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# STRINGY INVARIANTS FOR HOROSPHERICAL VARIETIES OF COMPLEXITY ONE

KEVIN LANGLOIS, CLÉLIA PECH AND MICHEL RAIBAUT

ABSTRACT. In this paper we determine the stringy motivic volume of log terminal horospherical  $G$ -varieties of complexity one, where  $G$  is a connected reductive linear algebraic group. The stringy motivic volume of a log terminal variety is an invariant of singularities which was introduced by Batyrev and plays an important role in mirror symmetry for Calabi–Yau varieties. A horospherical  $G$ -variety of complexity one is a normal  $G$ -variety which is equivariantly birational to a product  $C \times G/H$ , where  $C$  is a smooth projective curve and the closed subgroup  $H$  contains a maximal unipotent subgroup of  $G$ . The simplest example of such a variety is a normal surface with a non-trivial  $\mathbb{C}^*$ -action. Our formula extends the results of Batyrev–Moreau [BM13] on stringy invariants of horospherical embeddings. The proof involves the study of the arc space of a horospherical variety of complexity one and a combinatorial description of its orbits. In contrast to [BM13], the number of orbits is no longer countable, which adds significant difficulties to the problem. As a corollary of our main theorem, we obtain a smoothness criterion using a comparison of the stringy and usual Euler characteristics.

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## 1. INTRODUCTION

Motivic integration is a natural geometric counterpart to  $p$ -adic integration in number theory. It turned out to be a powerful tool for creating new invariants of the singularities of algebraic varieties. Stringy invariants, introduced by Batyrev in [Bat98], are an example of such constructions. In [BM13] they were instrumental in simplifying a smoothness criterion for embeddings of horospherical homogeneous spaces in terms of convex geometry; the formulation of this smoothness criterion using stringy invariants for general spherical varieties remains to this date a conjecture, see [BM13, Conjecture 6.7].

Thus it is tempting to continue this exploration for other kinds of reductive group actions. In this paper our main objects of interest are the stringy invariants of horospherical varieties of complexity one,  $\mathbb{Q}$ -Gorenstein, and with log terminal singularities. One of our results, Theorem 5.17, provides an explicit formula for computing the stringy invariants in terms of their combinatorial description. To prove this result we study the arc space of these varieties in Section 4, following the ideas underlying the study in [Ish04] of the case of toric varieties. We also give a description of the rational form of the stringy motivic volume and obtain precisely a finite set of candidate poles using convex geometry, see Theorem 5.27. Finally, under some extra assumptions we rephrase in Theorem 6.4 a smoothness criterion for these varieties in terms of the stringy Euler characteristic. To explain our results in more detail let us now recall some background on horospherical varieties.

Throughout this paper we denote by  $G$  a connected simply connected reductive linear algebraic group over  $\mathbb{C}$ . By ‘simply connected’ we mean that  $G = G^{ss} \times F$ , where  $G^{ss}$  is a simply connected semisimple group in the usual sense (see [MT11, Section 9.2, Definition 9.14]) and  $F$  is an algebraic torus. We say that a  $G$ -homogeneous space  $G/H$ , where  $H$  is a closed subgroup, is *horospherical* if  $H$  contains a maximal unipotent subgroup of  $G$ . These homogeneous spaces may be realized as Zariski-locally trivial torus fibre bundles on flag varieties. The fibration for the horospherical homogeneous space  $G/H$  is constructed via the natural projection  $G/H \rightarrow G/P$ , where  $P$  is the parabolic subgroup normalizing  $H$ . All the fibers of this projection are isomorphic to the algebraic torus  $T = P/H$ . The character lattice of  $T$  and the set of simple roots indexing the Schubert divisors in the flag variety  $G/P$  describe entirely the homogeneous space  $G/H$ , see Section 3.1 for a more precise statement.

More generally a *horospherical  $G$ -variety* is a normal variety endowed with a  $G$ -action such that every orbit is horospherical. We distinguish horospherical varieties depending on an integer called the *complexity*, which measures the size of the rational quotient by a Borel subgroup. In practice it coincides in the horospherical case with the minimum of the codimensions of the  $G$ -orbits. Horospherical varieties of complexity at most one admit a combinatorial description in terms of objects coming from convex geometry, almost analogous to the classical situation for toric varieties. This combinatorial description, introduced by Timashev in [Tim97] for the complexity one case, and adapted from the Luna–Vust theory [LV83], is either a generalization or a special case of similar descriptions for other types of  $G$ -varieties.

- *Spherical varieties*, that is, normal  $G$ -varieties where a Borel subgroup of  $G$  acts with an open orbit, see [Kno91, Bri95, Tim11, Per14]. This special case includes symmetric spaces and horospherical varieties of complexity zero;
- Normal varieties endowed with an algebraic torus action of complexity one, corresponding to the case  $G = T$ . See [KKMSD73] for a partial description, [Tim08] for the general case, and [Lan15] for a description on an arbitrary field. We also refer to [FZ03, AH06, AHS08, AIP<sup>+</sup>12] for actions of algebraic tori of arbitrary complexity.

The combinatorial description in [Tim97], see also [LT16], of horospherical varieties of complexity one, involves the following data. First of all we work with *simple  $G$ -varieties*, that is,  $G$ -varieties  $X$  which contain an open affine subset which is stable under the action of a Borel subgroup  $B \subset G$  and intersects every  $G$ -orbit. Such an open subset is called a  *$B$ -chart* of  $X$ . Any simple horospherical  $G$ -variety  $X$  of complexity one is described by a 4-tuple  $(C, G/H, \mathfrak{D}, \mathcal{F})$ , where  $C$  is a smooth projective curve over  $\mathbb{C}$  and  $G/H$  is a horospherical  $G$ -homogeneous space. These first two data encode the birational equivariant type of  $X$ . More precisely  $X$  is  $G$ -birational to the product  $C \times G/H$ , where  $G$  acts via  $g \cdot (y, x) = (y, g \cdot x)$  for  $g \in G, y \in C$ , and  $x \in G/H$ . Here  $G/H$  is equipped with its natural  $G$ -action. The third datum  $\mathfrak{D}$  is a Weil divisor on  $C$  whose coefficients are polyhedra. We say that  $\mathfrak{D}$  is a *polyhedral divisor*, see Definition 3.1 for a more precise description. Finally  $\mathcal{F}$  represents the set of irreducible  $B$ -stable divisors on  $X$  which are not  $G$ -stable and which contain a  $G$ -orbit of  $X$ .

Following the convention for actions of algebraic tori, we say that the pair  $(\mathfrak{D}, \mathcal{F})$  is a *colored polyhedral divisor*. In fact  $(\mathfrak{D}, \mathcal{F})$  defines a  $B$ -chart  $X_0$  of  $X$ , see [Tim11, Theorem 16.19], whose algebra of functions  $\mathbb{C}[X_0]$  is realized as the intersection of the ring  $\mathbb{C}(C) \otimes_{\mathbb{C}} \mathbb{C}[Bx_0]$  and of discrete valuation rings determined by  $(\mathfrak{D}, \mathcal{F})$ . Here  $Bx_0 \subset G/H$  denotes the open  $B$ -orbit.

It follows from a result of Sumihiro [Sum74] that any normal  $G$ -variety possesses a finite open covering by stable subsets which are simple  $G$ -varieties. Thus any horospherical  $G$ -variety of complexity one can be described by a triple  $(C, G/H, \mathcal{E})$ , where  $\mathcal{E}$  is a *colored divisorial fan* which consists in a finite collection of colored polyhedral divisors, see Definition 3.1 for a precise description.

The goal of this paper is to compute explicitly the stringy motivic volume of  $X$ , which is assumed to be  $\mathbb{Q}$ -Gorenstein with log terminal singularities, in terms of the combinatorial data  $(C, G/H, \mathcal{E})$ . Explicit combinatorial criteria in terms of  $(C, G/H, \mathcal{E})$  of the  $\mathbb{Q}$ -Gorenstein and log terminal conditions have been obtained in [LT16] from ideas coming from the case of torus actions, see [LS13].

Before stating our results let us first recall briefly the construction of the stringy motivic volume. Let  $V$  be a normal  $\mathbb{Q}$ -Gorenstein variety (which means that a canonical divisor  $K_V$  is  $\mathbb{Q}$ -Cartier) with log terminal singularities. By a result of Hironaka [Hir64] we may choose a resolution of singularities  $f : V' \rightarrow V$  and we consider the relative canonical divisor

$$K_{V'/V} = K_{V'} - f^*K_V.$$

Then the *stringy motivic volume* of  $V$  is defined as the integral

$$(1) \quad \mathcal{E}_{st}(V) = \int_{\mathcal{L}(V')} \mathbb{L}^{-\text{ord } K_{V'/V}} d\mu_{V'}.$$

The space  $\mathcal{L}(V')$  is the *arc space* of  $V'$  and can be interpreted as the ‘moduli space’ of formal arcs  $\text{Spec } K[[t]] \rightarrow V'$ , where  $K$  runs on field extensions of  $\mathbb{C}$ . The measure  $\mu_{V'}$  defined on  $\mathcal{L}(V')$  takes values in a modified version  $\hat{\mathcal{M}}_{\mathbb{C}}$  of the Grothendieck ring  $K_0(\mathbf{Var}_{\mathbb{C}})$  of complex algebraic varieties. Recall that the elements of  $K_0(\mathbf{Var}_{\mathbb{C}})$  are virtual complex algebraic varieties. They obey to some scissors relations and the multiplication in the ring  $K_0(\mathbf{Var}_{\mathbb{C}})$  is induced by the usual Cartesian product of varieties, see Definition 2.1 for a precise statement. Let  $\mathbb{L} = [\mathbb{A}_{\mathbb{C}}^1]$  be the class of the affine line in  $K_0(\mathbf{Var}_{\mathbb{C}})$ . The ring  $\hat{\mathcal{M}}_{\mathbb{C}}$  is constructed as the completion of the localization ring  $\mathcal{M}_{\mathbb{C}} := K_0(\mathbf{Var}_{\mathbb{C}})[\mathbb{L}^{-1}]$  with the respect to a certain norm.

The measure  $\mu_{V'}$  admits an explicit description on a class of subsets of  $\mathcal{L}(V')$  called cylinders. Denote by  $\pi_n$  the truncation morphism sending an arc of  $V'$  to its  $n$ -jet. A subset  $Z \subset \mathcal{L}(V')$  is a *cylinder* if there exists  $n_0 \in \mathbb{N}$  such that for any  $n \geq n_0$ , the subset  $\pi_n(Z)$  is a constructible set in the space of  $n$ -jets of  $V'$ , and  $Z = \pi_n^{-1}(\pi_n(Z))$ . If  $d = \dim V = \dim V'$ , then the measure  $\mu_{V'}(Z)$  of the cylinder  $Z$  is defined as

$$\mu_{V'}(Z) = [\pi_n(Z)]\mathbb{L}^{-nd} \in \mathcal{M}_{\mathbb{C}}.$$

It does not depend on the choice of the integer  $n \geq n_0$ . We know that the space  $\mathcal{L}(V')$  is itself a cylinder, see [Gre66, Corollary 2]. The measurable subsets of  $\mathcal{L}(V')$  are obtained as approximations of cylinders, see 2.11. The construction of the integral of a measurable function  $F : \mathcal{L}(V') \rightarrow \mathbb{Z}$  with respect to  $\mu_{V'}$  is obtained as the sum (if it exists) of the series  $\sum_{s \in \mathbb{Z}} \mu_{V'}(F^{-1}(s))\mathbb{L}^{-s}$  in the normed ring  $\hat{\mathcal{M}}_{\mathbb{C}}$ . Moreover the log terminal hypothesis on  $V$  implies that the integral of (1) converges in a ring extension  $\hat{\mathcal{M}}_{\mathbb{C}}(\mathbb{L}^{1/k}) \supset \hat{\mathcal{M}}_{\mathbb{C}}$ . It was shown in [Bat98] that the integral (1) does not depend on the choice of a resolution of singularities; this result is in fact a consequence of the *transformation rule*, which can itself be viewed as the ‘change of variables formula’ in motivic integration.

Let us mention few references on motivic integration. It first appears in [Kon95] and it was developed in the singular setting by Denef and Loeser, see [DL99]. Other generalizations have been obtained; see [Bat98] for a version involving stringy invariants, see [DL01a] for ‘arithmetic motivic integration’, see [Seb04] for motivic integration on formal schemes which contains motivic integration on a perfect field, and finally [Yas04, Yas06] for a construction in the setting of Deligne–Mumford stacks. A general approach including ‘arithmetic and geometric’ motivic integration and integrals with parameters is developed in [CL08] by Cluckers and Loeser. Finally, another approach using model theory of algebraic closed valued fields of equal characteristic 0 is given by Hrushovski and Kazhdan in [HK06]. In this paper we follow the conventions of [Bat98] and [Bli11]. We also refer to [DL01b, Cra04, Vey06, Loe09] for other expositions of this subject.

The stringy motivic volume  $\mathcal{E}_{st}(V)$  of the variety  $V$  has a rational form in terms of the data of the chosen resolution (see 2.15). It realizes on the motivic stringy invariants of Batyrev [Bat98]. For instance we may consider the function  $E$  which to a virtual variety associates its Hodge–Deligne polynomial. As  $\mathcal{E}_{st}(V)$  has a rational form, the *stringy E-function* (2.18) is obtained by the relation

$$E_{st}(V; u, v) = E(\mathcal{E}_{st}(V)).$$

Similarly the *stringy Euler characteristic* (2.18) is

$$e_{st}(V) = e(\mathcal{E}_{st}(V)).$$

Notice that when  $V$  is smooth,  $E_{st}(V; u, v)$ , respectively  $e_{st}(V)$ , coincides with the Hodge–Deligne polynomial, respectively the Euler characteristic of  $V$ , see Section 2.4. However in the general case these invariants are different. It is known (see [Bat98]) that if  $E_{st}(V; u, v) \neq E(V; u, v)$ , then  $V$  does not have any crepant resolution of singularities. This phenomenon already appears for locally factorial horospherical varieties of complexity zero, see [BM13, Example 4.6 and 5.4].

As a classical example of the computation of the stringy  $E$ -function, let us consider the situation of toric varieties. This important example will later on help us motivate our results. Let  $X$  be a normal toric  $\mathbb{Q}$ -Gorenstein variety defined by a fan  $\Sigma$  of a  $\mathbb{Q}$ -vector space  $N_{\mathbb{Q}} = \mathbb{Q} \otimes_{\mathbb{Z}} N$  associated with a lattice  $N \cong \mathbb{Z}^r$ . Then it is known that  $X$  is log terminal. Here we identify  $M = \text{Hom}(N, \mathbb{Z})$  with the lattice of characters of the torus  $T$  acting on  $X$ . The  $\mathbb{Q}$ -Gorenstein condition means that there exists a piecewise linear support function  $\theta_X : |\Sigma| \rightarrow \mathbb{Q}$  which takes values  $-1$  on each primitive generator of the one-dimensional cones of  $\Sigma$ , where  $|\Sigma|$  denotes the reunion in  $N_{\mathbb{Q}}$  of all cones of  $\Sigma$ . The stringy motivic volume of  $X$  is given by the formula

$$(2) \quad \mathcal{E}_{st}(X) = (\mathbb{L} - 1)^r \sum_{\nu \in |\Sigma| \cap N} \mathbb{L}^{\theta_X(\nu)},$$

and the  $E$ -function by

$$(3) \quad E_{st}(X; u, v) = (uv - 1)^r \sum_{\nu \in |\Sigma| \cap N} uv^{\theta_X(\nu)}.$$

The formula (2) is proved in [BM13, Theorem 4.3] in the more general setting of  $\mathbb{Q}$ -Gorenstein horospherical varieties of complexity zero. The formula (3) was initially obtained in [Bat98] using a characterization of stringy invariants via a resolution of singularities with smooth normal crossings exceptional divisors. We recall this in Section 2.4. The strategy which we use to obtain our main result is inspired from [BM13] that we divide into two parts. Let us recall it for the case of toric varieties.

Let  $S$  be a  $\mathbb{C}$ -scheme and  $R$  be a  $\mathbb{C}$ -algebra. We denote by  $S(R) = \text{Hom}(\text{Spec } R, S)$  the set of  $R$ -points in  $S$ . We start by considering the set  $\mathcal{L}(X)(K)$  of  $K$ -points in the arc space  $\mathcal{L}(X)$  of  $X$ , where  $K$  is a field extension of  $\mathbb{C}$ . The set  $\mathcal{L}(X)(K)$  identifies with  $X(\mathcal{O})$ , where  $\mathcal{O} = \mathcal{O}_K = \text{Spec } K[[t]]$  is the affine scheme associated with the ring of formal series with coefficients in  $K$ . Moreover the torus  $T(\mathcal{O})$  acts naturally in  $X(\mathcal{O}) \cap T(\mathcal{K})$ , where  $\mathcal{K}$  is the fraction field of  $\mathcal{O}$ , and  $X(\mathcal{O}), T(\mathcal{K})$  are viewed as subsets of  $X(\mathcal{K})$ . It has been observed in [Ish04] that the set of orbits of  $X(\mathcal{O}) \cap T(\mathcal{K})$  is in bijection with  $|\Sigma| \cap N$ . See also [BM13] for a generalization to the case of horospherical varieties of complexity zero. Denote by  $\mathcal{C}_{\nu}$  the functor of points of the orbit corresponding with  $\nu \in |\Sigma| \cap N$ . This first step gives us natural cylinders  $\mathcal{C}_{\nu}$  of the arc space  $\mathcal{L}(X)$ .

The second step consists in studying the integral  $\int \mathbb{L}^{-\text{ord } K_{X'/X}} d\mu_{X'}$  along  $\mathcal{C}_{\nu}$  for a desingularisation  $X' \rightarrow X$  given by a fan subdivision. Notice that  $X'(\mathcal{O}) \cap T(\mathcal{K})$  identifies with  $X(\mathcal{O}) \cap T(\mathcal{K})$  and that the complement (of the associated functor of points) in  $\mathcal{L}(X')$  has measure zero with respect to  $\mu_{X'}$ . Thus we obtain an equality

$$(4) \quad \int_{\mathcal{L}(X')} \mathbb{L}^{-\text{ord } K_{X'/X}} d\mu_{X'} = \sum_{\nu \in |\Sigma| \cap N} \int_{\mathcal{C}_{\nu}} \mathbb{L}^{-\text{ord } K_{X'/X}} d\mu_{X'}.$$

We conclude by computing explicitly each summand of the right-hand side.

When adapting these two steps in the context of horospherical varieties of complexity one we encounter significant differences. For simplicity let us assume in our explanation that  $G = T$  is an algebraic torus and  $H = \{e\}$  is trivial. Thus we consider a normal  $\mathbb{Q}$ -Gorenstein  $T$ -variety  $X$  of complexity one with log terminal singularities. We denote by  $(C, T, \mathcal{E})$  the combinatorial data associated with  $X$ . The first step

is modified as follows. The variety  $X$  contains an open subset of the form  $\Gamma \times T$ , where  $\Gamma \subseteq C$  is open dense subset. Thus comparing with the toric setting it is natural to consider the action of  $T(\mathcal{O})$  on

$$(5) \quad X(\mathcal{O}) \cap (\Gamma \times T)(\mathcal{K}).$$

However when we follow the almost similar approach of Ishii [Ish04] we find uncountably many  $T(\mathcal{O})$ -orbits. To solve this problem we cut the set (5) in two disjoint pieces; the subset of *horizontal arcs* and the subset of *vertical arcs*. Thus we obtain a partition of the set (5) by parts  $\mathcal{C}_{(y,\nu,\ell)}$  which are  $T(\mathcal{O})$ -stable and indexed by a countable set  $|\mathcal{E}|_\Gamma \cap \mathcal{N}$  combinatorially determined by  $\mathcal{E}$ .

The second step consists in checking that  $\mathcal{C}_{(y,\nu,\ell)}$  is measurable, see Lemmas 4.16 and 4.17, and to compute

$$(6) \quad \int_{\mathcal{L}(X')} \mathbb{L}^{-\text{ord } K_{X'/X}} d\mu_{X'} = \sum_{(y,\nu,\ell) \in |\mathcal{E}|_\Gamma \cap \mathcal{N}} \int_{\mathcal{C}_{(y,\nu,\ell)}} \mathbb{L}^{-\text{ord } K_{X'/X}} d\mu_{X'},$$

where  $X'$  is obtained by an explicit desingularization determined by  $\mathcal{E}$ , see Section 5.3.

It is important to notice that the function  $\theta_X : |\Sigma| \rightarrow \mathbb{Q}$  introduced above in the toric setting is a support function of a torus-invariant canonical divisor. This analogy between support functions and Cartier  $\mathbb{Q}$ -divisors persists for horospherical varieties of complexity one, see Section 5.2. However it is not possible, adapting the proofs in [BM13], to obtain directly a version involving the support function of an invariant canonical divisor. Instead we need to introduce an *auxiliary function*  $\omega_X$ , see Proposition-Definition 5.11. This function  $\omega_X$  does not define in the usual sense a Cartier  $\mathbb{Q}$ -divisor, as shown by the example of the hypersurface

$$x_2^3 - x_3^2 + x_1 x_4 = 0$$

in the affine space  $\mathbb{A}_{\mathbb{C}}^4$  with  $(\mathbb{C}^*)^2$ -action given by

$$(\lambda_1, \lambda_2) \cdot (x_1, x_2, x_3, x_4) = (\lambda_2 x_1, \lambda_1^2 x_2, \lambda_1^3 x_3, \lambda_1^6 \lambda_2^{-1} x_4).$$

This hypersurface appears in [LS13]. We specify the computation of the  $E$ -function of this example in Section 6.1. It will be interesting to find a natural geometric interpretation of the auxiliary function  $\omega_X$ .

Our main result can be stated as follows.

**Theorem 1.1.** *Let  $X$  be a  $\mathbb{Q}$ -Gorenstein horospherical variety of complexity one with log terminal singularities. Consider the combinatorial data  $(C, G/H, \mathcal{E})$  describing  $X$ . Assume that  $\Gamma \subset C$  is a dense open subset which does not contain any special point (see Definition 3.13). Then the stringy motivic volume is given by*

$$\mathcal{E}_{st}(X) = [G/H] \left[ \sum_{(y,\nu,\ell) \in |\mathcal{E}|_\Gamma \cap \mathcal{N}} [X_\ell] \mathbb{L}^{\omega_X(y,\nu,\ell)} \right],$$

where  $X_0 = \Gamma$  and  $X_\ell = \mathbb{A}^1 \setminus \{0\}$  if  $\ell \geq 1$ .

Let us give a brief summary of the contents of each section. In the second and third sections, we introduce notations from motivic integration and horospherical varieties that we use throughout this paper. In the fourth section, we study the arc space of a horospherical variety of complexity one. To do this we proceed into two steps. First, we give a description in the case where the acting group is an algebraic torus, this corresponds to Sections 4.1 and 4.2. The last step concerns the general case where we reduce to the case of torus actions by parabolic induction. We believe that the study in the first step, independently, might be interesting for the study of singularities of  $T$ -varieties.

In the fifth section, we prove our result which is the calculation of the stringy motivic volume of a log terminal horospherical variety  $X$  of complexity one. One of the main ingredients of the proof is the construction of an explicit desingularization by combinatorial methods of the variety  $X$ . This construction is explained in Section 5.3. Then we give a precise description of the rational form of the stringy motivic volume in term of the function  $\omega_X$ . In the last section, we provide examples and applications of our main result. In particular for the part concerning applications, we express the stringy Euler characteristic in a simple formula for a certain class of log terminal horospherical varieties of complexity one, see Lemma 6.1. This allows us for this class to characterize the smoothness condition in terms of the stringy Euler characteristic, see Theorem 6.4.

## CONVENTION

In this paper a *variety* will be an integral separated scheme of finite type over the field of complex numbers  $\mathbb{C}$ . If  $X$  is a variety, then we denote by  $\mathbb{C}[X]$  (resp.  $\mathbb{C}(X)$ ) its algebra of regular global functions (resp. its field of rational functions). Here an *algebraic group*  $G$  is a reduced affine group scheme of type over  $\mathbb{C}$ . In particular,  $G$  is generally not connected. We denote by  $\mathbb{X}(G)$  the character group of  $G$  which consists of morphisms of algebraic groups from  $G$  to  $\mathbb{C}^*$ . Subgroups of algebraic groups are assumed to be Zariski closed. A  *$G$ -variety* is a variety equipped with an algebraic action of the algebraic group  $G$ . If  $X$  is a  $G$ -variety, then  $G$  acts naturally as a linear representation on  $\mathbb{C}[X]$  and  $\mathbb{C}(X)$ .

## 2. ARC SPACES AND MOTIVIC INTEGRATION

The concept of *motivic integration* was invented by Kontsevich [Kon95] to show that birationally equivalent Calabi–Yau manifolds have the same Hodge numbers. He constructed a *motivic measure* on the *arc space* of an algebraic variety. This measure takes values in some localization and completion of the *Grothendieck ring of complex algebraic varieties*  $K_0(\mathbf{Var}_{\mathbb{C}})$ . This ring is generated by isomorphism classes of algebraic varieties with some scissors relations. In particular, if  $X$  is an algebraic variety, then its class  $[X]$  in  $K_0(\mathbf{Var}_{\mathbb{C}})$  contains all the additive and multiplicative invariants of  $X$ , and it is simply called the *motive* of  $X$ . Furthermore if  $X$  is smooth, then the motivic volume  $\mu_X(\mathcal{L}(X))$  of the arc space of  $X$  is equal to  $[X]$ .

The theory of motivic integration was developed by Batyrev in the smooth case [Bat98] and by Denef and Loeser [DL99] in full generality. In this section we sketch the construction of this measure. In 2.1 we recall the definition of the arc space of an algebraic variety. In 2.2 we define the Grothendieck ring of varieties  $K_0(\mathbf{Var}_{\mathbb{C}})$ . In 2.3 we give an introduction to the motivic measure on the arc space of a smooth variety. Finally in 2.4 we present the stringy invariants of Batyrev for log terminal varieties. We refer to the surveys [DL01b, Loe09, Vey04, Bli11].

**2.1. The arc space of a variety.** Here we introduce the  $m$ -jet scheme of arcs of a variety  $X$  and its arc space. Let  $\mathbf{Set}$  be the category of sets and  $\mathbf{Sch}_{\mathbb{C}}$  be the category of schemes over  $\mathbb{C}$ . Let  $m \in \mathbb{N}$  be an integer. The functor

$$\begin{aligned} (\mathbf{Sch}_{\mathbb{C}})^{op} &\rightarrow \mathbf{Set} \\ S &\mapsto \mathrm{Hom}_{\mathbf{Sch}_{\mathbb{C}}}(\mathrm{Spec} \mathbb{C}[t]/(t^{m+1}) \times_{\mathrm{Spec} \mathbb{C}} S, X) \end{aligned}$$

is represented by a  $\mathbb{C}$ -scheme  $\mathcal{L}_m(X)$  of finite type (cf. [BLR90], p. 276) which is unique up to isomorphism. This means that for any scheme  $S$  in  $\mathbf{Sch}_{\mathbb{C}}$  we have a bijection

$$\mathrm{Hom}_{\mathbf{Sch}_{\mathbb{C}}}(\mathrm{Spec} \mathbb{C}[t]/(t^{m+1}) \times_{\mathrm{Spec} \mathbb{C}} S, X) \simeq \mathrm{Hom}_{\mathbf{Sch}_{\mathbb{C}}}(S, \mathcal{L}_m(X)).$$

The scheme  $\mathcal{L}_m(X)$  is called the  *$m$ -jet scheme* of  $X$ .

When  $m \geq \ell \geq 0$  the transition morphisms

$$j^{m,\ell} : \mathcal{L}_m(X) \rightarrow \mathcal{L}_\ell(X)$$

induced by the Weil restriction morphisms

$$\mathbb{C}[t]/(t^{\ell+1}) \rightarrow \mathbb{C}[t]/(t^{m+1})$$

are affine (cf. [DL99, Lemma 2.8 and 4.1]). Hence the projective limit

$$\mathcal{L}(X) := \varprojlim \mathcal{L}_m(X)$$

is well-defined in  $\mathbf{Sch}_{\mathbb{C}}$ . This scheme is called the *arc space* of  $X$ .

