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Coeffects: A Calculus of Context-Dependent Computation

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Abstract

The notion of context in functional languages no longer refers just to variables in scope. Context can capture additional properties of variables (usage patterns in linear logics; caching requirements in dataflow languages) as well as additional resources or properties of the execution environment (rebindable resources; platform version in a cross-platform application). The recently introduced notion of coeffects captures the latter, whole-context properties, but it failed to capture fine-grained per-variable properties.

We remedy this by developing a generalized coeffect system with annotations indexed by a coeffect shape. By instantiating a concrete shape, our system captures previously studied flat (whole-context) coeffects, but also structural (per-variable) coeffects, making coeffect analyses more useful. We show that the structural system enjoys desirable syntactic properties and we give a categorical semantics using extended notions of indexed comonad.

The examples presented in this paper are based on analysis of established language features (liveness, linear logics, dataflow, dynamic scoping) and we argue that such context-aware properties will also be useful for future development of languages for increasingly heterogeneous and distributed platforms.

Categories and Subject Descriptors   D.3.1 [Programming Languages]: Formal Definitions and Theory

Keywords   Context; Types; Coeffects; Indexed comonads

1. Introduction

Context is important for defining meaning – not just in natural languages, but also in logics and programming languages. The standard notion of context in programming is an environment providing values for free variables. An open term with free variables is context dependent – its meaning depends on the free-variable context. The simply-typed \( \lambda \)-calculus famously analyses such context usage. Other systems go further. For example, bounded linear logic tracks the number of times a variable is used \[7\].

In software engineering, “context” often encompasses more than just free-variable values. For example, in a distributed system, the context provides resources that may be available on a particular device (e.g., a database on a server or a GPS sensor on a phone).

In this paper, we develop a calculus for capturing various notions of context in programming. A key feature and contribution of the calculus is its coeffect system which provides a static analysis for contextual properties (coeffects). The system follows the style of type and effect systems, but captures a different class of properties. Another key contribution of the calculus is its semantics which can be smoothly instantiated for specific notions of context.

Coeffect systems were previously introduced as a generic analysis of context dependence which can be instantiated for various notions of context \[15\]. The formalization was restricted to tracking a class of whole-context properties where terms have just one coeffect. This limited the applications and precision of any analysis. For example, a whole-context liveness analysis marks the entire free-variable context as live (some variable may be used) or dead (no variable is used), but it cannot record liveness per variable.

We develop a more general system which captures both per-variable coeffects, which we call structural, and whole-context coeffects, which we call flat, and more. Our key contributions are:

- We present the coeffect calculus which augments the simply-typed \( \lambda \)-calculus with a general coeffect type system (Section 3). We demonstrate the two classes of flat (whole-context) and structural (fine-grained, per-variable) systems.

- We show practical examples, instantiating the calculus for structural systems capturing variable usage based on bounded linear logic, dataflow caching, and precise liveness analysis. We also instantiate the calculus to flat systems, building on and extending previous examples from \[15\].

- We discuss the syntactic properties of flat and structural variants of the coeffect calculus (Section 4). Notably, structural systems satisfy type preservation under both \( \beta \)-reduction and \( \eta \)-expansion, allowing their use with both call-by-name and call-by-value languages. This important property distinguishes structural coeffects from both effect systems and flat coeffects.

- We provide a denotational semantics, revisiting and extending the notion of indexed comonads to the structural setting (Section 5). We prove soundness by showing the correspondence between syntactic and semantic properties of coeffect systems.

Coeffects can be approached from multiple directions (Section 2.5) including syntactic (effect systems), semantic, and proof-theoretic. We emphasize the syntactic view, though we also outline a categorical semantics and note the interesting technical details.

2. Why coeffects matter

Coeffects are a way to describe notions of context that keep turning up in programming. To illustrate this, we overview three systems tracking contextual properties that motivate our general coeffect system. Two systems track per-variable properties (bounded linear logic and dataflow) and one tracks whole-context properties (implicit parameters). We start with some background and finish with a brief overview of the literature leading to coeffects.
2.1 Background, scalars and vectors

The $\lambda$-calculus is asymmetric – it maps a context with multiple variables to a single result. An expression with $n$ free variables of types $\tau_1$ can be modelled by a function $\tau_1 \times \ldots \times \tau_n \to \tau$ with a product on the left, but a single value on the right. Effect systems attach effect annotations to the result $\tau$. In coeffect systems, we attach coeffects to the context $\tau_1 \times \ldots \times \tau_n$ and we often (but not always) have one coeffect per variable. We call the overall coeffect a vector consisting of scalar coeffects. This asymmetry explains why coeffect systems are not trivially dual to effect systems.

It is useful to clarify how vectors are used in this paper. Suppose we have a set $C$ of scalars. A vector $R$ over $C$ is a tuple $(\tau_1, \ldots, \tau_n)$ of scalars. We use letters like $R, S, T$ for vectors and $r, s, t$ for scalars. We also say that the shape of a vector $[R]$ (or more generally any container) is the set of positions in a vector. So, a vector of length $n$ has shape $[1, 2, \ldots, n].$

Just as in scalar-vector multiplication, we lift any binary operation $\bullet$ on scalars into a scalar-vector one: $s \bullet R = (s \bullet \tau_1, \ldots, s \bullet \tau_n).$ Given two vectors $R, S$ of the same shape, containing partially ordered scalars, we write $R \leq S$ for the pointwise extension of $\leq$ on scalars. Finally, the associative operation $\times$ concatenates vectors.

We note that an environment $\Gamma$ containing $n$ uniquely named, typed variables is also a vector, but we continue to write $\langle \rangle$ for the product, so $\Gamma_1, x : \tau, \Gamma_2$ should be seen as $\Gamma_1 \times (\langle x : \tau \rangle \times \Gamma_2$.

2.2 Bounded reuse

Bounded linear logic provides a modality that limits the number of times a proposition (variable) can be reused [7]. A type system corresponding to this logic can be used, for example, to restrict well-typed terms to polynomial-time algorithms. A proposition $I_n A$ means that $A$ can be used at most $k$ times. For uniformity with later notation, we write propositions $A$ as $\tau$. Our work attaches a coeffect annotation to a scalar-vector one: $s \bullet R = (s \bullet \tau_1, \ldots, s \bullet \tau_n).$ Given two vectors $R, S$ of the same shape, containing partially ordered scalars, we write $R \leq S$ for the pointwise extension of $\leq$ on scalars. Finally, the associative operation $\times$ concatenates vectors.

Bounded linear logic includes explicit weakening and contraction rules that affect the multiplicity. Following the original logical style (but with our notation), these are written as:

\[
\begin{align*}
\text{(weak)} & \quad \Gamma \not\vdash \tau \\
\text{(contr)} & \quad \Gamma, \tau_1, \tau_0, \Gamma_2 \not\vdash R \times (s, t) \not\vdash \tau \\
\text{(exch)} & \quad \Gamma, \tau_1, \tau_0, \Gamma_2 \not\vdash R \times (s, t) \not\vdash t
\end{align*}
\]

The context $\Gamma \not\vdash R$ includes a coeffect annotation $R$ which is a vector $(\tau_1, \ldots, \tau_n)$ of the same length as $\Gamma$ (a side-condition omitted for brevity). In weakening, unused propositions are annotated with $0$ (no uses), while in contraction, multiple occurrences of a proposition are joined by adding the number of uses.

Bounded linear coeffects. The system in Figure 1[1] flushes out the idea into a simple calculus. Variable access $(\text{var})$ has a singleton context with a coeffect coeffect vector $(1)$. Weakening (weak) extends the free-variable context with an unused variable and the coeffect with an associated scalar $0$. Explicit contraction (contr) and exchange (exch) rules manipulate variables in the context and modify the annotations accordingly – adding the number of uses in contraction and switching vector elements in exchange.

For abstraction (abs), we know the number of uses of the parameter variable $z$ and attach it to the function type $\tau_1 \to \tau_2$ as a latent coeffect. The remaining variables in $\Gamma$ are annotated with the remaining coeffect vector $R$, specifying immediate coeffects.

Application (app) describes call-by-name evaluation. Applying a function that uses its parameter $t$-times to an argument that uses variables in $\Gamma_2$ $S$-times means that, in total, the variables in $\Gamma_2$ will be used $(t \cdot S)$-times. Recall that $t \cdot S$ is a scalar multiplication of a vector. Meanwhile, the variables in $\Gamma_1$ are used just $R$-times when reducing the expression $e_1$ to a function value.

Finally, the sub-coeffecting rule (sub) safely overapproximates the number of uses by the pointwise $\leq$ relation. We can view any variable as being used a greater number of times than it actually is.

Example. To demonstrate, consider a term $(\lambda v. x \cdot v + v + v) (x + y)$. According to the call-by-name intuition, the variable $x$ is used three times – once directly inside the function and twice via the variable $v$ after substitution. Similarly, $y$ is used twice. Eliding the derivation of the function body’s coeffect, abstraction yields:

\[
\begin{align*}
\text{(abs)} & \quad \Gamma, x : \mathbb{Z}, v : \mathbb{Z} @ (1, 2) \not\vdash x + v + v : \mathbb{Z} \\
\text{(exch)} & \quad x : \mathbb{Z} @ (1) \not\vdash (\lambda v. x \cdot v + v + v) : \mathbb{Z} \rightarrow \mathbb{Z}
\end{align*}
\]

To avoid name clashes, we $\alpha$-rename $x$ to $x'$ and later join $x$ and $x'$ using contraction. Assuming $(x' + y)$ is checked in a context that marks $x'$ and $y$ as used once, the application rule yields a judgment that is simplified as follows:

\[
\begin{align*}
\text{(exch)} & \quad x : \mathbb{Z}, x' : \mathbb{Z}, y : \mathbb{Z} @ (1, 2) \not\vdash (\lambda v. x \cdot v + v + v) (x' + y) : \mathbb{Z} \\
\text{(exch)} & \quad x : \mathbb{Z} @ (1, 2) \not\vdash (\lambda v. x \cdot v + v + v) (x + y) : \mathbb{Z}
\end{align*}
\]

The first step performs scalar multiplication, producing the vector $(1, 2, 2)$. In the second step, we use contraction to join variables $x$ and $x'$ from the function and argument terms respectively.

It is worth pointing out that reduction by substitution yields $x + (x + y) + (x + y)$ which has the same coeffect as the original. We return to evaluation strategies in Section 4 and show that structural coeffect systems preserve types and coeffects under $\beta$-reduction.

