$G_n^{(j)} \widehat{Q}_j = G_m^{(j)} \widehat{K}_j.$$

According to Theorem 2, we know that there are no symmetries for equation (30) when $j \geq 4$, or when $j = 3$ and n is even. When $j = 3$ and n is odd, it can only have odd order symmetries. In this situation, one can remark that if the equation possesses symmetries for any m then it must admit a symmetry of order 3. \square

Only equations with nonzero quadratic or cubic terms have nontrivial symmetries. For each possible $\lambda > 0$, we must find a third order symmetry for a second order equation, a fifth order symmetry

n	λ									
2	1/2,	3/2,								
3	1/2,	1,	3/2,	5/2,						
5	1/2,	1,	3/2,	2,	5/2,	7/2,	9/2,			
7	1/2,	1,	3/2,	2,	5/2,	3,	7/2,	9/2,	11/2,	13/2

Table 1: All values of n and λ

for a third order equation, a seventh order symmetry for a fifth order equation with quadratic terms, and the thirteenth order symmetry for a seventh order equation with quadratic terms. Hence, there are finite possibilities for the order n and λ , as illustrated by Table 1.

For every pair of (n, λ) , equation (1) can be explicitly written out as supersymmetric differential polynomials in Φ with free coefficients. Let's take the second order equations as examples. For $\lambda = 3/2$, the generic second order equation is given by

$$\Phi_t = \Phi_2 + K_2 \quad \text{with} \quad K_2 = c\Phi(\mathcal{D}\Phi), \quad (38)$$

while for $\lambda = 1/2$ we have

$$\Phi_t = \Phi_2 + K_2 + K_3 \quad \text{with} \quad K_2 = c_1\Phi_1(\mathcal{D}\Phi) + c_2\Phi(\mathcal{D}\Phi_1), \quad K_3 = c_3\Phi(\mathcal{D}\Phi)^2, \quad (39)$$

where c and $c_i (i = 1, 2, 3)$ are parameters to be determined. To identify integrable cases, we have to find third order symmetries for second order equations. As we did for the supersymmetric KdV equation at the end of Section 4, we compute the third order homogeneous symmetries Q starting with the linear term Φ_3 and determine the other terms degree by degree.

Regrading equation (38), its third order symmetry is of the form $Q = \Phi_3 + Q_2$ ($Q_2 \in \mathfrak{R}^{(1,1)}$) such that

$$[\Phi_2, Q_2] + [K_2, \Phi_3] = 0, \quad (40)$$

$$[K_2, Q_2] = 0. \quad (41)$$

Following general symbolic formulas in section 4, we solve both equations in terms of symbols. Equation (40) is converted to the symbolic form as

$$\widehat{Q}_2 \left((\zeta_1 - z_1)^2 - \zeta_1^2 - z_1^2 \right) = \widehat{K}_2 \left((\zeta_1 - z_1)^3 - \zeta_1^3 - z_1^3 \right),$$

where $\widehat{K}_2 = c\phi u \zeta_1^0 z_1^0$. Now we have

$$\widehat{Q}_2 = \frac{3}{2}c\phi u (\zeta_1^1 z_1^0 + \zeta_1^0 z_1^1).$$

According to proposition 4, the symbolic expression of $[K_2, Q_2]$ is given by

$$[\widehat{K}_2, \widehat{Q}_2] = -\frac{3}{4}c^2\phi u^2 \left(2\zeta_1^1 z_1^0 z_2^0 + \zeta_1^0 z_1^1 z_2^0 + \zeta_1^0 z_2^1 z_1^0 \right).$$

Hence, only if $c = 0$, the symmetry conditions (40) and (41) hold. Thus in this case, we get the linear second order equation, which is trivial.

The situation of equation (39) is a little complicated. A third order symmetry of equation (39) is assumed as

$$Q = \Phi_3 + Q_2 + Q_3 + Q_4, \quad \left(Q_2 \in \mathfrak{R}^{(1,1)}, Q_3 \in \mathfrak{R}^{(3,0)} \oplus \mathfrak{R}^{(1,2)} \text{ and } Q_4 \in \mathfrak{R}^{(1,3)} \oplus \mathfrak{R}^{(3,1)} \right)$$

and determined by the conditions

$$[\Phi_2, Q_2] + [K_2, \Phi_3] = 0, \quad (42)$$

$$[\Phi_2, Q_3] + [K_2, Q_2] + [K_3, \Phi_3] = 0, \quad (43)$$

$$[\Phi_2, Q_4] + [K_2, Q_3] + [K_3, Q_2] = 0, \quad (44)$$

$$[K_2, Q_4] + [K_3, Q_3] = 0, \quad (45)$$

$$[K_3, \Phi_4] = 0. \quad (46)$$

From the symbolic form of (42), we get

$$\widehat{Q}_2 = \frac{3}{2}\phi u \left(c_1 \zeta_1^2 z_1^0 + (c_1 + c_2) \zeta_1^1 z_1^1 + c_2 \zeta_1^0 z_1^2 \right).$$