If  $S$  is a scheme over  $\mathbb{C}$ , by definition of  $\mathcal{L}(X)$  we have the functorial property

$$(7) \quad \mathrm{Hom}_{\mathbf{Sch}_{\mathbb{C}}}(S, \mathcal{L}(X)) \simeq \mathrm{Hom}_{\mathbf{Sch}_{\mathbb{C}}}(S \hat{\times}_{\mathrm{Spec} \mathbb{C}} \mathrm{Spec} \mathbb{C}[[t]], X),$$

where  $S \hat{\times}_{\mathrm{Spec} \mathbb{C}} \mathrm{Spec} \mathbb{C}[[t]]$  denotes the formal completion of  $S \times_{\mathrm{Spec} \mathbb{C}} \mathrm{Spec} \mathbb{C}[[t]]$  along the subscheme  $S \times_{\mathrm{Spec} \mathbb{C}} \mathrm{Spec} \mathbb{C}$ .

If  $K$  is a field extension of  $\mathbb{C}$ , a  $K$ -rational point of  $\mathcal{L}(X)$ , i.e., an element of  $\mathcal{L}(X)(K)$ , is called a  *$K$ -valued (formal) arc* of  $X$ . Because of the functoriality property (7), the  $K$ -points of  $\mathcal{L}(X)$  may be interpreted as elements of  $\mathrm{Hom}_{\mathbf{Sch}_{\mathbb{C}}}(\mathrm{Spec} K[[t]], X)$ .

We have truncation morphisms

$$(8) \quad \pi_\ell : \mathcal{L}(X) \rightarrow \mathcal{L}_\ell(X)$$

induced by the projection  $\mathbb{C}[[t]] \rightarrow \mathbb{C}[t]/(t^{\ell+1})$ . These morphisms commute with the transition morphisms:

$$\begin{array}{ccc} \mathcal{L}(X) & \xrightarrow{\pi_m} & \mathcal{L}_m(X) \\ & \searrow \pi_\ell & \swarrow j^{m,\ell} \\ & \mathcal{L}_\ell(X) & \end{array}$$

From now on we will consider the jet spaces  $\mathcal{L}_m(X)$  and the arc space  $\mathcal{L}(X)$  with their reduced structure. A point  $\gamma \in \mathcal{L}(X)$  can be viewed as a formal curve as follows. Let  $k_\gamma$  be the residue field of  $\gamma \in \mathcal{L}(X)$ . It is a field extension of  $\mathbb{C}$ . Consider the morphism in  $\mathrm{Hom}_{\mathbf{Sch}_{\mathbb{C}}}(\mathrm{Spec} k_\gamma, \mathcal{L}(X))$  with image  $\gamma$ . By the functorial property of  $\mathcal{L}(X)$  it corresponds to a unique  $\tilde{\gamma} : \mathrm{Spec} k_\gamma[[t]] \rightarrow X$ , which we call the *arc associated with  $\gamma$* .

**2.2. The Grothendieck ring  $K_0(\mathbf{Var}_{\mathbb{C}})$ .** In this section and below we sketch the construction of the *motivic volume* on the arc space of a smooth variety  $X$  over  $\mathbb{C}$ . The motivic volume will take values in a localization and completion of the Grothendieck ring of the category  $\mathbf{Var}_{\mathbb{C}}$  of reduced separated schemes of finite type over  $\mathbb{C}$ .

**Definition 2.1.** The *Grothendieck ring*  $K_0(\mathbf{Var}_{\mathbb{C}})$  of  $\mathbf{Var}_{\mathbb{C}}$  is the ring generated by the symbols  $[S]$  for any object  $S$  in  $\mathbf{Var}_{\mathbb{C}}$ , subject to the relations:

- $[S] = [S']$  if  $S$  and  $S'$  are isomorphic;
- $[S] = [S \setminus S'] + [S']$  if  $S'$  is a closed reduced subscheme of  $S$ ;
- $[S \times_{\mathrm{Spec} \mathbb{C}} S'] = [S] \times [S']$ .

*Remark 2.2.* The ring  $K_0(\mathbf{Var}_{\mathbb{C}})$  has been introduced by Grothendieck in a letter to Serre [CS01, letter of 16/8/1964]. For a complete survey on this topic we refer to [NS11].

We will denote by  $\mathbb{L}$  the class of the affine line  $\mathbb{A}_{\mathbb{C}}^1$ . It follows from the definition that  $1 = [\mathrm{Spec} \mathbb{C}]$ .

If  $f : S' \rightarrow S$  is a piecewise trivial fibration with fiber  $S''$  we have  $[S'] = [S] \times [S'']$ . Let us mention some properties of the ring  $K_0(\mathbf{Var}_{\mathbb{C}})$ .

- Remark 2.3.*
- (1) The Grothendieck ring  $K_0(\mathbf{Var}_{\mathbb{C}})$  is generated by the classes of smooth quasi-projective varieties with the relations  $[\emptyset] = 0$  and  $[X] - [Y] = [Bl_Y X] - [E]$ , where  $Y \subset X$  is a smooth closed subvariety,  $Bl_Y X$  denotes the blow-up of  $X$  along  $Y$ , and  $E$  denotes the exceptional divisor of this blow-up (cf. [Bit04, Theorem 3.1]).
  - (2) The Grothendieck ring contains a polynomial ring in infinitely many variables over  $\mathbb{Z}$  (cf. [Nau07, Theorem 12]).
  - (3) The Grothendieck ring is not integral (cf. [Poo02, Theorem 1]). In particular the class  $\mathbb{L}$  of the affine line is a zero divisor (cf. [Bor14, Theorem 2.12]). However we will see that  $\mathbb{L}$  is not nilpotent.

**Definition 2.4.** Let  $R$  be a ring. An *additive and multiplicative invariant* on  $\mathbf{Var}_{\mathbb{C}}$  is a map  $\lambda : \mathbf{Var}_{\mathbb{C}} \rightarrow R$  such that

- (1) if  $X$  and  $Y$  are two isomorphic varieties, then  $\lambda(X) = \lambda(Y)$ .
- (2) if  $F$  is a closed reduced subscheme of  $X$ , then  $\lambda(X) = \lambda(F) + \lambda(X \setminus F)$ .
- (3) if  $X$  and  $Y$  are two varieties, then  $\lambda(X \times_{\mathrm{Spec} \mathbb{C}} Y) = \lambda(X) \cdot \lambda(Y)$ .

It follows from the definitions that the map  $[\cdot] : \mathbf{Var}_{\mathbb{C}} \rightarrow K_0(\mathbf{Var}_{\mathbb{C}})$  is universal : for each additive and multiplicative invariant  $\lambda$  there is a unique ring morphism  $\tilde{\lambda} : K_0(\mathbf{Var}_{\mathbb{C}}) \rightarrow R$  such that  $\lambda = \tilde{\lambda} \circ [\cdot]$ . The morphism  $\tilde{\lambda}$  is called a *realization morphism*. In particular, in some sense, for any variety the class  $[X]$  contains all the additive and multiplicative invariants of  $X$  and can be called for that reason *motivic*.

**Example 2.5.** The Euler characteristic  $e : \mathbf{Var}_{\mathbb{C}} \rightarrow \mathbb{Z}$  defined by

$$e(X) = \sum_{n \geq 0} (-1)^n \dim_{\mathbb{C}}(H_c^n(X, \mathbb{C}))$$



is an additive and multiplicative invariant. Here  $H_c^n(X; \mathbb{C})$  is the  $n$ -th cohomology group with compact support of  $X$ . In particular its realization map  $\tilde{e} : K_0(\mathbf{Var}_{\mathbb{C}}) \rightarrow \mathbb{Z}$  satisfies

$$\tilde{e}(\mathbb{L}) = e(\mathbb{A}_{\mathbb{C}}^1) = 1,$$

which shows that  $\mathbb{L}$  is not nilpotent. In the sequel, we will simply denote by  $e$  the realization map  $\tilde{e}$ .

**Example 2.6.** If  $X$  is a variety of dimension  $d$ , its  $E$ -polynomial or *Hodge–Deligne polynomial* is defined by

$$(9) \quad E(X; u, v) := \sum_{p, q=0}^d \sum_{i=0}^{2d} (-1)^i h^{p, q}(H_c^i(X; \mathbb{C})) u^p v^q,$$

where  $h^{p, q}(H_c^i(X; \mathbb{C}))$  is the dimension of  $(p, q)$ -type Hodge component of the cohomology group  $H_c^i(X; \mathbb{C})$  (cf. [Del71, Del74]). The map  $E : \mathbf{Var}_{\mathbb{C}} \rightarrow \mathbb{Z}[u, v]$  is an additive and multiplicative invariant (cf. [DK86, Prop. 1.6 and 1.8]). In this paper, we still denote by  $E$  the associated realization map.

Note that if  $X$  is smooth and projective, then its Hodge–Deligne numbers  $h^{p, q}(H_c^i(X; \mathbb{C}))$  coincide with its usual Hodge numbers

$$h^{p, q}(X) := \dim H^p(X, \Omega^q(X)).$$

For instance the  $E$ -polynomial of the line  $\mathbb{A}_{\mathbb{C}}^1$  is  $uv$ .

*Remark 2.7.* Since  $\mathbb{L}$  is not nilpotent we may localize  $K_0(\mathbf{Var}_{\mathbb{C}})$  with respect to  $\mathbb{L}$ . The motivic measure will take values in the completion of this localization with respect to a filtration defined below.

**Definition 2.8.** By convention we set the dimension of the empty variety to be  $-\infty$ . An element  $\Gamma$  in  $K_0(\mathbf{Var}_{\mathbb{C}})$  is said to be  $d$ -dimensional if it may be written in  $K_0(\mathbf{Var}_{\mathbb{C}})$  as  $\Gamma = \sum a_i [X_i]$ , where  $a_i \in \mathbb{Z}$  and the  $X_i$  are varieties of dimension  $\leq d$ , and if it cannot be written in such a way that  $\dim X_i \leq d - 1$  for all indices  $i$ . Thus there is a well-defined map

$$\dim : K_0(\mathbf{Var}_{\mathbb{C}}) \rightarrow \mathbb{N} \cup \{-\infty\}.$$

- We denote by  $\mathcal{M}_{\mathbb{C}}$  the localization  $K_0(\mathbf{Var}_{\mathbb{C}})[\mathbb{L}^{-1}]$  of  $K_0(\mathbf{Var}_{\mathbb{C}})$  with respect to  $\mathbb{L}$ . The dimension map  $\dim$  extends to a map from  $\mathcal{M}_{\mathbb{C}}$  to  $\mathbb{Z} \cup \{-\infty\}$  by setting  $\dim \mathbb{L}^{-1} = -1$ .
- On  $\mathcal{M}_{\mathbb{C}}$  we define an increasing filtration by subgroups:

$$\dots \subset \mathcal{F}^{-2} \subset \mathcal{F}^{-1} \subset \mathcal{F}^0 \subset \mathcal{F}^1 \subset \mathcal{F}^2 \subset \dots,$$

where

$$\mathcal{F}^i = \mathcal{F}^i(\mathcal{M}_{\mathbb{C}}) := \{\Gamma \in \mathcal{M}_{\mathbb{C}} \mid \dim \Gamma \leq i\}.$$

We will denote by  $\hat{\mathcal{M}}_{\mathbb{C}}$  the completion of  $\mathcal{M}_{\mathbb{C}}$  with respect to this filtration.

*Remark 2.9.* The  $E$ -polynomial extends to a map from  $\mathcal{M}_{\mathbb{C}}$  to  $\mathbb{Z}[u, v, (uv)^{-1}]$  by setting

$$E(\mathbb{L}^{-1}; u, v) = (uv)^{-1}.$$

*Remark 2.10.* In fact the completion  $\hat{\mathcal{M}}_{\mathbb{C}}$  is also the completion with respect to the non-archimedean norm  $\|\cdot\|$  defined by composing the dimension map  $\mathcal{M}_{\mathbb{C}} \rightarrow \mathbb{Z} \cup \{-\infty\}$  with the exponential map  $\exp$  from  $\mathbb{Z} \cup \{-\infty\}$  to  $\mathbb{R}_+$ .

**2.3. Motivic integration on the arc space of a smooth variety.** In this section  $X$  is a smooth  $d$ -dimensional variety. We will say that a subset  $S$  of a variety  $Y$  is *constructible* if it is a finite union of locally closed subvarieties of  $Y$ . We now introduce a family of subsets of the arc space  $\mathcal{L}(X)$  on which the motivic measure can be explicitly defined.

A subset  $C$  of the arc space  $\mathcal{L}(X)$  will be called a *cylinder* or a *stable set* if there exists an integer  $m_0$  such that for any  $m \geq m_0$ , the set  $\pi_m(C)$  is constructible in  $\mathcal{L}_m(X)$ , and  $C = \pi_m^{-1}(\pi_m(C))$ . We call the set  $\pi_m(C)$  the  $m$ -basis of  $C$ . Note that  $\mathcal{L}(X)$  itself is a cylinder (cf. [Gre66, Corollary 2]). Cylinders will constitute a basis of measurable sets for the motivic measure. If  $C$  is a cylinder we set

$$\mu_X(C) = [\pi_m(C)] \mathbb{L}^{-md} \in \mathcal{M}_{\mathbb{C}}$$

for any  $m \geq m_0$ . This does not depend on the choice of  $m \geq m_0$  since if  $m \geq \ell \geq 0$ , the map

$$j^{m, \ell} |_{\pi_m(C)} : \pi_m(C) \rightarrow \pi_{\ell}(C)$$

is a piecewise trivial fibration with fiber  $\mathbb{A}^{(m-\ell)d}$ .

We also define a larger family of subsets which are measurable for the motivic measure. These sets are ‘approximations’ of cylinders.

**Definition 2.11.** A subset  $C \subset \mathcal{L}(X)$  is called *measurable* if for any  $m \in \mathbb{N}$  there exists a cylinder  $C_m$  and cylinders  $D_{m,i}$  for  $i \in \mathbb{N}$  such that the symmetric difference  $C \Delta C_m = C \setminus C_m \cup C_m \setminus C$  satisfies

$$C \Delta C_m \subset \bigcup_{i \in \mathbb{N}} D_{m,i}$$

and  $\dim \mu_X(D_{m,i}) \leq -m$  for any  $i$ . Note that the family of measurable subsets is an algebra of sets. If  $C$  is a measurable subset the limit  $\mu_X(C) := \lim_{m \rightarrow \infty} \mu(C_m)$  exists in  $\hat{\mathcal{M}}_{\mathbb{C}}$ . It is independent of the choice of the cylinders  $C_m$  and  $D_{m,i}$  (cf. [Bat98, DL99], [DL02, Appendix], and [Bli11] for a survey). We call it the *motivic measure* of  $C$ .

*Remark 2.12.* If  $Y \subsetneq X$  is a closed reduced subscheme, then  $\mathcal{L}(Y)$  is measurable, and  $\mu_X(\mathcal{L}(Y)) = 0$  (cf [DL99, Equation (3.2.2)] or [Bli11, Proposition 4,5]).

**Definition 2.13.** A function  $F : \mathcal{L}(X) \rightarrow \mathbb{Z} \cup \{\infty\}$  is *measurable* if for any  $s \in \mathbb{Z} \cup \{\infty\}$  the subset  $F^{-1}(s)$  is measurable.

Let us introduce an important example of a measurable function.

**Example 2.14** ([Bli11, Section 2.4]). Let  $Y$  be a proper closed subscheme of  $X$  defined by the sheaf of ideals  $\mathcal{I}_Y$ . We define a function

$$\text{ord}_Y : \mathcal{L}(X) \rightarrow \mathbb{N} \cup \{\infty\}$$

as follows. Let  $\gamma$  be a point in  $\mathcal{L}(X)$ . Its associated arc  $\tilde{\gamma} : \text{Spec } k_\gamma[[t]] \rightarrow X$  induces a sheaf homomorphism  $\gamma^* : \mathcal{O}_X \rightarrow \mathcal{O}_{\text{Spec}(k_\gamma[[t]])}$ . We define the *order of  $\gamma$  along  $Y$*  as

$$\text{ord}_Y(\gamma) := \sup \{e \in \mathbb{N} \cup \{\infty\} \mid \gamma^*(\mathcal{I}_Y) \subseteq (t^e)\}.$$

The function  $\text{ord}_Y$  is measurable and the set  $\text{ord}_Y^{-1}(\infty)$  has zero measure. Indeed  $\text{ord}_Y^{-1}(\infty) = \mathcal{L}(Y)$ , see Remark 2.12.

We may generalize this construction to define the order function of a  $\mathbb{Q}$ -divisor. Let  $D$  be a  $\mathbb{Q}$ -divisor in  $X$ . Write  $D = \sum_{i=1}^s a_i D_i$ , where  $a_i \in \mathbb{Q}$  and the  $D_i$  are prime divisors. We define

$$\text{ord}_D(\gamma) := \sum_{i=1}^s a_i \text{ord}_{D_i}(\gamma)$$

for an arc  $\gamma \in \mathcal{L}(X)$ . If any of the  $\text{ord}_{D_i}(\gamma)$  is infinite we set  $\text{ord}_D(\gamma) := \infty$ . As before the function  $\text{ord}_D$  is measurable and the set  $\text{ord}_D^{-1}(\infty)$  has zero measure.

Let  $C$  be a measurable subset of  $\mathcal{L}(X)$  and  $F : \mathcal{L}(X) \rightarrow \mathbb{Z} \cup \{\infty\}$  be a measurable function such that  $\mu_X(F^{-1}(\infty)) = 0$ . If the series

$$(10) \quad \sum_{s \in \mathbb{Z}} \mu_X(C \cap F^{-1}(s)) \mathbb{L}^{-s}$$

converges in  $(\hat{\mathcal{M}}_{\mathbb{C}}, \|\cdot\|)$ , then we say that  $\mathbb{L}^{-F}$  is *integrable*, and we define

$$\int_C \mathbb{L}^{-F} d\mu_X := \sum_{s \in \mathbb{Z}} \mu_X(C \cap F^{-1}(s)) \mathbb{L}^{-s}.$$

**2.4. Stringy invariants of varieties with log terminal singularities.** We introduce in this subsection some of our main objects of study, the *stringy motivic volume*  $\mathcal{E}_{st}$ , the *stringy E-function* and the *stringy Euler characteristic* of log terminal algebraic varieties. These invariants are defined by Batyrev ([Bat98, Bat99, Bat00]) and also studied by several authors such as Denef–Loeser [DL02], Yasuda [Yas04] and Schepers–Veys ([Vey03, Vey04, SV07, SV09]). Using these invariants Batyrev introduced in [Bat98] a topological mirror symmetry test for singular Calabi–Yau mirror pairs and a conjectural definition of stringy Hodge numbers for certain canonical Gorenstein varieties. Veys generalized these invariants to almost arbitrary  $\mathbb{Q}$ -Gorenstein singular varieties, assuming Mori’s Minimal Program ([Vey03, Vey04]). Batyrev in [Bat99], Denef–Loeser in [DL02] and Yasuda in [Yas04] gave also a proof of a version of Reid’s McKay correspondence conjecture.

Let  $X$  be an irreducible normal  $\mathbb{Q}$ -Gorenstein variety, namely  $X$  is a normal variety and that a (thus any) canonical divisor  $K_X$  is  $\mathbb{Q}$ -Cartier. We say that a morphism of varieties  $f : X' \rightarrow X$  is a *resolution of singularities* if  $X'$  is smooth,  $f$  is birational and proper, and the exceptional locus is the reunion of finitely many normal crossings irreducible smooth divisors. By Hironaka's theorem [Hir64] such resolutions exist.

Choose such a resolution of singularities  $f : X' \rightarrow X$  and denote by  $(E_i)_{i \in I}$  the irreducible components of the exceptional locus of  $f$ . If  $K_X$  (resp.  $K_{X'}$ ) is a canonical divisor of  $X$  (resp. of  $X'$ ) we define the *relative canonical divisor* as

$$K_{X'/X} := K_{X'} - f^*K_X = \sum_{i \in I} \nu_i E_i.$$

The multiplicities  $\nu_i$  will also be denoted by  $\nu(E_i)$ . By [KM98, Remark 2.23], if  $f' : Y' \rightarrow X$  is an other birational morphism and  $E' \subset Y'$  is an irreducible component of the exceptional locus such that the valuation  $v_{E'}$  is equal to the valuation  $v_{E_i}$  for some  $i \in I$ , then the multiplicities  $\nu(E_i)$  and  $\nu(E')$  are equal. We call *divisor over  $X$* , any irreducible component of the exceptional locus of some birational morphism to  $X$ . The rational number

$$\text{discrep}(X) := \inf_E \{ \nu(E) \mid E \text{ is a divisor over } X \}$$

is called *discrepancy* of  $X$ . We say that  $X$  is (*purely*) *log terminal* if  $\text{discrep}(X) > -1$ . For every subset  $J \subset I$  we write

$$E_J^\circ := \bigcap_{j \in J} E_j \setminus \bigcup_{i \in I \setminus J} E_i.$$

Clearly  $E_\emptyset^\circ = X' \setminus \bigcup_{i \in I} E_i$  and  $E_I^\circ = \bigcap_{i \in I} E_i$ .

**Definition 2.15.** Let  $X$  be a normal  $\mathbb{Q}$ -Gorenstein algebraic variety with log terminal singularities. Let  $f : X' \rightarrow X$  be a resolution of singularities of  $X$  and let  $(E_i)_{i \in I}$  be the set of irreducible components of its exceptional locus. For any  $i \in I$  we denote by  $\nu_i$  the multiplicity of  $K_{X'/X}$  along  $E_i$ .

The *stringy motivic volume*  $\mathcal{E}_{st}(X)$  is usually defined following [Bat98] as

$$\mathcal{E}_{st}(X) := \sum_{J \subset I} [E_J^\circ] \prod_{j \in J} \frac{\mathbb{L} - 1}{\mathbb{L}^{\nu_j + 1} - 1} \in \hat{\mathcal{M}}_{\mathbb{C}}(\mathbb{L}^{\frac{1}{m}}),$$

where  $m$  is the g.c.d of the denominators of the  $\nu_i$ , and by definition:

$$\hat{\mathcal{M}}_{\mathbb{C}}(\mathbb{L}^{\frac{1}{m}}) := \hat{\mathcal{M}}_{\mathbb{C}} \left[ f(\mathbb{L}^{\frac{1}{m}}) \mid f \in \mathbb{Q}(T) \right].$$

The theorem below follows from the appendix of [Bat98] :

**Theorem 2.16.** *The stringy motivic volume is equal to*

$$\mathcal{E}_{st}(X) = \int_{\mathcal{L}(X')} \mathbb{L}^{-\text{ord}_{K_{X'/X}}} d\mu_{X'} \in \hat{\mathcal{M}}_{\mathbb{C}}(\mathbb{L}^{\frac{1}{m}})$$

and does not depend on the resolution  $f : X' \rightarrow X$  of the singularities of  $X$ .

*Remark 2.17.* (i) If  $X$  is smooth, then  $\mathcal{E}_{st}(X)$  is equal to the class  $[X]$ .

(ii) Denef and Loeser proposed in [DL02] (see also [Yas04]) an intrinsic point of view of  $\mathcal{E}_{st}(X)$  in the Gorenstein case. Suppose  $X$  be log terminal Gorenstein normal with dimension  $d$ . Then there exists a differential form  $\omega_X$  in  $\Omega_X^d \otimes_{\mathbb{C}} \mathbb{C}(X)$  which generates  $\Omega_X^d$  at each smooth point of  $X$ . Since  $X$  is log terminal,  $\text{div } h^*(\omega_X)$  is effective for any resolution  $h : Y \rightarrow X$ . The function  $\mathbb{L}^{-\text{ord } \omega_X}$  is integrable on  $\mathcal{L}(X)$  with the respect to the motivic measure  $\mu_X$  of Denef–Loeser defined in [DL99] and

$$\mathcal{E}_{st}(X) = \int_{\mathcal{L}(X)} \mathbb{L}^{-\text{ord } \omega_X} d\mu_X.$$

In this form the stringy motivic volume does not depend on the choice of a resolution. The change variable formula of Kontsevich (see [DL99, Section 3]) gives the connection with the integral formula of Batyrev.

From  $\mathcal{E}_{st}(X)$  we can deduce two new invariants for normal algebraic varieties with at worst log terminal singularities.

**Definition 2.18.** The  $E$ -function of a log terminal variety  $X$  is

$$E_{st}(X; u, v) := \sum_{J \subset I} E(E_J; u, v) \prod_{j \in J} \frac{uv - 1}{(uv)^{\nu_j + 1} - 1}.$$

The *stringy Euler characteristic* of  $X$  is

$$e_{st}(X) := \sum_{J \subset I} e(E_J) \prod_{j \in J} \frac{1}{\nu_j + 1},$$

where  $e(E_J) = E(E_J; 1, 1)$  denotes the usual Euler characteristic.

When  $X$  is smooth this definition coincides with the definition of the  $E$ -polynomial  $E(X; u, v)$  in Equation (9).

### 3. HOROSPHERICAL VARIETIES OF COMPLEXITY ONE

In this section we recall some notions we need to study horospherical varieties of complexity one and describe them from combinatorial data, see [Tim11, Section 16], [LT16]. Section 3.1 deals with the general case of horospherical group actions of reductive groups in normal varieties with arbitrary complexity. In particular we recall the definition of germs, colors, and charts. Following [Pas08], we also explain how horospherical homogeneous spaces are parametrized. In 3.2 we focus on the classification of normal  $G$ -varieties of complexity one from [Tim97], specialized to the horospherical case. We adopt there the language of colored polyhedral divisors – and colored divisorial fans in 3.3. When the group  $G$  is an algebraic torus, this corresponds to the theory developed in [AH06, AHS08].

Throughout this section we let  $G$  be a connected simply connected reductive algebraic group over  $\mathbb{C}$  and  $B$  be a Borel subgroup. By ‘ $G$  simply connected’ we mean that it is the direct product of an algebraic torus and of a simply connected semisimple algebraic group. Fix a decomposition  $B = QU$ , where  $Q$  is a maximal torus and  $U$  a maximal unipotent subgroup.

**3.1. Horospherical transformations.** The *complexity* (cf [Vin86]) of an action of  $G$  on a variety  $Y$  is the transcendence degree over  $\mathbb{C}$  of the field extension  $\mathbb{C}(Y)^B$  of  $\mathbb{C}$ , where we denote by  $\mathbb{C}(Y)^B$  the subfield of  $B$ -invariant rational functions on  $Y$ . From [Ros63] we know that the complexity is the minimum of the codimensions of the  $B$ -orbits in  $Y$ .

We say that the action of  $G$  on  $Y$  is *horospherical* if for any  $x \in Y$  (or equivalently, for a generic  $x \in Y$ , see [Tim11, Remark 7.2]), the stabilizer  $G_x$  of  $x$  contains a maximal unipotent subgroup of  $G$ . If, moreover,  $Y$  is normal we say it is a *horospherical variety*.

The simplest example of a horospherical variety is a *horospherical homogeneous space*, that is, a horospherical variety of the form  $Y = G/H$ , where  $H$  is a closed subgroup of  $G$  containing a maximal unipotent subgroup and  $G$  acts by left multiplication. The subgroup  $H$  is called a *horospherical subgroup*. Note that due to the Bruhat decomposition the complexity of a horospherical homogeneous space is always zero.

More generally, an *embedding* of a horospherical homogeneous space  $G/H$  (or a  $G/H$ -embedding when one needs to emphasize the homogeneous space  $G/H$ ) is a pair  $(X, x)$  such that  $X$  is a normal  $G$ -variety containing  $x$ , the orbit  $G \cdot x$  of  $x$  is open, and the stabilizer  $G_x$  of  $x$  is  $H$ . For brevity, in the sequel we will denote an embedding of  $G/H$  simply by  $X$ , omitting the point  $x \in X$  which is attached to it.

**Description of the horospherical subgroups of  $G$ .** Let  $\Phi$  be the set of simple roots of  $G$  with respect to  $(B, Q)$ . Let  $W = N_G(Q)/Q$  be the Weyl group of  $(G, Q)$ . We denote by  $s_\alpha \in W$  the simple reflection associated with  $\alpha \in \Phi$  and by  $\dot{s}_\alpha$  a lift of  $s_\alpha$  to  $G$ . For every  $w \in W$  the *length* of  $w$  is the minimal number of simple reflections required to express  $w$  as a product. We also denote by  $w_0$  the longest element of  $W$ . If  $I$  is a subset of  $\Phi$  we let  $\dot{W}_I$  be the subgroup of  $G$  generated by the lifts of simple reflections  $\dot{s}_\alpha$  for  $\alpha \in I$ , and  $P_I$  be the parabolic subgroup generated by  $B$  and  $\dot{W}_I$ . The map  $I \mapsto P_I$  between subsets of  $\Phi$  and closed subgroups of  $G$  containing  $B$  is bijective.