2.3 Dataflow and data access

Dataflow languages, such as Lucid, describe computations over streams [20]. An expression is re-evaluated when new inputs are available (push) or when more output is demanded (pull). In causal dataflow, programs can access past values of a stream. We consider a language where $\text{prev} e$ returns the previous value of $e$. In the language, $\text{prev (prev e)}$ returns the second past value and so on.

An implementation of causal dataflow may cache past values of variables as an optimisation. The question is, how many past values should be cached? This can be approximated by a coeffect system.

Dataflow coeffects. The coeffect system for dataflow is similar to the one for bounded reuse, tracking a vector of natural numbers $R$ as part of the context $\Gamma \not\vdash R$. Here, coeffects represent the maximal number of past values (causality depth) required for a variable.
Weakening, exchange, abstraction and sub-coeffecting are the same as in bounded linear coeffects, but the remaining rules differ. In Figure 2, accessed variables (var) are annotated with 0 meaning that no past value is required (only the current one). The (prev) rule creates caching requirements – it increments the number of required values for all variables used in \( e \) using scalar-vector addition.

Application and contraction have the same structure as before, but use different operators. If two variables are contracted, requiring \( s \) and \( t \) past values, then at most \( \max(s, t) \) past values are needed (contr). That is, two caches are combined with the maximum of the two requirements, which satisfy the smaller requirements. In (app), the function requires \( t \) past values of its parameter. This means \( t \) past values of \( e_2 \) are needed which in turn requires \( S \) past values of its free variables \( F_2 \). Thus, we need \( t + S \) past values of \( F_2 \) to perform the call (e.g., we need \( 1 + S \) values to get 1 past value of the input \( \tau_1, 2 + S \) values to get 2 past values of \( \tau_1, \ldots, \)).

**Example.** As an example, consider a function \( \lambda x. \text{prev}(y + x) \) applied to an argument \( \text{prev}(\text{prev}y) \). The body of the function accesses the past value of two variables, one free and one bound:

\[
\begin{align*}
\text{(abs)} & : \Gamma, y : \tau, z : \tau, \Gamma @ R \times (s, t) \times Q \vdash e : \tau \\
\text{(app)} & : \Gamma @ R \vdash e_1 : \tau_1, \Gamma @ R \times (t + S) \vdash e_1 e_2 : \tau_2 \\
\text{(prev)} & : \Gamma @ R \vdash e : \tau \\
\end{align*}
\]

We previously used dataflow as an example of coeffects [13], but tracked caching requirements on the whole context. The system outlined here is more powerful and practically useful, with finer-grained coeffects tracking per-variable caching requirements.

**2.4 Implicit parameters**

As our third example, we revisit Haskell implicit parameters [9] used in our earlier coeffect work [15]. Implicit parameters are variables that mix aspects of dynamic and lexical scoping. Implicit parameters are a distinct syntactic category to variables and we write them as \( ?p \). For simplicity, we omit let-binding for implicit parameters and focus just on tracking requirements.

**Implicit parameters coeffects.** Implicit parameters are a whole-context coeffect not linked to ordinary variables. We keep track of sets of implicit parameters that are required by an expression (and their types). For example \( \Gamma \bowtie \{ ?p_1 : \tau_1, \ldots, ?p_n : \tau_n \} \) means that a context provides ordinary variables \( \Gamma \) and values for implicit parameters \( ?p \). Unlike in the previous examples, we no longer need to distinguish between coeffects attached to variables (scalars) and coeffects attached to contexts (vectors), so we write \( r, s, t \) for both.

Despite the differences, the type system in Figure 3 follows the same structure as the earlier two examples. Context requirements are created when accessing an implicit parameter, in a system-specific rule (param). Structural rules (exchange, weaken, contract) do not affect the coeffects. For example, parameters are reordered in (exch), but this has no effect as set union \( \cup \) is commutative.

In abstraction and application, the structural \( \times \) operator (previously vector concatenation) becomes \( \cup \). Sets of implicit parameters are not associated to individual variables and so they are unioned. The (app) rule uses \( \cup \) to combine the implicit parameters required by the function with the requirements of the argument itself.

We call this a flat coeffect system since coeffects have only one shape (there is no scalar/vector distinction). Other flat coeffect systems may use a richer structure [13]. In particular, the operations used in abstraction and application may differ (to accommodate over-approximation). We return to this in Section 2.5.

**Example.** Unlike structural (per-variable) coeffect systems, flat (whole-context) systems do not necessarily have principal coeffects. This arises from the (abs) rule which can freely split requirements between the function type and the declaring context. Consider a function \( \lambda(). ?p_1 + ?p_2 \). There are nine possible type and coeffect derivations, two of which are:

\[
\begin{align*}
\text{(abs)} & : \Gamma \vdash \lambda(). ?p_1 + ?p_2 : \text{unit} \\
\text{(contr)} & : \Gamma \vdash \lambda(). ?p_1 : ?p_2 : \text{unit} \\
\end{align*}
\]

We previously used dataflow as an example of coeffects [13], but tracked caching requirements on the whole context. The system outlined here is more powerful and practically useful, with finer-grained coeffects tracking per-variable caching requirements.

This paper largely follows work on effect systems and their link to categorical semantics. We briefly review this and other directions leading to coeffects. An eager reader can return to this section later.

**Effect systems.** Effect systems [6] track effectful operations of computations such as memory access or lock usage [4]. They are written as judgments \( \Gamma \vdash e : \tau & \rho \) associating effects \( \rho \) with the result. Effect systems capture output effects where, as Tate puts it, “all computations with [an] effect can be thunked as pure computations for a domain-specific notion of purity.” [13]. This thinking is typically a \( \lambda \)-abstraction. Given an effectful expression \( e \), the function \( \lambda x.e \) is an effect-free value that delays all effects:

\[
\begin{align*}
\text{(abs)} & : \Gamma, x : \tau_1 \vdash e : \tau_2 & \rho \\
\text{(contr)} & : \Gamma \vdash \lambda x.e : \tau_1 \rightarrow \tau_2 & \emptyset \\
\end{align*}
\]
Coeffects do not follow this pattern. In contrast to effect systems, context requirements cannot be easily “thunked” as pure values. Lambda abstraction can split context requirements between immediate and latent requirements. This is akin to how lambda abstraction splits a free-variable context into the bound parameter (call site) and the remaining free variables (declaration site).

Categorical semantics. Moggi models effectful computations as functions of type \( \tau_1 \to M \tau_2 \) where \( M \) is a monad providing composition of effectful computations. Walder and Thiennam link effect systems to monads via annotated monads \( \tau_1 \to M^r \tau_2 \), whose semantics has been provided by Katsumata.

Context-dependent computations require a different model. Uustalu and Vene use functions \( C_1 \tau_1 \to \tau_2 \) where \( C \) is a comonad. Our earlier work used indexed comonads with denotation \( C^r \tau_1 \to \tau_2 \). In Section 5, we extend indexed comonads to capture the general coeffect systems of this paper, in the style of Katsumata.

Language and meta-language. Moggi uses monads in two systems. In the first system, a monad is used to model an effectful language itself—the semantics of a language uses a specific monad. In the second system, monads are added as type constructors, together with syntax corresponding to unit and bind operations.

For context dependence, Uustalu and Vene follow the first approach using comonads for their semantics. Contextual-Modal Type Theory (CMTT) of Nanovesi et al. follows the latter approach, adding a comonad to the language via the modality of modal S4. We focus on concrete languages using the first approach. A “coeffect meta-language” is an interesting future work.

Sub-structural systems Sub-structural type systems restrict how a context is used. This is achieved by removing some of the structural typing rules (weakening, contraction, exchange). As the bounded linear logic example (Section 2.3) shows, our system can be viewed as a generalization of sub-structural type systems.

3. The coeffect calculus

The three calculi shown in the previous section track two kinds of contextual properties: bounded reuse and dataflow are structural (per-variable) systems, and implicit parameters and our earlier coeffect systems are flat (whole-context) systems. This section presents our primary contribution: the general coeffect calculus.

The calculus is parameterised by an algebraic structure of coeffects. To capture both structural and flat systems, coeffect annotations are indexed by a shape. In flat systems, the shape is a singleton set \( \{ \} \) and so annotations are scalar values. Structural systems use shapes matching the number of variables in a free-variable context \( \{ 1, \ldots, n \} \) and so annotations are vectors. However, the coeffect calculus could also use shapes describing trees and other structures.

3.1 Understanding coeffects: syntax and semantics

The coeffect calculus provides both an analysis of context dependence (its coeffect system) and a semantics for context (see Section 5). These two features of the calculus provide different perspectives on coeffect annotations \( R \) in a judgment \( \Gamma @ R \vdash e : \tau \).

- Syntactically, coeffects model contextual requirements and may be overapproximated, so that more capabilities are required than necessary at runtime.
- Semantically, coeffects model contextual capabilities and behave like containers of capabilities, such that the semantics may throw away capabilities that will not be needed.

Thus there are two dual ways to understand coeffect annotations. Each perspective implies an alternate reading of the typing rules.

- As contextual requirements, the rules should be read top-down. The requirements of multiple sub-terms are merged and the requirements of a function body are split between immediate (declaration-site) and latent (call-site) coeffects.
- As contextual capabilities, the rules should be read bottom-up. The capabilities provided to a larger term are split between sub-terms; for functions, the capabilities of declaration-site and call-site are merged and passed to the body.

The reason for this asymmetry follows from the fact that context appears in a negative position in the model. In Section 5, the denotation of a judgment \( \Gamma @ R \vdash e : \tau \) is a function of the form \( D_\tau[\Gamma] \to [\tau] \) where \( D_\tau[\Gamma] \) encodes the contextual capabilities used to evaluate a term. Similarly a function \( \tau_1 \to \tau_2 \) has a model of the form \( D[\tau_1] \to [\tau_2] \) with additional contextual capabilities attached to the input.

3.2 Structure of coeffects

We describe the algebraic structure of coeffects in three steps. First, we define a coeffect scalar structure which defines the basic building blocks of coeffect information; then we define coeffect shapes which determines how coeffect scalar values are related to the free-variable context. Finally, we define the coeffect algebra which consists of shape-indexed coeffect scalar values.

For example, in bounded reuse the coeffect scalar structure comprise natural numbers \( \mathbb{N} \) with \( + \) and \( \ast \) operators. The shape for bounded reuse is the length of the free-variable context and so the coeffect annotation is a vector of matching length. Finally, the coeffect algebra specifies how vectors are concatenated and split in abstraction and application.

