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# Quantum phase transition in the chirality of the (2+1)-dimensional Dirac oscillator

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(Dated: December 19, 2013)

## Abstract

We study the (2+1)-dimensional Dirac oscillator in the presence of an external uniform magnetic field ( $B$ ). We show how the change of the strength of  $B$  leads to the existence of a quantum phase transition in the chirality of the system. A critical value of the strength of the external magnetic field ( $B_c$ ) can be naturally defined in terms of physical parameters of the system. While for  $B = B_c$  the fermion can be considered as a free particle without defined chirality, for  $B < B_c$  ( $B > B_c$ ) the chirality is left (right) and there exist a net potential acting on the fermion. For the three regimes defined in the quantum phase transition of chirality, we observe that the energy spectra for each regime is drastically different. Then, we consider the  $z$ -component of the orbital angular momentum as an order parameter that characterizes the quantum phase transition.

*Keywords:* Dirac oscillator in (2+1) dimensions, quantum phase transition, chirality, orbital angular momentum.

*PACS Codes:* 03.65.Pm, 72.80.Vp, 71.70.Di.

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## I. INTRODUCTION

The system constituted by a relativistic fermion under the action of a linear vector potential is called the Dirac oscillator [1]. One the most interesting characteristic of the Dirac oscillator equation is that when its non-relativistic limit is taken, the associated Klein-Gordon equations describe an harmonic oscillator in the presence of a strong spin-orbit coupling [2]. Another characteristic of the Dirac oscillator equation is that it can be exactly solved in one, two and three dimensions. This is the reason why this system has been one of the most popular in the context of the relativistic quantum mechanics and mathematical physics during the last twenty years [3]-[24]. Fortunately, as stimulation to the search for physical applications of the Dirac oscillator, a first experimental realization of this system was reported recently [32].

On the other hand, some of the properties of the (2+1)-dimensional Dirac oscillator equation have been studied with the purpose of establishing a connection between this system and the (Anti)-Jaynes-Cummings model of the quantum optics [25]-[31]. In particular, by defining the chiral annihilation and creation operators, it has been possible to calculate the energy spectra of the Dirac oscillator without and with the presence of a uniform external magnetic field [25]-[28]. Using the same technique it has been shown that the linear term in the Dirac oscillator leads directly to a description of a fermion with an intrinsic left chirality, which is only present in the two-dimensional Dirac oscillator [25]. Additionally, with the same technique, it has been shown that the system constituted by the two-dimensional Dirac oscillator in presence of an uniform external magnetic field presents a chiral quantum phase transition [28]. Some of the characteristics of this quantum phase transition are known [28], however a study of how the energy spectra and orbital angular momentum change exactly in each regimes defined in this phase transition is lacking, nor has it been studied how these two physical observable change when the non-relativistic limit is taken.

The main goal of this work is to show precisely how the energy spectra and the  $z$ -component of orbital angular momentum change during the quantum phase transition in the chirality, which is defined for the (2+1)-dimensional Dirac oscillator in presence of an external uniform magnetic field ( $B$ ). We show how these changes both for the relativistic case and in the non-relativistic limit. We can show precisely how these physical observables change during the quantum phase transition because we introduce a new exact technique

that employs the number operators of right and left chirality. In this work, the change in the strength of  $B$  permits us to see how the quantum phase transition in the chirality is present in the system. To do this, we first identify a critical value of strength of the external magnetic field ( $B_c$ ). The value of  $B_c$  can be naturally defined in terms of physical parameters involved in the system, because for the regime  $B = B_c$  the fermion can be considered as a free particle without defined chirality. Then using the technique of the number operators of right and left chirality, we show that for the regime  $B < B_c$  the fermion has left chirality, while for the regime  $B > B_c$  the fermion has right chirality. For the three regimes associated to the quantum phase transition, we observe that the energy spectra in each regime is drastically different. Using the energy spectra previously obtained, we consider the  $z$ -component of the orbital angular momentum as an order parameter that characterizes the quantum phase transition.