By [Pas08, Proposition 2.4], there is a bijection between subgroups  $H$  of  $G$  containing  $U$  and pairs  $(M, I)$ , where:

- $M$  is a sublattice of the character group  $\mathbb{X}(Q)$  of the torus  $Q$ ,
- $I$  is a subset of the set  $\Phi$  of simple roots,

subject to the property that for any simple root  $\alpha \in I$ , the associated co-root  $\alpha^\vee$  satisfies  $\langle m, \alpha^\vee \rangle = 0$  for any  $m \in M$ . The bijection is constructed as follows.

First, we consider the normalizer  $N_G(H)$  of  $H$  in the group  $G$ . It is a parabolic subgroup  $P = P_I$ , where  $I \subset \Phi$ . The subset  $I$  constitutes the first part of the data parameterizing  $H$ . Now the quotient  $T := P/H$  is an algebraic torus, and our second data  $M$  is the character lattice of  $T$ .

Geometrically the pair  $(M, I)$  parameterizing  $H$  can be interpreted as follows. The natural projection  $p : G/H \rightarrow G/P$  is a torus fibration with fiber  $T$ . The set  $\Phi \setminus I$  corresponds to the indexing set for the Schubert divisors of  $G/P$ . More precisely, we denote by

$$X_\alpha = \overline{B\dot{w}_0\dot{s}_\alpha P/P}$$

the Schubert divisor associated with the simple root  $\alpha \in \Phi \setminus I$ .

**Germ, colors, and charts.** Let  $Y$  be a horospherical  $G$ -variety and  $L \subset G$  be a closed subgroup of  $G$ . An  $L$ -germ (or a *prime  $L$ -cycle*) of  $Y$  is an irreducible reduced non-empty  $L$ -stable subscheme of  $Y$ . A  $G$ -germ will be simply called a *germ*. An  $L$ -divisor is a codimension-one  $L$ -germ, and a *color* is a  $B$ -divisor which is not  $G$ -stable.

A  $B$ -chart of  $Y$  is a dense open  $B$ -stable affine subset of  $Y$ . We say that the horospherical variety  $Y$  is *simple* if  $Y$  possesses a  $B$ -chart which intersects every  $G$ -orbit of  $Y$  or, equivalently, every germ of  $Y$ . Since  $Y$  is normal, it is the reunion of simple  $G$ -stable open subsets, see [Sum74, Theorem 1] and [Kno91, Theorem 1.3]. Moreover from [Tim00, Lemma 2] it follows that every normal simple horospherical  $G$ -variety is quasi-projective.

**Models.** Let  $Y$  be a variety with a horospherical  $G$ -action. We say that a  $G$ -variety  $Z$  is a *model* of  $Y$  if  $Z$  is normal and there exists a birational  $G$ -equivariant map  $Z \dashrightarrow Y$ . For instance, any horospherical  $G$ -variety  $Y$  has a model of the form  $C \times G/H$ , where  $H$  is a horospherical subgroup of  $G$ ,  $C$  is a smooth variety, and  $G$  acts by left multiplication on  $G/H$  and trivially on  $C$  (see [Kno90, Satz 2.2]). This implies that the dimension of  $C$  is equal to the complexity of the action of  $G$  on  $Y$ . Thus equivariant birational classes of horospherical varieties have a simple description.

Let  $Y$  be a horospherical  $G$ -variety. Using a model  $Z = C \times G/H$  of  $Y$  we may now describe more precisely the colors of the  $G$ -variety  $Y$ . The variety  $Y$  contains a  $G$ -stable open subset which is identified with  $\Gamma \times G/H \subset Z$ , where  $\Gamma \subset C$  is open dense. Let  $(M, I)$  be the lattice/subset pair parameterizing the horospherical subgroup  $H$ . For any  $\alpha \in \Phi \setminus I$  we define a color by letting  $D_\alpha$  denote the set

$$\overline{\Gamma \times p^{-1}(X_\alpha)} \subset Y,$$

where  $p : G/H \rightarrow G/P$  is the natural projection and  $P = N_G(H)$ . The map  $\alpha \mapsto D_\alpha$  defines a bijective correspondence between  $\Phi \setminus I$  and the set  $\mathcal{F}_Y$  of colors of  $Y$ . We will use the same notation  $D_\alpha$  for the colors of  $G/H$  and the colors of  $Y$ .

**3.2. Colored polyhedral divisors.** In this subsection we consider a horospherical variety of complexity one. From the previous paragraph we know that it is  $G$ -equivariantly birational to a direct product  $Z = C \times G/H$ , where  $C$  is a smooth projective curve and  $H$  is a horospherical subgroup of  $G$ . Here we introduce some combinatorial data which describes all the models of  $Z$ . We refer to [Tim11, Theorem 3.5] and [AH06].

Recall that the horospherical subgroup  $H$  is associated with a unique pair  $(M, I)$ , where  $M$  is a sublattice of the character lattice  $\mathbb{X}(Q)$ , and  $I$  is a subset of the set  $\Phi$  of simple roots. Let  $N$  denote the dual lattice  $\text{Hom}(M, \mathbb{Z})$  and  $M_\mathbb{Q}$  and  $N_\mathbb{Q}$  denote the dual associated vector spaces  $\mathbb{Q} \otimes_\mathbb{Z} M$  and  $\mathbb{Q} \otimes_\mathbb{Z} N$ .

If  $\sigma \subset N_\mathbb{Q}$  is a polyhedral cone we define its *dual*  $\sigma^\vee \subset M_\mathbb{Q}$  by the equality

$$\sigma^\vee = \{m \in M_\mathbb{Q} \mid \forall v \in \sigma, \langle m, v \rangle \geq 0\}.$$

The polyhedral cone  $\sigma$  is *strongly convex* (i.e.  $\{0\}$  is a face of  $\sigma$ ) if and only if its dual  $\sigma^\vee$  generates  $M_\mathbb{Q}$ . Note that  $(\sigma^\vee)^\vee = \sigma$ .

Let us recall the notion of polyhedral divisors (introduced in [AH06]), specialized to the complexity one case.

**Definition 3.1.** Let  $\sigma$  be a strongly convex polyhedral cone in  $N_\mathbb{Q}$  and  $C_0$  be a smooth curve.

- (1) A  $\sigma$ -polyhedral divisor on  $C_0$  is a formal sum

$$\mathfrak{D} = \sum_{y \in C_0} \Delta_y \cdot [y],$$

where  $\Delta_y \subset N_{\mathbb{Q}}$  is a  $\sigma$ -polyhedron (that is, the Minkowski sum of the cone  $\sigma$  and a non-empty polytope  $\Pi \subset N_{\mathbb{Q}}$ ), with the property that  $\Delta_y = \sigma$  for all but finitely many  $y \in C_0$ . The set of special points of  $\mathfrak{D}$ , denoted by  $\text{Sp}(\mathfrak{D})$ , is the set

$$\text{Sp}(\mathfrak{D}) = \{y \in C_0 \mid \Delta_y \neq \sigma\}.$$

The cone  $\sigma$  will be called the *tail* of  $\mathfrak{D}$  and the curve  $C_0$  will be called its *locus*.

- (2) The *degree*  $\deg \mathfrak{D}$  of the polyhedral divisor  $\mathfrak{D}$  is defined as the Minkowski sum

$$\deg \mathfrak{D} = \sum_{y \in C_0} \Delta_y \subset N_{\mathbb{Q}},$$

if  $C_0$  is complete and as the empty set otherwise.

- (3) For  $m \in \sigma^{\vee}$  we define the *evaluation of  $\mathfrak{D}$  at  $m$*  by

$$\mathfrak{D}(m) := \sum_{y \in C_0} \min\{\langle m, v \rangle \mid v \in \Delta_y\} \cdot [y].$$

This is a  $\mathbb{Q}$ -divisor on the curve  $C_0$ .

**Definition 3.2.** (1) For any subsemigroup  $S$  of  $(M, +)$ , let us denote by

$$\mathbb{C}[S] = \bigoplus_{m \in S} \mathbb{C}\chi^m$$

the  $\mathbb{C}$ -algebra generated by the elements  $\chi^m$  for  $m \in S$ , subject to the relations  $\chi^m \cdot \chi^{m'} = \chi^{m+m'}$  for all  $m, m' \in S$ .

- (2) Let  $\mathfrak{D}$  be a polyhedral divisor as before. The *associated  $M$ -graded algebra* of  $\mathfrak{D}$  is defined as the subalgebra

$$A(C_0, \mathfrak{D}) := \bigoplus_{m \in \sigma^{\vee} \cap M} H^0(C_0, \mathcal{O}_{C_0}([\mathfrak{D}(m)])) \otimes \chi^m \subset \mathbb{C}(C_0) \otimes_{\mathbb{C}} \mathbb{C}[M],$$

where  $[\mathfrak{D}(m)]$  stands for the integral Weil divisor obtained by taking the integral part of each coefficient of  $\mathfrak{D}(m)$ .

In the next paragraph, we introduce the notion of properness of a polyhedral divisor  $\mathfrak{D}$ . This condition ensures that the associated algebra  $A(C_0, \mathfrak{D})$  and  $\mathbb{C}(C_0) \otimes_{\mathbb{C}} \mathbb{C}[M]$  have the same field of fractions.

**Definition 3.3.** Let  $\sigma$  be a strongly convex polyhedral cone in  $N_{\mathbb{Q}}$  and  $C_0$  be a smooth curve. A  $\sigma$ -polyhedral divisor  $\mathfrak{D}$  is *proper* if either  $C_0$  is affine, or it is projective and the two following conditions are satisfied:

- (i)  $\deg \mathfrak{D} \not\subset \sigma$ ;
- (ii) If  $\min\{\langle m, v \rangle \mid v \in \deg \mathfrak{D}\} = 0$  for some  $m \in \sigma^{\vee}$ , then  $\mathfrak{D}(rm)$  is principal for some  $r \in \mathbb{Z}_{>0}$ .

The next definition introduces the *coloration map*  $\varrho$  associated with the horospherical homogeneous space  $G/H$ .

**Definition 3.4.** Let  $(M, I)$  be the lattice/subset pair associated with the horospherical subgroup  $H$ . For any color  $D$  of  $G/H$  there exists a unique root  $\alpha \in \Phi$  such that  $D = D_{\alpha}$ . In particular, the coroot  $\alpha^{\vee}$  is an element of  $\text{Hom}(\mathbb{X}(Q), \mathbb{Z})$ . Thus its restriction to  $M$ ,  $\alpha|_M^{\vee} : M \rightarrow \mathbb{Z}$ , is an element of  $N = \text{Hom}(M, \mathbb{Z})$ , and we define  $\varrho(D)$  as  $\alpha|_M^{\vee}$ .

**Definition 3.5.** Let  $\sigma$  be a strongly convex polyhedral cone in  $N_{\mathbb{Q}}$  and  $C_0$  be a smooth curve. A pair  $(\mathfrak{D}, \mathcal{F})$  is a *colored  $\sigma$ -polyhedral divisor* (cf [LT16, Section 1.3]) on  $C_0$  if  $\mathcal{F}$  is a subset of  $\mathcal{F}_{G/H}$  such that  $\varrho(\mathcal{F}) \subset \sigma$ ,  $0 \notin \varrho(\mathcal{F})$ , and if  $\mathfrak{D}$  is a proper  $\sigma$ -polyhedral divisor on  $C_0$ .

Using previous definitions, the following result is a combinatorial description of the simple models of the product  $C \times G/H$ . It was proved for  $G = T$  in [Tim08, Theorem 2] and [AH06, Theorems 3.1 and 3.4]. It is a consequence of a general classification result for normal  $G$ -varieties of complexity one (cf [Tim97, Theorem 3.1]).

**Theorem 3.6.** *Let  $G$  be a connected simply connected reductive algebraic group and  $H$  be a horospherical subgroup associated with a pair  $(M, I)$ . Let  $Z = C \times G/H$ , where  $C$  is a smooth projective curve.*

- (i) *Let  $(\mathfrak{D}, \mathcal{F})$  be a colored polyhedral divisor on an open dense subset  $C_0 \subset C$ . Then one can associate a simple  $G$ -model  $X(\mathfrak{D}) := X(\mathfrak{D}, \mathcal{F})$  of  $Z$  containing a  $B$ -chart which is determined by  $(\mathfrak{D}, \mathcal{F})$ . The construction of  $X(\mathfrak{D})$  is specified in Equation (12).*
- (ii) *Let  $X$  be a simple  $G$ -model of  $Z$ . Then there exists an open dense subset  $C_0 \subset C$ , a colored polyhedral divisor  $(\mathfrak{D}, \mathcal{F})$  on  $C_0$ , and a  $G$ -isomorphism  $X \rightarrow X(\mathfrak{D})$ .*

We now give the main ideas of the construction of the  $G$ -variety  $X(\mathfrak{D})$  in the first part of Theorem 3.6.

*Construction.* If  $(\mathfrak{D}, \mathcal{F})$  is a colored  $\sigma$ -polyhedral divisor on  $C_0 \subset C$ , following [Tim11, §28] we may associate to it a  $G$ -model  $X(\mathfrak{D}) = X(\mathfrak{D}, \mathcal{F})$  of  $Z = C \times G/H$  as follows. Denote by  $P_{\mathcal{F}}$  the parabolic subgroup of  $G$  containing the Borel subgroup  $B$  and corresponding to the set of roots

$$I_{\mathcal{F}} = \{\alpha \in \Phi \setminus I \mid D_{\alpha} \in \mathcal{F}\} \cup I.$$

Let  $L$  be the Levi subgroup of  $P_{\mathcal{F}}$  containing the maximal torus  $Q$  and  $B_L$  be the Borel subgroup of  $L$  containing  $Q$  such that  $I_{\mathcal{F}}$  is the set of simple roots of  $L$  with respect to  $(B_L, Q)$ . We also let  $H_L$  (resp.  $U_L$ ) denote the intersection  $H \cap L$  (resp.  $U \cap L$ ). The subgroup  $H_L$  contains  $U_L$ , hence the homogeneous space  $L/H_L$  is horospherical. It is quasi-affine (cf. [Tim11, Corollary 15.6]). Note that  $L/H_L$  is constructed so that the set of  $B$ -divisors in  $L/H_L$  exactly identifies with  $\mathcal{F}$ . Moreover the lattice of  $B_L$ -weights in  $\mathbb{C}[L/H_L]$  is the lattice  $M$ .

Denote by  $\mathbb{X}_+(Q) \subset M_{\mathbb{Q}}$  the cone generated by the dominant weights of  $\mathbb{C}[L/H_L]$ . Recall that the rational  $L$ -module  $\mathbb{C}[L/H_L]$  admits a gradation of vector spaces

$$\mathbb{C}[L/H_L] = \bigoplus_{m \in \mathbb{X}_+(Q) \cap M} V(m),$$

where  $V(m)$  denotes the irreducible rational representation corresponding to  $m \in \mathbb{X}_+(Q) \cap M$ . This gradation is also a  $\mathbb{C}$ -algebra gradation (cf [Tim11, Proposition 7.6]).

Since by definition of  $(\mathfrak{D}, \mathcal{F})$  we have  $\varrho(\mathcal{F}) \subset \sigma$ , it follows that  $\sigma^{\vee} \subset \mathbb{X}_+(Q)$ . Thus we may define a subalgebra of  $\mathbb{C}(C) \otimes_{\mathbb{C}} \mathbb{C}[L/H_L]$  by

$$(11) \quad A(C_0, \mathfrak{D}, \mathcal{F}) := \bigoplus_{m \in \sigma^{\vee} \cap M} H^0(C_0, \mathcal{O}_{C_0}([\mathfrak{D}(m)])) \otimes_{\mathbb{C}} V(m)$$

Set  $Y(\mathfrak{D}, \mathcal{F}) := \text{Spec } A(C_0, \mathfrak{D}, \mathcal{F})$ . Since  $\mathfrak{D}$  is proper,  $Y(\mathfrak{D}, \mathcal{F})$  is a well-defined affine  $L$ -model of  $C \times L/H_L$ . To conclude, consider the parabolic induction:

$$(12) \quad X(\mathfrak{D}) = X(\mathfrak{D}, \mathcal{F}) := G \times^{P_{\mathcal{F}}} Y(\mathfrak{D}, \mathcal{F}).$$

The notation  $G \times^{P_{\mathcal{F}}} Y(\mathfrak{D}, \mathcal{F})$  means that  $X(\mathfrak{D})$  is the quotient  $(G \times Y(\mathfrak{D}, \mathcal{F}))/P_{\mathcal{F}}$ , where  $P_{\mathcal{F}}$  acts as follows

$$p \cdot (g, y) := (gp^{-1}, \phi(p) \cdot y)$$

for all  $p \in P_{\mathcal{F}}$ ,  $g \in G$ , and  $y \in Y(\mathfrak{D}, \mathcal{F})$ . Here  $\phi : P_{\mathcal{F}} \rightarrow L$  is the usual projection. Note that  $X(\mathfrak{D})$  is a locally trivial fiber bundle over  $G/P_{\mathcal{F}}$  with fiber  $Y(\mathfrak{D}, \mathcal{F})$ , cf [Jan87, §I.5].  $\square$

Let us illustrate the parabolic induction process explained in the preceding paragraph with the following example. Note that this process for horospherical homogeneous spaces can be translated into a localization of spherical systems. This is a special case of [Gag15, Proposition 3.2].

**Example 3.7.** Let  $G = \text{GL}_4(\mathbb{C})$  with Borel subgroup the subgroup of upper triangular matrices and maximal torus the subgroup of diagonal matrices. We consider the horospherical subgroup

$$H = \left( \begin{array}{c|c} \text{SL}_2(\mathbb{C}) & M_2(\mathbb{C}) \\ \hline 0 & \text{GL}_2(\mathbb{C}) \end{array} \right)$$

associated with the parabolic subgroup

$$P = N_G(H) = \left( \begin{array}{c|c} \mathrm{GL}_2(\mathbb{C}) & M_2(\mathbb{C}) \\ \hline 0 & \mathrm{GL}_2(\mathbb{C}) \end{array} \right)$$

and the lattice  $M = \mathbb{X}(T)$ , where

$$T = P/H = \begin{pmatrix} t & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The indexing set  $I$  associated with  $P$  is  $\Phi \setminus \{\alpha_2\} = \{\alpha_1, \alpha_3\}$ , where the  $\alpha_i = e_i - e_{i+1}$  are the simple roots of  $G$ , expressed in the canonical basis of  $\mathbb{C}^4$ . Let  $L$  be the Levi subgroup of  $P$  described above:

$$L = \left( \begin{array}{c|c} \mathrm{GL}_2(\mathbb{C}) & 0 \\ \hline 0 & \mathrm{GL}_2(\mathbb{C}) \end{array} \right).$$

Then clearly the subgroup

$$H_L = \left( \begin{array}{c|c} \mathrm{SL}_2(\mathbb{C}) & 0 \\ \hline 0 & \mathrm{GL}_2(\mathbb{C}) \end{array} \right)$$

has normalizer  $N_L(H_L) = L$ , and the lattice is  $\mathbb{X}(L/H_L) \simeq M$ . The indexing set corresponding to the parabolic subgroup  $L$  is  $\{\alpha_1, \alpha_3\} = I$ , so the pair associated with  $L/H_L$  is the same as the pair associated with  $G/H$ .

*Remark 3.8.* For a colored polyhedral divisor  $(\mathfrak{D}, \mathcal{F})$ , there is a geometric description of  $\mathcal{F}$ , namely

$$\mathcal{F} = \{D \in \mathcal{F}_{G/H} \mid D \supset G \cdot x \text{ for some } x \in X(\mathfrak{D})\},$$

where we see the elements of  $\mathcal{F}_{G/H}$  as colors on  $X(\mathfrak{D})$ . It is a consequence of the description in terms of ‘colored data’ in [Tim11, Section 16].

A more conceptual construction of the simple  $G$ -models  $X(\mathfrak{D})$  was initially given in [Tim97], generalizing the approach of [LV83] in complexity one. Let us recall here how this construction relates to the one explained above.

First we introduce the scheme of geometric localities (cf. [LV83, Section 1] for more details).

**Definition 3.9.** Let  $Y$  be a horospherical  $G$ -variety. A *locality* of  $\mathbb{C}(Y)$  is a local ring  $R_{\mathfrak{p}}$ , where  $\mathfrak{p} \subset R$  is a prime ideal in a subalgebra  $R$  of  $\mathbb{C}(Y)$  of finite type, such that the fraction field of  $R$  is  $\mathbb{C}(Y)$ . The set of localities  $\mathfrak{S}(Y)$  is naturally endowed with a  $\mathbb{C}$ -scheme structure, and it is called the *scheme of geometric localities*. It can be interpreted as the geometric object obtained by ‘gluing all the varieties birational to  $Y$ ’. We denote by  $\mathfrak{S}_G(Y) \subset \mathfrak{S}(Y)$  the maximal normal open subset where the  $G$ -action (induced by the  $G$ -action on  $\mathbb{C}(Y)$ ) is a morphism of  $\mathbb{C}$ -schemes; we will call it the *equivariant scheme of geometric localities*.

Thus according to the previous definition, a  $G$ -model of a horospherical  $G$ -variety  $Y$  is nothing but a  $G$ -stable separated open dense subset of  $\mathfrak{S}_G(Y)$  of finite type over  $\mathbb{C}$ .

Now we consider a horospherical  $G$ -variety of the form  $Z = C \times G/H$ , where  $C$  is a smooth projective curve and  $H$  is a horospherical subgroup. We introduce two further ingredients involved in the construction of the scheme of geometric localities, namely, the hyperspace  $\mathcal{N}_{\mathbb{Q}}$  associated with  $Z$  and the hypercone of a polyhedral divisor.

**Definition 3.10.** The *hyperspace*  $\mathcal{N}_{\mathbb{Q}}$  is the set  $C \times N_{\mathbb{Q}} \times \mathbb{Q}_{\geq 0}$  modulo the equivalence relation

$$(y, \nu, \ell) \sim (y', \nu', \ell') \Leftrightarrow (y = y', \nu = \nu', \ell = \ell') \text{ or } (\ell = \ell' = 0, \nu = \nu').$$



Geometrically, it may be viewed as the disjoint union over  $C$  of all upper half-spaces  $N_{\mathbb{Q}} \times \mathbb{Q}_{\geq 0}$  glued along their boundary  $N_{\mathbb{Q}} \times \{0\}$ . Since an element  $(y, \nu, 0) \in \mathcal{N}_{\mathbb{Q}}$  does not depend on  $y \in C$ , in the sequel we will denote it by  $(C, \nu, 0)$ . We also denote by  $\mathcal{N} \subset \mathcal{N}_{\mathbb{Q}}$  the subset of integral points

$$\mathcal{N} = \{(y, \nu, \ell) \in \mathcal{N}_{\mathbb{Q}} \mid \nu \in N, \ell \in \mathbb{Z}_{\geq 0}\}.$$

Let  $\mathfrak{D} = \sum_{y \in C_0} \Delta_y \cdot [y]$  be a  $\sigma$ -polyhedral divisor on an open dense subset  $C_0 \subset C$ . For  $y \in C_0$ , the *Cayley cone*  $C_y(\mathfrak{D})$  is defined as the cone generated by the subsets  $\sigma \times \{0\}$  and  $\Delta_y \times \{1\}$  in  $N_{\mathbb{Q}} \times \mathbb{Q}_{\geq 0}$ . The *hypercone* associated with  $\mathfrak{D}$  is the subset  $C(\mathfrak{D}) \subset \mathcal{N}_{\mathbb{Q}}$  defined by

$$C(\mathfrak{D}) = \bigcup_{y \in C_0} \{y\} \times C_y(\mathfrak{D}) / \sim$$

Denote by  $C(\mathfrak{D})(1)$  the subset of elements  $[y, \nu, \ell] \in C(\mathfrak{D}) \cap \mathcal{N}$  such that  $(\nu, \ell)$  is an integral primitive generator of a one-dimensional face of  $C_y(\mathfrak{D})$ .

Each triple  $\xi = (y, \nu, \ell) \in \mathcal{N}_{\mathbb{Q}}$  induces a unique discrete  $G$ -invariant valuation  $w(\xi) : \mathbb{C}(Z)^* \rightarrow \mathbb{Q}$  whose restriction to the subalgebra  $\mathbb{C}(C) \otimes_{\mathbb{C}} \mathbb{C}[M] \subset \mathbb{C}(Z)$  satisfies

$$w(\xi)(f \otimes \chi^m) = \ell \operatorname{ord}_y(f) + \langle m, \nu \rangle$$

for all  $f \in \mathbb{C}(C)^*$  and  $m \in M$ . Here we identify the semigroup algebra  $\mathbb{C}[M]$  with  $\mathbb{C}[Bx_0]^U$ , where  $Bx_0$  is the open  $B$ -orbit in  $G/H$ . From this description it follows that the set of  $G$ -invariant discrete valuations of  $\mathbb{C}(Z)$  with values in  $\mathbb{Q}$  is in one-to-one correspondence with the hyperspace  $\mathcal{N}_{\mathbb{Q}}$  (cf. [Tim11, Corollary 19.13, Theorems 20.3 and 21.10]).

Let us now consider the morphism  $X(\mathfrak{D}) \rightarrow G/P_{\mathcal{F}}$  induced by the projection to the first factor in Equation (12), whose fibers are isomorphic to  $Y(\mathfrak{D}, \mathcal{F})$ . The inverse image of the open  $B$ -orbit  $By_0$  of  $G/P_{\mathcal{F}}$  is a  $B$ -chart which we denote by  $X_0$ . This  $B$ -chart intersects all the  $G$ -orbits in  $X(\mathfrak{D})$ . Note that  $By_0$  is isomorphic to the affine space  $\mathbb{A}_{\mathbb{C}}^r$ , where  $r = \dim G/P_{\mathcal{F}}$ . Since the fibration restricted to  $By_0$  is trivial, the chart  $X_0$  is identified with  $Y(\mathfrak{D}, \mathcal{F}) \times \mathbb{A}_{\mathbb{C}}^r$ . The ring of functions on  $X_0$  can be described as

$$(13) \quad \mathbb{C}[X_0] = (\mathbb{C}(C) \otimes_{\mathbb{C}} \mathbb{C}[Bx_0]) \cap \bigcap_{\xi \in C(\mathfrak{D})(1)} \mathcal{O}_{v(\xi)} \cap \bigcap_{D \in \mathcal{F}} \mathcal{O}_{v_D},$$

where  $v_D$  is the valuation associated with the divisor  $D \in \mathcal{F}$  and  $\mathcal{O}_{v_D}, \mathcal{O}_{v(\xi)}$  are the valuation rings respectively of  $v_D$  and  $v(\xi)$ .

*Remark 3.11.* (1) Equation (13) is a consequence of the correspondence between  $B$ -charts and colored polyhedral divisors from [Tim11, Section 16].

(2) The coordinate ring of the open orbit  $Bx_0$  in  $G/H$  can be interpreted as

$$\mathbb{C}[Bx_0] = \mathbb{C}[M] \otimes_{\mathbb{C}} \mathbb{C}[t_1, \dots, t_s],$$

where  $s = \dim G/P_I$  and  $P_I = N_G(H)$ . In addition

$$\mathbb{C}[X_0] = A(C_0, \mathfrak{D}, \mathcal{F}) \otimes_{\mathbb{C}} \mathbb{C}[t_1, \dots, t_r].$$

Here  $r \leq s$ .

The variety  $X(\mathfrak{D})$  is a reunion of  $G$ -orbits

$$(14) \quad X(\mathfrak{D}) = G \cdot X_0 = \{g \cdot x \mid g \in G, x \in X_0\} \subset \mathfrak{S}_G(Z).$$

Thus Equations (13) and (14) yield another more conceptual construction of the  $G$ -variety  $X(\mathfrak{D})$  (cf. [Tim11, Section 16]). From now on we will identify  $X(\mathfrak{D})$  with an open subset of  $\mathfrak{S}_G(Z)$ .

**3.3. Colored divisorial fans.** Here we introduce the combinatorial objects describing the horospherical  $G$ -varieties of complexity one. The idea is to consider ‘fans of colored polyhedral divisors’ (called later colored divisorial fans) in order to describe these varieties as a gluing of simple  $G$ -models.