In the coeffect system of the calculus, contexts are annotated with shape-indexed coeffects (e.g., vectors) as in \( \Gamma @ R \vdash e : \tau \). However, functions take just a single input parameter and so are annotated with scalar coeffect values as in \( \sigma \vdash \tau \). From now on, we write \( \sigma \) for the source and \( \tau \) for the target of function types.

Coeffect scalar. Coeffect scalar structures are equipped with two operations. In bounded reuse, these were \( \ast \) for sequencing (in function application) and \( + \) for context sharing (in contraction). Additional structure is needed for variable access and sub-coeffecting.

Definition 1. A coeffect scalar \( (C, \odot, \oplus, \text{use}, \text{ign}, \leq) \) comprises a set \( C \) together with elements \( \text{use}, \text{ign} \in C \), relation \( \leq \) and binary operations \( \odot, \oplus \) such that \( (C, \odot, \text{use}) \) and \( (C, \oplus, \text{ign}) \) are monoids, \( (C, \leq) \) is a pre-order, and the following distributivity axioms hold:

\[
(r \odot s) \odot t = (r \odot t) \odot (s \odot t) \\
(t \odot (r \odot s)) = ((t \odot r) \odot (t \odot s))
\]

The operation \( \odot \) must form a monoid with use to guarantee an underlying category in the semantics (Section 5). It models sequential composition with variable access (use) as the identity. The other element (ign) is used for variables that are not accessed. The operation \( \oplus \) combines coeffects for contexts used in multiple places (contraction). The notation is inspired by the bounded reuse rule, which uses coeffect scalar structure \( (\mathbb{N}, +, +, 0, \leq) \), but be aware that \( \odot \) and \( \oplus \) are not always multiplication and addition.

Coeffect annotations \( R \) can be viewed as containers of scalar coeffects. For structural coeffects, the container is a vector, while for flat coeffects it is a trivial singleton container. The following definition takes inspiration from the work of Abbott et al. which describes containers in terms of shapes and a set of positions.

Coeffect shapes. The coeffect system is parameterised by a set of shapes \( S \). A coeffect annotation is indexed by a shape \( s \in S \) calculated from the shape of the free-variable context \( \Gamma \). The correspondence is not necessarily bijective. For example, flat coeffect systems have just a single shape \( S = \{ \ast \} \).
Thus, in the judgment $\Gamma @ R \vdash e : \tau$, the coeffect annotation $R$ is drawn from the set of coeffect scalars $\mathcal{C}$ indexed by the shape of $\Gamma$. We write $s = \Gamma$ for the shape corresponding to $\Gamma$. We define shapes by a set of positions and so we can define $R \in s \rightarrow C$ as a mapping from positions (defined by the shape) to scalar coeffects. We usually write this as the exponent $R \in C^s$.

The set of shapes is equipped with an operation that combines shapes (when we combine variable contexts), an operation that computes shape from the free-variable contexts, and two special shapes in $\mathcal{S}$ representing empty context and singleton context.

**Definition 2.** A coeffect shape structure $(\mathcal{S}, [-], \circ, 0, \hat{1})$ comprises a set $\mathcal{S}$ with a binary operation $\circ$ on $\mathcal{S}$ for shape composition, a mapping from contexts to shapes $[\Gamma] \in \mathcal{S}$, and elements $0, \hat{1} \in \mathcal{S}$ such that $(\mathcal{S}, \circ, 0)$ is a monoid and $[-]$ is partially specified on empty and singleton free-variable contexts by:

$$[\emptyset] = 0 \quad [v : \tau] = \hat{1}$$

This means that the elements $0$ and $\hat{1}$ represent the shapes of empty and singleton free-variable contexts respectively. As said earlier, we use two kinds of shape structure:

- **Structural coeffect shape** is defined as $[N, [-], +, 0, 1]$. We treat numbers as sets $0 = \{\}$, $1 = \{\emptyset\}$, $2 = \{0, 1\}$, $3 = \{0, 1, 2\}$, and so on (that is a number is a set of positions). The shape mapping $[\Gamma]$ returns the number of variables in $\Gamma$. Empty and singleton contexts are annotated with 0 and 1, respectively, and shapes of combined contexts are added so that $[\Gamma_1, \Gamma_2] = [\Gamma_1] + [\Gamma_2]$. Therefore, a coeffect annotation is a vector $R \in C^n$ and assigns a coeffect scalar $R(i) \in C$ for each variable $x_i$ in the context.

- **Flat coeffect shape** is defined as $[(\times), \star, \circ, \times, \circ, 0, \hat{1}]$ where $\star(\Gamma) = \times$ and $\circ \circ \circ \star = \times$. That is, there is a single shape $\star$ with a single position and all free-variable contexts have the same shape. Therefore, a coeffect annotation is drawn from $C^n$ which is isomorphic to $C$ and so a coeffect scalar $\tau \in C$ is associated with every free-variable context.

Using a shape with no positions reduces our system to the simply-typed $\lambda$-calculus with no context annotations. Trees can also be used to build a system akin to bunched typing [12].

**Coeffect algebra.** The coeffect calculus annotates judgments with shape-indexed, or shaped, coeffects. The coeffect algebra structure combines a coeffect scalar and coeffect shape structure to define shaped coeffects and operations for combining these. In Section 3.2, shaped coeffects were combined by the tensor $\times$ in structural examples and $\cup$ in the implicit parameters example. To capture the examples so far and those described previously [15], we distinguish two operators for combining shaped coeffects.

**Definition 3.** Given a coeffect scalar $(\mathcal{C}, \circ, \cup, use, ign, \leq)$ and a coeffect shape $(\mathcal{S}, [-], \circ, 0, \hat{1})$, a coeffect algebra extends the two structures with $(\times, \times, \times, \cup, \circ, 0, \hat{1})$ such that $\times \in \mathcal{S}$ and a coeffect algebra $(\mathcal{S}, [-], \circ, 0, \hat{1})$ is a coeffect algebra.

A coeffect algebra induces the following two additional operations:

$$m : \mathcal{S} \times \mathcal{S} \rightarrow \mathcal{S}$$

$$(\times) : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$$

$$(\circ) : \mathcal{C} \times \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$$

$$(\cup) : \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$$

The operators $\times$ and $\cup$ combine shaped coeffects associated with two contexts. For example, assume we have $\Gamma_1$ and $\Gamma_2$ with coeffects $R \in C^n$ and $S \in C^n$. In the structural system, the context shapes $m, n$ denote the number of variables in the two contexts. The combined context $\Gamma_1, \Gamma_2$ has a shape $m \circ n$ and the combined coeffects $R \times S, R \times S \in C^{m \circ n}$ are indexed by that shape.

The operators $\times$ and $\cup$ combine shaped coeffects associated with two contexts. For example, assume we have $\Gamma_1$ and $\Gamma_2$ with coeffects $R \in C^n$ and $S \in C^n$. In the structural system, the context shapes $m, n$ denote the number of variables in the two contexts. The combined context $\Gamma_1, \Gamma_2$ has a shape $m \circ n$ and the combined coeffects $R \times S, R \times S \in C^{m \circ n}$ are indexed by that shape.

**3.3 General coeffect type system**

In the previous section, we developed an algebraic structure capable of capturing different concrete context-dependent properties discussed in Section 2. Now, we use the structure to define the general coeffect calculus in Figure 4. Coeffect annotations on free-variable contexts are shape-indexed, where for some shape $s \in \mathcal{S}$ then $R, S, T \in C^s$. Function types are annotated with coeffects scalars $r, s, t \in C$. The rules of Figure 4 manipulate coeffect annotations using the coeffect algebra.
operations \((\times, \times, \bot)\) and the derived constructs \((-\cdot)\) and \(\oplus\). Free-variable contexts \(\Gamma\) are treated as vectors modulo duplicate use of variables – associativity is built-in. Variable order matters, but can be changed using the structural rules. Structural rules are expressed using a helper relation, written \(\rightsquigarrow\).

**Typing rules.** Constants \((\text{const})\) and variables \((\text{var})\) annotate the context with special values. The empty unused context is annotated with \(\bot \in C^\emptyset\) and the singleton context with \((\text{use}) \in C^1\). Note that the shapes \(0, 1, \bot\) match the shape of the variable contexts.

Lambda abstraction \(\rightsquigarrow\) splits the context requirements using \(\times\) into a coeffect \(R\) and a coeffect \((s)\) of a shape \(1\) (semantically, it merges capabilities provided by the declaration-site and call-site contexts). In structural systems such as bounded reuse, this identifies coeffect associated with the bound variable, because \(\times\) is not commutative.

The \((\text{app})\) rule follows the patterns seen earlier – it uses the scalar-vector multiplication \((t \otimes S)\) of the coeffects \(S\) from the argument (associated with \(\Gamma_2\)) and the latent coeffect \(t\) of the function. Using the syntactic reading, it then merges context requirements for \(\Gamma_1\) and \(\Gamma_2\). In the dual semantic reading, it splits the provided context into two parts passed to the sub-expressions.

The typing of let-binding \((let\langle\rangle\rangle\) corresponds to the typing of an expression \((\lambda x.e_2)\ e_1\). Syntactically, the context requirements are first split using \(\times\) and then re-combined using \(\times\).

**Structural rules.** The coeffect-annotated context can be transformed using structural rules that are not syntax-directed. These are captured by \((\text{ctx})\), which uses a helper relation representing context transformations \(\Gamma \otimes R' \rightsquigarrow \Gamma \otimes R\). The \(\text{ctx}\) rules model that a context used in the rule conclusion \(\Gamma \otimes R'_1\) can be transformed to a context required by the premise \(\Gamma \otimes R\) (using the semantic bottom-up reading). In the rule, \(\theta\) is a variable substitution generated by the transformation, which is used in the \((\text{con})\) rule.

Exchange and contraction decompose and reconstruct coeffect annotations using \(\times m, n\) (in assumption) and \(\times m, n\) (in conclusion). The shape subscripts are omitted, but we require the shapes to match using \(m = [\Gamma_1]\) and \(n = [\Gamma_2]\).

The \((\text{weak})\) rule drops an ignored variable annotated with \((\text{ign})\) (compare with \((\text{var})\) annotated using \((\text{use})\)). The \((\text{exch})\) rule swaps variables/coeffects while \((\text{con})\) combines coeffects using \(\oplus\) to represent sharing of the context. Finally, \((\text{sub})\) represents subcoeffecting and can be applied (pointwise) to any scalar coeffect.