The structure of this paper is the following. First, in the section II, for the Dirac oscillator in presence of an external uniform magnetic field, we obtain two coupled equations that permit us to define the regime of critical external magnetic field. Next, in the section III, by introducing the right and left chirality number operators, we study the regime of weak external magnetic fields which define the left chirality phase. Then, in section IV, we study the regime of strong external magnetic fields which define the right chirality phase. In section V, we show that the energy spectra are a signature of the quantum phase transition in the chirality in both the relativistic case and the non-relativistic limit. Next, in section VI, we consider the  $z$ -component of the orbital angular momentum as an order parameter, showing its value for each of the two phases and how at the critical point it is an undefined parameter. Finally, in section VII we present some conclusions.

## II. REGIME OF CRITICAL EXTERNAL MAGNETIC FIELD

The system constituted by a (2+1)-dimensional Dirac oscillator in presence of an external uniform magnetic field is described by the following equation

$$i\hbar\frac{\partial}{\partial t}|\psi\rangle = \left[ c \sum_{j=1}^2 \sigma_j (p_j - im\omega\sigma_z r_j + eA_j) + \sigma_z mc^2 \right] |\psi\rangle, \quad (1)$$

where  $c$  is the light speed,  $m$  is the fermion mass,  $\omega$  is the Dirac oscillator frequency,  $A_j$  are the components of the vector potential,  $p_j$  are the components of the linear momentum,

$r_j$  are the spatial coordinates in the  $(x, y)$  plane with respect to the origin of the potential,  $\sigma_j$  are the non-diagonal Pauli matrices,  $\sigma_z$  is the diagonal Pauli matrix. The spinor  $|\psi\rangle$  is written as

$$|\psi\rangle = \begin{pmatrix} |\psi_1\rangle \\ |\psi_2\rangle \end{pmatrix} \exp(-iEt/\hbar), \quad (2)$$

The external uniform magnetic field acting over the (2+1)-dimensional Dirac oscillator is defined perpendicular to the plane in the form  $\vec{B} = -B\hat{e}_z$ . The associated vector potential  $\vec{A}$ , then has the form  $\vec{A} = (A_x, A_y, A_z) = \frac{B}{2}(y, -x, 0)$ . Substituting the spinor expressed by (2) in Eq. (1) and using the explicit form of the Pauli matrices  $\sigma_x$ ,  $\sigma_y$  and  $\sigma_z$ , we obtain the following two coupled equations

$$(E - mc^2)|\psi_1\rangle = c[(p_x + im\omega x - im\tilde{\omega}x) - i(p_y + im\omega y - im\tilde{\omega}y)]|\psi_2\rangle, \quad (3)$$

$$(E + mc^2)|\psi_2\rangle = c[(p_x - im\omega x + im\tilde{\omega}x) + i(p_y - im\omega y + im\tilde{\omega}y)]|\psi_1\rangle, \quad (4)$$

where  $\tilde{\omega} = \omega_c/2$ , with  $\omega_c = eB/m$  denoting the cyclotron frequency. For the case in which  $\omega = \tilde{\omega}$ , *i.e.*  $\omega_c = 2\omega$ , the Eqs. (3) and (4) lead to

$$(E - mc^2)|\psi_1\rangle = c[p_x - ip_y]|\psi_2\rangle, \quad (5)$$

$$(E + mc^2)|\psi_2\rangle = c[p_x + ip_y]|\psi_1\rangle, \quad (6)$$

and Eq. (1) then becomes

$$i\hbar\frac{\partial}{\partial t}|\psi\rangle = \left[ c \sum_{i=1}^2 \sigma_i p_i + \sigma_z mc^2 \right] |\psi\rangle, \quad (7)$$

that corresponds to the (2+1)-dimensional Dirac equation describing a relativistic fermion of mass  $m$  which moves freely over the  $xy$ -plane. Starting from Eqs. (5) and (6), we substitute one into the other and we obtain the associated Klein-Gordon equations