**Definition 3.12** ([LT16, §1.3.4]). Let  $C$  be a smooth projective curve and  $G/H$  be a horospherical homogeneous space. A *colored divisorial fan* on the pair  $(C, G/H)$  is a finite set  $\mathcal{E} = \{(\mathfrak{D}^i, \mathcal{F}^i) \mid i \in J\}$  of colored polyhedral divisors, where

$$\mathfrak{D}^i = \sum_{y \in C^i} \Delta_y \cdot [y],$$

$C^i \subset C$  is an open dense subset, and  $\mathcal{F}^i \subset \mathcal{F}_{G/H}$ , are subject to the following conditions.

(i) For all  $i, j \in J$ , we have  $(\mathfrak{D}^i \cap \mathfrak{D}^j, \mathcal{F}^i \cap \mathcal{F}^j) \in \mathcal{E}$ , where the intersection is defined as

$$\mathfrak{D}^i \cap \mathfrak{D}^j := \sum_{y \in C_{i,j}} (\Delta_y^i \cap \Delta_y^j) \cdot [y],$$

and the curve  $C_{i,j}$  by the equality

$$C_{i,j} := \{y \in C^i \cap C^j \mid \Delta_y^i \cap \Delta_y^j \neq \emptyset\}.$$

(ii) For all  $i, j \in J$ ,  $y \in C_{i,j}$ , the polyhedron  $\Delta_y^i \cap \Delta_y^j$  is a common face to  $\Delta_y^i$  and  $\Delta_y^j$ .

(iii) For all  $i, j \in J$  we have

$$\mathcal{F}^i \cap \mathcal{F}^j = \varrho^{-1}(\sigma_i \cap \sigma_j) \cap \mathcal{F}^i = \varrho^{-1}(\sigma_i \cap \sigma_j) \cap \mathcal{F}^j,$$

where  $\sigma_i, \sigma_j$  are the respective tails of  $\mathfrak{D}^i, \mathfrak{D}^j$ .

(iv) The intersection of the degree of  $\mathfrak{D}^i$  with the tail  $\sigma_{ij}$  of  $\mathfrak{D}^i \cap \mathfrak{D}^j$  is equal to the intersection of the degree of  $\mathfrak{D}^j$  with  $\sigma_{ij}$ .

To any colored divisorial fan  $\mathcal{E}$  one can attach some combinatorial objects as follows.

**Definition 3.13.** The set

$$H(\mathcal{E}) := \{(C(\mathfrak{D}), \mathcal{F}) \mid (\mathfrak{D}, \mathcal{F}) \in \mathcal{E}\}$$

is the associated *colored hyperfan* (see [Tim11, Definition 16.18]). The *support* of  $\mathcal{E}$  is defined as

$$|\mathcal{E}| := \bigcup_{(\mathfrak{D}, \mathcal{F}) \in \mathcal{E}} C(\mathfrak{D}) \subset \mathcal{N}_{\mathbb{Q}}$$

and the *set of special points* as

$$\mathrm{Sp}(\mathcal{E}) := \bigcup_{(\mathfrak{D}, \mathcal{F}) \in \mathcal{E}} \mathrm{Sp}(\mathfrak{D}).$$

See 3.1 for the definition of  $\mathrm{Sp}(\mathfrak{D})$ . We denote by  $\Sigma(\mathcal{E})$  the *tail fan* of  $\mathcal{E}$ , which is the fan generated by all the tails  $\sigma$  of polyhedral divisors  $\mathfrak{D}$  with  $(\mathfrak{D}, \mathcal{F}) \in \mathcal{E}$ .

When  $G = T$  is a torus, we write  $\mathcal{E} = \{\mathfrak{D}^i \mid i \in J\}$  instead of  $\mathcal{E} = \{(\mathfrak{D}^i, \emptyset) \mid i \in J\}$ , and as in [AHS08, Definition 5.2] we say that  $\mathcal{E}$  is a *divisorial fan*. The following result provides a description of horospherical varieties of complexity one in terms of colored divisorial fans.

**Theorem 3.14** ([Tim11, Theorems 12.13 and 16.19]). *Let  $C$  be a smooth projective curve,  $G/H$  be a horospherical homogeneous space, and  $Z$  be the product  $C \times G/H$ . Denote by  $\mathcal{E}$  a colored divisorial fan on  $(C, G/H)$ . Then the open reunion*

$$X(\mathcal{E}) := \bigcup_{(\mathfrak{D}, \mathcal{F}) \in \mathcal{E}} X(\mathfrak{D}, \mathcal{F})$$

*inside the equivariant scheme of geometric localities  $\mathfrak{S}_G(Z)$  is a  $G$ -model of  $Z$ . Conversely, every  $G$ -model of  $Z$  arises in this way. Moreover  $X(\mathcal{E})$  is a complete variety if and only if  $|\mathcal{E}| = \mathcal{N}_{\mathbb{Q}}$ .*

*Remark 3.15.* The uniqueness of the description of the above theorem can be characterized. Two colored divisorial fans define the same  $G$ -model in  $\mathfrak{S}_G(Z)$  if and only if they have the same colored data (we refer to [Tim11, Section 16.4] for more details).

Let us now recall the construction of the discoloration morphism associated with a colored divisorial fan (cf. [LT16, Section 2.2]). This construction was inspired from the case of spherical varieties in [Bri91, Section 3.3]. It will allow us to provide specific equivariant desingularizations of  $X(\mathcal{E})$  in a combinatorial way. If  $\mathcal{E}$  is a colored divisorial fan, we denote by

$$\mathcal{E}_{dis} := \{(\mathfrak{D}, \emptyset) \mid (\mathfrak{D}, \mathcal{F}) \in \mathcal{E}\}$$

its *discoloration*. The set  $\mathcal{E}_{dis}$  is a colored divisorial fan with trivial coloration. For any  $(\mathfrak{D}, \mathcal{F})$  in  $\mathcal{E}$  (respectively  $(\mathfrak{D}, \emptyset)$  in  $\mathcal{E}_{dis}$ ) we consider  $X_0$  (respectively  $X_{dis}$ ) the  $B$ -chart associated with  $(\mathfrak{D}, \mathcal{F})$  (respectively with  $(\mathfrak{D}, \emptyset)$ ). The inclusions of  $\mathbb{C}$ -algebras  $\mathbb{C}[X_0] \subset \mathbb{C}[X_{dis}]$  induce morphisms  $X(\mathfrak{D}, \emptyset) \rightarrow X(\mathfrak{D}, \mathcal{F})$  which glue into a birational proper  $G$ -equivariant morphism

$$(15) \quad \pi_{dis} : X(\mathcal{E}_{dis}) \rightarrow X(\mathcal{E}),$$

see [LT16, Proposition 2.9].

Writing  $P = P_I = N_G(H)$ ,  $T = P/H$  and  $M = \mathbb{X}(T)$ , we define the  $T$ -variety associated with  $\mathcal{E}$  by

$$(16) \quad V(\mathcal{E}) := \bigcup_{i \in J} \text{Spec } A(C^i, \mathfrak{D}^i) \subset \mathfrak{S}_T(C \times T),$$

where  $\mathfrak{S}_T(C \times T)$  is the  $T$ -equivariant scheme of geometric localities of  $C \times T$ , and  $A(C^i, \mathfrak{D}^i)$  denotes the  $M$ -graded algebra associated with  $\mathfrak{D}^i$  (see Definition 3.2). From [LT16, Proposition 2.9] it follows that  $X(\mathcal{E}_{dis})$  is  $G$ -isomorphic to  $G \times^P V(\mathcal{E})$ , where  $P$  acts on  $V(\mathcal{E})$  via the canonical surjection  $P \rightarrow T$ .

#### 4. THE ARC SPACE OF A HOROSPHERICAL VARIETY OF COMPLEXITY ONE

The purpose of this section is to obtain a precise description of the arc space of a horospherical variety  $X$  of complexity one in order to compute their stringy  $E$ -functions via motivic integration in Section 5.

From Section 2.1 we know that an arc  $\alpha$  in  $\mathcal{L}(X)$  uniquely defines a  $K$ -valued point  $\tilde{\alpha}$  in  $\mathcal{L}(X)$  for some field extension  $K$  of  $\mathbb{C}$  depending on  $\alpha$ . Let  $\mathcal{O}_K$  be the ring of power series in one variable with coefficients in  $K$ , and  $\mathcal{K}_K$  be the fraction field of  $\mathcal{O}_K$ . We also denote by  $0$  and  $\eta$  the closed and generic point of  $\text{Spec } \mathcal{O}_K$ , respectively. In this section, we will describe the sets of  $K$ -valued points of  $\mathcal{L}(X)$  – that is, the sets  $X(\mathcal{O}_K)$  – for all field extensions  $K$  of  $\mathbb{C}$ . So from now, we fix an extension  $K$  of  $\mathbb{C}$  and simply denote  $\mathcal{O}$  and  $\mathcal{K}$  for  $\mathcal{O}_K$  and  $\mathcal{K}_K$ .

Our strategy is to restrict ourselves to the study of a subset  $\mathcal{L}_\Gamma(X)$  of  $\mathcal{L}(X)$  which has the same motivic measure as  $\mathcal{L}(X)$ , and which we decompose into *horizontal* and *vertical* arcs, see 4.1 for the case of affine  $T$ -varieties of complexity one. We also detail the two subsets further by looking at the  $T(\mathcal{O})$ -orbits in  $X_\Gamma := \mathcal{L}_\Gamma(X)(K)$ . We extend this decomposition to general  $T$ -varieties of complexity one in 4.2. In 4.3 we generalize our results to all horospherical  $G$ -varieties of complexity one. We give a parametrization of the  $G(\mathcal{O})$ -orbits in Theorem 4.14. Finally, we show in 4.4 that these  $\mathcal{L}(G)$ -orbits in  $\mathcal{L}_\Gamma(X)$  are cylinders, hence measurable, and we compute their motivic measure in Theorem 4.19.

**Notation 4.1.** Throughout this section we let  $\mathcal{E}$  be a colored divisorial fan on  $(C, G/H)$ , where  $C$  is a smooth projective curve and  $G/H$  is a horospherical homogeneous space. We denote by  $P$  the parabolic subgroup  $N_G(H) \subset G$ , by  $T$  the torus  $P/H$ , by  $M := \mathbb{X}(T)$  its character lattice, and by  $N := \text{Hom}(M, \mathbb{Z})$  the dual of  $M$ . We write  $M_{\mathbb{Q}}$  and  $N_{\mathbb{Q}}$  for the associated  $\mathbb{Q}$ -vector spaces. The symbol  $\mathcal{F}_{G/H}$  stands for the set of colors of  $G/H$ .

We denote by  $(\mathfrak{D}^i, \mathcal{F}^i)_{i \in J}$  the colored polyhedral divisors in  $\mathcal{E}$ , and for any  $i \in J$ , we write

$$\mathfrak{D}^i = \sum_{y \in C^i} \Delta_y^i \cdot [y],$$

where  $C^i \subset C$  is open dense, and we let  $\sigma_i$  be the tail of  $\mathfrak{D}^i$ . We let  $\Gamma$  be an open dense affine subset of  $C$  which does not contain the subset  $\text{Sp}(\mathcal{E})$  of special points of  $\mathcal{E}$ .

Finally, we let  $X := X(\mathcal{E})$  be the  $G$ -variety associated with  $\mathcal{E}$ .

**4.1. Horizontal and vertical arcs.** We place ourselves in the situation of Notation 4.1, and we also assume that  $G = T$  is a torus,  $H = \{e\}$  is trivial, and the colored divisorial fan  $\mathcal{E}$  consists in a single colored divisor  $\mathfrak{D}$  with *affine* locus  $C_0$  containing  $\Gamma$  and tail  $\sigma$ . Following the construction after theorem 3.6,  $X$  is the affine  $T$ -variety  $\text{Spec } A(C_0, \mathfrak{D})$ . Note that the curve  $C_0$  is identified with the good quotient  $X // T$  since  $\mathbb{C}[C_0]$  is the subalgebra of  $T$ -invariant of  $A(C_0, \mathfrak{D})$ . We will describe a decomposition of the arc space of  $X$ .

We denote by  $X_\sigma$  the toric variety  $\text{Spec } \mathbb{C}[\sigma^\vee \cap M]$  associated with the cone  $\sigma$ . The subset  $\Gamma \times X_\sigma$  is a dense open  $T$ -stable subset of  $X$ , and it contains another dense open  $T$ -stable subset of  $X$ , namely  $\Gamma \times T$ . Define

$$X_\Gamma = X_\Gamma^K := X(\mathcal{O}) \cap (\Gamma \times T)(\mathcal{K}) := \{\alpha : \text{Spec } \mathcal{O}_K \rightarrow X \mid \alpha(\eta) \in \Gamma \times T\} \subset X(\mathcal{O}),$$

where  $X(\mathcal{O})$  and  $(\Gamma \times T)(\mathcal{K})$  are both viewed as subset of  $X(\mathcal{K})$ . Here the image of  $\eta$  is a schematic point of  $\Gamma \times T$ . Instead of studying the whole arc space  $\mathcal{L}(X)$ , for purposes of motivic integration we may restrict ourselves to studying the following particular subspace.

*Remark 4.2.* The subspace

$$\mathcal{L}_\Gamma(X) := \{\alpha \in \mathcal{L}(X) \mid \tilde{\alpha}(\eta) \in \Gamma \times T\}$$

satisfies the equality  $\mu_X(\mathcal{L}_\Gamma(X)) = \mu_X(\mathcal{L}(X))$ . Indeed, its complement in  $\mathcal{L}(X)$  corresponds to the arc space of  $X_1 := X \setminus (\Gamma \times T)$  via the closed immersion  $X_1 \hookrightarrow X$ , which has zero measure by Remark 2.12. Hence for our purposes we may study  $\mathcal{L}_\Gamma(X)$  instead of  $\mathcal{L}(X)$ . Note that  $\mathcal{L}_\Gamma(X)(K)$  identifies with  $X_\Gamma^K$ .

Any  $K$ -valued arc  $\alpha$  in  $\mathcal{L}_\Gamma(X)(K)$  induces morphisms  $\mathbb{C}[M] \rightarrow \mathcal{K}$  and  $\mathbb{C}[\Gamma] \rightarrow \mathcal{K}$ . Indeed, as  $\Gamma \times T$  is an open subset of  $X$  we have a commutative diagram

$$\begin{array}{ccc} \mathbb{C}[\Gamma] \otimes_{\mathbb{C}} \mathbb{C}[M] & \xrightarrow{\bar{\alpha}^*} & \mathcal{K} \\ \uparrow & & \uparrow \\ \mathbb{C}[X] & \xrightarrow{\alpha^*} & \mathcal{O}. \end{array}$$

We denote by  $\alpha_\Gamma^*$  (resp. by  $\alpha_M^*$ ) the restriction of  $\bar{\alpha}^*$  to  $\mathbb{C}[\Gamma]$  (resp. to  $\mathbb{C}[M]$ ). Clearly  $\alpha$  is uniquely determined by the data of  $\alpha_\Gamma^*$  and  $\alpha_M^*$ . Conversely, if we are given two  $\mathbb{C}$ -algebra morphisms  $\beta_\Gamma : \mathbb{C}[\Gamma] \rightarrow \mathcal{K}$  and  $\beta_M : \mathbb{C}[M] \rightarrow \mathcal{K}$  such that  $\beta_\Gamma \otimes \beta_M(\mathbb{C}[X]) \subset \mathcal{O}$ , then there exists an arc  $\alpha \in X_\Gamma$  such that  $\alpha_\Gamma^* = \beta_\Gamma$  and  $\alpha_M^* = \beta_M$ .

*Remark 4.3.* • We parametrize the map  $\alpha_M^*$  associated with an arc  $\alpha \in X_\Gamma$  in the same way as in [Ish04, Proof of Theorem 4.1]. More precisely, we consider the group homomorphisms  $\nu_\alpha : M \rightarrow \mathbb{Z}$  and  $\omega_\alpha : M \rightarrow \mathcal{O}^*$  defined by the equality

$$\alpha_M^*(\chi^m) = t^{\langle m, \nu_\alpha \rangle} \omega_\alpha(m)$$

for any lattice vector  $m \in M$ .

- We may also parametrize  $\alpha_\Gamma^*$  in a similar fashion:

$$\alpha_\Gamma^*(f) = t^{v_\alpha(f)} R_\alpha(f)$$

where  $v_\alpha : \mathbb{C}[\Gamma] \rightarrow \mathbb{Z}$  is a valuation and  $R_\alpha : \mathbb{C}[\Gamma] \rightarrow \mathcal{O}^*$  is a function. The valuation  $v_\alpha$  being a discrete valuation on an algebraic function field of transcendence degree one, by [Har77, Chap. 1, Corollary 6.6] there exists a unique integer  $\ell_\alpha$  in  $\mathbb{Z}_{\geq 0}$  and a unique point  $y_\alpha$  in  $\bar{\Gamma} = C$  such that

$$v_\alpha = \ell_\alpha \text{ord}_{y_\alpha}.$$

The following lemma is well-known. However, for the sake of completeness we give a short proof here.

**Lemma 4.4.** *With notation 4.1, the following statements are equivalent. Let  $w : \mathbb{C}(X)^* \rightarrow \mathbb{Q}$  be a  $T$ -invariant discrete valuation.*

- (i) *The restriction  $w|_{\mathbb{C}[X] \setminus \{0\}}$  is non-negative.*
- (ii) *There exists  $(y, \nu, \ell) \in C(\mathfrak{D})$  such that for any homogeneous element  $f \otimes \chi^m$  in  $\mathbb{C}[X]$ , we have*

$$w(f \otimes \chi^m) = \ell \text{ord}_y(f) + \langle m, \nu \rangle.$$

*Hence if  $\alpha \in X_\Gamma$  is an arc, then the triple  $(y_\alpha, \nu_\alpha, \ell_\alpha)$  defined above is a point of the hypercone  $C(\mathfrak{D}) \cap \mathcal{N}$ .*

*Remark 4.5.* In the sequel, we will use the notation  $w = \ell \text{ord}_y + \nu$  for the valuation associated with a point  $(y, \nu, \ell)$  of the hypercone  $C(\mathfrak{D}) \cap \mathcal{N}$ .

*Proof.* Write  $\mathfrak{D} := \sum_{z \in C_0} \Delta_z \cdot [z]$ .

(ii)  $\Rightarrow$  (i) If  $\ell = 0$ , then by definition of  $C(\mathfrak{D})$  (see Definition 3.10) we have  $\nu \in \sigma \cap N$ , so that  $w(f \otimes \chi^m) = \langle m, \nu \rangle \geq 0$  for any homogeneous element  $f \otimes \chi^m \in \mathbb{C}[X]$  of degree  $m$ . Thus  $w$  satisfies Condition (i). Now if  $\ell \neq 0$  write

$$\frac{w}{\ell} = \text{ord}_y + \frac{\nu}{\ell}.$$

Since  $(y, \nu, \ell) \in C(\mathfrak{D})$  we have  $\frac{\nu}{\ell} \in \Delta_y$ . By definition of  $\mathbb{C}[X] = A(C_0, \mathfrak{D})$  this again implies that  $w$  is non-negative on any homogeneous element  $f \otimes \chi^m$ , hence Condition (i) is satisfied.

(i)  $\Rightarrow$  (ii) Without loss of generality we may assume that  $w = \text{ord}_y + \nu$  for some  $y \in C$  and  $\nu \in N_{\mathbb{Q}}$ . Assume by contradiction that  $\nu \notin \Delta_y$ . Define a new  $\sigma$ -polyhedral divisor

$$\mathfrak{D}' := \sum_{z \in C_0 \cup \{y\}} \Delta'_z \cdot [z],$$

where

$$\Delta'_z = \begin{cases} \Delta_z & \text{if } z \neq y, \\ \text{Conv}(\{\nu\} \cup \Delta_y(0)) + \sigma & \text{if } z = y, \end{cases}$$

and  $\Delta_y(0)$  denotes the set of vertices of the  $\sigma$ -polyhedron  $\Delta_y$ . If  $y \notin C_0$ , then we set  $\Delta_y = \emptyset$  in the preceding formula. Clearly  $\mathfrak{D}' \neq \mathfrak{D}$ , but  $A(C_0 \cup \{y\}, \mathfrak{D}') = A(C_0, \mathfrak{D})$  since  $w$  satisfies Condition (i). The polyhedral divisor  $\mathfrak{D}$  being proper, it follows that so is  $\mathfrak{D}'$ . Hence by [AH06, Lemma 9.1] we have  $\mathfrak{D} = \mathfrak{D}'$ , which is a contradiction.

Finally the last claim in the lemma is an immediate consequence of the rest of the lemma.  $\square$

Let us describe two disjoint subsets of  $X_\Gamma$ , according to the position of the image of the closed point  $0 \in \text{Spec } \mathcal{O}$ .

**Definition 4.6.** Define the *horizontal part* of  $X_\Gamma$  by

$$\mathcal{X}_{\text{hor}} := \{\alpha : \text{Spec } \mathcal{O} \rightarrow X \mid \alpha(0) \in \Gamma \times X_\sigma \text{ and } \alpha(\eta) \in \Gamma \times T\}$$

and its *vertical part* by  $\mathcal{X}_{\text{ver}} := X_\Gamma \setminus \mathcal{X}_{\text{hor}}$ , that is

$$\mathcal{X}_{\text{ver}} = \{\alpha : \text{Spec } \mathcal{O} \rightarrow X \mid \alpha(0) \notin \Gamma \times X_\sigma \text{ and } \alpha(\eta) \in \Gamma \times T\}.$$

The condition  $\alpha(0) \notin \Gamma \times X_\sigma$  means that the closed point  $\alpha(0)$  belongs to the fiber over a point of  $C_0 \setminus \Gamma$  of the quotient map  $X \rightarrow C_0$ .

We may characterize elements  $\alpha \in \mathcal{X}_{\text{ver}}$  in terms of their associated maps  $\alpha_\Gamma^*$ .

**Lemma 4.7.** *Let  $\alpha \in X_\Gamma$  be an arc. Then*

$$\alpha \in \mathcal{X}_{\text{ver}} \Leftrightarrow \alpha_\Gamma^*(\mathbb{C}[\Gamma]) \not\subset \mathcal{O} \Leftrightarrow y_\alpha \in C_0 \setminus \Gamma \text{ and } \ell_\alpha \neq 0,$$

where  $y_\alpha$  and  $\ell_\alpha$  are as in Remark 4.3. In particular, if one of the above conditions is satisfied, then  $\alpha_\Gamma^*$  is injective.

*Proof.* The equivalence  $\alpha_\Gamma^*(\mathbb{C}[\Gamma]) \not\subset \mathcal{O} \Leftrightarrow y_\alpha \in C_0 \setminus \Gamma, \ell_\alpha \neq 0$  is clear. We now prove the equivalence between  $\alpha \in \mathcal{X}_{\text{ver}}$  and  $(y_\alpha \in C_0 \setminus \Gamma \text{ and } \ell_\alpha \neq 0)$ .

If  $y_\alpha \in \Gamma$ , then  $\nu_\alpha \in \sigma \cap N$  by definition of  $C(\mathfrak{D})$ . Moreover  $\text{ord}_{y_\alpha}$  is non-negative on  $\mathbb{C}[\Gamma]$ , so that the image of  $\mathbb{C}[\Gamma] \otimes_{\mathbb{C}} \mathbb{C}[X_\sigma]$  by  $\alpha_\Gamma^* \otimes \alpha_M^*$  is contained in  $\mathcal{O}$ , hence  $\alpha \in \mathcal{X}_{\text{hor}}$ . So  $\alpha \in \mathcal{X}_{\text{ver}}$  implies  $y_\alpha \in C_0 \setminus \Gamma$ . Conversely if  $y_\alpha \in C_0 \setminus \Gamma$  and  $\ell_\alpha \neq 0$ , then there exists  $f \in \mathbb{C}[\Gamma]$  such that  $\ell_\alpha \text{ord}_{y_\alpha}(f) < 0$ , so that  $\alpha_\Gamma^*(\mathbb{C}[\Gamma]) \not\subset \mathcal{O}$ . In particular  $\alpha_\Gamma^* \otimes \alpha_M^*(\mathbb{C}[\Gamma] \otimes_{\mathbb{C}} \mathbb{C}[X_\sigma]) \not\subset \mathcal{O}$ , hence  $\alpha \in \mathcal{X}_{\text{ver}}$ .

Now assume  $\alpha_\Gamma^*$  is not injective. Then its kernel is the ideal of a  $\mathbb{C}$ -point, so that  $\ell_\alpha = 0$  and hence  $\nu_\alpha \in \sigma$ . This implies that  $\alpha_\Gamma^* \otimes \alpha_M^*(\mathbb{C}[\Gamma] \otimes_{\mathbb{C}} \mathbb{C}[X_\sigma]) \subset \mathcal{O}$ , which means that  $\alpha \in \mathcal{X}_{\text{hor}}$ .  $\square$

Let us now study the  $T(\mathcal{O})$ -orbits in the space  $X_\Gamma$ . Note that both  $X(\mathcal{O})$  and  $X_\Gamma$  are endowed with an action of  $T(\mathcal{O})$  coming from the  $T$ -action on  $X$ . We consider the well-defined map

$$\underline{\text{val}} : X_\Gamma \rightarrow C(\mathfrak{D}) \cap \mathcal{N},$$

which to an arc  $\alpha$  associates the point  $(y_\alpha, \nu_\alpha, \ell_\alpha)$  from Lemma 4.4. We want to study the fibers of  $\underline{\text{val}}$  above points of  $\mathcal{X}_{\text{hor}}$  and  $\mathcal{X}_{\text{ver}}$  respectively, and decompose them into  $T(\mathcal{O})$ -orbits. In the case of  $\mathcal{X}_{\text{hor}}$  the result is an immediate consequence of [Ish04, Theorem 4.1].

**Lemma 4.8.** *We have a decomposition*

$$\mathcal{X}_{\text{hor}} = \Gamma(\mathcal{O}) \times (X_\sigma(\mathcal{O}) \cap T(\mathcal{K})) = \Gamma(\mathcal{O}) \times \bigsqcup_{\nu \in \sigma \cap N} \mathcal{C}_\nu,$$

where  $\alpha \in \mathcal{X}_{\text{hor}}$  is identified with the pair  $(\alpha_\Gamma^*, \alpha_M^*) \in \Gamma(\mathcal{O}) \times (X_\sigma(\mathcal{O}) \cap T(\mathcal{K}))$ , and

$$\mathcal{C}_\nu := \{\alpha_M^* \mid \alpha \in \mathcal{X}_{\text{hor}}, \nu_\alpha = \nu\}$$

is a  $T(\mathcal{O})$ -orbit of  $X_\sigma(\mathcal{O}) \cap T(\mathcal{K})$ .

Moreover

$$\underline{\text{val}}(\mathcal{X}_{\text{hor}}) = \{(C, \nu, 0) \mid \nu \in \sigma \cap N\}$$

and  $\underline{\text{val}}^{-1}(C, \nu, 0) = \Gamma(\mathcal{O}) \times \mathcal{C}_\nu$  for any  $\nu \in \sigma \cap N$ .

We now state a similar result for the subset  $\mathcal{X}_{\text{ver}}$ .

**Lemma 4.9.** *Let  $\alpha$  be an arc in  $\mathcal{X}_{\text{ver}}$  and  $\pi \in \mathbb{C}(C)^*$  be a uniformizer of  $y_\alpha$ . There exists a one-to-one correspondence between*

$$\mathcal{X}_{\text{ver},\alpha} := \underline{\text{val}}^{-1}(\underline{\text{val}}(\alpha)) = \{\beta \in \mathcal{X}_{\text{ver}} \mid y_\beta = y_\alpha, \nu_\beta = \nu_\alpha, l_\beta = l_\alpha\}$$

and the set of pairs  $(\omega, u)$ , where

- $\omega : M \rightarrow \mathcal{O}^*$  is a group homomorphism,
- $u \in \mathcal{O}^*$ .

The correspondence is given by

$$\beta \mapsto (\omega_\beta, R_\beta(\pi)),$$

where  $R_\beta$  is the function from Remark 4.3. It identifies the fiber  $\mathcal{X}_{\text{ver},\alpha}$  with the product  $C_{\nu_\alpha} \times \mathcal{O}^*$ .