### 3.4 Structural coeffects

The coeffect system uses a general notion of context shape, but it was designed with structural and flat systems in mind. The structural system is new in this paper and so we look at it first.

Recall the coeffect shapes that characterise structural systems: the shape is formed by natural numbers (with addition) modelling the number of variables in the context. The coeffect algebra is therefore formed by the free monoid (lists/sockets) over a coeffect scalar. This means that the system keeps a vector of coeffect scalar annotations – one for each variable. An empty context (e.g., in the \((\text{const})\) rule) is annotated with a zero-length vector.

**Definition 4.** Given a coeffect scalar \((C, \oplus, \otimes, \text{use}, \text{ign}, \leq)\) a structural coeffect system has:

- Coeffect shape \((\mathbb{N}, [-], +, 0, 1)\) formed by natural numbers
- Coeffect algebra \((\times, \times, \{\})\) where \(\times\) and \(\{\}\) are shape-indexed versions of the binary operation and the unit of a free monoid over \(C\). That is \(\times : C^m \times C^n \to C^{m+n}\) appends vectors (lists) and \(\{\} : C^0 \to C^0\) represents empty vectors (lists).

The definition is valid since the shape operations form a monoid \((\mathbb{N}, +, 0)\) and \([-]\) (calculating the length of a list) is a monoid homomorphism from the free monoid to the monoid of shapes.

**Examples.** Defining a concrete structural coeffect system is easy, we just provide the coeffect scalar structure and the rest is free.

- To recreate the system for bounded reuse, we use coeffect scalars formed by \((\mathbb{N}, +, 0, 0, \leq)\). As in the system of Figure 1, \textit{used} variables are therefore annotated with 1 and \textit{unused} with 0. Contraction adds the number of uses via \(+\) and application (sequencing) multiplies the uses.
- Dataflow uses natural numbers (of past values), but differently: \((\mathbb{N}, +, 0, 0, \leq)\). Variables are initially annotated with 0 (and can be incremented using the \texttt{prev} keyword). Annotations of a shared variable are combined by taking maximum (of past values needed) and sequencing uses +.
- Another use of the system is to track \textit{variable liveness}. The annotations are formed by \(C = \{D, L\}\) where \(L\) represents a \textit{live} (used) variable and \(D\) represents a \textit{dead} (unused) variable. The coeffect scalar structure is \((C, \cap, \cup, \bot, \top)\) where \(D \subseteq L\).

In sequential composition \((\otimes)\), a variable is live only if it is required by both of the computations \((L \cap L = L)\), otherwise it is marked as dead \((D)\). A computation is not evaluated if its result is not needed. A shared variable \((\cup)\) is live if either of the uses is live \((D \cup D = D, \text{otherwise} L)\).

Structural liveness is a practically useful, precise version of an example from our earlier work, which was a flat system overapproximating liveness of the entire context \([15]\). Since \(\times = \times = \times\), structural rules (weaken, contract, exchange) are freely permitted, modifying the coeffects accordingly.

### 3.5 Flat coeffects

The same general coeffect system can be used to define systems that track whole-context coeffects as in the implicit parameters example (Section 2.4). Flat coeffect systems are characterised by a singleton set of shapes, such as \(\{\}\). In this setting, the context annotations \(C^*\) coincide with coeffect scalars \(C\).

In addition to the coeffect scalar structure, we also need to define \(\text{x}\) and \(\text{x}\). Our examples of flat coeffects use \(\oplus\) (merging of scalar coeffects) for \(\text{x}\) (merging of shaped coeffect annotations). However, the \(\times\) operation needs to be provided explicitly. Thus the general form of flat coeffect system is defined as follows.

**Definition 5.** Given a coeffect scalar \((C, \oplus, \otimes, \text{use}, \text{ign}, \leq)\) and an operation \(\land : C \times C \to C\) such that \((r \land s) \leq (r \oplus s)\), we define:

- Flat coeffect shape \((\{\}, \text{const}, \otimes, \bullet, \star, \times)\) where \(\star \bullet \star = \star\)
- Flat coeffect algebra \((\land, \otimes, \text{ign})\), i.e., the \(\times = \otimes\) and \(\bot = \text{ign}\) with the additional binary operation \(\times = \land\).

The additional axiom \((r \land s) \leq (r \oplus s)\) is required for \(\beta\)-equivalence in flat systems (see later Theorem 11, Section 4.2).

If \(\times\) is idempotent, then structural rules (weaken, contract, exchange) are freely permitted since any flat coeffect annotation \(r\) can be expanded to \(r \times r\). This property holds for all examples here, hence structural rules can always be applied.

If \(\times\) is also idempotent (as in all our examples), then exchange and contraction rules preserve the coeffects of the assumption in the conclusion. Otherwise \((r \land s) \leq (r \oplus s)\) means that exchange and contraction behave as the \((\text{sub})\) rule for subcoeffecting.

**Examples.** Implicit parameters are the prime example of a flat coeffect system, but other examples include rebindable resources \([17]\) and Haskell type classes \([13]\).

In the implicit parameters system (Section 2.4), coeffect scalars are sets of name-type pairs \(C = \mathcal{P}(\text{Name} \times \text{Types})\). Variables are annotated with \(\emptyset\) and coeffects are combined or split (in the top-down reading for \((\text{abs})\)) using set union \(\cup\). Thus, the coeffect scalar structure is \((\mathcal{P}(\text{Name} \times \text{Types}), \cup, \cup, \emptyset, \emptyset, \subseteq)\) with \(\land = \cup\).
Remark 6. We previously described flat systems for liveness and dataflow [15]. Turning a structural system to a flat system requires finding \( \Lambda \) that underapproximates the capabilities of combined contexts. For dataflow, this is given by the min function, which satisfies the requirement because \( \min(r,s) \leq \max(r,s) \).

In flat dataflow, we annotate the entire context with the maximal number of past elements required overall. We use the same coeffect scalars \( \langle N, +, \max, 0, 0, \leq \rangle \) as in the structural version, but with \( \Lambda = \min \). Abstraction (which is the only rule using \( \Lambda \)) becomes:

\[
\Gamma, x : \sigma @ \min(r,s) \vdash e : \tau \\
\Gamma @ R \vdash \lambda x.e : \sigma \rightarrow \tau
\]

Both the declaration-site and call-site must provide at least the number of past values required by the body. The overapproximation means both \( r \) and \( s \) can be greater than actually required. For dataflow, we could enforce that immediate and latent coeffects are identical, but that would require treating \( x \) as a partial function.

4. Equational theory

Each of the concrete coeffect systems discussed in this paper has a different notion of context dependence, much like various effectful languages have different notions of effects (such as state or exceptions). However, there are common equational properties that hold for all (or some) of the systems we consider.

The equational theory in this section illuminates the axioms of coeffect algebra and the semantics of the calculus. We discuss syntactic substitution as it can form the basis for reduction in a concrete operational semantics. We consider structural and flat systems separately. This provides better insight into how the two systems work and differ. In particular, call-by-name evaluation is coeffect preserving for all structural, but only some flat systems.

The properties and proofs in this section are syntactic. In Section [5,6], we show that our denotational model of the coeffect calculus is sound with respect to the equational theory here.

We use standard syntactic substitution written as \( e_1[x \leftarrow e_2] \), \( \beta \)-reduction and \( \eta \)-expansion, written as \( \sim_{\beta} \) and \( \sim_{\eta} \). Equality of terms \( e_1 \) and \( e_2 \), written as \( \equiv \) is defined w.r.t. their contexts, types and coeffects and is written \( \Gamma @ R \vdash e_1 \equiv e_2 : \tau \).

4.1 Structural coeffect systems

For structural coeffect systems, recall that coeffects are vectors with \( x = x = x \times \) (vector concatenation) and \( \bot = () \) (the empty vector), thus coeffect annotations comprise the free monoid over scalars.

We first show substitution:

**Lemma 7 (Substitution lemma).** In a structural coeffect calculus with a coeffect scalar structure \( (C, \oplus, \ominus, use, \text{ign}, \leq) \):

\[
\Gamma @ S \vdash e_\sigma : \sigma \land \Gamma, x : \sigma, \Gamma_2 @ R_1 \times (r) \times R_2 \vdash e_r : \tau \\
\Rightarrow \Gamma_1, \Gamma_2 @ R_1 \times (r @ S) \times R_2 \vdash e_r[x \leftarrow e_\sigma \ominus e_r] : \tau
\]

**Proof.** By induction over the derivation for \( e_r \) using the free monoid \((C, \times, \emptyset)\) and coeffect scalar axioms (full proof [15]).

Because of the vector (free monoid) structure, coeffects \( R_1, R_2, \) and \( (r) \) for the receiving term \( e_r \) are uniquely associated with \( \Gamma_1, \Gamma_2, \) and \( x \) respectively. Therefore, substituting \( e_\sigma \) (which has coeffects \( S \)) for \( x \) introduces the context dependencies specified by \( S \) which are composed with the requirements \( r \) on \( x \).

Using the substitution lemma, we can demonstrate \( \beta \)-equality:

\[
\Gamma_1, x : \sigma @ R \times (r) \vdash e_1 : \tau \\
\Gamma_1 @ R \vdash \lambda x.e_1 : \sigma \rightarrow \tau \\
\Gamma_2 @ S \vdash e_2 : \sigma \\
\Gamma_1, \Gamma_2 @ R \times (r @ S) \vdash (\lambda x.e_1) e_2 \equiv e_1[x \leftarrow e_2] : \tau
\]

As a result, \( \beta \)-reduction preserves the type and coeffects of a term. This gives the following subject reduction property:

**Theorem 8 (Subject reduction).** In a structural coeffect calculus, if \( \Gamma @ R \vdash e : \tau \) and \( e \sim_{\beta} e' \) then \( \Gamma @ R \vdash e' : \tau \).

**Proof.** Following from Lemma 7 and \( \beta \)-equality.

Structural coeffect systems also exhibit \( \eta \)-equality, therefore satisfying both the local soundness and local completeness conditions of Pfenning and Davies [16]. This means that abstraction does not introduce too much, and application does not eliminate too much.

\[
\Gamma @ R \vdash e : \sigma \rightarrow \tau \\
\Gamma, x : \sigma @ R \times (s @ (use)) \vdash e \times x : \tau \\
\Gamma @ R \vdash \lambda x.e \equiv e : \sigma \rightarrow \tau
\]

The last step uses the equalities \( s @ (use) = (s @ use) = (s) \) arising from the monoid \((C, \oplus, use)\) of the scalar coeffect structure.