Then according to corollary 1 and proposition 4, we have the symbolic expression of the second and third terms in the left hand side of equation (43)

$$\begin{aligned} [\widehat{Q}_2, \widehat{K}_2] + [\widehat{\Phi}_3, \widehat{K}_3] &= \frac{3}{4}(2c_1^2 + c_1 c_2 + 4c_3)\phi u^2 \zeta_1^2 (z_1^1 z_2^0 + z_2^1 z_1^0) + \frac{3}{4}(2c_1 c_2 + c_2^2 + 4c_3)\phi u^2 \zeta_1^1 (z_1^2 z_2^0 + z_2^2 z_1^0) \\ &\quad + \frac{3}{2}(c_1^2 + 2c_1 c_2 + c_2^2 + 4c_3)\phi u^2 \zeta_1^1 z_1^1 z_2^1 + \frac{3}{4}(c_1 c_2 + c_2^2 + 4c_3)\phi u^2 \zeta_1^0 (z_1^2 z_2^1 + z_2^2 z_1^1) \\ &\quad - \frac{1}{4}c_2^2 \phi^3 (\zeta_1^3 \zeta_2^1 \zeta_3^0 + \zeta_3^3 \zeta_1^1 \zeta_2^0 + \zeta_2^3 \zeta_3^1 \zeta_1^0 - \zeta_1^3 \zeta_3^1 \zeta_2^0 - \zeta_3^3 \zeta_2^1 \zeta_1^0 - \zeta_2^3 \zeta_1^1 \zeta_3^0) \\ &= \frac{3}{2}c_1^2 \phi u^2 \zeta_1 (\zeta_1 z_1 + \zeta_1 z_2 + z_1 z_2) + \frac{3}{4}c_1 c_2 \phi u^2 (z_1 + z_2) (\zeta_1^2 + z_1 z_2 + 2\zeta_1 z_1 + 2\zeta_1 z_2) \\ &\quad + \frac{3}{4}c_2^2 \phi u^2 (z_1 + z_2) (\zeta_1 z_1 + \zeta_1 z_2 + z_1 z_2) + 3c_3 \phi u^2 (\zeta_1 + z_1) (\zeta_1 + z_2) (z_1 + z_2) \\ &\quad - \frac{1}{4}c_2^2 \phi^3 (\zeta_1 - \zeta_2) (\zeta_1 - \zeta_3) (\zeta_2 - \zeta_3) (\zeta_1 + \zeta_2 + \zeta_3) \end{aligned}$$

Suppose $Q_3 = Q_{3,1} + Q_{3,2}$, where $Q_{3,1} \in \mathfrak{R}^{(3,0)}$ and $Q_{3,2} \in \mathfrak{R}^{(1,2)}$, then the first term in the left hand side of equation (43) is formulated in terms of symbols as

$$\begin{aligned} [\widehat{\Phi}_2, \widehat{Q}_3] &= \widehat{Q}_{3,1} \left((\zeta_1 + \zeta_2 + \zeta_3)^2 - \zeta_1^2 - \zeta_2^2 - \zeta_3^2 \right) + \widehat{Q}_{3,2} \left((\zeta_1 + z_1 + z_2)^2 - \zeta_1^2 - z_1^2 - z_2^2 \right) \\ &= 2\widehat{Q}_{3,1} \left(\zeta_1 \zeta_2 + \zeta_2 \zeta_3 + \zeta_1 \zeta_3 \right) + 2\widehat{Q}_{3,2} \left(\zeta_1 z_1 + \zeta_1 z_2 + z_1 z_2 \right). \end{aligned}$$

Inferred from polynomial factors of symbolic expressions, equation (43) holds if and only if

$$c_2 = c_3 = 0, \quad \widehat{Q}_{3,1} = 0 \quad \text{and} \quad \widehat{Q}_{3,2} = \frac{3}{4}c_1^2 \phi u^2 \zeta_1^1 z_1^0 z_2^0.$$

Now from these symbolic results we obtain the quadratic and cubic terms of the symmetry Q , given by

$$Q_2 = \frac{3}{2}c_1 \Phi_2(\mathcal{D}\Phi) + \frac{3}{2}c_1 \Phi_1(\mathcal{D}\Phi_1) \quad \text{and} \quad Q_3 = \frac{3}{4}c_1^2 \Phi_1(\mathcal{D}\Phi)^2.$$