Furthermore

$$(17) \quad \underline{\text{val}}(\mathcal{X}_{\text{ver}}) = \{(y, \nu, \ell) \in C(\mathfrak{D}) \cap \mathcal{N} \mid y \in C_0 \setminus \Gamma \text{ and } \ell \geq 1\}.$$

*Proof.* Let us first prove the surjectivity of the correspondence. Fix a pair  $(\omega, u) \in T(\mathcal{O}) \times \mathcal{O}^*$  and write  $y = y_\alpha$ ,  $\nu = \nu_\alpha$ , and  $\ell = \ell_\alpha$ . From [Eis95, Theorem 7.16], we know that there exists a unique morphism  $\phi : \hat{\mathcal{O}}_y = \mathbb{C}[[\pi]] \rightarrow \mathcal{O}$  such that  $\phi(\pi) = t^\ell u$ , where  $\hat{\mathcal{O}}_y$  is the formal completion of  $(\mathcal{O}_y, \pi \mathcal{O}_y)$ . By restriction to  $\mathbb{C}[C_0]$ , since  $\ell \neq 0$ ,  $\phi$  induces an injective homomorphism  $\mathbb{C}[C_0] \rightarrow \mathcal{O}$  (see Lemma 4.7). Extending it to the fraction field we obtain an homomorphism  $\mathbb{C}(C_0) \rightarrow \mathcal{K}$ . Finally, let us denote by  $\lambda$  the restriction of the previous morphism to  $\mathbb{C}[\Gamma]$ . We define an arc  $\beta \in \mathcal{X}_{\text{ver}}$  by setting

$$\bar{\beta}^*(f \otimes \chi^m) = \lambda(f)t^{\langle m, \nu \rangle} \omega(m)$$

for any homogeneous element  $f \otimes \chi^m \in \mathbb{C}[\Gamma] \otimes_{\mathbb{C}} \mathbb{C}[\sigma^\vee \cap M]$ . The arc  $\beta$  is in  $\mathcal{X}_{\text{ver}}$  since  $\ell_\beta = \ell \neq 0$  and  $y_\beta = y \in C_0 \setminus \Gamma$ .

Let us now prove that the correspondence is injective. It is enough to check that for  $\beta \in \mathcal{X}_{\text{ver}}$ , the map  $\lambda : \mathbb{C}[\Gamma] \rightarrow \mathcal{K}$  introduced above is uniquely determined by  $\ell = \ell_\beta$ ,  $y = y_\beta$ , and  $R_\beta(\pi)$ , where  $\pi \in \mathbb{C}(C)^*$  is a uniformizer of  $y$ . Clearly  $\lambda$  induces a continuous morphism

$$\bar{\lambda} : (\mathcal{O}_y, \pi \mathcal{O}_y) \rightarrow (\mathcal{O}, t\mathcal{O}),$$

i.e., a morphism such that  $\bar{\lambda}(\pi \mathcal{O}_y) \subset t\mathcal{O}$ , as well as a morphism  $\tilde{\lambda} : \hat{\mathcal{O}}_y = \mathbb{C}[[\pi]] \rightarrow \mathcal{O}$  by taking formal completions (see [Mat80, §(23.H)]). Using again [Eis95, Theorem 7.16] we see that  $\tilde{\lambda}$  is the unique morphism sending  $\pi$  to  $t^\ell R_\beta(\pi)$ . We conclude by noticing that  $\lambda$  is uniquely determined by  $\tilde{\lambda}$ , which proves the injectivity.

Let us now conclude the proof by checking Equation (17). The direct inclusion of  $\underline{\text{val}}(\mathcal{X}_{\text{ver}})$  has been proved by combining Lemmas 4.4 and 4.7. For the other inclusion, consider  $(y, \nu, \ell) \in C(\mathfrak{D}) \cap \mathcal{N}$  with  $y \in C_0 \setminus \Gamma$  and  $\ell \neq 0$ . Choose a uniformizer  $\pi$  of  $y$  and consider the unique morphism  $\hat{\mathcal{O}}_y = \mathbb{C}[[\pi]] \rightarrow \mathcal{O}$  which sends  $\pi$  to  $t^\ell$ . By the same argument as before it implies the existence of a morphism  $\bar{\beta}^* : A(C_0, \mathfrak{D}) \rightarrow \mathcal{O}$  such that

$$\bar{\beta}^*(f \otimes \chi^m) := t^{\ell \text{ord}_y(f) + \langle m, \nu \rangle},$$

hence the existence of an arc  $\beta \in \mathcal{X}_{\text{ver}}$  such that  $\underline{\text{val}}(\beta) = (y, \nu, \ell)$ . □

*Remark 4.10.* Via the correspondence of Lemma 4.9, the  $T(\mathcal{O})$ -action on the fiber  $\mathcal{X}_{\text{ver},\alpha}$  identifies with the action by left multiplication on the first factor of the product  $T(\mathcal{O}) \times \mathcal{O}^*$ .

**4.2. The arc space of a normal  $T$ -variety of complexity one.** In this section we generalize the results of the previous section to the non-affine case. Our setting is that of Notation 4.1 and we assume furthermore that  $G = T$  is a torus and  $H = \{e\}$  is trivial.

We want to describe the arc space of  $X$  in terms of the arc spaces of the affine charts  $X(\mathfrak{D})$  for polyhedral divisors  $\mathfrak{D} \in \mathcal{E}$ . However, some of these polyhedral divisors may have non-affine loci, see Definition 3.1. So to apply the preceding results, we need to replace  $X$  with a  $T$ -birationally equivalent variety  $\tilde{X}$  associated with a divisorial fan  $\tilde{\mathcal{E}}$  such that all the polyhedral divisors in  $\tilde{\mathcal{E}}$  are defined on open dense *affine* subsets of  $C$ . Furthermore we require the important property that the birational map  $\tilde{X} \dashrightarrow X$  be a proper morphism. We describe this process below and call it *affinization*.

**Notation 4.11.** Write  $\mathcal{E} =: \{\mathfrak{D}^i \mid i \in J\}$ . For any  $\mathfrak{D}^i = \sum_{y \in C^i} \Delta_y^i \cdot [y]$  in  $\mathcal{E}$ , we consider a finite open dense affine covering  $(C_j^i)_{j \in J_i}$  of the curve  $C^i$ . Denote by

$$\mathfrak{D}_j^i := \sum_{y \in C_j^i} \Delta_y^i \cdot [y]$$

the polyhedral divisor obtained as the restriction of  $\mathfrak{D}^i$  to the open affine subset  $C_j^i$ . Denote also by  $\tilde{\mathcal{E}}$  the divisorial fan constituted of all the  $\mathfrak{D}_j^i$ . Then the  $T$ -variety  $\tilde{X} = X(\tilde{\mathcal{E}})$  does not depend on the choices of the affine coverings, and its support  $|\tilde{\mathcal{E}}|$  coincides with that of  $\mathcal{E}$ .

From [AH06, Theorem 3.1] it follows that the inclusions of  $\mathbb{C}$ -algebras  $A(C^i, \mathfrak{D}^i) \hookrightarrow A(C_j^i, \mathfrak{D}_j^i)$  induce a proper  $T$ -equivariant birational morphism

$$(18) \quad q : X(\tilde{\mathcal{E}}) \rightarrow X(\mathcal{E}).$$

As in Section 4.1, we introduce a morphism  $\underline{\text{val}}$  defined on  $X_\Gamma$ . The image of this morphism will be contained in a particular subset of the hyperspace  $\mathcal{N}_\mathbb{Q}$ , namely

$$(19) \quad |\mathcal{E}|_\Gamma := \bigcup_{\mathfrak{D} \in \mathcal{E}} C_\Gamma(\mathfrak{D}),$$

where

$$C_\Gamma(\mathfrak{D}) = C(\mathfrak{D}) \setminus \{(y, \nu, \ell) \mid y \in \Gamma, \nu \in N_\mathbb{Q}, \ell > 0\}.$$

Clearly this subset is the same for the divisorial fan  $\mathcal{E}$  and its affinization  $\tilde{\mathcal{E}}$ , i.e.  $|\mathcal{E}|_\Gamma = |\tilde{\mathcal{E}}|_\Gamma$ .

The next result extends Lemmas 4.8 and 4.9 to the  $T$ -variety  $X$ .

**Proposition 4.12.** *Let  $\mathcal{E}$  be a divisorial fan on  $(C, T)$  and write  $X = X(\mathcal{E})$ . Consider a dense open subset  $\Gamma \subset C$  which does not contain any special point. Denote by  $X_\Gamma$  the sets of  $K$ -valued formal arcs on  $X$  whose generic point is contained in the open subset  $\Gamma \times T$  of  $X$ . There exists a surjective map*

$$(20) \quad \begin{array}{ccc} \underline{\text{val}} : X_\Gamma & \rightarrow & |\mathcal{E}|_\Gamma \cap \mathcal{N} \\ \alpha & \mapsto & (y_\alpha, \nu_\alpha, \ell_\alpha), \end{array}$$

where  $|\mathcal{E}|_\Gamma$  is as in Equation (19), and  $\mathcal{N}$  as in Definition 3.10. Moreover

$$\underline{\text{val}}^{-1}(y_\alpha, \nu_\alpha, \ell_\alpha) = \begin{cases} \Gamma(\mathcal{O}) \times \mathcal{C}_{\nu_\alpha} & \text{if } \ell_\alpha = 0, \\ \mathcal{O}^* \times \mathcal{C}_{\nu_\alpha} & \text{if } \ell_\alpha \geq 1. \end{cases}$$

*Proof.* Recall the proper birational morphism  $q : \tilde{X} \rightarrow X$  from Equation (18). The first step of the proof is to check that we may identify the space  $X_\Gamma$  with

$$\tilde{X}_\Gamma := \{\alpha \in \tilde{X}(\mathcal{O}) \mid \alpha(\eta) \in \Gamma \times T\}.$$

We know that  $q|_{\Gamma \times T}$  is the identity map. Write  $X_1 := X \setminus (\Gamma \times T)$ . The morphism  $q$  induces a natural bijection

$$\mathcal{L}(\tilde{X}) \setminus \mathcal{L}(q^{-1}(X_1)) \xrightarrow{\sim} \mathcal{L}(X) \setminus \mathcal{L}(X_1).$$

Indeed, by the valuative criterion of properness (see [Har77, Chapter II, Theorem 4.7]), we have a commutative diagram

$$\begin{array}{ccc} \text{Spec } \mathcal{O} & \xrightarrow{\bar{\alpha}} & X \\ \uparrow & & \uparrow q \\ \text{Spec } \mathcal{K} & \xrightarrow{\tau} & \tilde{X} \end{array}$$

where the map  $\tau : \text{Spec } \mathcal{K} \rightarrow \tilde{X}$  is a lift of  $\bar{\alpha}$ . By a similar argument as in Remark 4.2, the arc  $\alpha$  is in  $\mathcal{L}(X) \setminus \mathcal{L}(X_1)$  if and only if the map  $\bar{\alpha}$  factorizes through  $X \setminus X_1$ . Finally,  $\tau$  is unique. Indeed  $q|_{\Gamma \times T}$  is the identity, so  $\tau$  factorizes through  $\Gamma \times T \subset \tilde{X}$  via  $\alpha : \text{Spec } \mathcal{O} \rightarrow \Gamma \times T$ . Since  $\tilde{X} \setminus q^{-1}(X_1) = \Gamma \times T$  (cf. [AH06, Theorem 10.1]), we have  $\tilde{X}_\Gamma = \mathcal{L}(\tilde{X}) \setminus \mathcal{L}(q^{-1}(X_1))$ , so that we can indeed identify  $\tilde{X}_\Gamma$  and  $X_\Gamma$ . Hence we will assume in the rest of the proof that  $\mathcal{E} = \tilde{\mathcal{E}}$ .

Let us now prove that  $\underline{\text{val}}$  is well-defined and surjective. Clearly the arc space has the following covering

$$X_\Gamma = \bigcup_{\mathfrak{D} \in \mathcal{E}} X(\mathfrak{D})_\Gamma,$$

where  $X(\mathfrak{D})_\Gamma := \{\alpha \in X_\Gamma \mid \alpha(0) \in X(\mathfrak{D})\}$ . After fixing a uniformizer for each point of  $C \setminus \Gamma$ , the maps

$$\underline{\text{val}}_{\mathfrak{D}} : X(\mathfrak{D})_\Gamma \rightarrow C_\Gamma(\mathfrak{D}) \cap \mathcal{N}$$

are well-defined and surjective by a combination of Lemmas 4.8 and 4.9. Moreover they are clearly compatible and glue into  $\underline{\text{val}}$ .

Finally the description of the fibers of  $\underline{\text{val}}$  is a consequence of Lemma 4.8 when  $\ell_\alpha = 0$  and of Lemma 4.9 when  $\ell_\alpha \geq 1$ .  $\square$

This concludes our decomposition into  $T(\mathcal{O})$ -orbits of the arc space of a  $T$ -variety of complexity one.

**4.3. The general case.** Let us now treat the case of a horospherical  $G$ -variety  $X$  of complexity one. We start by recalling the complexity zero case.

Let  $H$  be a closed subgroup of  $G$  containing the unipotent radical of  $B$ . We write  $P = N_G(H)$  and we denote by  $I \subset \Phi$  the subset of the set of simple roots parameterizing  $P$ . We also consider  $T := P/H$ ,  $M := \mathbb{X}(T)$  the character group of the torus  $T$ , and  $N := \text{Hom}(M, \mathbb{Z})$  its dual. A horospherical embedding  $Y$  of the homogeneous space  $G/H$  is parametrized by a colored fan  $\Sigma$  contained in the vector space  $N_{\mathbb{Q}}$  (cf. [Kno91, Theorem 3.3]). Let  $|\Sigma|$  be the support of  $\Sigma$ , that is, the reunion  $|\Sigma| := \bigcup_{(\sigma, \mathcal{F}) \in \Sigma} \sigma$ , where  $\sigma$  is a strongly convex cone and  $\mathcal{F} \subset \mathcal{F}_{G/H}$  is a subset of colors. The following theorem describes the arc space of  $Y$  in terms of  $|\Sigma|$ . We refer to [LV83, Section 4], [GN10, §8.2], and [BM13, Section 3] for more information.

**Theorem 4.13** ([BM13, Theorem 3.1]). *Let  $Y$  be a horospherical  $G/H$ -embedding defined by a colored fan  $\Sigma$ . Then there exists a surjective map*

$$\mathcal{V} : Y(\mathcal{O}) \cap (G/H)(\mathcal{K}) \rightarrow |\Sigma| \cap N$$

whose fiber over  $\nu \in |\Sigma| \cap N$  is a  $G(\mathcal{O})$ -orbit  $\Omega_\nu$ . In particular we obtain a one-to-one correspondence between the lattice points in  $|\Sigma| \cap N$  and the  $G(\mathcal{O})$ -orbits in  $Y(\mathcal{O}) \cap (G/H)(\mathcal{K})$ .

Let us sketch the construction of the map  $\mathcal{V}$ . After discoloring, using the valuative criterion of properness we may assume that  $Y = G \times^P Y_\Sigma$ , where  $Y_\Sigma$  denotes the  $T$ -variety corresponding to the (uncolored) fan  $\Sigma$ . Since  $T = P/H \subset G/H$ , the closure  $Y_\Sigma = \bar{T}$  embeds into  $Y$ . We have a commutative diagram:

$$(21) \quad \begin{array}{ccc} Y_\Sigma(\mathcal{O}) \cap T(\mathcal{K}) & \xrightarrow{\Psi} & |\Sigma| \cap N \\ \downarrow & \nearrow \mathcal{V} & \\ Y(\mathcal{O}) \cap (G/H)(\mathcal{K}) & & \end{array}$$

where the map  $\Psi$  comes from [Ish04, Corollary 4.3]. Let us denote by  $\mathcal{C}_\nu$  the fiber over  $\nu \in |\Sigma| \cap N$  of  $\Psi$ . The map  $\mathcal{V}$  extends  $\Psi$  in such a way that  $\Omega_\nu$  is the unique  $G(\mathcal{O})$ -orbit containing  $\mathcal{C}_\nu$ .

The following theorem, which deals with the complexity one case, is the main result of this section. We use the setting of Notation 4.1.

**Theorem 4.14.** *Let  $C$  be a smooth projective curve and  $G/H$  be a horospherical  $G$ -homogeneous space. Write  $X := X(\mathcal{E})$ , where  $\mathcal{E}$  be a colored divisorial fan on  $(C, G/H)$ . Consider a dense open affine subset  $\Gamma$  of  $C$  such that  $\Gamma \cap \text{Sp}(\mathcal{E}) = \emptyset$ . Denote by  $X_\Gamma$  the set of  $K$ -valued formal arcs on  $X$  whose generic point is contained in the open subset  $\Gamma \times G/H$  of  $X$ . There exists a surjective map*

$$\underline{\text{val}} : \begin{array}{ccc} X_\Gamma & \rightarrow & |\mathcal{E}|_\Gamma \cap \mathcal{N} \\ \alpha & \mapsto & (y_\alpha, \nu_\alpha, \ell_\alpha) \end{array},$$



where

$$|\mathcal{E}|_\Gamma := \bigcup_{(\mathfrak{D}, \mathcal{F}) \in \mathcal{E}} C_\Gamma(\mathfrak{D}) \subset \mathcal{N}_\mathbb{Q}$$

and  $\mathcal{N}, \mathcal{N}_\mathbb{Q}$  are as in Definition 3.10. Moreover it satisfies

$$\underline{\text{val}}^{-1}(y_\alpha, \nu_\alpha, \ell_\alpha) = \begin{cases} \Gamma(\mathcal{O}) \times \Omega_{\nu_\alpha} & \text{if } \ell_\alpha = 0, \\ \mathcal{O}^* \times \Omega_{\nu_\alpha} & \text{if } \ell_\alpha \geq 1, \end{cases}$$

where the  $\Omega_{\nu_\alpha}$  are as in Theorem 4.13.

*Proof.* Our first step is to modify  $X$  via proper birational transformations. We start by discoloring  $X$  using the proper birational map  $\pi_{dis} : X_{dis} \rightarrow X$  from Equation (15). We have a decomposition  $X_{dis} = G \times^P V(\mathcal{E})$ , where  $V(\mathcal{E})$  is defined in Equation (16) and  $P = N_G(H)$ . Then we replace the colored divisorial fan  $\mathcal{E}$  with another colored divisorial fan  $\tilde{\mathcal{E}}$  using the map  $q$  introduced in Equation (18). Thus we finally obtain a proper birational morphism

$$\Theta : G \times^P V(\tilde{\mathcal{E}}) \rightarrow X,$$

which restricts to the identity on  $\Gamma \times G/H$ . By the valuative criterion of properness, it is equivalent to consider arcs on  $X$  or on  $G \times^P V(\tilde{\mathcal{E}})$ , so from now on we assume  $X = G \times^P V(\tilde{\mathcal{E}})$ .

Next we compare  $X_\Gamma$  with

$$V(\tilde{\mathcal{E}})_\Gamma = \{\alpha : \text{Spec } \mathcal{O} \rightarrow V(\tilde{\mathcal{E}}) \mid \alpha(\eta) \in \Gamma \times T\} \subset \mathcal{L}(V(\tilde{\mathcal{E}}))(K).$$

By construction of  $X = G \times^P V(\tilde{\mathcal{E}})$ , the closure of  $\Gamma \times T$  embeds in  $X$  and identifies with  $V(\tilde{\mathcal{E}})$ . The variety  $V(\tilde{\mathcal{E}})$  will play the same role as  $Y_\Sigma$  in Equation (21). We have a map

$$\Psi : V(\tilde{\mathcal{E}})_\Gamma \rightarrow |\tilde{\mathcal{E}}|_\Gamma \cap \mathcal{N},$$

which is the map  $\underline{\text{val}}$  from Proposition 4.12.

From the proof of [BM13, Theorem 3.1] we obtain a map

$$\mathcal{V} : (G/H)(\mathcal{K}) \rightarrow N,$$

which is constant on the  $G(\mathcal{O})$ -orbits and whose restriction to  $T(\mathcal{K}) \subset (G/H)(\mathcal{K})$  is constructed from the standard valuation map  $\text{ord} : \mathcal{K}^* \rightarrow \mathbb{Z}$  as in [BM13, Lemma 3.2]. We use it to build part of a map from  $(\Gamma \times G/H)(\mathcal{K})$  to  $\mathcal{N}$ , which we will then restrict to  $X_\Gamma$ .

To get the other part of the map, as in Remark 4.3 we think of an element  $\alpha \in \Gamma(\mathcal{K})$  as associated with a pair  $(y_\alpha, \ell_\alpha)$ , where  $\alpha^*(f) = t^{\ell_\alpha \text{ ord}_{\nu_\alpha}(f)} R_\alpha(f)$  for any  $f \in \mathbb{C}[\Gamma]$ . Finally we obtain

$$\begin{array}{ccc} \underline{\text{val}} : & \Gamma(\mathcal{K}) \times (G/H)(\mathcal{K}) & \rightarrow \mathcal{N} \\ & (\alpha, \beta) & \mapsto (y_\alpha, \nu_\beta, \ell_\alpha), \end{array}$$

where  $\nu_\beta = \mathcal{V}(\beta)$ . By construction this map restricted to  $V(\tilde{\mathcal{E}})_\Gamma$  is simply  $\Psi$ , and it is constant on the  $G(\mathcal{O})$ -orbits.

Let us now study the restriction of  $\underline{\text{val}}$  to  $X_\Gamma$ , which we denote again by  $\underline{\text{val}}$ . It is a well-defined map  $\underline{\text{val}} : X_\Gamma \rightarrow \mathcal{N}$ . Consider the quotient map  $\varphi : X \rightarrow G/P$  and the corresponding map  $X(\mathcal{O}) \rightarrow (G/P)(\mathcal{O})$  on the set of  $\mathcal{O}$ -valued points. We denote by

$$\varphi_\infty : X_\Gamma \rightarrow (G/P)(\mathcal{O})$$

the restriction to  $X_\Gamma$ . It extends to the natural map

$$\bar{\varphi}_\infty : \Gamma(\mathcal{K}) \times (G/H)(\mathcal{K}) \rightarrow (G/P)(\mathcal{K}).$$

Since  $(G/P)(\mathcal{K}) = (G/P)(\mathcal{O})$  by the valuative criterion of properness, the image of  $\bar{\varphi}_\infty$  is in fact contained in  $(G/P)(\mathcal{O})$ , which is equal to  $G(\mathcal{O})/P(\mathcal{O})$  by the local triviality of the quotient map  $G \rightarrow G/P$ . Moreover the map  $\varphi_\infty$  is  $G(\mathcal{O})$ -equivariant. If  $\alpha$  is a  $K$ -valued arc in  $X_\Gamma$ , then by transitivity and  $G(\mathcal{O})$ -equivariance there exists  $g \in G(\mathcal{O})$  such that  $\varphi_\infty(g \cdot \alpha) = p_0$ , where  $p_0 = P(\mathcal{O}) \in (G/P)(\mathcal{O})$ . Hence

$$g \cdot \alpha \in \varphi_\infty^{-1}(p_0) \subset \bar{\varphi}_\infty^{-1}(p_0) = (\Gamma \times T)(\mathcal{K}).$$

It follows that  $g \cdot \alpha \in (\Gamma \times T)(\mathcal{K}) \cap X_\Gamma$ , and

$$(\Gamma \times T)(\mathcal{K}) \cap X_\Gamma \subset V(\tilde{\mathcal{E}})_\Gamma.$$

Indeed if  $\beta$  belongs to the set on the right-hand side, then

$$\beta(0) \in \overline{\beta(\eta)} \subset \overline{\Gamma \times T} = V(\tilde{\mathcal{E}}),$$

thus  $\beta$  is an element of  $V(\tilde{\mathcal{E}})_\Gamma$ . As a result  $g \cdot \alpha \in V(\tilde{\mathcal{E}})_\Gamma$ , and  $\underline{\text{val}}(\alpha) = \underline{\text{val}}(g \cdot \alpha) = \Psi(g \cdot \alpha)$ . This implies that  $\underline{\text{val}}(X_\Gamma) = \Psi(V(\tilde{\mathcal{E}})_\Gamma) = |\tilde{\mathcal{E}}|_\Gamma \cap \mathcal{N}$ .

To conclude the proof it now remains to compute the fibers of  $\underline{\text{val}} : X_\Gamma \rightarrow |\tilde{\mathcal{E}}|_\Gamma \cap \mathcal{N}$ . Consider  $(y, \nu, \ell) \in |\tilde{\mathcal{E}}|_\Gamma \cap \mathcal{N}$ . For any  $\alpha \in \underline{\text{val}}^{-1}(y, \nu, \ell)$ , by a previous argument there exists  $g \in G(\mathcal{O})$  such that  $g \cdot \alpha \in V(\tilde{\mathcal{E}})_\Gamma \cap \underline{\text{val}}^{-1}(y, \nu, \ell)$ . Finally, by Proposition 4.12, we know that

$$V(\tilde{\mathcal{E}})_\Gamma \cap \underline{\text{val}}^{-1}(y, \nu, \ell) = \begin{cases} \Gamma(\mathcal{O}) \times \mathcal{C}_\nu & \text{if } \ell = 0, \\ \mathcal{O}^\times \times \mathcal{C}_\nu & \text{if } \ell \geq 1, \end{cases}$$

which concludes the proof and our decomposition into  $G(\mathcal{O})$ -orbits of the arc space of a horospherical  $G$ -variety of complexity one.  $\square$

**4.4. Motivic volumes.** We use the setting of Notation 4.1. Assuming that  $X$  is smooth, we will compute the motivic measure of the fibers of  $\underline{\text{val}}$ .

We start by studying the truncations of arcs in the set  $X_\Gamma$  of  $K$ -valued formal arcs on  $X$  whose generic point is contained in the open subset  $\Gamma \times G/H$  of  $X$ . Using the discoloration morphism, we may assume that  $\mathcal{E}$  has trivial coloration, so that  $X = G \times^P V(\mathcal{E})$ . Recall that there is a surjective quotient map  $\varphi : X \rightarrow G/P$ . The next result, obtained from Theorem 4.14, gives a comparison between the jet spaces of  $X$  and those of the flag variety  $G/P$ . For the truncations of arcs we use the same notation as in Equation (8).

**Lemma 4.15.** *Consider  $\xi := (y, \nu, \ell) \in |\mathcal{E}|_\Gamma \cap \mathcal{N}$  and  $q \in \mathbb{N}$ , where  $|\mathcal{E}|_\Gamma$  is as in Equation (19). Then the restriction to  $\pi_q(\underline{\text{val}}^{-1}(\xi))$  of the bundle of  $q$ -jets*

$$\varphi_q : \mathcal{L}_q(X) \rightarrow \mathcal{L}_q(G/P)$$

*is a bundle, and its fiber is isomorphic to  $\pi'_q(V(\mathcal{E})_\Gamma \cap \underline{\text{val}}^{-1}(\xi))$ , where  $\pi'_q : \mathcal{L}(V(\mathcal{E})) \rightarrow \mathcal{L}_q(V(\mathcal{E}))$  is the truncation map.*

*Proof.* We have a commutative diagram

$$\begin{array}{ccc} \underline{\text{val}}^{-1}(\xi) & \longrightarrow & \mathcal{L}(G/P) \\ \downarrow & & \downarrow \\ \pi_q(\underline{\text{val}}^{-1}(\xi)) & \longrightarrow & \mathcal{L}_q(G/P) \end{array}$$

where the vertical maps are arc truncations and the horizontal maps are induced by  $\varphi : X \rightarrow G/P$ . The vertical map on the left-hand side is obviously surjective, while the other vertical map is surjective since  $G/P$  is smooth (see [Gre66]). Finally, the top horizontal map is surjective by the proof of Theorem 4.14, hence the last map is also surjective. The last claim in the lemma follows from the description of the fibers of the top map.  $\square$

Using Lemma 4.15 we refine our study of the fibers of  $\underline{\text{val}}$  by expressing them as cylinders, showing in particular that they are indeed measurable sets. Lemma 4.16 deals with the horizontal fibers, while Lemma 4.17 takes care of the vertical fibers. Note that the smoothness of the variety  $X$  will play a crucial role in both lemmas.