This highlights another difference between coeffects and effects, as \( \eta \)-equality does not hold for many notions of effect. For example, in a language with output effects, \( e = (\text{print} \ "x", (\lambda x.x)) \) has different effects to its \( \eta \)-converted form \( \lambda x.ex \) because the immediate effects of \( e \) are hidden by the purity of \( \lambda \)-abstraction. In the coeffect calculus, the \( (abs) \) rule rule allows immediate contextual requirements of \( e \) to “float outside” of the enclosing \( \lambda \). Furthermore, the free monoid of \( x \) in structural coeffect systems allows the exact immediate requirements of \( \lambda x.ex \) to match those of \( e \).

4.2 Flat coeffect systems

The equational theory for flat coeffect systems is somewhat similar to effect systems where (co)effects are not linked to individual variables. In effectful languages, substituting an effectful computation for \( y \) in \( \lambda x.g \) changes the latent effect associated with the function. Similarly, for some of the flat coeffect systems, substituting a context-dependent computation for \( y \) in \( \lambda x.(\lambda x.y) \) adds latent context requirements to the function type. However, this is not the case for all flat coeffect systems – for example, call-by-name reduction preserves types and coeffects for the implicit parameters system (which makes it a suitable model for Haskell). For other systems, we first briefly consider call-by-value reduction.

**Call-by-value.** The notion of value in coeffect systems differs from the usual syntactic understanding. As discussed earlier, a function \((\lambda x.e)\) is not necessarily a value in coeffect calculi, because it may not delay all context requirements of \( e \). Thus a syntactic value \( v \) is a value if it has no immediate context requirements.

**Definition 9.** A syntactic value \( v \) is a pure value if \( \Gamma @ Val \vdash v : \tau \) where \( Val : C^{[\tau]} \) is a coeffect indexed by the shape of \( \Gamma \) that always returns use. That is \( Val = \text{An.use} \).

In call-by-value, the right-hand side of an application is evaluated to a pure value, which is then substituted for a variable. However, the discharging of coeffects prior to substitution is different for each coeffect system.

Recall that a flat coeffect system consists of coeffect scalars \((C, \oplus, \ominus, use, \text{ign}, \leq)\) together with a binary operation \( \land \) on \( C \) such that the coeffect algebra structure is \((C, \ominus, use, \text{ign})\).

**Lemma 10 (Call-by-value substitution).** In a flat coeffect calculus with coeffect scalars \((C, \oplus, \ominus, use, \text{ign}, \leq)\) and the \( \land \) operator:

\[
\Gamma @ Val \vdash e_\sigma : \sigma \land \Gamma, x : \sigma, \Gamma_2 @ r \vdash e_r : \tau \\
\Rightarrow \Gamma_1, \Gamma_2 @ r \vdash e_\sigma[x \leftarrow e_r] : \tau
\]

**Proof.** By induction over the coeffect derivation, using the fact that both \( x \) and \( e_r \) are annotated with use.

**Lemma 10** holds for all flat coeffect systems, but it is weak. To use it, the operational semantics must provide a way of partially
evaluating a term with requirements \( R \) to a value. Assuming a call-by-value reduction \( \leadsto_{cbv} \), using the above definition of value:

**Theorem 11** (Call-by-value subject reduction). In a flat coeffect calculus, if \( \Gamma \bowtie e : \tau \) and \( e \leadsto_{cbv} e' \) then \( \Gamma \bowtie e' : \tau \).

**Proof.** A direct consequence of Lemma 10 using the flat coeffect system requirement \((\tau \land s) \leq (\tau \land \bot)\) to prove \( \beta \)-equality. 

**Call-by-name.** A term \((\lambda x.e_1) e_2\) can be \( \beta \)-reduced in the call-by-name strategy even if both sub-expressions have contextual requirements.

We call a flat coeffect algebra **top-pointed** if \( \text{use} \) (the coeffect of variable use) is the greatest (top) coeffect scalar \( \Gamma \) and bottom-pointed if it is the smallest (bottom) coeffect scalar with respect to the order \( \leq \). Liveness analysis is an example of top-pointed coeffects as \( \text{use} = L \) and \( D \leq L \).

**Lemma 12** (Top-pointed substitution). In a top-pointed flat coeffect calculus with \((\Gamma, \circ, \circ, \text{use}, \text{ign}, \leq)\) and the \( \otimes \) operator:

\[
\Gamma \circ s \vdash e_s : \sigma \quad \Gamma_1, x : \sigma, \Gamma_2 \circ r \vdash e_r : \tau \\
\Rightarrow \quad \Gamma, \Gamma_1, \Gamma_2 \circ s \circ r \vdash e_r[x \leftarrow e_s] : \tau
\]

**Proof.** Using sub-coeffecting \((s \leq u)\) and Lemma 10.

As variables are annotated with the top element \( \text{use} \), we can substitute a term \( e_r \) for any variable and use sub-coeffecting to get the original typing (because \( s \leq \text{use} \)).

In a bottom-pointed coeffect system, substituting \( e \) for \( x \) increases the context requirements. However, if the system satisfies the condition that \( \land = \bot = \circ \) then the context requirements arising from the substitution can be associated with the context \( \Gamma \). As a result, substitution does not break soundness as in effect systems. The requirement \( \land = \bot = \circ \) holds for our implicit parameters example (all three operators are set union) and allows the following substitution lemma:

**Lemma 13** (Bottom-pointed substitution). In a bottom-pointed flat coeffect calculus with \((\Gamma, \circ, \circ, \text{use}, \text{ign}, \leq)\) and the \( \otimes \) operator where \( \land = \bot = \circ \) is idempotent and commutative:

\[
\Gamma \circ s \vdash e_s : \sigma \quad \Gamma_1, x : \sigma, \Gamma_2 \circ r \vdash e_r : \tau \\
\Rightarrow \quad \Gamma, \Gamma_1, \Gamma_2 \circ s \circ r \vdash e_r[x \leftarrow e_s] : \tau
\]

**Proof.** By induction over \( r \), using the idempotent, commutative monoid structure to keep \( s \) with the free-variable context.

The structural system is precise enough to keep distinct coeffects associated with each concrete variable. The flat variant described here is flexible enough to let us always re-associate new context requirements with the free-variable context.

The two substitution lemmas show that the call-by-name evaluation strategy can be used for certain coeffect calculi, including liveness and implicit parameters. Assuming \( \leadsto_{cbn} \) is the standard call-by-name reduction, the following theorem holds:

**Theorem 14** (Call-by-name subject reduction). In a flat coeffect system that satisfies the conditions for Lemma 12 or Lemma 13 if \( \Gamma \bowtie e : \tau \) and \( e \leadsto_{cbn} e' \) then \( \Gamma \bowtie e' : \tau \).

**Proof.** Direct consequence of Lemma 12 or Lemma 13.

5. Semantics

Coeffects provide a unified description of context dependence. In the previous sections, we used this to define a unified coeffect calculus. We now define a unified (categorical) semantics for the coeffect calculus. The semantics can be instantiated for different notions of context dependence and thus can model a wide range of context-aware languages (both for flat and structural systems).

We relate the semantics to the equational theory and show that it is sound with respect to term equality. For a variant of the flat system, a similar result has already been shown in the second author’s PhD dissertation [13]. The semantics is introduced in pieces:

- **Section 5.1** describes the signature (range and domain) of the interpretation \([\cdot]_I\), gives the interpretations for types and free-variable contexts (in flat and structural systems), and defines the signature of functors \( D \) which encode contexts.
- The first part of the semantics (Section 5.2) defines sequential composition of context-dependent computations via indexed comonads (introduced briefly in our previous work [13]) and the indexed structural comonad structure (new here).
- More structure is needed for the semantics of application and abstraction. Section 5.3 defines indexed monoidal operations for splitting and merging contexts. Concrete structures are given throughout for the semantics of the structural bounded reuse and flat implicit parameter systems.
- Section 5.4 puts the pieces together, defining the semantics of the coeffect calculus. The semantics is illustrated by executing an example bounded-reuse program (Example 25).

**Section 5.5** shows our semantics sound with respect to the syntactic equational theory of Section 4. This uses the derivation of the categorical structures for the semantics as \( \lambda \text{axioms} \) and the base category \( C \).

In this section, \( \Gamma, C, D, I \) range over categories. The objects of a category \( C \) are written \( \text{obj}(C) \). The category of functors between \( C \) and \( D \) is written \([C, D]\). Exponential objects, representing function types in our model, are written in two ways, either \( B^A \) or \( A \Rightarrow B \).

5.1 Interpreting contexts and judgments

The semantics is parameterised by a coeffect algebra, with scalar coeffects \((\Gamma, \circ, \circ, \text{use}, \text{ign}, \leq)\), coeffect shape \((S, [\cdot], \circ, 0, 1)\), and \((s, x, \bot)\). An interpretation \([\cdot]_I\) is given to types, free-variable contexts, and type and coeffect judgments, with a base Cartesian-closed category \( C \) for denotations and a category \( \Gamma \) of scalar coeffects, where \( \text{obj}(\Gamma) \to C \). Since \( C \) is Cartesian-closed, we use the \( -\) calculus as the syntax for giving concrete definitions in \( C \).

The interpretation \([\cdot]_I\) is parameterised by categorical structures which model a particular notion of context. The interpretation of free-variable contexts depends on shape, for which we give concrete definitions for flat and structural shapes.

**Interpreting judgments.** Type and coeffect judgments are interpreted (given denotations) as morphisms in \( C \), of the form:

\[
[\Gamma \bowtie e : \tau] : D^{\tau}_I[\Gamma] \to [\tau]
\]

The interpretation is a morphism from an interpretation of the context \( \Gamma \) to the interpretation of the result. The functor \( D^{\tau}_I \) over \( \Gamma \) encodes the semantic notion of context and is indexed by the free-variable context shape \([\Gamma]_I\) and coeffect annotation \( R \).

The structure \( D \) can be thought of as a dependent product of functors \( D^n \) over possible shapes \( n \in S \)

\[
D : \Pi_{n : S}.D^n \quad \text{where} \quad D^n : \Gamma^n \to [C^n, C]
\]

For a fixed context shape \( n \) the functor \( D^n : \Gamma^n \to [C^n, C] \) maps an \( n \)-indexed coeffect (think positions) to a functor from a context \( C^n \) to an object in \( C \). This is, given a coeffect annotation (matching the shape of the context), we get a functor \( C^n \to C \).