$$(E^2 - m^2c^4) |\psi_1\rangle = c^2 p^2 |\psi_1\rangle, \quad (8)$$

$$(E^2 - m^2c^4) |\psi_2\rangle = c^2 p^2 |\psi_2\rangle, \quad (9)$$

where  $p^2 = p_x^2 + p_y^2$ . Therefore, the Eqs. (8) and (9) are describing a same free fermion with relativistic energy given by

$$E = \pm\sqrt{\hbar^2 k^2 c^2 + m^2 c^4}, \quad (10)$$

where we have written  $p^2 = \hbar^2 k^2$ , with  $k$  being the wavevector associated with the momentum  $p$ . This means that for the case in which the strength of an external magnetic field satisfies the critical condition  $\omega_c = eB_c/m = 2\omega$ , or equivalently

$$B_c = 2m\omega/e, \quad (11)$$

then the effect of the linear potential  $im\omega\sigma_z r_j$  on the fermion in Eq. (1) is annulled by the effect of the vector potential  $eA_j$ . In this form, it is possible to speculate that the quantity  $m\omega$  that appears in the linear potential of the Dirac oscillator defined by Eq. (12) can be written as  $m\omega = eB_I/2$ , where  $B_I$  can be interpreted as an effective magnetic field acting on the fermion. In other words, the effect of the linear potential on the fermion described by the Dirac oscillator is equivalent to the effect that an uniform magnetic field  $\vec{B}_I = B_I\hat{e}_z$  has on the fermion. Therefore, for the case  $\omega = \tilde{\omega}$ , the effective magnetic field  $\vec{B}_I = B_I\hat{e}_z$  is canceled by the external magnetic field  $\vec{B}_c = -B_I\hat{e}_z$ , and the Eq. (1) describes a free fermion.

If the external magnetic field vanishes in Eq. (1), this means that  $A_j = 0$ , then we obtain the (2+1)-dimensional Dirac oscillator equation given by

$$i\hbar\frac{\partial}{\partial t}|\psi\rangle = \left[ c \sum_{i=1}^2 \sigma_j (p_j - im\omega\sigma_z r_j) + \sigma_z mc^2 \right] |\psi\rangle. \quad (12)$$

This equation has been extensively studied and many of its properties are very well known [25]-[31], [34]. After the Eq. (12) is solved, the energy spectrum is given by [25]

$$E_{n_l} = \pm \sqrt{\hbar^2 k_{n_l}^2 c^2 + m^2 c^4}, \quad (13)$$

where the wave length  $k_{n_l}$  has been defined as

$$k_{n_l}^2 = k^2 \left( n_l + \frac{1}{2} \mp \frac{1}{2} \right), \quad (14)$$

with  $k^2 = 4m\omega/\hbar$  and  $n_l = 0, 1, 2, \dots$ . We can observe the energy spectrum (13) has the same form that the energy for the case of free fermion (10), but now the wave length takes discrete values as is shown in (14). Thus, we can observe that the main effect of the linear potential that appears in Eq. (12) is to quantize the energy spectrum.

The strength of an external magnetic field  $B$ , in relation with the value of the critical magnetic field  $B_c$ , is the parameter that will determine the chirality of the states describe

by the Eq. (1). In the next two sections, we will solve the coupled system given by Eqs. (3) and (4) for two different regimes: (i) weak external magnetic fields  $\omega > \tilde{\omega}$ , or  $B < B_c$ ; (ii) strong external magnetic fields  $\omega < \tilde{\omega}$ , or  $B > B_c$ .