**Lemma 4.16.** *Let  $X$  be as in Notation 4.1. Assume that  $X$  is smooth of dimension  $d$  and that  $\mathcal{E} = \tilde{\mathcal{E}}$  has trivial coloration. Consider  $\xi := (y, \nu, 0) \in |\mathcal{E}|_\Gamma \cap \mathcal{N}$ . Let  $n$  be the rank of the lattice  $N$ . Let  $\sigma$  be a cone containing  $\nu$  and  $r$  be its dimension. The linear part  $\sigma^\vee \cap (-\sigma^\vee)$  of  $\sigma^\vee$  is a  $(d-1-r)$ -dimensional vector space, which we denote by  $V_\mathbb{Q}$ . Let  $W_\mathbb{Q}$  be a complement of  $V_\mathbb{Q}$  in  $M_\mathbb{Q}$  and  $u_1, \dots, u_r$  be an integral basis of  $\sigma^\vee \cap W_\mathbb{Q}$ .*

Then for any  $q \geq \max(\{\langle u_j, \nu \rangle \mid 1 \leq j \leq r\})$ , the set  $\underline{\text{val}}^{-1}(\xi) \cap V(\mathcal{E})_\Gamma$  is a cylinder with  $q$ -basis the  $K$ -point set of

$$\mathcal{L}_q(\Gamma) \times (\mathbb{A}^1 \setminus 0)^n \times \mathbb{A}^{qn - \sum_{j=1}^r \langle u_j, \nu \rangle}.$$

*Proof.* By the smoothness criterion of [KKMSD73, Chap. II], any cone  $\sigma \in \Sigma(\mathcal{E})$  is generated by part of a basis of  $N$ . Let  $\alpha$  be an arc in  $\underline{\text{val}}^{-1}(\xi) \cap V(\mathcal{E})_\Gamma$ . We consider the morphisms  $\alpha_\Gamma^* : \mathbb{C}[\Gamma] \rightarrow \mathcal{O}$  and  $\alpha_M^* : \mathbb{C}[\sigma^\vee \cap M] \rightarrow \mathcal{O}$  as in Remark 4.3. The maps  $\alpha_\Gamma^*$  and  $\alpha_M^*$  induce arcs on  $\Gamma$  and  $X_\sigma$ , which we can then truncate to  $q$ -jets  $\alpha'_\Gamma \in \mathcal{L}_q(\Gamma)$  and  $\alpha'_M \in \mathcal{L}_q(X_\sigma)$ . Moreover

$$\{\alpha'_\Gamma \mid \alpha \in \underline{\text{val}}^{-1}(\xi) \cap V(\mathcal{E})_\Gamma\} \simeq \mathcal{L}_q(\Gamma)(K)$$

since  $\Gamma$  is smooth. To conclude we use an adapted version of [BM13, Lemma 3.4] where we do not assume the cone to be full-dimensional. The result of the lemma implies that

$$\{\alpha'_M \mid \alpha \in \underline{\text{val}}^{-1}(\xi) \cap V(\mathcal{E})_\Gamma\} \simeq (\mathbb{A}^1(K) \setminus 0)^n \times \mathbb{A}^{qn - \sum_{j=1}^r \langle u_j, \nu \rangle}(K). \quad \square$$

**Lemma 4.17.** *Let  $X$  be as in Notation 4.1. Assume that  $X$  is smooth of dimension  $d$  and that  $\mathcal{E} = \tilde{\mathcal{E}}$  has trivial coloration. Let  $n$  be the rank of the lattice  $N$ . Consider  $\xi := (y, \nu, \ell) \in |\mathcal{E}|_\Gamma \cap \mathcal{N}$  such that  $\ell \geq 1$ . Let  $(\mathfrak{D}, \mathcal{F}) \in \mathcal{E}$  be such that  $\xi \in C_\Gamma(\mathfrak{D})$  and let  $\sigma$  be the tail of  $\mathfrak{D}$ . Denote by  $r$  the dimension of  $\sigma$  and by  $s \geq r + 1$  the dimension of the Cayley cone  $C_y(\mathfrak{D})$ . The linear part  $C_y(\mathfrak{D})^\vee \cap (-C_y(\mathfrak{D}))^\vee$  of  $C_y(\mathfrak{D})^\vee$  is a  $(d - s)$ -dimensional vector space, which we denote by  $V_\mathbb{Q}$ . Let  $W_\mathbb{Q}$  be a complement of  $V_\mathbb{Q}$  in  $\Lambda_\mathbb{Q} = M_\mathbb{Q} \oplus \mathbb{Q}$  and denote by*

$$(u_1, 0), \dots, (u_r, 0), w_1, \dots, w_{s-r}$$

an integral basis of  $C_y(\mathfrak{D})^\vee \cap W_\mathbb{Q}$ , where the  $u_i$  are in  $\sigma^\vee \cap M$ .

Then for any  $q$  greater than

$$\max(\{\langle u_j, \nu \rangle \mid 1 \leq j \leq r\} \cup \{\langle w_j, (\nu, \ell) \rangle \mid 1 \leq j \leq s - r\}),$$

the set  $\underline{\text{val}}^{-1}(\xi) \cap V(\mathcal{E})_\Gamma$  is a cylinder with  $q$ -basis the  $K$ -point set of

$$(\mathbb{A}^1 \setminus 0)^{n+1} \times \mathbb{A}^{q(n+1) - (\sum_{j=1}^r \langle u_j, \nu \rangle + \sum_{j=1}^{s-r} \langle w_j, (\nu, \ell) \rangle)}.$$

*Proof.* Let  $\pi \in \mathbb{C}(C)^*$  be a uniformizer of  $y$ . We have

$$\mathbb{C}[C_y(\mathfrak{D})^\vee \cap \Lambda] = \{\pi^k \otimes \chi^m \mid (m, k) \in C_y(\mathfrak{D})^\vee \cap \Lambda\},$$

where  $\Lambda = M \oplus \mathbb{Z}$ . We denote by  $X_{C_y(\mathfrak{D})} := \text{Spec } \mathbb{C}[C_y(\mathfrak{D})^\vee \cap \Lambda]$  the associated toric variety. The restricted fiber  $\underline{\text{val}}^{-1}(\xi) \cap V(\mathcal{E})_\Gamma$  can be identified with the  $(T \times \mathbb{G}_m)(\mathcal{O})$ -orbit  $C_{\nu, \ell}$  of  $X_{C_y(\mathfrak{D})}(\mathcal{O}) \cap (T \times \mathbb{G}_m)(\mathcal{K})$  given in [Ish04, Theorem 4.1].

Indeed, as in the proof of Lemma 4.9, any arc  $\alpha$  in the restricted fiber induces a morphism  $\mathbb{C}[[\pi]] \otimes_{\mathbb{C}} \mathbb{C}[M] \rightarrow \mathcal{K}$ , which restricts to  $\mathbb{C}[C_y(\mathfrak{D})^\vee \cap \Lambda] \rightarrow \mathcal{O}$ . Conversely, if we have an arc  $\beta \in \mathcal{L}(X_{C_y(\mathfrak{D})})(\mathcal{K})$  with the property above, it induces a morphism

$$\mathbb{C}[\pi, \pi^{-1}] \otimes_{\mathbb{C}} \mathbb{C}[M] \rightarrow \mathcal{K}.$$

Since the completion of  $(\mathbb{C}[\pi, \pi^{-1}], \pi\mathbb{C}[\pi, \pi^{-1}])$  is  $\mathbb{C}((\pi))$ , by [Eis95, Theorem 7.16]  $\beta$  gives a morphism  $\mathbb{C}[[\pi]] \otimes_{\mathbb{C}} \mathbb{C}[M] \rightarrow \mathcal{K}$  which restricts to a co-morphism of an arc in  $\underline{\text{val}}^{-1}(\xi) \cap V(\mathcal{E})_\Gamma$ . The rest of the proof follows from the suitably modified version of [BM13, Lemma 3.4] already used in Lemma 4.16.  $\square$

*Remark 4.18.* One may check that all the isomorphisms of  $K$ -point sets in the proofs of Lemmas 4.16 and 4.17 are functorial.

The following result concludes our study of the arc space of horospherical varieties of complexity one. It gives the motivic volume of the fibers of the surjective map  $\underline{\text{val}} : X_\Gamma \rightarrow |\mathcal{E}|_\Gamma \cap \mathcal{N}$ . Thus we obtain, up to a subset of motivic measure zero, a decomposition of the arc space of  $X$  as a disjoint union of measurable subsets of known motivic volume.

**Theorem 4.19.** *Let  $\mathcal{E}$  be a colored divisorial fan on  $(C, G/H)$  such that  $X = X(\mathcal{E})$  is smooth of dimension  $d$ . Without loss of generality, we may suppose that  $\mathcal{E} = \tilde{\mathcal{E}}$  has trivial coloration. Consider  $\xi := (y, \nu, \ell) \in |\mathcal{E}|_\Gamma \cap \mathcal{N}$ , where*

$$|\mathcal{E}|_\Gamma := \bigcup_{(\mathfrak{D}, \mathcal{F}) \in \mathcal{E}} C_\Gamma(\mathfrak{D}) \subset \mathcal{N}_\mathbb{Q}$$

and  $\mathcal{N}, \mathcal{N}_{\mathbb{Q}}$  are as in Definition 3.10. Let  $(\mathfrak{D}, \mathcal{F}) \in \mathcal{E}$  be such that  $\xi \in C_{\Gamma}(\mathfrak{D})$  and let  $\sigma$  be the tail of  $\mathfrak{D}$ . Denote by  $r$  the dimension of  $\sigma$  and by  $s \geq r + 1$  be the dimension of the Cayley cone  $C_y(\mathfrak{D})$ . The linear part  $C_y(\mathfrak{D})^{\vee} \cap (-C_y(\mathfrak{D}))^{\vee}$  of  $C_y(\mathfrak{D})^{\vee}$  is a  $(d - s)$ -dimensional vector space, which we denote by  $V_{\mathbb{Q}}$ . Let  $W_{\mathbb{Q}}$  be a complement of  $V_{\mathbb{Q}}$  in  $\Lambda_{\mathbb{Q}} = M_{\mathbb{Q}} \oplus \mathbb{Q}$  and denote by

$$(u_1, 0), \dots, (u_r, 0), w_1, \dots, w_{s-r}$$

an integral basis of  $C_y(\mathfrak{D})^{\vee} \cap W_{\mathbb{Q}}$ , where the  $u_i$  are in  $\sigma^{\vee} \cap M$ . Then

$$\mu_X(\underline{\text{val}}^{-1}(\xi)) = \begin{cases} [\Gamma][G/H]\mathbb{L}^{-\sum_{j=1}^r \langle u_j, \nu \rangle} & \text{if } \ell = 0, \\ [G/H](\mathbb{L} - 1)\mathbb{L}^{-\sum_{j=1}^s \langle u_j, \nu \rangle - \sum_{j=1}^{s-r} \langle w_j, (\nu, \ell) \rangle} & \text{if } \ell \geq 1, \end{cases}$$

*Proof.* Let us show the formula for  $\ell = 0$ . Since  $G/P$  is a smooth variety, by Lemmas 4.15 and 4.16 it follows that

$$\begin{aligned} \mu_X(\underline{\text{val}}^{-1}(C, \nu, 0)) &= [\pi'_q(V(\mathcal{E})_{\Gamma} \cap \underline{\text{val}}^{-1}(\xi))][\mathcal{L}_q(G/P)]\mathbb{L}^{-qd} \\ &= [\mathcal{L}_q(\Gamma)](\mathbb{L} - 1)^n [G/P]\mathbb{L}^{q(n + \dim(G/P) - d) - \sum_{j=1}^r \langle u_j, \nu \rangle} \end{aligned}$$

for any sufficient large  $q \geq 0$ . Using the equalities

$$\dim(G/P) = d - 1 - n, [G/H] = (\mathbb{L} - 1)^n [G/P] \text{ and } \mathcal{L}_q(\Gamma) = [\Gamma]\mathbb{L}^q,$$

we obtain the result of the theorem for  $\ell = 0$ . For the case  $\ell \geq 1$ , we have by Lemmas 4.15 and 4.17 the equality

$$\mu_X(\underline{\text{val}}^{-1}(\xi)) = (\mathbb{L} - 1)^{n+1} [G/P]\mathbb{L}^{q(n+1 + \dim(G/P) - d) - \sum_{j=1}^s \langle u_j, \nu \rangle - \sum_{j=1}^{s-r} \langle w_j, (\nu, \ell) \rangle}$$

which simplifies to our desired formula.  $\square$

## 5. THE $E$ -FUNCTION OF A HOROSPHERICAL VARIETY OF COMPLEXITY ONE

We now use our study of the arc space from the previous section to compute the stringy invariants of a (log terminal) horospherical variety  $X$  of complexity one.

To obtain our main result, Theorem 5.17, we use the description of the stringy motivic volume as a motivic integral from Theorem 2.16. This requires a good understanding of the discrepancy of  $X$  (relatively to a desingularization), and thus of its canonical divisor, which is the object of 5.1. As was already the case in complexity zero, cf. [BM13], Theorem 5.17 requires  $X$  to be  $\mathbb{Q}$ -Gorenstein and log terminal; we expose the related notions in 5.2. In 5.3 we construct a desingularization of  $X$  in terms of its colored divisorial fan, and in 5.4, we put the previous results to use by computing the discrepancy. This enables us to prove Theorem 5.17 in 5.5. Throughout this section we will consider a colored divisorial fan  $\mathcal{E}$  describing  $X$  and we will follow the conventions of 4.1.

**5.1. Canonical class.** Here we recall a combinatorial description of the canonical class of the horospherical  $G$ -variety  $X$  of complexity one, see [LT16, Sections 2.3, 2.4]. The canonical class  $K_X$  of  $X$  will be expressed as a linear combination of  $B$ -invariant divisors.

To do this we follow the description of  $G$ -invariant divisors on  $X$  from [LT16, Section 2.3]. By [Tim11, Section 16] any germ  $Y \subset X(\mathfrak{D}^i, \mathcal{F}^i)$  is described by a pair  $(\Pi_Y, \mathcal{F}_Y)$ , where  $\Pi_Y$  is a hyperface of  $C(\mathfrak{D}^i)$  (cf [Tim11, Definition 16.18]) and

$$\mathcal{F}_Y = \{D \in \mathcal{F}^i \mid \varrho(D) \in \Pi_Y\}.$$

There is a geometric characterization of  $\mathcal{F}_Y$  as

$$\mathcal{F}_Y = \{D \in \mathcal{F}_{G/H} \mid D \supset Y\},$$

where we view elements of  $\mathcal{F}_{G/H}$ , i.e. colors, as  $B$ -invariant divisors on  $X(\mathfrak{D}^i, \mathcal{F}^i)$  which are not  $G$ -invariant. Using this description we can consider two types of  $G$ -divisors called *vertical* and *horizontal* divisors according that their  $G$ -actions on it are of complexity zero or one.

For the vertical case, we consider the following datas. For any  $i \in J$ , denote by  $\text{Vert}(\mathfrak{D}^i, \mathcal{F}^i) = \text{Vert}(\mathfrak{D}^i)$  the set of pairs  $(y, p)$ , where  $y \in C^i$  and  $p$  is a vertex of  $\Delta_y^i$ . Then

$$\text{Vert}(\mathcal{E}) := \bigcup_{i \in J} \text{Vert}(\mathfrak{D}^i)$$

will parametrize a set of vertical  $G$ -divisors on  $X$ .

The other  $G$ -divisors on  $X$  that are horizontal can be described as follows. Define  $\text{Ray}(\mathfrak{D}^i, \mathcal{F}^i)$  as the set of rays  $\rho$  of  $\sigma_i$  such that

- $\rho \cap \varrho(\mathcal{F}^i) = \emptyset$  (i.e., the ray  $\rho$  is uncolored),
- $\rho \cap \deg(\mathfrak{D}^i) = \emptyset$  if  $C^i$  is projective.

Here we used the same notation  $\rho$  for a ray and for the corresponding primitive generator. We denote by  $\text{Ray}(\mathcal{E})$  the reunion

$$\text{Ray}(\mathcal{E}) := \bigcup_{i \in J} \text{Ray}(\mathfrak{D}^i, \mathcal{F}^i),$$

which will parametrize the rest of the  $G$ -invariant divisors of  $X$ .

**Theorem 5.1** ([LT16, Theorem 2.11]). *Let  $\text{Div}(\mathcal{E})$  denote the set of  $G$ -divisors of  $X$ . The map*

$$\begin{array}{ccc} \text{Vert}(\mathcal{E}) \sqcup \text{Ray}(\mathcal{E}) & \rightarrow & \text{Div}(\mathcal{E}) \\ (y, p) & \mapsto & D_{(y,p)} \\ \rho & \mapsto & D_\rho \end{array}$$

which to the pair  $(y, p)$  – respectively to the ray  $\rho$  – associates the germ  $D_{(y,p)}$  of  $X$  defined by the colored datum  $([(y, \mathbb{Q}_{\geq 0}(p, 1))], \emptyset)$  – respectively the germ  $D_\rho$  of  $X$  defined by the colored datum  $(\rho, \emptyset) = ([C, \rho, 0], \emptyset)$ , is well-defined and bijective.

We may now describe the canonical class of  $X$  as a Weil divisor.

**Theorem 5.2** ([LT16, Theorem 2.18]). *With the same notation as in Theorem 5.1, the divisor*

$$K_X = \sum_{(y,p) \in \text{Vert}(\mathcal{E})} (\kappa(p)b_y + \kappa(p) - 1) \cdot D_{(y,p)} - \sum_{\rho \in \text{Ray}(\mathcal{E})} D_\rho - \sum_{\alpha \in \Phi \setminus I} a_\alpha \cdot D_\alpha$$

is a canonical divisor of  $X$ . Here  $K_C = \sum_{y \in C} b_y \cdot [y]$  is a canonical divisor on  $C$ , and

$$a_\alpha := \sum_{\beta \in R^+ \setminus R_I} \langle \beta, \alpha^\vee \rangle,$$

where  $R^+$  is the set of positive roots of  $G$  and  $R_I$  is the set of roots of  $P = N_G(H)$ .

**5.2. Cartier divisors and support functions.** We begin by recalling the combinatorial description of invariant Cartier divisors on horospherical varieties of complexity one from [Tim00, §4] and [LT16, Corollary 2.17], via functions on hypercones called *support functions*. See [PS11] for a description in the setting of torus actions. Then we give conditions for these varieties to have  $\mathbb{Q}$ -Gorenstein or log terminal singularities.

Denote by  $\mathbb{C}(X)^{(B)}$  the set of  $B$ -eigenfunctions in the function field  $\mathbb{C}(X)$  of  $X$ , which identifies with a subset of the tensor product  $\mathbb{C}(C) \otimes_{\mathbb{C}} \mathbb{C}[M]$  (see Remark 3.11). The principal divisor associated with a rational function  $f \otimes \chi^m$  in  $\mathbb{C}(C) \otimes_{\mathbb{C}} \mathbb{C}[M]$  is given by

$$\text{div}(f \otimes \chi^m) = \sum_{(y,p) \in \text{Vert}(\mathcal{E})} \kappa(p) (\langle m, p \rangle + \text{ord}_y(f)) \cdot D_{(y,p)} + \sum_{\rho \in \text{Ray}(\mathcal{E})} \langle m, \rho \rangle \cdot D_\rho + \sum_{D \in \mathcal{F}_{G/H}} \langle m, \varrho(D) \rangle \cdot D,$$

where  $\kappa(p) = \min\{\lambda \in \mathbb{Z}_{>0} \mid \lambda \cdot p \in N\}$ .

Cartier divisors are locally principal divisors. On  $X$ , they will be associated to some specific *support functions* defined on the Cayley cones  $C_y(\mathfrak{D}^i)$ .

**Definition 5.3.** An *integral linear function* on  $\mathfrak{D}^i$  is a map  $\vartheta : C(\mathfrak{D}^i) \rightarrow \mathbb{Q}$  satisfying the following conditions.

- (i) For every  $y \in C^i$  there exists  $m_y \in M$  and  $b_y \in \mathbb{Z}$  such that

$$\vartheta(y, \nu, \ell) = \langle m_y, \nu \rangle + \ell c_y$$

for any  $(\nu, \ell) \in C_y(\mathfrak{D}^i)$ .

- (ii) If  $C^i$  is projective, there exists  $m \in M$  such that  $m_y = m$  for any  $y \in C$ , and a function  $f \in \mathbb{C}(C)^*$  such that

$$\text{div } f = \sum_{y \in C} c_y \cdot [y].$$

Denote by  $\mathcal{F}_\mathcal{E}$  the reunion of all the sets  $\mathcal{F}^i$  for  $i \in J$ . A *colored integral piecewise linear function* on  $\mathcal{E}$  is a pair  $\theta = (\vartheta, (r_\alpha))$ , where  $\vartheta : |\mathcal{E}| \rightarrow \mathbb{Q}$  is a function such that the restriction  $\vartheta|_{C(\mathfrak{D}^i)}$  is integral linear for every  $i \in J$ , and where  $(r_\alpha)$  is a sequence of integers with  $\alpha$  running over simple roots in  $\Phi \setminus I$  such that  $D_\alpha \notin \mathcal{F}_\mathcal{E}$ . We also require that the linear functions  $\vartheta|_{C(\mathfrak{D}^i)}$  and  $\vartheta|_{C(\mathfrak{D}^j)}$  coincide on  $C(\mathfrak{D}^i) \cap C(\mathfrak{D}^j)$  for all  $i, j \in J$ .

More generally we say  $\theta = (\vartheta, (r_\alpha))$  is a *colored piecewise linear function* on  $\mathcal{E}$  if there exists  $k \in \mathbb{Z}_{>0}$  such that  $k\theta$  is a colored integral piecewise linear function. We denote by  $\text{PL}(\mathcal{E})$  (resp.  $\text{PL}(\mathcal{E}, \mathbb{Q})$ ) the set of colored integral piecewise linear functions (resp. colored piecewise linear functions) on  $\mathcal{E}$ .

Now the Cartier divisor associated with a colored piecewise linear function  $\theta \in \text{PL}(\mathcal{E})$  is given by the formula

$$D_\theta = \sum_{(y,p) \in \text{Vert}(\mathcal{E})} \vartheta(y, \kappa(p)p, \kappa(p)) D_{(y,p)} + \sum_{\rho \in \text{Ray}(\mathcal{E})} \vartheta(C, \rho, 0) D_\rho + \sum_{D \in \mathcal{F}_\mathcal{E}} \vartheta(C, \varrho(D), 0) D + \sum_{D_\alpha \notin \mathcal{F}_\mathcal{E}} r_\alpha D_\alpha.$$

More precisely, the map  $\theta \mapsto D_\theta$  is an isomorphism between the group  $\text{PL}(\mathcal{E})$  and the group of  $B$ -stable Cartier divisors on  $X$ . We may now give a criterion for  $X$  to be  $\mathbb{Q}$ -Gorenstein, i.e., for the canonical divisor  $K_X$  to be  $\mathbb{Q}$ -Cartier.

**Proposition 5.4** ([LT16, Corollary 2.19]). *The variety  $X(\mathcal{E})$  is  $\mathbb{Q}$ -Gorenstein if and only if there exists  $\theta \in \text{PL}(\mathcal{E}, \mathbb{Q})$  such that the following conditions are satisfied.*

(i) *There exists a canonical divisor  $K_C = \sum_{y \in C} b_y \cdot [y]$  on  $C$  such that for every  $(y, p) \in \text{Vert}(\mathcal{E})$  we have*

$$\vartheta(y, \kappa(p)p, \kappa(p)) = \kappa(p)b_y + \kappa(p) - 1.$$

(ii) *For every  $\rho \in \text{Ray}(\mathcal{E})$  we have  $\vartheta(C, \rho, 0) = -1$ .*

(iii) *For every  $D_\alpha \in \mathcal{F}_\mathcal{E}$  we have  $\vartheta(C, \varrho(D_\alpha), 0) = -a_\alpha$ .*

We will denote by  $\theta_X$  the colored piecewise linear function on  $X$  satisfying the conditions of Proposition 5.4 and such that  $r_\alpha = -a_\alpha$  for any  $\alpha$  with  $D_\alpha \notin \mathcal{F}_\mathcal{E}$ . We refer to [Tim11, Equations (17.1-2)] for the uniqueness of  $\theta_X$ .

In fact in our main result we will not use the support function  $\theta$ , but another related support function  $\varpi$ , which will be constructed by gluing the linear functions introduced in the next lemma.

**Lemma 5.5.** *Let  $(\mathfrak{D}, \mathcal{F})$  be a colored  $\sigma$ -polyhedral divisor on  $(C, G/H)$  such that  $Y = X(\mathfrak{D}, \mathcal{F})$  is  $\mathbb{Q}$ -Gorenstein. Let  $z$  be a point in the locus of  $\mathfrak{D}$ . Denote by  $X(C_z(\mathfrak{D}), \mathcal{F})$  the horospherical  $(G \times \mathbb{C}^*)$ -variety associated with the pair  $(C_z(\mathfrak{D}), \mathcal{F})$  and the horospherical homogeneous space  $G/H \times \mathbb{C}^*$ . There exists a linear function  $\omega_{Y,z}$  on  $C_z(\mathfrak{D})$  such that*

(i)  $\omega_{Y,z}(\rho, 0) = \vartheta_Y(z, \rho, 0)$  for any uncolored ray  $\rho$  of  $\sigma$ ,

(ii)  $\omega_{Y,z}(\tau) = -1$  for any ray  $\tau$  of  $C_z(\mathfrak{D})$  not contained in  $N_{\mathbb{Q}}$ ,

(iii)  $\omega_{Y,z}(\varrho(D_\alpha), 0) = -a_\alpha$  for any color  $D_\alpha \in \mathcal{F}$ .

In particular, if the locus of  $\mathfrak{D}$  is affine, we have

$$X(\mathfrak{D}, \mathcal{F}) \mathbb{Q}\text{-Gorenstein} \Rightarrow X(C_z(\mathfrak{D}), \mathcal{F}) \mathbb{Q}\text{-Gorenstein}.$$

*Proof.* Since  $Y$  is  $\mathbb{Q}$ -Gorenstein, by Proposition 5.4 there exists a colored piecewise linear function  $\theta_Y = (\vartheta_Y, (r_\alpha))$  on  $(\mathfrak{D}, \mathcal{F})$  satisfying the conditions (i)-(iii) of the proposition.

We will construct the function  $\omega_{Y,z}$  by modifying  $\theta_Y$ . More precisely we find a function  $f \in \mathbb{C}(C)^*$  such that

$$\omega_{Y,z}(\kappa(p)p, \kappa(p)) = \vartheta(z, \kappa(p)p, \kappa(p)) + \kappa(p) \text{ord}_z(f),$$

or equivalently

$$-\text{ord}_z(f) = b_z + 1$$

for any  $(z, p) \in \text{Vert}(\mathfrak{D})$ , where  $K_C = \sum_{y \in C} b_y \cdot [y]$  is a canonical divisor of  $C$ .

Since  $C$  is smooth any divisor on  $C$  is locally principal. Hence there exists an open neighborhood  $C'$  of  $z$  in  $C$  and a function  $f \in \mathbb{C}(C')^* = \mathbb{C}(C)^*$  such that

$$(K_C + [z])|_{C'} = (\text{div } f^{-1})|_{C'}.$$

The divisor  $D_{\theta_Y} + \text{div}(f)$  is a Cartier  $\mathbb{Q}$ -divisor in  $X$ . Clearly the restriction of its support function to  $C_z(\mathfrak{D})$  is the required function  $\omega_{Y,z}$ .

Let us now assume the locus of  $\mathfrak{D}$  is affine. The variety  $X(C_z(\mathfrak{D}), \mathcal{F})$  being horospherical, it is  $\mathbb{Q}$ -Gorenstein if and only if there exists a (restricted) linear function  $\omega : C_z(\mathfrak{D}) \rightarrow \mathbb{Q}$  satisfying the following properties (cf. [Bri93, Proposition 4.1]):

- $\omega(\tau) = -1$  for any uncolored ray  $\tau$  of  $C_z(\mathfrak{D})$  (i.e.  $\tau \cap \varrho(\mathcal{F}) = \emptyset$ ),
- $\omega(\varrho(D_\alpha), 0) = -a_\alpha$  for any color  $D_\alpha \in \mathcal{F}$ .

These conditions are satisfied by the function  $\omega_{Y,z}$ . Indeed, since the locus of  $\mathfrak{D}$  is affine, we have

$$\omega_{Y,z}(\rho, 0) = \vartheta_Y(z, \rho, 0) = -1$$

for any uncolored ray  $\rho$  of  $\sigma$ . □

Finally, we recall a criterion for  $X$  to have only log terminal singularities. Later on we will need to assume  $X$  is log terminal to ensure the motivic integral involves in the computation of the stringy motivic volume converges.