From a programming perspective, this functor defines a data structure that models the additional context provided to the pro-
Types. Types are interpreted as objects of $C$, that is $[\tau] : \text{obj}(C)$ where function types have the interpretation as exponents: $$[\sigma \to \tau] = \text{D}^1([\sigma]) \Rightarrow [\tau]$$ The parameter of a function is wrapped by a functor $\text{D}^1_{\mathcal{L}}$, that defines a context with singleton shape $1$, matching the single value that it contains. This interpretation is shared by all coeffect calculi.

Free-variable contexts. As described above, free-variable contexts $I$ are given an interpretation as objects in $C[I]$. Thus, the interpretation of contexts is shape dependent.

We define $[\mathcal{L}]$ on free-variable contexts for structural and flat systems. For flat systems, there is only a single shape, so the interpretation is product of objects in structural interpretations of free-variable contexts is that flat uses $\Gamma$ on free-variable contexts for structural and flat systems. For structural systems, the shape matches the number of variables and so the model is a value in the product category $C \times \ldots \times C$.

Flat coeffects. Recall that $S = \{\}$ and $[\Gamma] = \emptyset$. Since the set of positions $\ast$ is a singleton, then $C^\ast$ is isomorphic to $C$. Therefore $[\Gamma] : \text{obj}(C)$, which is defined as:

$$[[x_1 : \tau_1, \ldots, x_n : \tau_n] = [\tau_1] \times \ldots \times [\tau_n]$$

Denotations of typing judgments in a flat coeffect system are thus of the form (where $r \in D$):

$$[[x_1 : \tau_1, \ldots, x_n : \tau_n, \Gamma \vdash e : \tau] : \text{D}^n_\mathcal{L}([[\tau_1] \times \ldots \times [\tau_n]]) \Rightarrow [\tau]$$

Structural coeffects. Recall that $S = C$ and $[\Gamma] = C$ (number of free variables), thus $[\Gamma] : \text{obj}(C[I])$. This is defined similarly to the above, but instead of using products in $C$, we use the product of categories. Thus, denotations have the form:

$$[[x_1 : \tau_1, \ldots, x_n : \tau_n, \Gamma \vdash e : \tau] : \text{D}^n_\mathcal{L}([[\tau_1] \times \ldots \times [\tau_n]]) \Rightarrow [\tau]$$

where $[R] = n$ and we use comma (instead of $\times$) to denote the product of categories. This means that $D^n : I^n \Rightarrow [C^n, C]$ is a functor between the $n$-length vector of coeffects indices and an $n$-ary endofunctor. Thus, the key difference between the flat and structural interpretations of free-variable contexts is that flat uses products of objects in $C$ and the structural uses products of $C$ in the category of categories.

Example 15 (Bounded reuse). Recall bounded reuse has coeffect scalars $C = N$ and shapes $S = N$. We model contexts by replicating the value of each variable so there is a value for each use. This matches the model used by Girard et al. [7]. Contexts are described by $B : \Pi_{\mathcal{L},\mathcal{N}}(I^\mathcal{L} \Rightarrow [C^\mathcal{N}, C])$, for where $R = \{r_1, \ldots, r_n\}$:

$$B^n_R(A_1, \ldots, A_n) = A^n_1 \times \ldots \times A^n_n$$

$$B^\mathcal{N}_R(f_1, \ldots, f_n) = \lambda(a_1, \ldots, a_n).((f_1 \circ a_1), \ldots, (f_n \circ a_n))$$

Thus each object in the free-variable context $A_1$ is exponentiated by its associated coeffect $r_1$. For the morphism mapping part, $f_1 : A_1 \to B_1$ and $a_1 : A_1^\mathcal{N}$, thus $(f_1 \circ a_1) : B_1^\mathcal{N}$. The exponent $A_1^\mathcal{N}$ can be read as a product of $r_1$ copies of $A_1$, e.g.

$$B^\mathcal{N}_{0,2}(A, B, C) = A^1 \times B^0 \times C^2 = (A \times 1 \times (C \times C))$$

Example 16 (Implicit parameters). Recall the implicit parameter calculus with scalars as sets of names paired with types $C = \mathcal{P}(\mathcal{T} \times \mathcal{N})$ and flat shape with singleton $S = \{\ast\}$. Its contexts are defined by $I^\mathcal{L} : \Pi_{\mathcal{L},\mathcal{E}}(I^\mathcal{L} \Rightarrow \text{Set}^\mathcal{N}, \text{Set})$ which is equivalent to $I \Rightarrow \{\text{Set}, \text{Set}\}$ and defined as follows:

$$I^\mathcal{L}_A = A \times [R]$$

$$I^\mathcal{L}_f = \lambda(a, \Gamma). (f \ a, \ r)$$

The interpretation $[R]$ maps a set of variable-type pairs to an object representing a set of variable-value pairs in $\text{Set}$.

5.2 Sequential composition

Following the usual semantics approach, we require a notion of sequential composition for our denotations. We first consider a special case for $\text{D}^1$ (where $\mathcal{L} = \emptyset$ and $C = C_1 = C$) in both flat and structural systems and thus $\text{D}^1 : I \Rightarrow [C, C]$ and $\text{D}^1_\mathcal{L} : I \Rightarrow [C[I], C[I]]$. Composition of morphisms $f : \text{D}^1_\mathcal{L} A \to B$ and $g : \text{D}^1 A \to C$ is defined by an indexed comonad (which we introduced previously [13,15]).

Definition 17. An indexed comonad $F : I \Rightarrow [C, C]$ comprises a strict monoidal category $(\mathcal{I}, \mathcal{I}, I)$ and a functor $F : I \Rightarrow [C, C]$ with two natural transformations (where we write $(F R) A$ as $F R A$):

$$(\delta_{X,Y})_A : F_{(X \cdot Y)} A \Rightarrow F_X(F_Y A) \quad (\varepsilon_1)_A : FA \Rightarrow A$$

where $\delta$ is called coaddition and $\varepsilon$ is called counit. We require indexed analogues of the usual comonad axioms (cf. [19]):

$$F R \delta_{R \cdot I} \Rightarrow F F R I F T \quad \delta_{R \cdot S \cdot T} = F R S \delta_{S \cdot T} \Rightarrow F F R S T F T$$

An indexed comonad $F : I \Rightarrow [C, C]$ induces a notion of composition for all $f : F S A \to B$ and $g : F R B \Rightarrow C$:

$$g \circ f = \delta_{R,S} \circ g \circ \delta_{R,S} : F R S A \Rightarrow C$$

with the identity $i_d A = (\varepsilon_1)_A : F I A \Rightarrow A$ for all $A$. Thus indexed comonads induce a category which has the same objects as $C$ and morphisms $C F_{(A,B)} = \bigcup_{R \in \mathcal{I}} C(F R A, B)$. Note that an indexed comonad is not a family of (ordinary) comonads, because identity need only be defined for the functor $F I$.

Therefore, if $\text{D}^1$ is an indexed comonad, there is a notion of composition for denotations with a single coeffect index.

Example 18 (Bounded reuse). $\text{B}^\mathcal{N}_R$ (Example 15) has an indexed comonad structure, where the monoid $(\mathcal{N}, \ast, I)$ from the coeffect scalar for bounded reuse induces a monoidal category structure on $\mathcal{I}$ with $1 : \emptyset$ and the bifunctor $\ast : \mathcal{I} \times \mathcal{I} \Rightarrow \emptyset$, with operations:

$$\varepsilon_1 = \lambda(a_1, a_1) \quad \delta_{R,S} = \lambda(a_1, \ldots, a_{R \cdot S}) \quad ((a_1, \ldots, a_S), (a_{S+1}, \ldots, a_{S+R}), \ldots, (a_{R \cdot (R-1) + 1}, \ldots, a_{R \cdot (R-1) + R}))$$

Indexed comonads essentially model single-variable contexts. Count here requires a single copy of the value from the context. Multimultiplication splits $R$ times $S$ copies of a value into $R$ copies of a context where each context contains just $S$ copies of the value.

Remark 19. A semantics for dataflow coeffects is similar to bounded reuse with $\text{D}^1_\mathcal{L}(A_1, \ldots, A_n) = (A_1 \times \ldots \times (A_n \times \text{Set}^\mathcal{N})_R)$, i.e., each free-variable has an extra value representing the “current” value. A dataflow indexed comonad is similar to the above but with additive rather than multiplicative behaviour.

Example 20 (Implicit parameters). For the coeffect scalar monoid $\mathcal{P}(\mathcal{N} \times \mathcal{X})$ and $\emptyset$ of implicit parameters, $\mathcal{I}^\mathcal{L}$ (Example 16) has an indexed comonad structure, with operations:

$$\varepsilon_\emptyset = \lambda(\emptyset, \emptyset, a) \quad \delta_{R,S} = \lambda(a, \gamma_\emptyset).((a, \gamma_\emptyset), \gamma_{(R)})$$

where $\gamma_{(R)} = \{(x, v) \mid (x, v) \in \gamma, (x, t) \in R\}$ filters incoming implicit parameters to those variable-value pairs where the variable is in the coeffect $R$.

2Since $\mathcal{I} = 1$ in structural and $\mathcal{I} \leq 1$ in flat, i.e., $\ast$ is isomorphic to 1.
These two examples (which are new here) provide composition for context-dependent computations indexed by coeffects in a flat calculus. For structural coeffects, we need to compose morphisms which have more than a single coeffect annotation. For this, we introduce the new notion of structural indexed comonads.

Definition 21. A structural indexed comonad comprises a functor \( D : \Pi_{n,S} \Rightarrow [C^n, C] \) where \( (I, \bullet, I) \) is a strict monoidal category, \( I \in S \) which is terminal (e.g., a singleton set), an indexed comonad over \( D^1 : \mathbb{I} \Rightarrow [C^1, C] \) and a structural comultiplication natural transformation:

\[
(\delta^n_{R,S})_{A^n} : D^i_{n,S} A^n \Rightarrow D^1 D^2 A^n
\]

where \( A^n \in C^n, r \in I, S \in I^n \) and \( r: I \times I^n \Rightarrow I^n \) is the monoid left action that \( \bullet \)-lifts scalar coeffects to shaped coeffects (i.e., the, the scalar-vector version of \( \bullet \)). Analogous laws to monoid left actions for unitality and associativity hold for structural comultiplication:

\[
\begin{align*}
D^i_{n,S} & \xrightarrow{\delta^n_{R,S}} D^1 D^2 \quad D^i_{n,S} \xrightarrow{\delta^n_{s,T}} D^1 D^2 \\
D^i & \xrightarrow{\delta^n_{R,S}} D^1 D^2 \quad D^i \xrightarrow{\delta^n_{R,S}} D^1 D^2
\end{align*}
\]

using axioms \( I \bullet R = R \) and \((r \bullet s) \ast T = r \ast (s \bullet T)\) on coeffects respectively which are the monoid left action axioms for the scalar-vector application of \( \bullet \). Note the use of indexed comonad comultiplication \( \delta^1 \) for associativity [SC2].