### III. REGIME $\omega - \tilde{\omega} > 0$ : WEAK EXTERNAL MAGNETIC FIELDS

If the external magnetic field satisfies the condition  $B < B_c$ , *i.e.*  $\omega - \tilde{\omega} > 0$ , then Eqs. (3) and (4) can be written as

$$(E - mc^2)|\psi_1\rangle = c[(p_x + im\omega_T x) - i(p_y + im\omega_T y)]|\psi_2\rangle, \quad (15)$$

$$(E + mc^2)|\psi_2\rangle = c[(p_x - im\omega_T x) + i(p_y - im\omega_T y)]|\psi_1\rangle, \quad (16)$$

where

$$\omega_T = \omega - \tilde{\omega}. \quad (17)$$

Using Eqs. (15) and (16), it is possible to substitute one into the other and obtain the associated Klein-Gordon equations

$$(E^2 - m^2c^4)|\psi_1\rangle = 2mc^2 [H_{ho} - \hbar\omega_T - \omega_T L_z]|\psi_1\rangle, \quad (18)$$

$$(E^2 - m^2c^4)|\psi_2\rangle = 2mc^2 [H_{ho} + \hbar\omega_T - \omega_T L_z]|\psi_2\rangle, \quad (19)$$

where we have used the quantum mechanics commutation relations  $[x, p_x] = [y, p_y] = i\hbar$ ,  $[x, y] = [p_x, p_y] = 0$ . In Eqs. (18) and (19),  $H_{ho}$  and  $L_z$  represent respectively the two-dimensional harmonic oscillator Hamiltonian and the  $z$ -component of the orbital angular momentum

$$H_{ho} = \frac{\mathbf{p}^2}{2m} + \frac{m\omega_T^2}{2}\mathbf{r}^2, \quad (20)$$

$$L_z = xp_y - p_yx, \quad (21)$$

where  $\mathbf{r}^2 = x^2 + y^2$  and  $\mathbf{p}^2 = p_x^2 + p_y^2$ . Now we introduce the right chiral annihilation and creation operators given respectively by  $a_r = \frac{1}{\sqrt{2}}(a_x - ia_y)$ ,  $a_r^\dagger = \frac{1}{\sqrt{2}}(a_x^\dagger + ia_y^\dagger)$  and the left chiral annihilation and creation operators given respectively by  $a_l = \frac{1}{\sqrt{2}}(a_x + ia_y)$ ,  $a_l^\dagger = \frac{1}{\sqrt{2}}(a_x^\dagger - ia_y^\dagger)$ , where  $a_x$ ,  $a_y$ ,  $a_x^\dagger$  and  $a_y^\dagger$  are the usual annihilation and creation operators of the harmonic oscillator defined respectively as  $a_j = \frac{1}{\sqrt{2}}\left(\frac{1}{\Delta}r_j + i\frac{\Delta}{\hbar}p_j\right)$  and  $a_j^\dagger = \frac{1}{\sqrt{2}}\left(\frac{1}{\Delta}r_j - i\frac{\Delta}{\hbar}p_j\right)$ ,

with  $\Delta = v_f/\omega$  representing the ground-state oscillator width. Because the orbital angular momentum  $L_z$  is written in terms of the right and left chiral operators as  $L_z = \hbar(a_r^\dagger a_r - a_l^\dagger a_l)$ , then  $a_r^\dagger$  ( $a_l^\dagger$ ) is interpreted as the operator that create a right (left) quantum of angular momentum [25]. If we introduce the number operators of right and left chirality  $N_r$  and  $N_l$  defined by

$$N_r = a_r^\dagger a_r, \quad (22)$$

$$N_l = a_l^\dagger a_l, \quad (23)$$

then expressions (20) and (21) become [33]

$$H_{ho} = \hbar\omega_T(N_r + N_l + 1), \quad (24)$$

$$L_z = \hbar(N_r - N_l). \quad (25)$$

If we substitute expressions (24) and (25) in Eqs. (18) and (19), we obtain

$$(E^2 - m^2 c^4)|\psi_1\rangle = 4m^2 c^2 \hbar\omega_T N_l |\psi_1\rangle, \quad (26)$$

$$(E^2 - m^2 c^4)|\psi_2\rangle = 4m^2 c^2 \hbar\omega_T (N_l + 1) |\psi_2\rangle, \quad (27)$$

where we observe that only the left chiral number operator  $N_l$  appears, implying that for the regime  $B < B_c$  the eigenstates  $|+ E_{n_l}\rangle$  and  $|- E_{n_l}\rangle$  describe quantum states of left chirality. From Eqs. (26) and (27) and working in the base of eigenstates of the number operator  $N_l$ , we obtain the energy spectrum for fermions of left chirality

