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The semantic marriage of monads and effects

Extended abstract

Dominic Orchard  Tomas Petricek  Alan Mycroft
Computer Laboratory, University of Cambridge
{firstname.lastname}@cl.cam.ac.uk

Abstract

Wadler and Thiemann unified type-and-effect systems with monadic semantics via a syntactic correspondence and soundness results with respect to an operational semantics. They conjecture that a general, “coherent” denotational semantics can be given to unify effect systems with a monadic-style semantics. We provide such a semantics based on the novel structure of an indexed monad, which we introduce. We redefine the semantics of Moggi’s computational λ-calculus in terms of (strong) indexed monads which gives a one-to-one correspondence between indices of the denotations and the effect annotations of traditional effect systems. Dually, this approach yields indexed comonads which provides a unified semantics and effect system to contextual notions of effect (called coeffects), which we have previously described [9].

Previously, Wadler and Thiemann established a syntactic correspondence between type-and-effect systems and the monadic semantics approach by annotating monadic type constructors with the effect set of the type-and-effect system [10]. They established soundness results between the effect system and an operational semantics, and conjectured a “coherent semantics” of effects and monads in a denotational style. One suggestion was to associate to each effect set σ a different monad Tσ.

We take a different approach to a coherent semantics, unifying effect systems with a monadic-style semantics in terms of the novel notion of indexed monads, which generalises monads [1].

Indexed monads

Indexed monads comprise a functor

\[ T : I \rightarrow [C, C] \]

(i.e., an indexed family of endofunctors) where \( I \) is a strict monoidal category \((I, \otimes, 1)\) and \( T \) is a lax monoidal functor, mapping the strict monoidal structure on \( I \) to the strict monoid of endofunctor composition \([C, C], \circ, I_C\).

The operations of the lax monoidal structure are thus:

\[ \eta_I : I_C \rightarrow T1 \quad \mu_{F,G} : TF \circ TG \rightarrow T(F \otimes G) \]

Note this differs to Johnstone’s notion of indexed monad in the context of topos theory, the indexed monads seen in the work of McBride [4], and parameterised monads by Atkey [1].

These lax monoidal operations of \( T \) match the shape of the regular monad operations. Furthermore, the standard associativity and unitality conditions of the lax monoidal functor give coherence conditions to \( \eta_I \) and \( \mu_{F,G} \) which are analogous to the regular monad laws, but with added indices, e.g., \( \mu_{1,G} \circ (\eta_I)_TG = id_{TG} \).

Example

(Indexed exponent/reader monad) Given the monoid \((P(X), \cup, \emptyset)\) (for some set \( X \)), the indexed family of Set endofunctors where \( TXA = X \Rightarrow A \) (with \( \Rightarrow \) denoting exponents) and \( TXf = \lambda k.f \circ k \), is an indexed monad with:

\[ \eta_0 = \lambda x.a \]

\[ \mu_{F,G}k = \lambda x.(k(x - (G - F)))(x - (F - G)) \]

where \( x : F \cup G \) and \( k : F \Rightarrow (G \Rightarrow A) \) thus \( k \) takes two arguments, the \( F \)-only subset of \( x \) (written \( x - (G - F) \)) and the \( G \)-only subset of \( x \) (written \( x - (F - G) \)) where \((-\) \) is set difference.

The indexed reader monad models the composition of computations with implicit parameters, where the required implicit parameters of subcomputations are combined in their composition. This provides a more refined model to the notion of implicitly parameterised computations than the traditional reader monad, where implicit parameters are uniform throughout a computation and its subcomputations.

Relating indexed monads and monads

Indexed monads collapse to regular monads when \( I \) is a single-object monoidal category. Thus, indexed monads generalise monads.

Note that indexed monads are not indexed families of monads. That is, for all indices \( F \in obj(I) \) then \( TF \) may not be a monad.

An indexed monadic semantics for \( \lambda \)

We extend indexed monads to strong indexed monads, with an indexed strength operation (and analogous laws to usual monadic strength): \((\tau_F)_{A,B} : (A \times TF B) \rightarrow TF(A \times B)\)

We replay Moggi’s categorical semantics for the computational λ-calculus (\( \lambda \)) [5], replacing the regular strong monad operations with the analogous operations of an indexed strong monad. This provides an indexed semantics. For example, the semantics of λ- abstraction becomes the following (where we write the parameter to \( T \) as a subscript for notational clarity below):

\[ [\Gamma, x : \sigma : e : \tau] = g : [\Gamma] \times [\sigma] \rightarrow TF \tau \]

\[ [F + \lambda x : \sigma, e : \sigma : e : \tau] = \eta_I \circ (\Lambda g) : [\Gamma] \rightarrow T1 (\sigma : \Rightarrow TF \tau) \]

(where for \( g : A \times B \rightarrow C, \Lambda g : A \rightarrow (B \Rightarrow C) \)).

Coherent semantics

In this indexed monadic semantics, the indices of denotations have exactly the same structure as the effect annotations of a traditional effect system (with judgments \( \Gamma \vdash e : \tau, F \) for an expression \( e \) with effects \( F \)).

We unify effect systems with indexed monadic semantics, so that \([\Gamma] \vdash e : \tau, F : [\Gamma] \rightarrow TF [\tau] \), taking \( obj(I) \) as the effect
sets of a traditional effect system, with the strict monoidal structure on \( I \) provided by the effect lattice, with \( 1 = \bot \) and \( \otimes = \sqcup \), and morphisms \( f : X \to Y \) in \( I \) iff \( X \subseteq Y \) in the effect lattice. Pleasingly, the usual equational theory for \( \lambda_c \) (such as \( \beta \)-equality for values) follows directly from the strict indexed monad axioms.

The morphism mapping of \( T \) defines natural transformations \( \iota_{X,Y} : TX \to TY \) when \( X \subseteq Y \) which provides a semantics to sub-effecting:

\[
\text{(sub)}: \frac{\Gamma \vdash e : \tau, F' \in \rho}{\Gamma \vdash \iota_{F,F'} : \tau, T \rho \to \iota_{F,F'} \in \rho} \quad \frac{\Gamma \vdash e : \tau, F \in \rho}{\Gamma \vdash \iota_{F,F} : \tau, T \rho \to \iota_{F,F} \in \rho}
\]

Therefore strong indexed monads neatly unify a (categorial) semantics of effects with traditional effect systems. The indexed monad structure arises simply from the standard category theory semantics of effects with traditional effect systems. The indexed strong monad can be shown to provide the notion of a (categorial) comonads since they relax the usual requirements (written \( \rightarrow \)). Thus, with respect to contextual requirements, \( \lambda \)-abstraction is not “pure” as it is for effects. In an indexed semantics unifying a coeffect system with an indexed comonad, the semantics of \( \lambda \)-abstraction requires the additional structure of an indexed (semi-)monoidal comonad with the operation:

\[
(m \rho, C)_{A,B} : DF \times DG \to D(F \times G)(A \times B)
\]

where \( \vee \) is an associative binary operation over \( I \).

2. Nielson and Nielson defined a more general effect system with a richer algebraic effect structure, separating the traditional approach of an effect lattice into operations for sequential composition, alternation, and fixed-points \( \mathbb{F} \). Relatedly on the semantic side, the structure of a cojoinad has been proposed to give the semantics of sequencing, alternation and parallelism in an effectful language \( \mathbb{F} \), adding additional monoidal structures to a monad. Similarly to indexed monads, cojoinads can be generalised to indexed cojoinads, giving a correspondence between the richer effect systems of Nielson and Nielson and a cojoinad-based semantics. This is future work.

References