Structural indexed comonads provide composition for morphisms \( f : D^1 A^n \Rightarrow B \) and singleton-shaped morphisms \( g : D^1 B \Rightarrow C \):

\[ g \circ f \Rightarrow D^1 f \circ \delta^n_{R,S} : D^2 A^n \Rightarrow C \]

Note that this composition is asymmetric: the left morphism and right morphisms have different shapes. To compose morphisms which both have non-trivial context shapes requires additional structure for manipulating contexts (shown in the next section).

Example 22 (Bounded reuse). \( B : \Pi_{n,N} \Rightarrow [C^n, C] \) has a structural indexed comonad structure with the indexed comonad \( B^1 \) (Example 15) and the following structural comultiplication:

\[
\delta^n_{R,S} = \lambda((a_1, ..., a_n), (a_1^1, ..., a_n^1)) : (\ast, (a_1, ..., a_n)) \mapsto \mathbb{I} (\ast, (a_1, ..., a_n))
\]

\[
\begin{align*}
\cdots \to (\ast, (a_1^1, ..., a_n^1)) \mapsto (\ast, (a_1^1, ..., a_n^1))
\end{align*}
\]

The input is an n-variable context containing \( r \) times \( S_i \) copies of \( a^i \) for each variable. The output has \( r \) copies of a single n-variable context containing \( S_i \) copies of \( a^i \) for each variable. Thus, \( \delta^n_{R,S} \) partitions the incoming context into \( r \)-sized contexts.

Note that in the case of the flat system, a structural indexed comonad collapses to a standard indexed comonad on \( D^1 \).

5.3 Splitting and merging contexts

Indexed comonads and structural indexed comonads give a semantics for sequential composition of contextual computations. However, this does not provide enough structure for a semantics of the full coeffect calculus. Core to the semantics of abstraction and application is the merging and splitting of contexts. Recall the free-variable contexts and coeffects in the (abs) and (app) rules:

\[
\begin{align*}
\text{(app)} & \quad \Gamma_1 @ R \vdash e_1 \ldots \Gamma_2 @ S \vdash e_2 \ldots \\
& \quad \Gamma_1, \Gamma_2 @ \text{RK}(t:S) \vdash e_1 \ldots \Gamma, x : \sigma @ \text{RK}(s) \vdash e_2 \ldots \\
& \quad \Gamma @ R \vdash \lambda x.e : \sigma \Rightarrow \ldots \\
\end{align*}
\]

Reading (app) bottom-up, the context of the application is split into two contexts for each subterm \( e_1 \) and \( e_2 \). Reading (abs) bottom-up, the context of the abstraction is merged with the singleton context of the parameter. Capturing these notions in the denotational semantics requires some additional structure.

A (non-indexed) comonadic semantics for the \( \lambda \)-calculus requires a monoidal comonad with operation \( m_{A,B} : FA \times FB \Rightarrow FA \times FB \) [19]. Previously, we defined a similar operation for the semantics of a flat coeffect system, with an indexed monoidal operation \( m_{R,S}^{n,m} \) for merging contexts. Dually, contexts were split with \( n_{R,S}^{m,n} \). We used two operations for combining and splitting the coeffect annotations, respectively. Here we generalize these to shape-indexed versions using \( x \) and \( \bar{x} \).

Definition 23. A functor \( D : \Pi_{n,S} \Rightarrow [C^n, C] \) is an indexed lax (semimonoidal) functor and/or colax (semimonoidal) functor if it has the following natural transformations respectively:

\[
\begin{align*}
\mathbf{m}_{R,S}^{n,m} : D^R A \times D^S B \Rightarrow D^{R \ast S}(A \times B) \\
\mathbf{n}_{R,S}^{m,n} : D^{n \ast m}(A \times B) \Rightarrow D^R A \times D^S B
\end{align*}
\]

satisfying associativity coherence conditions. In both, shape descriptions are combined by \( \otimes \). The first operation models context merging and combines coeffects using \( \times \). The second models context splitting, with \( \times \) for the pre-split coeffect.

Example 24 (Bounded reuse). For bounded reuse, \( B \) is an indexed lax and colax semimonoidal functor with the following operations:

\[
\begin{align*}
\mathbf{m}_{R,S}^{n,m} = \lambda((a_1, ..., a_n) \times (b_1, ..., b_m)), ((a_1, ..., a_n), (b_1, ..., b_m)) \\
\mathbf{n}_{R,S}^{m,n} = \lambda((a_1, ..., a_n), (b_1, ..., b_m)), ((a_1, ..., a_n) \times (b_1, ..., b_m))
\end{align*}
\]

Here \( \mathbf{m}_{R,S}^{n,m} \) takes a pair of contexts and merges them simply by replacing the product in \( C \) which pairs the two arguments (written using \( \times \)) with products inside of \( B \) (written using tuple notation \( (x, y) \)). The operation \( \mathbf{n}_{R,S}^{m,n} \) is the inverse.

Example 25 (Implicit parameters). For implicit parameters, \( I \) is an indexed lax and colax semimonoidal functor with operations:

\[
\begin{align*}
\mathbf{m}_{R,S}^{n,m} = \lambda((a, \gamma(R)), (b, \gamma(S))), ((a, b), \gamma(R) \cup \gamma(S)) \\
\mathbf{n}_{R,S}^{m,n} = \lambda((a, b), \gamma(S)), ((a, \gamma(R)), (b, \gamma(S)))
\end{align*}
\]

As in Example 22, \( \gamma(R) \) and \( \gamma(S) \) restrict the set of implicit parameters \( \gamma \) to variable-value pairs for variables in \( R \) and \( S \).

5.4 Putting it together

The semantics of the general coeffect calculus \( \llbracket \cdot \rrbracket \) is defined in Figure 5 using the structures described in the previous sections.

Core rules. The denotation in (var) maps a context of the singleton shape \( \mathbb{I} \) containing just a single variable \( \tau \) (with coeffect \( I \)) to a \( \tau \) value using the count operm operation. The premise of (abs) takes a context of shape \( n \mathbb{I} \) with coeffects \( R \times (s) \) and a free-variables context consisting of \( \Gamma \) and an additional variable \( x \). The denotation \( g : D^{n \mathbb{I}}(R \times (s)) \Rightarrow [\tau] \) is pre-composed with \( m \), such that its context is obtained by merging the declaration-site context \( \Gamma \) and call-site context \( \sigma \):

\[ g \circ m_{R,S}^{n \mathbb{I}} : (D^{n \mathbb{I}}(\Gamma) \times D^I(\sigma)) \Rightarrow [\tau] \]

This is uncurred to give a denotation from a context to an exponental object representing the abstraction, where the singleton-shaped context becomes the source of the exponential.

The application rule (app) has two sub-expressions for the function and argument, with denotations requiring two distinct contexts:

\[ g_1 : D^{n \mathbb{I}}(\Gamma_1) \Rightarrow (D^I(\sigma) \Rightarrow [\tau]) \]

\[ g_2 : D^m(\Gamma_2) \Rightarrow [\sigma] \]
The target of \( g_1 \) is an exponential object with singleton shape for the parameter of type \( \sigma \). To evaluate \( g_1 \) and \( g_2 \), the semantics of (app) splits the incoming context over \( \Gamma_1, \Gamma_2 \) using \( n \):

\[
D_{R \times (s \otimes S)}^{n,m} \left( (\Gamma_1 \times [\Gamma_2]) \right) \xrightarrow{\text{nm}} D^n_R [\Gamma_1] \times D^m_S [\Gamma_2]
\]

Since \( e_2 \) computes the argument for function \( e_1 \), the denotation \( g_2 \) is sequentially composed with the parameter part of \( g_1 \). Thus, the structural indexed comonad (where \( \odot = \otimes \)) is used with \( g_2 \) to compute the correct context for the parameter of the function denotation \( g_1 \):

\[
D^m_S [\Gamma_2] \xrightarrow{\text{dm}} D^1_{(t)} D^m_S [\Gamma_2] \xrightarrow{\text{Df}} D^{1}_{[\sigma]}
\]

This is composed with the previous equation by lifting to the right-component of the product:

\[
D^n_R [\Gamma_1] \times D^{n,m}_S [\Gamma_2] \xrightarrow{\text{id} \times (D^1_{(t)} \circ \text{dm})} D^n_R [\Gamma_1] \times D^1_{[\sigma]}
\]

This equation computes the calling context and parameter context for the function \( e_1 \), which is then composed with the uncurried \( g_1 \) denotation as shown in the (app) rule in Figure 5.