From equations (44) it follows that Q_4 is zero since $[K_2, Q_3] + [K_3, \Phi_2] = 0$. Equations (45) and (46) are trivially satisfied due to $K_3 = Q_4 = 0$. Hence, the second order equation

$$\Phi_t = \Phi_2 + c_1 \Phi_1(\mathcal{D}\Phi) \quad (47)$$

has the third order symmetry

$$\Phi_3 + \frac{3}{2}c_1\Phi_2(\mathcal{D}\Phi) + \frac{3}{2}c_1\Phi_1(\mathcal{D}\Phi_1) + \frac{3}{4}c_1^2\Phi_1(\mathcal{D}\Phi)^2.$$

(The parameter c_1 can be fixed as an arbitrary nonzero number through a scaling transformation). In appendix A, equation (47) ($c_1 = 2$) is rewritten as the component form, which is a decoupled system, so equation (47) is a trivial supersymmetric integrable equation, which is indeed equivalent to the classical potential Burgers equation. Therefore, there are no nontrivial integrable cases of second order supersymmetric equations (38) and (39).

Other cases in Table 1 are discussed in the same way as above, and we omit cumbersome details. For seventh order equations having thirteenth order symmetries, a fact indicated by their quadratic terms is that they all have fifth order symmetries. So there are no seventh order equations in our classification result.

Finally we remark that our computations are also independently checked using some computer algebra systems, for instance, SUSY2 [39], which directly deal with calculations with super functions instead of symbols.

6 Conclusions and discussions

As an efficient tool to study the integrability, the symbolic method is successfully extended to the $N = 1$ supersymmetric case in this paper. Any $N = 1$ supersymmetric differential polynomial is associated to a unique multi-variable polynomial which is anti-symmetric with respect to symbols of fermionic variables and meanwhile symmetric with respect to symbols of bosonic variables. All operations on the ring of $N = 1$ supersymmetric differential polynomials can be carried out in the symbolic way. On the basis of these results, we give a global classification for scalar λ -homogeneous $N = 1$ supersymmetric evolutionary equations when $\lambda > 0$, and identify eight leading equations having infinitely many higher order symmetries. Eight prototypes are respectively nontrivial supersymmetric integrable extensions of KdV, potential KdV, modified KdV, third order Burgers, Sawada-Kotera, fifth order KdV, Fordy-Gibbons and fifth order modified KdV equations. Two facts are noticed that there is no nontrivial scalar λ -homogeneous ($\lambda > 0$) supersymmetric integrable counterparts of the form (1) for the second order Burgers equation, the Ibragimov-Shabat equation and the Kaup-Kupershmidt equation, but there are two nonequivalent nontrivial supersymmetric integrable versions respectively for the fifth order KdV and modified KdV equations. Supersymmetrization both decreases and increases the number of integrable models.

Besides supersymmetric integrable evolutionary equations, non-evolutionary ones, especially equations of Camassa-Holm type, attract much attention in recent years. Various supersymmetric Camassa-Holm equations have been proposed through different approaches [40, 41], but their integrability has not been proved. In classical case, the symmetry approach in symbolic representation formulated in [42] has been successfully applied to classification of Camassa-Holm type equations. We expect that the formulation of symmetry approach in symbolic method developed in this paper can be applied to settle these problems.

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A Scalar λ -homogeneous ($\lambda > 0$) $N = 1$ supersymmetric integrable equations: trivial cases

For completeness, trivial supersymmetric integrable equations are presented, and all of them are converted into classical integrable equation by the same potential transformation.

- (i). Trivial supersymmetric Burgers equation [43] ($\lambda = \frac{1}{2}$)

$$\Phi_t = \Phi_2 + 2\Phi_1(\mathcal{D}\Phi), \quad (48)$$

and it is rewritten in components as

$$\begin{cases} v_t = v_2 + 2v_1v \\ \xi_t = \xi_2 + 2\xi_1v. \end{cases}$$

Under the super Cole-Hopf transformation $\Phi = \mathcal{D} \ln V$, where $V = V(x, \theta, t)$ is bosonic, equation (48) is linearized [43] to the heat equation $V_t = V_2$, and its infinitely many symmetries can be easily deduced from those of the heat equation via the Cole-Hopf transformation .