$$E_{n_l} = \pm \sqrt{F_w \hbar^2 k_{n_l}^2 c^2 + m^2 c^4}, \quad (28)$$

where the function  $F_w$  is given by

$$F_w = 1 - \frac{eB}{2m\omega}, \quad (29)$$

and  $k_{n_l}^2 = k^2 (n_l + \frac{1}{2} \mp \frac{1}{2})$ , with  $k^2 = 4m\omega/\hbar$  and  $n_l = 0, 1, 2, \dots$

For the case in which the external magnetic field vanishes ( $B = 0$ ), then  $F_w = 1$  and the energy spectrum (28) leads consistently to the energy spectrum of the Dirac oscillator given by the expression (13).

Now we will obtain the energy spectrum in the non-relativistic limit. For the case  $E > 0$ , the non-relativistic limit is taken by means of the approximation

$$E^2 - m^2 c^4 \simeq 2m c^2 E^+. \quad (30)$$

Substituting (30) in (26), we obtain that the energy spectrum is

$$E_{n_l}^+ = F_w \frac{\hbar^2 k^2}{2m} n_l, \quad (31)$$

with  $n_l = 0, 1, 2, \dots$ . For the case  $E < 0$ , the non-relativistic limit is taken by means of the approximation

$$E^2 - m^2 c^4 \simeq -2mc^2 E^-. \quad (32)$$

Substituting (32) in (27), we obtain that the energy spectrum is

$$E_{n_l}^- = -F_w \frac{\hbar^2 k^2}{2m} (n_l + 1), \quad (33)$$

with  $n_l = 0, 1, 2, \dots$

#### IV. REGIME $\tilde{\omega} - \omega > 0$ : STRONG EXTERNAL MAGNETIC FIELDS

If the external magnetic field satisfies the condition  $B > B_c$ , *i.e.*  $\tilde{\omega} - \omega > 0$ , then the Eqs. (3) and (4) can be written as

$$(E - mc^2)|\psi_1\rangle = c[(p_x - im\tilde{\omega}_T x) - i(p_y - im\tilde{\omega}_T y)]|\psi_2\rangle, \quad (34)$$

$$(E + mc^2)|\psi_2\rangle = c[(p_x + im\tilde{\omega}_T x) + i(p_y + im\tilde{\omega}_T y)]|\psi_1\rangle, \quad (35)$$

where

$$\tilde{\omega}_T = \tilde{\omega} - \omega. \quad (36)$$

From Eqs. (34) and (35), we substitute one into the other and we obtain the associated Klein-Gordon equations

$$(E^2 - m^2 c^4)|\psi_1\rangle = 2mc^2 \left[ \tilde{H}_{ho} + \hbar\tilde{\omega}_T + \tilde{\omega}L_z \right] |\psi_1\rangle, \quad (37)$$

$$(E^2 - m^2 c^4)|\psi_2\rangle = 2mc^2 \left[ \tilde{H}_{ho} - \hbar\tilde{\omega}_T + \tilde{\omega}L_z \right] |\psi_2\rangle, \quad (38)$$

where we have used the usual quantum mechanics commutation relations and  $L_z$  is given by the expression (21). In Eqs. (37) and (38),  $\tilde{H}_{ho}$  is the two-dimensional harmonic oscillator Hamiltonian described by

$$\tilde{H}_{ho} = \frac{\mathbf{p}^2}{2m} + \frac{m\tilde{\omega}_T^2}{2} \mathbf{r}^2. \quad (39)$$