**Structural rules.** In Figure 5 (ctx) composes the denotation of an expression with a transformation \( c \) providing the semantic structural rules. The semantics of structural rules are defined by using \( n_{R,S}^{n,m} \) to split contexts, transforming the components, and merging the transformed contexts using \( m_{R,S}^{n} \). The (contr) rule uses an additional operation which duplicates a variable inside a context:

\[
\Delta_{R,S} : D^{n}_{(r) \otimes S} A \rightarrow D_{(r) \otimes (r) \otimes S}^{1} (A \times A)
\]

**Example 26.** We demonstrate the semantics with a concrete example for the bounded reusue calculus. Consider the following term:

\[
f : \mathbb{Z} \rightarrow \mathbb{Z}, \ x : \mathbb{Z} \otimes (2,4) \vdash (\lambda z. z + z) (fx)
\]

Let the denotation of the function body, prior to contraction, be \( g = [x : Z, y : Z \otimes (1,1) \vdash (+y) y : Z] \). The example term’s denotation is then constructed as follows:

\[
[g] \vdash \lambda x. e : \sigma \rightarrow \tau = \Lambda (g \circ m_{(n)}^{1;(1)} (\Lambda x. e)) \rightarrow (D^1_{[\sigma]}) [\tau]
\]

\[
[\Gamma \vdash R \vdash e : \sigma \rightarrow \tau] = \text{Lazy eval}(g, m_{(n)}^{1;(1)} (\Lambda x. e)) \rightarrow (D^1_{[\sigma]}) [\tau]
\]

\[
[\Gamma \vdash R \vdash e : \sigma \rightarrow \tau] = \text{Lazy eval}(g, m_{(n)}^{1;(1)} (\Lambda x. e)) \rightarrow (D^1_{[\sigma]}) [\tau]
\]

5.5 **Soundness, with respect to the equational theory.**

Our denotational semantics for the coeffect calculus is sound with respect to the equational theory of Section 4. That is:

**Theorem 27.** (Soundness).

\[
\Gamma \vdash R \vdash e \equiv e' : \tau \rightarrow [\Gamma \vdash R \vdash e : \tau] \equiv [\Gamma \vdash R \vdash e' : \tau]
\]

Proof of this follows from an interesting result which we first unpack: determining whether \([\Gamma \vdash R \vdash e : \tau] \equiv [\Gamma \vdash S \vdash e : \tau]\) follows from a proof (on coeffect annotations) that \( R = S \).
Lemma 28. Every coeffect algebra axiom corresponds to an axiom of one of the categorical structures introduced here (indexed (structural) comonad or indexed (colax) monoidal functor).

For example, the monoid axiom $X \triangleright_{\text{use}} X$ for scalar coeffects corresponds to indexed comonad axiom $\delta_{X, \text{use}} X \cdot \Delta_{X, \text{use}} X = id_{\Delta_{X, \text{use}} X}$ (which requires the monoid axiom to hold). This lemma follows from our derivation of the indexed categorical structures here. They are not derived ad hoc but systematically as (lax) homomorphisms (structure-preserving maps) between the structure of coeffect annotations in $\mathcal{I}$ and the structure of denotations in $\mathcal{C}$.

Proposition 29. An indexed comonad on $\mathcal{D}$ witnesses that $\mathcal{D}$ is a colax monoid homomorphism between the (strict) monoidal categories $(\mathcal{I}, \bullet, I)$ and $(\mathcal{C}, \odot, 1_{C})$ (endofunctor composition).

Unpacking this, a monoid homomorphism maps between the underlying sets of two monoids, preserving the monoid structure of one into the other, i.e., given monoids $(X, \bullet, I)$ and $(Y, \odot, E)$ then a monoid homomorphism is a mapping $F : X \rightarrow Y$ such that:

$$F X \odot FY \equiv F (X \bullet Y) \quad E \equiv FI \tag{5}$$

The axioms of each monoid are preserved trivially by these equalities, e.g., $F X \equiv F (X \bullet I) \equiv F X \odot FI \equiv FX \odot E \equiv FX$.

A homomorphism is lax if the above equalities are instead morphisms (which we say witness the homomorphism) and colax if these morphisms go in the opposite direction. Thus, a colax monoid homomorphism is witnessed by:

$$\delta : F X \odot FY \rightarrow F (X \bullet Y) \quad \varepsilon : E \rightarrow FI$$

Note our choice of morphism names. $F$ no longer preserves the monoid axioms up to equality but has axions on $\delta$ and $\varepsilon$, e.g.,

$$F(X \bullet I) = F X \odot FI \quad F X \odot E = FX \quad \text{id}_{ FX} = FX$$

Our indexed comonad definition is equivalent to $D$ being a colax homomorphism between strict monoidal category $(X, \bullet, I)$ and the monoidal category of $\mathcal{C}$-endofunctors $(Y, \odot, E) = (\mathcal{C}, \odot, 1_{C})$, with endofunctor composition $\odot$ and the trivial endofunctor $1_{C}$; monoids are now at the level of categories. The indexed comonad axioms are the axioms of the colax homomorphism. Equivalently, $\mathcal{D}$ is a colax monoidal functor.

A similar approach is taken to deriving the remaining structures below, though we give less detail for brevity.

Proposition 30. A structural indexed comonad provides $\delta^n_{R,S} : D^n_1 D^n_A \odot A^n \rightarrow D^n_2 A^n$ which is a family of morphisms (indexed by shapes $n$) witnessing that $\mathcal{D}$ is a colax homomorphism between the following monoid left-actions: $(\mathcal{I}^n, \triangleright)$ for $(\mathcal{I}, \triangleright, \text{ign})$ and $(\mathcal{C}^n, \odot)$ for $(\mathcal{C}, \odot, 1_{C})$, defined:

$$\begin{align*}
\triangleright & : (r : I) \odot (s_1, \ldots, s_n : \mathsf{I}^n) \\
& \rightarrow (r \triangleright s_1, \ldots, r \triangleright s_n : \mathsf{I}^n) \\
\odot & : (D^n_1 : \mathcal{C}^n, \odot) \odot D^n_2 : \mathcal{C}^n, \odot) \\
& \rightarrow D^n_1 \odot D^n_2 : \mathcal{C}^n, \odot)
\end{align*}$$

The axioms are the lax versions of the monoid left-action laws.

The lax and colax indexed monoidal operations $m^n_{R,S} \triangleleft$ and $m^n_{R,S} \triangleright$ follow a similar derivation but as lax and colax monoid homomorphisms between composite monoids on coeffect annotations and shapes and $X$ in $\mathcal{C}$. The details are elided here.

Returning to soundness, our semantics is therefore defined in terms of structures whose axioms correspond to axioms of the syntactic equational theory. Consequently, semantic proofs correspond to syntactic proofs, modulo naturality laws and product/exponent laws in $\mathcal{C}$. This result holds in the general coeffect calculus and semantics since every semantic structure has a unique corresponding structure on coeffect annotations (i.e., $(\mathcal{C}, \odot, \text{use})$ for sequential composition of unary denotations, $(\mathcal{C}, \times)$ for joining contexts, $(\mathcal{C}, X)$ for joining contexts).

Example 31. Section 4 showed $\eta$-equality for structural systems, which uses the properties (1) $x = x \equiv x$ for structural systems and (2) $s \odot (\text{use}) \equiv (s \odot \text{use}) \equiv (\text{use})$. The semantics here is sound with respect to $\eta$-equality; the proof uses the corresponding axioms (1) $n^n_{R,S} \odot m^n_{R,S} \equiv id$ and (2) $\varepsilon_{n_{C}} D^n_1 \odot D^n_2 \equiv id$ (structural indexed comonad unit law [SC1]; Definition 11).

The full semantic proofs of $\beta\eta$-equality then correspond to syntactic proofs on coeffect annotations. For brevity, we omit the full proofs here.

6. Related work

We expand on the overview of related work in Section 2.5.

Bounded reuse. The (storage) rule for bounded linear logic explains the contextual requirements induced by proposition reuse [7].

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Bounded reuse. The (storage) rule for bounded linear logic explains the contextual requirements induced by proposition reuse [7].

The necessity modality $\Box$ in S4 logic corresponds to a comonad with lax monoidal functor structure $m : \Box A \times \Box B \rightarrow \Box (A \times B)$.

Bierman and de Paiva [2] defined a term language corresponding to a natural deduction S4, where contexts contain sequences of wrapped assumptions $x_1 : \Box A_1, \ldots, x_n : \Box A_n$. Modelling these judgments does not require a context-splitting operation unlike in our approach, which uses the $\eta$ operation of the form $n : \Box (A \times B) \rightarrow \Box A \times \Box B$. Our approach can be thought of as having a single $\Box$ modality over the context which can represent both flat whole-context dependence and structural per-variable dependence.

Coeffect-like calculi Recent works have also developed coeffect systems, following related approaches. Brunel, Gaboardi, Mazza, and Zdancewic derive a kind of general structural coeffect system, taking inspiration from bounded linear logic [3]. Their work provides an operational semantics and proves soundness of its coeffect system with respect to the semantics. Their coeffect system provides a coeffect-indexed $\vdash$-modality as a type constructor in the language, and explicit coeffect-inducing expressions for a particular notion of coeffect.

This differs to our approach where coeffects are implicit: they are not attached to a type constructor and can be introduced through variable use, e.g. the $(\forall x$) rule. The coeffect systems of Brunel et al. allow “local coeffects”, which we called structural (per-variable), but not “global coeffects” (flat or per-context). Our previous work provided just global/flat (per context) coeffects. Our new system here reconciles both kinds into one system.

There are a number of encouraging similarities in the approach of Brunel et al. Their semantics is similar (even isomorphic in some parts) to ours, defining an indexed comonad like structure
in terms of a positive action \( \bullet : S \times A \to A \) on a monoidal category of coeffect annotations \( S \) (and base category \( A \)) and an exponential action providing related operations to our indexed monoidal operations \( m \) and \( n \).

There has been some other related bounded linear logic work, by Ghica and Smith, with “resource-aware types” annotated by a semiring of resource bounds \([5]\). This allows reuse bounds for BLL to be tracked, as well as other kinds of concurrency information. A categorical semantics is provided which is similar in style to that of Brunel et al. and which loosely resembles our indexed comonad approach (but does not require the additional indexed monoidal structures we used here). Included in their work, which this paper lacks, is a procedure for type-inference (using a decision procedure on semirings). Future work for us is to adopt a similar approach for inference of coeffects in our system.

Both the works of Brunel et al. and Ghica et al. use a semiring structure for annotations. Our scalar coeffect structure is similar: it is a semiring without commutativity of the \( + \) operation (although all our examples here have a commutative \( + \)) and without the absorption law for multiplication (which only some of our systems have).

Future work is to unify the approaches of this paper and the co-effect systems of Brunel et al. and Ghica et al. Initial comparisons show several similarities, suggesting that unification is plausible.

7. Conclusions

In this paper, we looked at two forms of context-dependence analysis – flat coeffect systems that track whole-context requirements (such as implicit parameters, resources, or platform version) and structural coeffects that track per-variable requirements (such as usage or data access patterns). The newly introduced structural system makes applications such as liveness, bounded reuse, and dataflow analysis (from our earlier work) practical. With the move towards cross-platform systems running in diverse environments, analysing context dependence is vital for reasoning and compilation. The coeffect calculus provides a foundation for further study, similar to the type-and-effect discipline.

We presented the system together with its syntactic equational theory and categorical semantics. The equational theory is presented in order to explain how the systems work, but it also provides a basis for an operational semantics for concrete systems. Exploring these, and their connection to the denotational semantics, is further work.

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