- (ii). Trivial supersymmetric KdV equation [35] ($\lambda = \frac{3}{2}$)

$$\Phi_t = \Phi_3 + 6\Phi_1(\mathcal{D}\Phi), \quad (49)$$

and in components it is of the form

$$\begin{cases} v_t = v_3 + 6v_1v \\ \xi_t = \xi_3 + 6\xi_1v. \end{cases}$$

Equation (49) was claimed to be relevant to supersymmetric extensions of matrix models, or conformal field theories coupled to gravity [35].

- (iii). Trivial supersymmetric modified KdV equation ($\lambda = \frac{1}{2}$)

$$\Phi_t = \Phi_3 + 6\Phi_1(\mathcal{D}\Phi)^2, \quad (50)$$

and its component form is given by

$$\begin{cases} v_t = v_3 + 6v_1v^2 \\ \xi_t = \xi_3 + 6\xi_1v^2. \end{cases}$$

(iv). Trivial supersymmetric Sawada-Kotera equation [31] ($\lambda = \frac{3}{2}$)

$$\Phi_t = \Phi_5 + 5\Phi_3(\mathcal{D}\Phi) + 5\Phi_1(\mathcal{D}\Phi_2) + 5\Phi_1(\mathcal{D}\Phi)^2, \quad (51)$$

and in components it is rewritten as

$$\begin{cases} v_t = v_5 + 5v_3v + 5v_2v_x + 5v_1v^2 \\ \xi_t = \xi_5 + 5\xi_3v + 5\xi_1v_{2x} + 5\xi_1v^2. \end{cases}$$

(v). Trivial supersymmetric Kaup-Kupershmidt equation [31] ($\lambda = \frac{3}{2}$)

$$\Phi_t = \Phi_5 + 10\Phi_3(\mathcal{D}\Phi) + 15\Phi_2(\mathcal{D}\Phi_1) + 10\Phi_1(\mathcal{D}\Phi_2) + 20\Phi_1(\mathcal{D}\Phi)^2, \quad (52)$$

and its component form is

$$\begin{cases} v_t = v_5 + 10v_3v + 25v_2v_1 + 20v_1v^2 \\ \xi_t = \xi_5 + 10\xi_3v + 15\xi_2v_1 + 10\xi_1v_2 + 20\xi_1v^2. \end{cases}$$

(vi). Trivial supersymmetric Fordy-Gibbons equation ($\lambda = \frac{1}{2}$)

$$\begin{aligned} \Phi_t = \Phi_5 - 5\Phi_3(\mathcal{D}\Phi_1) - 5\Phi_3(\mathcal{D}\Phi)^2 - 5\Phi_2(\mathcal{D}\Phi_2) - 10\Phi_2(\mathcal{D}\Phi_1)(\mathcal{D}\Phi) \\ - 10\Phi_1(\mathcal{D}\Phi_2)(\mathcal{D}\Phi) - 5\Phi_1(\mathcal{D}\Phi_1)^2 + 5\Phi_1(\mathcal{D}\Phi)^4, \end{aligned} \quad (53)$$

and it is rewritten in components as

$$\begin{cases} v_t = v_5 - 5v_3v_1 - 5v_3v^2 - 5v_2^2 - 20v_2v_1v - 5v_1^3 + 5v_1v^4 \\ \xi_t = \xi_5 - 5\xi_3(v_1 + v^2) - 5\xi_2(v_2 + 2v_1v) - 5\xi_1(2v_2v + v_1^2 - v^4). \end{cases}$$

As a common remark to six trivial extensions, their triviality can be alternatively understood by introducing a potential, i.e. $\Phi = (\mathcal{D}W)$, where $W = W(x, \theta, t)$ is bosonic. Then it is straightforward to show equations (48)–(53) are respectively converted into

$$\text{Potential Burgers equation: } W_t = W_2 + W_1^2,$$

$$\text{Potential KdV equation: } W_t = W_3 + 3W_1^2,$$

$$\text{Potential modified KdV equation: } W_t = W_3 + 2W_1^3,$$

$$\text{Potential Sawada-Kotera equation: } W_t = W_5 + 5W_3W_1 + \frac{5}{3}W_1^3,$$

$$\text{Potential Kaup-Kupershmidt equation: } W_t = W_5 + 10W_3W_1 + \frac{15}{2}W_2^2 + \frac{20}{3}W_1^3,$$

$$\text{Potential Fordy-Gibbons equation: } W_t = W_5 - 5W_3W_2 - 5W_3W_1^2 - 5W_2^2W_1 + W_1^5.$$

Through the potential transformation, recursion operators and/or master symmetries of equations (48)–(53) are easily constructed from the relevant results about these potential equations [44] (or references therein), and hence will not be presented here.

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