We again introduce the number operators of right and left chirality  $N_r$  and  $N_l$ , respectively. In terms of these operators,  $L_z$  is written as expression (25), while  $\tilde{H}_{ho}$  is

$$\tilde{H}_{ho} = \hbar\tilde{\omega}_T(N_r + N_l + 1), \quad (40)$$

After expressions (40) and (25) are substituted in Eqs. (37) and (38), we obtain

$$(E^2 - m^2c^4)|\psi_1\rangle = 4mc^2\hbar\tilde{\omega}_T(N_r + 1)|\psi_1\rangle, \quad (41)$$

$$(E^2 - m^2c^4)|\psi_2\rangle = 4mc^2\hbar\tilde{\omega}_TN_r|\psi_2\rangle, \quad (42)$$

where we now observe that only the right chiral number operator  $N_r$  appears, implying that for the regime  $B > B_c$  the eigenstates  $|+E_{n_r}\rangle$  and  $|-E_{n_r}\rangle$  describe quantum states of right chirality. From Eqs. (41) and (42) and working in the base of eigenstates of the number operator  $N_r$ , we obtain the energy spectrum for fermions of right chirality

$$E_{n_r} = \pm\sqrt{F_s2\hbar c^2eB\left(n_r + \frac{1}{2} \mp \frac{1}{2}\right) + m^2c^4}, \quad (43)$$

where the function  $F_s$  is given by

$$F_s = 1 - \frac{2m\omega}{eB}, \quad (44)$$

and  $n_r = 0, 1, 2, \dots$

For the case in which the linear potential in Eq. (1) vanishes ( $\omega = 0$ ), the system that we are considering corresponds to a massive fermion having electric charge in presence of a uniform magnetic field  $B$ . For this case,  $F_s = 1$  and the energy spectrum (43) is written as

$$E_{n_r} = \pm\sqrt{2\hbar c^2eB\left(n_r + \frac{1}{2} \pm \frac{1}{2}\right) + m^2c^4}, \quad (45)$$

with  $n_r = 0, 1, 2, \dots$ , which corresponds to the Landau-level spectrum.

For the case  $E > 0$ , we find that the energy spectrum in the non-relativistic limit is

$$E_{n_r}^+ = F_s\frac{\hbar eB}{m}(n_r + 1), \quad (46)$$

while for the case  $E < 0$ , we find that this spectrum is

$$E_{n_r}^- = -F_s\frac{\hbar eB}{m}n_r. \quad (47)$$

## V. ENERGY SPECTRA AS SIGNATURE OF THE QUANTUM PHASE TRANSITION

We have shown in the previous sections that the chirality of the fermion described by Eq. (1) changes drastically accordingly to the strength of the external magnetic field  $B$  relative to the critical value  $B_c$ . To show this, we have defined three different regimes: (i) weak external magnetic fields ( $B < B_c$ ); (ii) a critical external magnetic field ( $B = B_c$ ); (iii) strong external magnetic fields ( $B > B_c$ ). The change of chirality that we have observed is: For  $B < B_c$ , the fermion has left chirality; for  $B = B_c$ , the fermion has not a defined chirality; for  $B > B_c$ , the fermion has right chirality.

In this section we will show that the energy is a good signature with which to characterize the quantum phase transition in the chirality, because the energy spectrum of the fermion changes drastically with the regime considered. We will show also that that the origin of this phase transition of chirality is not a consequence of relativistic effects because the changes in the energy spectrum appear also in the non-relativistic limit.

For  $B < B_c$ , from the energy spectrum (28), we obtain that the relativistic discrete energy spectra are

$$E_{n_l} = \pm \sqrt{4\hbar c^2 m \omega \left(1 - \frac{eB}{2m\omega}\right) \left(n_l + \frac{1}{2} \mp \frac{1}{2}\right)}, \quad (48)$$

with  $n_l = 0, 1, 2, \dots$ . For  $B = B_c$ , from (10), the relativistic continuum energy spectra are

$$E = \pm \sqrt{p^2 c^2 + m^2 c^4}. \quad (49)$$

For  $B > B_c$ , from the energy spectrum (43), the relativistic discrete energy spectra are

$$E_{n_r} = \pm \sqrt{2\hbar c^2 e B \left(1 - \frac{2m\omega}{eB}\right) \left(n_r + \frac{1}{2} \pm \frac{1}{2}\right) + m^2 c^4}, \quad (50)$$