**Theorem 5.6** ([LT16, Theorem 2.22]). *Suppose  $X = X(\mathcal{E})$  is  $\mathbb{Q}$ -Gorenstein. Then  $X$  has only log terminal singularities if and only if for any  $(\mathfrak{D}, \mathcal{F}) \in \mathcal{E}$ , where  $\mathfrak{D} := \sum_{y \in C_0} \Delta_y \cdot [y]$ , one of the following assertions holds.*

- (i) *The locus of  $\mathfrak{D}$  is affine.*
- (ii) *The locus of  $\mathfrak{D}$  is the projective line  $\mathbb{P}^1$  and  $\sum_{y \in \mathbb{P}^1} \left(1 - \frac{1}{\kappa_y}\right) < 2$ , where*

$$\kappa_y := \max(\{\kappa(p) \mid p \text{ is a vertex of } \Delta_y\}).$$

**5.3. Desingularization.** Here we explain how to desingularize  $X$  in terms of combinatorial data. The desingularization involves three proper birational morphisms:

$$(22) \quad \begin{array}{ccccccc} X(\mathcal{E}') & \xrightarrow{q'} & X(\tilde{\mathcal{E}}) & \xrightarrow{q} & X(\mathcal{E}_{dis}) & \xrightarrow{\pi_{dis}} & X(\mathcal{E}) = X \\ & & & & \searrow & & \\ & & & & & \psi & \end{array}$$

The discoloration morphism  $\pi_{dis}$  and the affinization morphism  $q$  have already been defined in Equations (15) and (18).

Let us now construct the morphism  $q'$ . Let  $\text{Sp}(\tilde{\mathcal{E}})$  be the set of special points of  $\tilde{\mathcal{E}}$  and denote its elements by  $y_1, \dots, y_r$ . Up to refining the affine coverings  $(C_j^i)_{j \in J_i}$  in the construction outlined in Notation 4.11, we may assume that each element  $\tilde{\mathfrak{D}}_j^i$  of the fan  $\tilde{\mathcal{E}}$  contains at most one special point. For  $1 \leq i \leq r$ , we let  $\tilde{\mathcal{E}}_i$  be the set of all polyhedral divisors  $\mathfrak{D} \in \mathcal{E}$  which have  $y_i$  as a special point and we define a fan  $\Sigma_i \subset N_{\mathbb{Q}} \oplus \mathbb{Q}$  as the fan generated by the Cayley cones  $C_{y_i}(\mathfrak{D})$  for any  $\mathfrak{D} \in \tilde{\mathcal{E}}$ . Clearly the fans  $\Sigma_i$  all contain the tail fan  $\Sigma(\tilde{\mathcal{E}})$  as a subfan. Now following [CLS11, Section 11.1] we will consider a star subdivision of  $\Sigma(\tilde{\mathcal{E}})$  and compatible star subdivisions of the  $\Sigma_i$ .

**Definition 5.7.** Consider a fan  $\Sigma$  in  $N_{\mathbb{Q}}$  and an element  $\nu$  of  $|\Sigma| \cap N$  in the support of  $\Sigma$ . The *star subdivision* of  $\Sigma$  associated with  $\nu$  is the fan consisting of the cones

- (i)  $\sigma \in \Sigma$  such that  $\nu \notin \sigma$ ,
- (ii)  $\text{Cone}(\tau, \nu)$  for  $\tau \in \Sigma$  such that  $\nu \notin \tau$  and there exists  $\sigma \in \Sigma$  with  $\{\nu\} \cup \tau \subset \sigma$ .

A star subdivision is a subdivision of the original fan  $\Sigma$  (cf [CLS11, Lemma 11.1.3]).

We say a fan  $\Sigma$  is *smooth* if any cone in  $\Sigma$  is generated by a subset of a lattice basis of  $N$ . This is equivalent to the toric variety  $X_{\Sigma}$  being smooth. From [CLS11, Theorem 11.1.9] we have that any fan  $\Sigma$  has a smooth refinement  $\Sigma'$  such that

- $\Sigma'$  contains every smooth cone of  $\Sigma$ ,
- $\Sigma'$  is obtained from  $\Sigma$  by a sequence of star subdivisions.

Geometrically this implies that there is a projective desingularization of toric varieties  $X_{\Sigma'} \rightarrow X_{\Sigma}$ .

Coming back to our problem, let  $\Sigma'(\tilde{\mathcal{E}})$  be a smooth refinement of the tail fan  $\Sigma(\tilde{\mathcal{E}})$ , corresponding to a sequence of star subdivisions associated with an element  $(\nu_1, \dots, \nu_s) \in N^s$ . For any  $1 \leq i \leq r$  define  $\hat{\Sigma}_i$  as the fan obtained from  $\Sigma$  after applying star subdivisions with respect to  $((\nu_1, 0), \dots, (\nu_s, 0)) \in (N \oplus \mathbb{Z})^s$ .

Clearly  $\Sigma'(\tilde{\mathcal{E}})$  is a smooth subfan of  $\hat{\Sigma}_i$ . Finally, there exists a refinement  $\Sigma'_i$  of  $\hat{\Sigma}_i$  such that  $\Sigma'_i$  contains every smooth cone of  $\hat{\Sigma}_i$  – including in particular every cone of  $\Sigma'(\tilde{\mathcal{E}})$ .

For any  $\mathfrak{D} \in \tilde{\mathcal{E}}_i$  consider the set  $\Sigma_{i,\mathfrak{D}}$  of cones  $\tau \in \Sigma'_i$  such that  $\tau \subset C_{y_i}(\mathfrak{D})$ . For  $\tau \in \Sigma_{i,\mathfrak{D}}$  we denote by  $\sigma_{\tau,i}$  the cone  $\tau \cap (N_{\mathbb{Q}} \times \{0\})$ , and by  $\Delta_{\tau,i}$  the set  $\tau \cap (N_{\mathbb{Q}} \times \{1\})$  when  $\tau \not\subset N_{\mathbb{Q}} \times \{0\}$ . Since  $\tau \subset C_{y_i}(\mathfrak{D})$  it follows that  $\tau \cap (N_{\mathbb{Q}} \times \{1\})$  is a  $\sigma_{\tau,i}$ -polyhedron. Let  $C_0 \subset C$  denote the locus of  $\mathfrak{D}$ . We will define a  $\sigma_{\tau,i}$ -polyhedral divisor

$$\mathfrak{D}_{\tau,i} := \sum_{y \in C_0} \Delta_y \cdot [y]$$

as follows. For any  $y \in C_0 \setminus \{y_i\}$  we set  $\Delta_y = \sigma_{\tau,i}$ , and

$$\Delta_{y_i} = \begin{cases} \sigma_{\tau,i} & \text{if } \tau \subset N_{\mathbb{Q}} \times \{0\}, \\ \Delta_{\tau,i} & \text{otherwise.} \end{cases}$$

The set  $\tilde{\mathcal{E}}_{\mathfrak{D}} := \{\mathfrak{D}_{\tau,i} \mid \tau \in \Sigma_{i,\mathfrak{D}}\}$  is a divisorial fan. Moreover, the  $\mathbb{C}$ -algebras embeddings  $A(C_0, \mathfrak{D}) \subset A(C_0, \mathfrak{D}_{\tau,i})$  for  $\mathfrak{D}_{\tau,i} \in \tilde{\mathcal{E}}_{\mathfrak{D}}$  induce a proper  $T$ -equivariant birational morphism  $V(\tilde{\mathcal{E}}_{\mathfrak{D}}) \rightarrow V(\mathfrak{D})$ , see [Tim11, Theorem 12.13]. Since the set  $\mathcal{E}' := \bigcup_{\mathfrak{D} \in \tilde{\mathcal{E}}} \tilde{\mathcal{E}}_{\mathfrak{D}}$  is a divisorial fan, the previous morphisms glue together into proper birational morphism  $q' : X(\mathcal{E}') \rightarrow X(\tilde{\mathcal{E}})$ . To prove that  $X(\mathcal{E}')$  is indeed smooth we use [KKMSD73, Chap. II], [LS13, Proof of 2.6], [LT16, Theorem 2.5].

This concludes our construction of the desingularization of  $X$ .

**5.4. Discrepancy.** The aim of this section is to compute the discrepancy of the  $G$ -equivariant morphism  $\psi : X' \rightarrow X$  from Equation (22), whose definition we recall below.

$$X' := X(\mathcal{E}') \xrightarrow{q'} X(\tilde{\mathcal{E}}) \xrightarrow{q} X(\mathcal{E}_{dis}) \xrightarrow{\pi_{dis}} X.$$

$\psi$

Since the morphism  $\psi$  is equivariant, the exceptional divisors of  $\psi$  are particular  $G$ -divisors of  $X'$ , that is, particular elements of  $\text{Vert}(\mathcal{E}') \sqcup \text{Ray}(\mathcal{E}')$ .

**Exceptional divisors of  $\pi_{dis}$ .** Clearly we have  $\text{Vert}(\mathcal{E}_{dis}) = \text{Vert}(\mathcal{E})$ , thus the exceptional divisors of  $\pi_{dis}$  are in one-to-one correspondence with the elements of  $\text{Ray}(\mathcal{E}_{dis})$  which do not come from  $\text{Ray}(\mathcal{E})$ . Recall that

$$\text{Ray}(\mathcal{E}_{dis}) = \bigcup_{i \in J} \text{Ray}(\mathfrak{D}^i, \emptyset),$$

hence

**Lemma 5.8.** *The elements of  $\text{Ray}(\mathcal{E}_{dis})$  which do not come from  $\text{Ray}(\mathcal{E})$  can be expressed as*

$$\bigcup_{i \in J} \text{Ray}(\mathfrak{D}^i, \emptyset) \setminus \text{Ray}(\mathfrak{D}^i, \mathcal{F}^i).$$

*Proof.* The inclusion from the left to the right is obvious. Let us consider an element  $\rho \in \text{Ray}(\mathfrak{D}^i, \emptyset) \setminus \text{Ray}(\mathfrak{D}^i, \mathcal{F}^i)$  for some  $i \in J$ . This means that  $\rho$  is a ray of the tail  $\sigma_i$  of  $\mathfrak{D}^i$ , and that moreover

- $\rho \cap \varrho(\mathcal{F}^i) \neq \emptyset$ ,
- if the locus  $C^i$  of  $\mathfrak{D}^i$  is projective, then  $\rho \cap \text{deg}(\mathfrak{D}^i) = \emptyset$ .

It is enough to check that  $\rho \notin \text{Ray}(\mathfrak{D}^j, \mathcal{F}^j)$  for  $j \in J$  such that  $\rho$  is a ray of  $\sigma_j$ . Consider such a  $j \in J$ . Then  $\rho$  is a ray of the tail  $\sigma_i \cap \sigma_j$  of  $\mathfrak{D}^i \cap \mathfrak{D}^j$ . By definition of a colored polyhedral divisor we have

$$\mathcal{F}^i \cap \mathcal{F}^j = \varrho^{-1}(\sigma_i \cap \sigma_j) \cap \mathcal{F}^i = \varrho^{-1}(\sigma_i \cap \sigma_j) \cap \mathcal{F}^j.$$

Hence  $\varrho(\mathcal{F}^j) \cap \rho = \varrho(\mathcal{F}^i) \cap \rho \neq \emptyset$ , which concludes the proof.  $\square$

Write

$$\bigcup_{i \in J} (\text{Ray}(\mathfrak{D}^i, \emptyset) \setminus \text{Ray}(\mathfrak{D}^i, \mathcal{F}^i)) =: \{\rho_1, \dots, \rho_t\}.$$

The lemma implies that the associated divisors  $D_{\rho_1}, \dots, D_{\rho_t} \subset X(\mathcal{E}_{dis})$  are the exceptional divisors of  $\pi_{dis}$ .



**Exceptional divisors of  $q$ .** The morphism  $q : X(\tilde{\mathcal{E}}) \rightarrow X(\mathcal{E}_{dis})$  is obtained via parabolic induction on the  $T$ -equivariant morphism  $V(\tilde{\mathcal{E}}) \rightarrow V(\mathcal{E})$  from Equation (18), so the exceptional divisors of  $q$  are the parabolic inductions of the exceptional divisors of the latter map. From [AH06, Theorem 10.1] we know that the set  $\text{Ray}(\tilde{\mathcal{E}}) \setminus \text{Ray}(\mathcal{E}_{dis})$  is equal to the reunion over all colored polyhedral divisors  $(\mathfrak{D}, \emptyset)$  in  $\tilde{\mathcal{E}}$  with projective locus, of the rays  $\rho$  of the tails of polyhedral divisors  $\mathfrak{D}$  with the property that  $\deg \mathfrak{D} \cap \rho \neq \emptyset$ . We denote the associated exceptional divisors of  $q$  by  $D_{\rho_{t+1}}, \dots, D_{\rho_s} \subset X(\tilde{\mathcal{E}})$ .

**Exceptional divisors of  $q'$ .** As before the map  $q' : X' = X(\mathcal{E}') \rightarrow X(\tilde{\mathcal{E}})$  is obtained from the map  $q'' : V(\mathcal{E}') \rightarrow V(\tilde{\mathcal{E}})$ . Denote by  $\{y_1, \dots, y_r\}$  the set of special points of  $\tilde{\mathcal{E}}$ . For  $1 \leq i \leq r$  consider the fans  $\Sigma_i$  and  $\Sigma'_i$  constructed in Section 5.3. The exceptional divisors of  $q''$  (or equivalently of  $q'$ ) are in bijection with the triples  $[y_i, \nu, \ell]$ , where  $(\nu, \ell)$  is a ray of  $\Sigma'_i \setminus \Sigma_i$ . When  $\ell = 0$  the triple  $(y_i, \nu, 0)$  corresponds to an element of  $\text{Ray}(\mathcal{E}')$ , otherwise it corresponds to an element of  $\text{Vert}(\mathcal{E}')$ . We denote the associated exceptional divisors by  $D'_{\rho_{s+1}}, \dots, D'_{\rho_e}$  and  $D'_{(y_1, p_1)}, \dots, D'_{(y_u, p_u)}$ , respectively.

**Relative canonical class of  $\psi$ .** Since  $\Sigma(\tilde{\mathcal{E}}) = \Sigma(\mathcal{E}_{dis}) = \Sigma(\mathcal{E})$  and  $\Sigma(\mathcal{E}')$  is a fan subdivision of  $\Sigma(\tilde{\mathcal{E}})$ , we may identify the pullbacks of  $D_{\rho_1}, \dots, D_{\rho_s}$  with the corresponding divisors  $D'_{\rho_1}, \dots, D'_{\rho_s}$  in  $X'$ . Hence combining the previous paragraphs, we obtain that the exceptional divisors of  $\psi$  are the  $(D'_{\rho_i})_{1 \leq i \leq e}$  and the  $(D'_{(y_j, p_j)})_{1 \leq j \leq u}$ .

**Proposition 5.9.** *Assume that  $X(\mathcal{E})$  is  $\mathbb{Q}$ -Gorenstein. Then the relative canonical class of  $\psi : X' \rightarrow X$  is represented by*

$$K_{X'/X} = \sum_{j=1}^u (\kappa(p_j) b_{y_j} + \kappa(p_j) - 1 - \vartheta_X(y, \kappa(p_j) p_j, \kappa(p_j))) D'_{(y_j, p_j)} + \sum_{i=1}^e (-1 - \vartheta_X(C, \rho_i, 0)) D'_{\rho_i},$$

where  $\sum_{y \in C} b_y \cdot [y]$  is a canonical divisor of  $C$ ,  $\theta_X = (\vartheta_X, (r_\alpha))$  is the support function from Section 5.2, and  $\kappa : N_{\mathbb{Q}} \rightarrow \mathbb{Z}$  is defined by

$$\kappa(p) = \min\{\lambda \in \mathbb{Z}_{>0} \mid \lambda \cdot p \in N\}.$$

*Proof.* A canonical divisor of  $X'$  is given by

$$K_{X'} = \sum_{(y,p) \in \text{Vert}(\mathcal{E}')} \vartheta_{X'}(y, \kappa(p)p, \kappa(p)) D'_{(y,p)} + \sum_{\rho \in \text{Ray}(\mathcal{E}')} \vartheta_{X'}(C, \rho, 0) D'_\rho - \sum_{\alpha \in \Phi \setminus I} a_\alpha D'_\alpha,$$

while the pullback by  $\psi$  of a canonical divisor of  $X$  is

$$\psi^* K_X = \sum_{(y,p) \in \text{Vert}(\mathcal{E}')} \vartheta_X(y, \kappa(p)p, \kappa(p)) D'_{(y,p)} + \sum_{\rho \in \text{Ray}(\mathcal{E}')} \vartheta_X(C, \rho, 0) D'_\rho - \sum_{\alpha \in \Phi \setminus I} a_\alpha D'_\alpha,$$

where we choose  $K_X$  as in Theorem 5.6. Hence

$$K_{X'/X} = \sum_{j=1}^u (\vartheta_{X'} - \vartheta_X)(y, \kappa(p_j) p_j, \kappa(p_j)) D'_{(y_j, p_j)} + \sum_{i=1}^e (\vartheta_{X'} - \vartheta_X)(C, \rho_i, 0) D'_{\rho_i}.$$

We conclude by computing the values of  $\vartheta_{X'}$  using Proposition 5.4.  $\square$

**5.5. Stringy invariants.** We combine the discrepancy calculation from Section 5.4 and the motivic volume results from Section 4.4 to obtain a formula for the stringy  $E$ -function of a  $\mathbb{Q}$ -Gorenstein horospherical variety  $X$  of complexity one with log terminal singularities. We reuse the notation of the previous section, and we denote by  $\Gamma \subset C$  an open dense affine subset which does not contain any special point. Let  $d$  be the dimension of  $X$ .

**Decomposition of the motivic integral.** First we decompose the motivic volume  $\mathcal{E}_{st}(X)$  along the  $\mathcal{L}(G)$ -orbits in  $\mathcal{L}_\Gamma(X')$ . We will use the set  $|\mathcal{E}'|_\Gamma$  from Equation (19) and the map  $\underline{\text{val}} : X'_\Gamma \rightarrow |\mathcal{E}'|_\Gamma \cap \mathcal{N}$  from Equation (20). This latter induces a map from  $\mathcal{L}_\Gamma(X')$  to the set  $|\mathcal{E}'|_\Gamma \cap \mathcal{N}$  that we will denote by  $\underline{\text{val}}$ .

**Lemma 5.10.**

$$\mathcal{E}_{st}(X) = \sum_{(y,\nu,\ell) \in |\mathcal{E}'|_\Gamma \cap \mathcal{N}} \int_{\underline{\text{val}}^{-1}(y,\nu,\ell)} \mathbb{L}^{-\text{ord}_{K_{X'/X}}} d\mu_{X'}.$$

*Proof.* We have

$$\mathcal{E}_{st}(X) = \int_{\mathcal{L}(X')} \mathbb{L}^{-\text{ord}_{K_{X'/X}}} d\mu_{X'} = \int_{\mathcal{L}_\Gamma(X')} \mathbb{L}^{-\text{ord}_{K_{X'/X}}} d\mu_{X'}$$

since we have seen in Remark 4.2 that the subset  $\mathcal{L}(X') \setminus \mathcal{L}_\Gamma(X')$  has zero motivic measure. We have a decomposition

$$\mathcal{L}_\Gamma(X') = \bigsqcup_{(y,\nu,\ell) \in |\mathcal{E}'|_\Gamma} \underline{\text{val}}^{-1}(y,\nu,\ell).$$

We have seen in Lemmas 4.16 and 4.17 that the fibers  $\underline{\text{val}}^{-1}(y,\nu,\ell)$  interpreted as subspaces of  $\mathcal{L}_\Gamma(X')$  are measurable. By [BM13, Proposition 1.3 (i)] it follows that

$$\int_{\mathcal{L}_\Gamma(X')} \mathbb{L}^{-\text{ord}_{K_{X'/X}}} d\mu_{X'} = \sum_{(y,\nu,\ell) \in |\mathcal{E}'|_\Gamma} \int_{\underline{\text{val}}^{-1}(y,\nu,\ell)} \mathbb{L}^{-\text{ord}_{K_{X'/X}}} d\mu_{X'},$$

which concludes the proof.  $\square$

**The stringy support function.** Now we want to decompose the right-hand side of the identity in Lemma 5.10, depending on whether  $\ell = 0$  or  $\ell \geq 1$ . We need to introduce a new support function  $\omega_X$  inspired from the definition of the function  $\theta_X$ .

**Proposition-Definition 5.11.** *Denote by  $(\mathfrak{D}^i, \mathcal{F}^i)$  for  $i \in J$  the elements of  $\mathcal{E}$ . There exists a pair  $\varpi_X = (\omega_X, (r_\alpha))$ , where  $\omega_X : |\mathcal{E}| \rightarrow \mathbb{Q}$  is a function and  $r_\alpha \in \mathbb{Z}$  for any  $\alpha \in \Phi \setminus I$ , satisfying the following properties*

- for any  $i \in J$  and any  $y$  in the locus of  $\mathfrak{D}^i$ , we have
  - (i)  $\omega_X(y, \rho, 0) = \vartheta_X(y, \rho, 0)$  for any uncolored ray  $\rho$  of the tail of  $\mathfrak{D}^i$ ,
  - (ii)  $\omega_X(y, \tau) = -1$  for any ray  $\tau$  of  $C_y(\mathfrak{D}^i)$  not contained in  $N_{\mathbb{Q}}$ ,
  - (iii)  $\omega_X(y, \rho(D_\alpha), 0) = -a_\alpha$  for any color  $D_\alpha \in \mathcal{F}^i$ ,
  - (iv)  $r_\alpha = -a_\alpha$  for any  $\alpha \in \Phi \setminus I$  such that  $D_\alpha \notin \mathcal{F}_{\mathcal{E}}$ , and
  - (v)  $\omega_X(y, \nu, \ell) = \langle m_y^i, \nu \rangle + \ell c_y^i$  for some  $m_y^i \in M$  and  $c_y^i \in \mathbb{Q}$ .
- If  $(y, \nu, \ell) \in C(\mathfrak{D}^i) \cap C(\mathfrak{D}^j)$ , then

$$\langle m_y^i, \nu \rangle + \ell c_y^i = \langle m_y^j, \nu \rangle + \ell c_y^j.$$

The pair  $\varpi_X = (\omega_X, (r_\alpha))$  will be called the stringy support function.

*Proof.* Since  $X$  is  $\mathbb{Q}$ -Gorenstein, by Lemma 5.5, for any  $i \in J$  and  $y$  in the locus of  $\mathfrak{D}^i$ , there exists a linear function  $\omega_{i,y}$  on  $C_y(\mathfrak{D}^i)$  satisfying Conditions (i)-(iii) above. It is enough to check that  $\omega_{i,y}$  and  $\omega_{j,y}$  coincide on  $C_y(\mathfrak{D}^i) \cap C_y(\mathfrak{D}^j)$ .

Indeed  $C_y(\mathfrak{D}^i) \cap C_y(\mathfrak{D}^j)$  is a common face to both  $C_y(\mathfrak{D}^i)$  and  $C_y(\mathfrak{D}^j)$ . It generated by colored or uncolored rays. Moreover both  $\omega_{i,y}$  and  $\omega_{j,y}$  are linear on the common face and coincide on the rays, which concludes the proof.  $\square$

**Motivic volume for horizontal arcs.** Using the stringy support function, we compute the motivic integral over all subsets of the form  $\underline{\text{val}}^{-1}(C, \nu, 0)$ , i.e., subsets corresponding to horizontal arcs, see Section 4.1.

**Lemma 5.12.** *Let  $(y, \nu, 0)$  be an element of  $|\mathcal{E}'|_\Gamma = |\mathcal{E}'|_\Gamma$ . We have*

$$\int_{\underline{\text{val}}^{-1}(C, \nu, 0)} \mathbb{L}^{-\text{ord}_{K_{X'/X}}} d\mu_{X'} = [G/H][\Gamma] \mathbb{L}^{\vartheta_X(C, \nu, 0)}.$$

*Proof.* We only need to prove the result for a  $T$ -variety, as the general case is obtained by parabolic induction. Let  $\sigma \in \Sigma(\mathcal{E}')$  be a cone containing  $\nu$  and  $\mathfrak{D}$  be a  $\sigma$ -polyhedral divisor of  $\mathcal{E}'$  with (affine) locus  $C_0$ . The rays of  $\sigma$  either correspond to exceptional divisors of  $\psi : X' \rightarrow X$ , or are elements of  $\text{Ray}(\mathcal{E})$ . For the exceptional divisors we use the notation of Section 5.4. Up to renumbering we may assume the exceptional rays of  $\sigma$  are  $\rho_1, \dots, \rho_a$  and we denote the remaining rays by  $\tau_{a+1}, \dots, \tau_r$ . Write  $V_{\mathbb{Q}} := \sigma^{\vee} \cap (-\sigma^{\vee})$  and choose a basis  $(v_1, \dots, v_{d-1-r})$  of  $V_{\mathbb{Q}} \cap M$ . Let  $u_1, \dots, u_a, u_{a+1}, \dots, u_r$  be the duals of  $\rho_1, \dots, \rho_a, \tau_{a+1}, \dots, \tau_r$ .

From [PS11, Remark 3.16 (2)] we know that the ideal of a  $T$ -stable divisor  $D'_{\rho_j}$  is given by

$$(23) \quad I(D'_{\rho_j}) = \bigoplus_{m \in (\sigma^{\vee} \setminus \rho_j^{\perp}) \cap M} H^0(C_0, \mathcal{O}_{C_0}([\mathfrak{D}(m)])) \otimes \chi^m.$$

If  $\alpha \in \underline{\text{val}}^{-1}(C, \nu, 0)$  and  $f \otimes \chi^m$  is a homogeneous element of degree  $m$  in  $A(C_0, \mathfrak{D})$ , we have

$$\alpha^*(f \otimes \chi^m) = t^{\nu(m)} \omega(m) R(f),$$

where we use the notation of Remark 4.3. Hence

$$\sup \{k \in \mathbb{N} \mid \alpha^*(f \otimes \chi^m) \in (t^k)\} = \langle m, \nu \rangle.$$

Write

$$\nu = \nu_1 \rho_1 + \dots + \nu_a \rho_a + \nu_{a+1} \tau_{a+1} + \dots + \nu_r \tau_r.$$

We have  $\nu_i \in \mathbb{N}$  for  $1 \leq i \leq r$ . An element  $m$  of  $(\sigma^{\vee} \setminus \rho_j^{\perp}) \cap M$  can be decomposed as

$$m = m_1 u_1 + \dots + m_r u_r + m'_1 v_1 + \dots + m'_{d-1-r} v_{d-1-r}$$

with  $m_i \in \mathbb{N}$  for  $1 \leq i \leq r$ ,  $m_j \neq 0$ , and  $m'_1, \dots, m'_{d-1-r}$  in  $\mathbb{Z}$ . Hence  $\langle m, \nu \rangle = \sum_{i=1}^r m_i \langle u_i, \nu \rangle$ . Finally

$$\min \{ \langle m, \nu \rangle \mid m \in (\sigma^{\vee} \setminus \rho_j^{\perp}) \cap M \} = \langle u_j, \nu \rangle.$$

It follows that  $\text{ord}_{D'_{\rho_j}} \alpha = \langle u_j, \nu \rangle$ . Moreover  $\text{ord}_{D'_{(\rho_i, \rho_i)}} \alpha = 0$  since  $\alpha \in \underline{\text{val}}^{-1}(C, \nu, 0)$ . Indeed  $\alpha(0) \notin D'_{(y_i, \rho_i)}$ , see [Bli11, Section 2.4]. We obtain

$$(24) \quad \text{ord}_{K_{X'/X}}(\alpha) = \sum_{i=1}^a (-1 - \vartheta_X(C, \rho_i, 0)) \langle u_i, \nu \rangle.$$

Now

$$\begin{aligned} \int_{\underline{\text{val}}^{-1}(y, \nu, 0)} \mathbb{L}^{-\text{ord}_{K_{X'/X}}} d\mu_{X'} &= \sum_{s \in \mathbb{Q}} \mu_{X'} \left( \underline{\text{val}}^{-1}(y, \nu, 0) \cap \text{ord}_{K_{X'/X}}^{-1}(s) \right) \mathbb{L}^{-s} \\ &= \mu_{X'}(\underline{\text{val}}^{-1}(C, \nu, 0)) \mathbb{L}^{\sum_{i=1}^a (1 + \vartheta_X(C, \rho_i, 0)) \langle u_i, \nu \rangle} \\ &= [\Gamma][G/H] \mathbb{L}^{-\sum_{j=1}^r \langle u_j, \nu \rangle + \sum_{i=1}^a (1 + \vartheta_X(C, \rho_i, 0)) \langle u_i, \nu \rangle} \end{aligned}$$

Here the second equality follows from Equation (24), and the last one from Theorem 4.19. To conclude, it only remains to check that

$$\vartheta_X(C, \nu, 0) = - \sum_{j=1}^r \langle u_j, \nu \rangle + \sum_{i=1}^a (1 + \vartheta_X(C, \rho_i, 0)) \langle u_i, \nu \rangle.$$

Using the linearity of  $\vartheta_X$  on each Cayley cone and the decomposition of  $\nu$  on the integral basis, we obtain

$$\vartheta_X(C, \nu, 0) = \sum_{i=1}^a \langle u_i, \nu \rangle \vartheta_X(C, \rho_i, 0) + \sum_{j=a+1}^r \langle u_j, \nu \rangle \vartheta_X(C, \tau_j, 0).$$

Finally, by definition of  $\vartheta_X$ , we have  $\vartheta_X(C, \tau_j, 0) = -1$  for  $a+1 \leq j \leq r$ , which concludes the proof.  $\square$

**Corollary 5.13.** *With the notations of Lemma 5.12, we have*

$$\int_{\underline{\text{val}}^{-1}(C, \nu, 0)} \mathbb{L}^{-\text{ord}_{K_{X'/X}}} d\mu_{X'} = [G/H][\Gamma] \mathbb{L}^{\omega_X(C, \nu, 0)}.$$

*Proof.* The corollary follows directly from Lemma 5.12 by noticing that the functions  $\vartheta_X$  and  $\omega_X$  coincide on rays contained in  $N_{\mathbb{Q}}$ , then using linearity.  $\square$

**Motivic volume for vertical arcs.** Let us now use the stringy support function  $\omega_X$  to compute motivic integrals over vertical components of the arc space  $\mathcal{L}_\Gamma(X')$ .