with  $n_r = 0, 1, 2, \dots$ . If  $B \ll B_c$ , *i.e.*  $B \simeq 0$ , from (49) we obtain the energy spectra of the (2+1)-dimensional Dirac oscillator [34]

$$E_{n_l} = \pm \sqrt{4\hbar c^2 m \omega \left(n_l + \frac{1}{2} \mp \frac{1}{2}\right)}. \quad (51)$$

with  $n_l = 0, 1, 2, \dots$ . If  $B \gg B_c$ , *i.e.*  $\omega \simeq 0$ , from (51) we obtain the Landau-level spectra [34]

$$E_{n_r} = \pm \sqrt{2\hbar c^2 e B \left(n_r + \frac{1}{2} \mp \frac{1}{2}\right) + m^2 c^4}, \quad (52)$$

with  $n_r = 0, 1, 2, \dots$ . We observe that the relativistic energy spectra for the three regimes are very different.

For  $B < B_c$ , we rewrite the expressions (31) and (33) and we obtain the non-relativistic discrete spectra for positive and negative energies

$$E_{n_l}^+ = \frac{\hbar^2 k^2}{2m} \left(1 - \frac{eB}{2m\omega}\right) n_l, \quad (53)$$

$$E_{n_l}^- = -\frac{\hbar^2 k^2}{2m} \left(1 - \frac{eB}{2m\omega}\right) (n_l + 1), \quad (54)$$

respectively, with  $n_l = 0, 1, 2, \dots$ . For  $B = B_c$ , the non-relativistic continuum spectra for positive and negative energies are

$$E^\pm = \pm \frac{p^2}{2m^2}. \quad (55)$$

For  $B > B_c$ , we rewrite the expressions (46) and (47) and we obtain the non-relativistic discrete spectra for positive and negative energies

$$E_{n_r}^+ = 2\hbar c^2 eB \left(1 - \frac{2m\omega}{eB}\right) (n_r + 1), \quad (56)$$

$$E_{n_r}^- = -2\hbar c^2 eB \left(1 - \frac{2m\omega}{eB}\right) n_r, \quad (57)$$

respectively, with  $n_r = 0, 1, 2, \dots$

As happens in the relativistic case, the non-relativistic limit the energy spectra are very different in the three regimes. This means that the phase transition in the chirality does not have a relativistic origin. This quantum phase transition exists in the relativistic and non-relativistic cases.

## VI. ORBITAL ANGULAR MOMENTUM AS AN ORDER PARAMETER

Now we will consider the  $z$ -component of the orbital angular momentum  $L_z$  as an order parameter that characterizes the quantum phase transition in the chirality. We will show that the physical magnitude of this quantity takes on different values in the two phases and is undetermined in the critical point. For  $B < B_c$ , the system is in the left chirality phase and the expectation value of  $L_z$  denoted as  $\langle L_z \rangle$  is negative, *i.e.*  $\langle L_z \rangle < 0$ , while for  $B > B_c$ , the system is in the right chirality phase and  $\langle L_z \rangle > 0$ .

For the left chirality phase, *i.e.* for  $B < B_c$ , from the associated Klein-Gordon equations given by Eqs. (18) and (19), and using the expressions (24) and (25), we obtain

$$\langle L_z^+ \rangle_{n_l} = \langle +E_{n_l} | L_z | +E_{n_l} \rangle = -\frac{E_{n_l}^2 - m^2 c^4}{2c^2 m \omega_T} + \hbar n_l, \quad (58)$$

$$\langle L_z^- \rangle_{n_l} = \langle -E_{n_l} | L_z | -E_{n_l} \rangle = -\frac{E_{n_l}^2 - m^2 c^4}{2c^2 m \omega_T} + \hbar(n_l + 2), \quad (59)$$

with  $n_l = 0, 1, 2, \dots$ . For the right chirality phase, *i.e.* for  $B > B_c$ , from the associated Klein-Gordon equations given by Eqs. (37) and (38), and using the expressions (40) and (25), we obtain

$$\langle L_z^+ \rangle_{n_r} = \langle +E_{n_r} | L_z | +E_{n_r} \rangle = \frac{E_{n_r}^2 - m^2 c^4}{2c^2 m \tilde{\omega}_T} - \hbar(n_r + 2), \quad (60)$$

$$\langle L_z^- \rangle_{n_r} = \langle -E_{n_r} | L_z | -E_{n_r} \rangle = \frac{E_{n_r}^2 - m^2 c^4}{2c^2 m \tilde{\omega}_T} - \hbar n_r, \quad (61)$$

with  $n_r = 0, 1, 2, \dots$ . For the critical point, *i.e.* for  $B = B_c$ , from the associated Klein-Gordon equations given by Eqs. (8) and (9), we observe that  $\langle L_z \rangle$  is undetermined.

The corresponding expectation values of the orbital angular momentum in the non-relativistic limit are the following. For the left chirality phase ( $B < B_c$ )

$$\langle L_z^+ \rangle_{n_l} = \langle +E_{n_l} | L_z | +E_{n_l} \rangle = -\frac{1}{\omega_T} E_{n_l} + \hbar n_l, \quad (62)$$

$$\langle L_z^- \rangle_{n_l} = \langle -E_{n_l} | L_z | -E_{n_l} \rangle = \frac{1}{\omega_T} E_{n_l} + \hbar(n_l + 2), \quad (63)$$

with  $n_l = 0, 1, 2, \dots$ . For the right chirality phase ( $B > B_c$ )

$$\langle L_z^+ \rangle_{n_r} = \langle +E_{n_r} | L_z | +E_{n_r} \rangle = \frac{1}{\tilde{\omega}_T} E_{n_r} - \hbar(n_r + 2), \quad (64)$$

$$\langle L_z^- \rangle_{n_r} = \langle -E_{n_r} | L_z | -E_{n_r} \rangle = -\frac{1}{\tilde{\omega}_T} E_{n_r} - \hbar n_r, \quad (65)$$

with  $n_r = 0, 1, 2, \dots$ . For the critical point  $B = B_c$ , these values are undetermined.

As happens in the relativistic case, in the non-relativistic limit the expectation values of the orbital angular momentum are drastically different between the two phases and are undetermined in the critical point.

## VII. CONCLUSIONS

We have shown how the energy spectra and the  $z$ -component of the orbital angular momentum change exactly during the quantum phase transition of chirality which is defined for

the (2+1)-dimensional Dirac oscillator in presence of an external uniform magnetic field ( $B$ ). These changes have been obtained both in the relativistic case and in the non-relativistic limit. After introducing a new technique that works with the number operators of right and left chirality, we have performed an explicit and exact calculation to show how these physical observables change during the quantum phase transition. We have demonstrated that the changing the strength of  $B$  leads to a quantum phase transition in the chirality of this system.

We have first identified a critical value of strength of the external magnetic field ( $B_c$ ) which is naturally defined in terms of physical parameters involved in the system, because for the regime  $B = B_c$  the fermion can be considered as a free particle without defined chirality. We have used the number operator technique to show that for the regime  $B < B_c$  the fermion has left chirality, while for the regime  $B > B_c$  the fermion has right chirality. We have observed that, for the three regimes associated with the quantum phase transition, the energy spectra in each of the three regimes is drastically different. By using the energy spectra previously obtained, we have considered the  $z$ -component of the orbital angular momentum as an order parameter that characterizes the quantum phase transition.

The advance made in this work has been to characterize the quantum phase transition in the chirality in an exact form. The new technique, which uses the number operators for right and left chirality, that we have introduced in this work has permitted us to calculate exactly the energy spectra and the  $z$ -component of the angular momentum. A possible application of the results that we have presented in this work can be in the context of the description of the electronic properties of monolayer and bilayer graphene [34].

## ACKNOWLEDGMENTS

C. Quimbay thanks the School of Physical Sciences at the University of Kent, Canterbury, for their hospitality during his visit there.

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