More precisely, let  $\xi = (y, \nu, \ell)$  be an element of  $|\mathcal{E}'|_\Gamma \cap \mathcal{N}$  such that  $\ell \geq 1$ . As in the previous lemma we assume  $G = T$  and  $H = \{e\}$ . Let  $\mathfrak{D}$  be a polyhedral divisor of  $\mathcal{E}'$  with (affine) locus  $C_0$  such that  $(\nu, \ell) \in C_y(\mathfrak{D})$ , and denote by  $\sigma$  the tail of  $\mathfrak{D}$ . Let  $r$  be the dimension of  $\sigma$  and  $s$  be the dimension of  $C_y(\mathfrak{D})$ . Up to renumbering of the exceptional divisors of Section 5.4, we may assume that the rays of  $\sigma$  either correspond to exceptional divisors  $D'_{\rho_1}, \dots, D'_{\rho_a}$  of  $\psi : X' \rightarrow X$ , or are elements  $\tau_{a+1}, \dots, \tau_r$  of  $\text{Ray}(\mathcal{E})$ . Similarly the other rays

$$\lambda_1 := (\kappa(p_1)p_1, \kappa(p_1)), \dots, \lambda_{s-r} := (\kappa(p_{s-r})p_{s-r}, \kappa(p_{s-r}))$$

of  $C_y(\mathfrak{D})$  comprise the exceptional divisors  $D'_{(y,p_1)}, \dots, D'_{(y,p_u)}$ .

Let  $\Lambda = M \oplus \mathbb{Z}$ . Write  $V_{\mathbb{Q}} := C_y(\mathfrak{D})^\vee \cap (-C_y(\mathfrak{D}))^\vee$  and choose a basis  $(v_1, \dots, v_{d-s})$  of  $V_{\mathbb{Q}} \cap \Lambda$ . Let  $u_1, \dots, u_a, u_{a+1}, \dots, u_r$  be the duals of  $\rho_1, \dots, \rho_a, \tau_{a+1}, \dots, \tau_r$  in  $N$  and  $w_1, \dots, w_{s-r}$  be the duals of  $\lambda_1, \dots, \lambda_{s-r}$ . The elements

$$(u_1, 0), \dots, (u_r, 0); \pm v_1, \dots, \pm v_{d-s}; w_1, \dots, w_{s-r}$$

constitute a set of generators of the semigroup  $C_y(\mathfrak{D})^\vee \cap \Lambda$ .

**Lemma 5.14.** *We have*

$$\int_{\underline{\text{val}}^{-1}(\xi)} \mathbb{L}^{-\text{ord}_{K_{X'/X}} d\mu_{X'}} = [G/H](\mathbb{L} - 1) \mathbb{L}^{\vartheta_X(\xi) - \sum_{i=1}^{s-r} (\kappa(p_i)b_y + \kappa(p_i)) \langle w_i, (\nu, \ell) \rangle}.$$

*Proof.* Let  $\pi \in \mathbb{C}(C)^*$  be a uniformizer of  $y$ . Recalling the expression of the ideal  $I(D'_{\rho_j})$  from Equation (23) we can compute  $\text{ord}_{D'_{\rho_j}}(\alpha)$  by studying

$$\alpha^*(\pi^k \otimes \chi^m) \text{ for all } (m, k) \in (C_y(\mathfrak{D})^\vee \setminus (\rho_j, 0)^\perp) \cap \Lambda.$$

We may then use the same argument as in Lemma 5.12. We obtain

$$\text{ord}_{D'_{\rho_j}}(\alpha) = \langle u_j, \nu \rangle$$

for any  $\alpha \in \underline{\text{val}}^{-1}(\xi)$ .

Let us now recall the description of  $I(D'_{(y,p_i)})$  from [PS11, Remark 3.16 (i)]:

$$I(D'_{(y,p_i)}) = \bigoplus_{m \in \sigma^\vee \cap M} H^0(C_0, \mathcal{O}_{C_0}([\mathfrak{D}(m)])) \cap \{f \in \mathbb{C}(C_0) \mid \text{ord}_y f + \langle m, p_i \rangle > 0\} \otimes \chi^m.$$

As before we can compute  $\text{ord}_{D'_{(y,p_i)}}(\alpha)$  from the computation of  $\alpha^*(\pi^k \otimes \chi^m)$  for any

$$(m, k) \in (C_y(\mathfrak{D})^\vee \setminus (p_i, 1)^\perp) \cap \Lambda.$$

We find

$$\text{ord}_{D'_{(y,p_i)}}(\alpha) = \langle w_i, (\nu, \ell) \rangle.$$

It follows that

$$(25) \quad \text{ord}_{K_{X'/X}}(\alpha) = \sum_{j=1}^a c_j \langle u_j, \nu \rangle + \sum_{i=1}^{s-r} d_i \langle w_i, (\nu, \ell) \rangle$$

where  $c_j = -1 - \vartheta_X(C, \rho_j, 0)$  and

$$d_i = \kappa(p_i)b_y + \kappa(p_i) - 1 - \vartheta_X(y, \kappa(p_i)p_i, \kappa(p_i)).$$

Note that  $d_i = 0$  for  $u < i \leq s-r$ . Now

$$\begin{aligned} \int_{\underline{\text{val}}^{-1}(\xi)} \mathbb{L}^{-\text{ord}_{K_{X'/X}} d\mu_{X'}} &= \sum_{s \in \mathbb{Q}} \mu_{X'} \left( \underline{\text{val}}^{-1}(\xi) \cap \text{ord}_{K_{X'/X}}^{-1}(s) \right) \mathbb{L}^{-s} \\ &= \mu_{X'}(\underline{\text{val}}^{-1}(\xi)) \mathbb{L}^{-\sum_{j=1}^a c_j \langle u_j, \nu \rangle - \sum_{i=1}^{s-r} d_i \langle w_i, (\nu, \ell) \rangle} \\ &= [G/H](\mathbb{L} - 1) \mathbb{L}^{-\sum_{j=1}^r \langle u_j, \nu \rangle - \sum_{i=1}^{s-r} \langle w_i, (\nu, \ell) \rangle - \sum_{j=1}^a c_j \langle u_j, \nu \rangle - \sum_{i=1}^{s-r} d_i \langle w_i, (\nu, \ell) \rangle}. \end{aligned}$$

Here the second equality follows from Equation (25), and the last one from Theorem 4.19. To conclude, it only remains to check that

$$\vartheta_X(\xi) - \sum_{i=1}^{s-r} (\kappa(p_i)b_y + \kappa(p_i)) \langle w_i, (\nu, \ell) \rangle = - \sum_{j=1}^r \langle u_j, \nu \rangle - \sum_{i=1}^{s-r} \langle w_i, (\nu, \ell) \rangle - \sum_{j=1}^a c_j \langle u_j, \nu \rangle - \sum_{i=1}^{s-r} d_i \langle w_i, (\nu, \ell) \rangle.$$

Using the linearity of  $\vartheta_X$  on each Cayley cone and the decomposition of  $\nu$  on the integral basis, we obtain

$$\begin{aligned} \vartheta_X(\xi) - \sum_{i=1}^{s-r} (\kappa(p_i)b_y + \kappa(p_i)) \langle w_i, (\nu, \ell) \rangle &= \sum_{j=1}^a \langle u_j, \nu \rangle \vartheta_X(C, \rho_j, 0) + \sum_{j=a+1}^r \langle u_j, \nu \rangle \vartheta_X(C, \tau_j, 0) \\ &\quad + \sum_{i=1}^{s-r} \langle (\nu, \ell), w_i \rangle \vartheta_X(y, \kappa(p_i)p_i, \kappa(p_i)) \\ &\quad - \sum_{i=1}^{s-r} (\kappa(p_i)b_y + \kappa(p_i)) \langle w_i, (\nu, \ell) \rangle \\ &= \sum_{j=1}^a (-1 - c_j) \langle u_j, \nu \rangle - \sum_{j=a+1}^r \langle u_j, \nu \rangle - \sum_{i=1}^{s-r} (1 + d_i) \langle w_i, (\nu, \ell) \rangle, \end{aligned}$$

which concludes the proof.  $\square$

The following proposition rephrases the formula of Lemma 5.14 in terms of the support function  $\varpi_X$ . As such it is a key step in the proof of our main result, Theorem 5.17.

**Proposition 5.15.** *Let  $\mathcal{E}$  be a colored divisorial fan on  $(C, G/H)$ . Assume  $X := X(\mathcal{E})$  is  $\mathbb{Q}$ -Gorenstein and let  $X' = X(\mathcal{E}')$  be the desingularization of  $X$  as defined in Equation (22). Then for any  $(y, p)$  in  $\text{Vert}(\mathcal{E}') \setminus \text{Vert}(\mathcal{E})$  we have*

$$(\vartheta_{X'} - \vartheta_X)(y, \kappa(p)p, \kappa(p)) = -1 - \omega_X(y, \kappa(p)p, \kappa(p)).$$

*Proof.* Since the problem is local we may assume that  $\mathcal{E} = \{(\mathfrak{D}, \mathcal{F})\}$ . We first show the statement when  $\mathcal{E} = \tilde{\mathcal{E}}$ , that is, that the coloration  $\mathcal{F}$  is trivial and the locus  $C_0$  of  $\mathfrak{D}$  is affine. By parabolic induction we may also assume that  $G = T$  and  $H = \{e\}$ . Write

$$\mathfrak{D} = \sum_{z \in C_0} \Delta_z \cdot [z]$$

and let  $y$  be a point of  $C_0$ . Denote by  $(\mathfrak{D}^i)_{i \in J}$  the set of polyhedral divisors in  $\mathcal{E}'$ .

The cones  $C_y(\mathfrak{D}^i)$  for  $i \in J$  are cones in  $N_{\mathbb{Q}} \oplus \mathbb{Q}$ . We may then consider the fan  $\Sigma'$  they generate in  $N_{\mathbb{Q}} \oplus \mathbb{Q}$ . Let us denote by  $X_{tor}$  (resp. by  $X'_{tor}$ ) the toric  $(T \times \mathbb{C}^*)$ -variety associated with the cone  $C_y(\mathfrak{D}) \subset N_{\mathbb{Q}} \oplus \mathbb{Q}$  (resp. with the fan  $\Sigma'$ ). By Lemma 5.5 we now that  $X_{tor}$  is  $\mathbb{Q}$ -Gorenstein. Moreover  $X'_{tor}$  is smooth, and the proper birational map  $X'_{tor} \rightarrow X_{tor}$  is a  $(T \times \mathbb{C}^*)$ -equivariant resolution of singularities, obtained by a sequence of star subdivisions corresponding to the star subdivisions used in desingularizing  $X$  (see Section 5.3).

Define a  $\sigma$ -polyhedral divisor  $\hat{\mathfrak{D}}$  with locus  $\mathbb{A}^1$  on  $(\mathbb{P}^1, T)$  by setting  $\hat{\mathfrak{D}} := \sum_{z \in \mathbb{A}^1} \hat{\Delta}_z \cdot [z]$ , where

$$\hat{\Delta}_z = \begin{cases} \Delta_y & \text{if } z = 0, \\ \sigma & \text{otherwise.} \end{cases}$$

Similarly, for any  $i \in J$ , we define a polyhedral divisor  $\hat{\mathfrak{D}}^i$  with locus  $\mathbb{A}^1$  associated with  $\mathfrak{D}^i$ , and we let  $\hat{\mathcal{E}}'$  be the corresponding divisorial fan.

We have a diagram

$$\begin{array}{ccc} X'_{tor} & \xrightarrow{q'} & X_{tor} \\ \parallel & & \parallel \\ X(\hat{\mathcal{E}}') & \xrightarrow{q'} & X(\hat{\mathfrak{D}}). \end{array}$$

Indeed, the equality on the right comes from the description of the  $\mathbb{C}$ -algebra  $A(\mathbb{A}^1, \hat{\mathfrak{D}})$  viewed as a  $T \times \mathbb{C}^*$ -algebra. More precisely,  $A(\mathbb{A}^1, \hat{\mathfrak{D}})$  identifies with the semigroup algebra  $\mathbb{C}[C_0(\hat{\mathfrak{D}})^\vee \cap (M \oplus \mathbb{Z})]$ . The other equality are similar.

Consider the colored piecewise linear functions  $\theta_X = (\vartheta_X, (-a_\alpha)) \in \text{PL}(\mathfrak{D})$  and  $\theta_{X'} = (\vartheta_{X'}, (-a_\alpha)) \in \text{PL}(\mathcal{E}')$  satisfying the hypotheses of Proposition 5.4, which induce functions respectively on  $C_y(\mathfrak{D})$  and  $\Sigma'$ . To prove the proposition we compute the discrepancy of  $q'$  in two different ways.

Let us first compute the discrepancy of  $q' : X(\hat{\mathcal{E}}') \rightarrow X(\hat{\mathfrak{D}})$  using the functions  $\vartheta_{X'}$  and  $\vartheta_X$ . By definition the Weil  $\mathbb{Q}$ -divisor

$$\begin{aligned} \hat{K} &:= \sum_{(0,p) \in \text{Vert}(\hat{\mathcal{E}}') \setminus \text{Vert}(\hat{\mathfrak{D}})} (\vartheta_{X(\hat{\mathcal{E}}')} - \vartheta_{X(\hat{\mathfrak{D}})})(0, \kappa(p)p, \kappa(p)) D'_{(0,p)} + \sum_{\rho \in \text{Ray}(\hat{\mathcal{E}}') \setminus \text{Ray}(\hat{\mathfrak{D}})} (\vartheta_{X(\hat{\mathcal{E}}')} - \vartheta_{X(\hat{\mathfrak{D}})})(\mathbb{P}^1, \rho, 0) D'_\rho \\ &= \sum_{(y,p) \in \text{Vert}(\mathcal{E}') \setminus \text{Vert}(\mathfrak{D})} (\vartheta_{X'} - \vartheta_X)(y, \kappa(p)p, \kappa(p)) D'_{(0,p)} + \sum_{\rho \in \text{Ray}(\mathcal{E}') \setminus \text{Ray}(\mathfrak{D})} (\vartheta_{X'} - \vartheta_X)(C, \rho, 0) D'_\rho \end{aligned}$$

is a relative canonical divisor. Next we compute the discrepancy of  $q' : X'_{\text{tor}} \rightarrow X_{\text{tor}}$  using the function  $\omega_X$ . By [BM13, Proposition 4.2] the Weil  $\mathbb{Q}$ -divisor

$$K := \sum_{(y,p) \in \text{Vert}(\mathcal{E}') \setminus \text{Vert}(\mathfrak{D})} (-1 - \omega_X(y, \kappa(p)p, \kappa(p))) D'_{(0,p)} + \sum_{\rho \in \text{Ray}(\mathcal{E}') \setminus \text{Ray}(\mathfrak{D})} (-1 - \omega_X(y, \rho, 0)) D'_\rho$$

is a relative canonical divisor. The divisors  $K$  and  $\hat{K}$  are linearly equivalent. By [KM98, Lemma 3.39] we know that they are in fact equal. Thus

$$(26) \quad -1 - \omega_X(y, \kappa(p)p, \kappa(p)) = (\vartheta_{X(\hat{\mathcal{E}}')} - \vartheta_{X(\hat{\mathfrak{D}})})(0, \kappa(p)p, \kappa(p))$$

for any  $(y, p) \in \text{Vert}(\mathcal{E}') \setminus \text{Vert}(\mathfrak{D})$ , which concludes the case where  $\mathcal{E} = \hat{\mathcal{E}}$ .

For the case where  $\mathcal{E}$  is general, we remark that the functions  $\omega_X(\hat{\mathcal{E}}) + \vartheta_X - \vartheta_{X(\hat{\mathcal{E}})}$ ,  $\omega_X$  coincide. Indeed, they are linear on each Cayley cone of  $\mathcal{E}$  and they have the same values on the rays. Hence taking this into account we obtain Equation (26) in the general case, which concludes the proof of the proposition.  $\square$

**Corollary 5.16.** *With the notations of Lemma 5.14, when  $\ell \geq 1$ , we have*

$$\int_{\text{val}^{-1}(y, \nu, \ell)} \mathbb{L}^{-\text{ord}_{\kappa_{X'/X}}} d\mu_{X'} = [G/H](\mathbb{L} - 1) \mathbb{L}^{\omega_X(y, \nu, \ell)}.$$

*Proof.* We proved in Lemma 5.14 that

$$(27) \quad \int_{\text{val}^{-1}(y, \nu, \ell)} \mathbb{L}^{-\text{ord}_{\kappa_{X'/X}}} d\mu_{X'} = [G/H](\mathbb{L} - 1) \mathbb{L}^{\vartheta_X(y, \nu, \ell) - \sum_{i=1}^{s-r} (\kappa(p_i) b_y + \kappa(p_i)) \langle w_i, (\nu, \ell) \rangle}.$$

By definition of  $\vartheta_{X'}$  and  $\vartheta_X$ , we have

$$\vartheta_X(y, \nu, \ell) - \sum_{i=1}^{s-r} (\kappa(p_i) b_y + \kappa(p_i)) \langle w_i, (\nu, \ell) \rangle = (\vartheta_X - \vartheta_{X'})(y, \nu, \ell) - \sum_{i=1}^r \langle \nu, u_i \rangle - \sum_{j=1}^r \langle (\nu, \ell), w_j \rangle.$$

Moreover by linearity

$$(28) \quad (\vartheta_X - \vartheta_{X'})(y, \nu, \ell) = \sum_{i=1}^r \langle \nu, u_i \rangle (\vartheta_X - \vartheta_{X'})(y, \rho_i, 0) + \sum_{j=1}^{s-r} \langle (\nu, \ell), w_j \rangle (\vartheta_X - \vartheta_{X'})(y, \kappa(p_j) p_j, \kappa(p_j)).$$

By the proof of Corollary 5.13 we have

$$(\vartheta_X - \vartheta_{X'})(y, \rho_i, 0) = \omega_X(y, \rho_i, 0) - 1,$$

and by Proposition 5.15 we have

$$(\vartheta_X - \vartheta_{X'})(y, \kappa(p_j) p_j, \kappa(p_j)) = 1 + \omega_X(y, \kappa(p_j) p_j, \kappa(p_j)).$$

We obtain

$$(29) \quad (\vartheta_X - \vartheta_{X'})(y, \nu, \ell) - \sum_{i=1}^r \langle \nu, u_i \rangle - \sum_{j=1}^{s-r} \langle (\nu, \ell), w_j \rangle = \omega_X(y, \nu, \ell).$$

Replacing the exponent of  $\mathbb{L}$  in the right-hand side of Equation (27) by  $\omega_X(y, \nu, \ell)$  using Equations (28) and (29), we obtain the stated result.  $\square$

**The stringy motivic volume.** Using the previous lemmas and corollaries, we obtain the main result of the section.

**Theorem 5.17.** *Let  $\mathcal{E}$  be a colored divisorial fan on  $(C, G/H)$  such that  $X = X(\mathcal{E})$  is  $\mathbb{Q}$ -Gorenstein with log terminal singularities. Then for any open dense subset  $\Gamma$  in  $C \setminus \text{Sp}(\mathcal{E})$  we have*

$$\mathcal{E}_{st}(X) = [G/H] \left[ \sum_{(y, \nu, \ell) \in |\mathcal{E}|_{\Gamma \cap \mathcal{N}}} [X_\ell] \mathbb{L}^{\omega_X(y, \nu, \ell)} \right],$$

where  $X_0 = \Gamma$  and  $X_\ell = \mathbb{A}^1 \setminus \{0\}$  if  $\ell \geq 1$ .

*Proof.* By combining Lemma 5.10 and Corollaries 5.13 and 5.16, we obtain

$$\mathcal{E}_{st}(X) = [G/H][\Gamma] \left( \sum_{\nu \in |\Sigma(\mathcal{E})| \cap N} \mathbb{L}^{\omega_X(C, \nu, 0)} \right) + [G/H](\mathbb{L} - 1) \left( \sum_{(y, \nu, \ell) \in (|\mathcal{E}|_{\Gamma} \setminus |\Sigma(\mathcal{E})|) \cap \mathcal{N}} \mathbb{L}^{\omega_X(y, \nu, \ell)} \right),$$

which concludes the proof.  $\square$

The following result is an immediate consequence of Theorem 5.17 and of the definition of the stringy  $E$ -polynomial in Definition 2.18.

**Corollary 5.18.** *Under the hypotheses of Theorem 5.17, the stringy  $E$ -function of  $X$  is computed as follows*

$$E_{st}(X; u, v) = E(G/H; u, v) \left[ \sum_{(y, \nu, \ell) \in |\mathcal{E}|_{\Gamma \cap \mathcal{N}}} E(X_\ell; u, v) (uv)^{\omega_X(y, \nu, \ell)} \right],$$

where

$$E(X_\ell; u, v) = \begin{cases} E(\Gamma; u, v) & \text{if } \ell = 0, \\ uv - 1 & \text{otherwise.} \end{cases}$$

**Rational form and candidate poles.** Let us end this section by expressing the rational form of the stringy volume  $\mathcal{E}_{st}(X)$  (see Definition 2.15) in terms of the combinatorial object  $\mathcal{E}$ .

For a special point  $y \in \text{Sp}(\mathcal{E})$ , we denote by  $\mathcal{E}_y$  the fan generated by the Cayley cones of the form  $C_y(\mathfrak{D}^i)$ , where  $i$  is in  $J$ . Let  $\tau \in \mathcal{E}_y$ . Fix a fan  $\Sigma_\tau$  with support  $\tau$  such that every cone of  $\Sigma_\tau$  is simplicial and the cones of dimension one in  $\Sigma_\tau$  are exactly the faces of dimension one of  $\tau$ . Such a fan always exists according to [Ewa96, V.4].

For an arbitrary polyhedral cone  $\lambda \subset N_{\mathbb{Q}} \oplus \mathbb{Q}$ , we denote by  $\lambda(1)$  the set of primitive generators in  $N \oplus \mathbb{Z}$  of the rays of  $\lambda$ . The *fundamental parallelootope* of  $\lambda$  is the set

$$P_\lambda := \left\{ \sum_{\rho \in \lambda(1)} \mu_\rho \rho \in N \oplus \mathbb{Z} \mid 0 \leq \mu_\rho < 1, \rho \in \lambda(1) \right\} \subset N_{\mathbb{Q}}.$$

By the symbol  $\text{Prop}(\lambda)$  we denote the set of proper faces of  $\lambda$ .

For an element  $v \in N \oplus \mathbb{Z}$  we denote by  $\chi^v$  the Laurent monomial associated with  $v$ . We define two functions  $L_1$  and  $L_2$  via the equalities

$$L_1(\gamma) = \left( \sum_{v \in P_\gamma} \chi^v \right) \prod_{\rho \in \tau(1) \setminus \gamma(1)} (1 - \chi^\rho)$$

and

$$L_2(\gamma) = L_1(\gamma) - \sum_{\gamma' \in \text{Prop}(\gamma)} L_1(\gamma'),$$

for every cone  $\gamma \in \Sigma_\tau$ . We also introduce

$$Q(\tau, \Sigma_\tau) := \sum_{\gamma \in \Sigma_\tau \setminus \text{Prop}(\tau)} L_2(\gamma).$$

The next theorem is a direct consequence of the proof of the Gordan Lemma in [CLS11, Proposition 1.2.17] (see also [Bri88, Section 2]).

**Theorem 5.19.** *Let  $\tau \subset N_{\mathbb{Q}} \oplus \mathbb{Q}$  be a strongly convex simplicial polyhedral cone. Then the series  $\sum_{u \in \tau \cap N \oplus \mathbb{Z}} \chi^u$  has a rational form*

$$\sum_{u \in \tau \cap N \oplus \mathbb{Z}} \chi^u = \frac{\sum_{n \in P_{\tau}} \chi^n}{\prod_{\rho \in \tau(1)} (1 - \chi^{\rho})}.$$

The following lemma gives an interpretation of the polynomial  $Q(\tau, \Sigma_{\tau})$  in term of the following generating function:

$$S(\tau) = \sum_{u \in \tau^{\circ} \cap (N \oplus \mathbb{Z})} \chi^u,$$

where  $\tau^{\circ}$  is the relative interior of  $\tau$ .

**Lemma 5.20.** *With the same notation as above, the series  $S(\tau)$  admits a rational form*

$$S(\tau) = Q(\tau, \Sigma_{\tau}) \prod_{\rho \in \tau(1)} (1 - \chi^{\rho})^{-1}.$$

*Proof.* Let  $\gamma \in \Sigma_{\tau}$ . By Theorem 5.19 we have

$$\sum_{u \in \gamma \cap (N \oplus \mathbb{Z})} \chi^u = L_1(\gamma) \prod_{\rho \in \tau(1)} (1 - \chi^{\rho})^{-1}.$$

This implies that

$$S(\gamma) = L_2(\gamma) \prod_{\rho \in \tau(1)} (1 - \chi^{\rho})^{-1}.$$

We conclude by using the equality  $S(\tau) = \sum_{\gamma \in \Sigma_{\tau} \setminus \text{Prop}(\tau)} S(\gamma)$ .  $\square$

*Remark 5.21.* Lemma 5.20 implies that the polynomial  $Q(\tau, \Sigma_{\tau})$  does not depend on the choice of the simplicial fan  $\Sigma_{\tau}$ . In the sequel, we denote it by  $Q(\tau)$ .

**Lemma 5.22.** *Let  $\mathfrak{D}$  be a proper  $\sigma$ -polyhedral divisor on  $\mathbb{P}^1$ . If  $\rho$  is a ray of  $\sigma$  such that  $\deg(\mathfrak{D}) \cap \rho \neq \emptyset$ , then there exists  $\lambda \in \mathbb{Q}_{>0}$  and a vertex  $v \in \deg(\mathfrak{D})$  such that  $\rho = \lambda v$ .*

*Proof.* Recall that a face  $F$  of  $\deg(\mathfrak{D})$  is by definition the intersection of  $\deg(\mathfrak{D})$  with a hyperplane  $\Pi' \subset N_{\mathbb{Q}}$  such that a halfspace  $\Pi'_+$  with boundary  $\Pi'$  satisfies  $\deg(\mathfrak{D}) \subset \Pi'_+$ . Since  $\rho$  is a face of  $\sigma$ , there exists a halfspace  $\Pi_+ \subset N_{\mathbb{Q}}$  with proper face a hyperplane  $\Pi$  such that  $\sigma \subset \Pi_+$  and  $\Pi \cap \sigma = \rho$ . By properness of  $\mathfrak{D}$  we have  $\deg(\mathfrak{D}) \subset \sigma$ . Hence  $\Pi \cap \deg(\mathfrak{D})$  is a face of  $\deg(\mathfrak{D})$  contained in  $\rho$ . This finishes the proof of the lemma.  $\square$

In the next lemma we study the sign of the function  $\omega_X$  on each cone of the fan  $\mathcal{E}_y$ , where  $y \in \text{Sp}(\mathcal{E})$  is a special point of  $\mathcal{E}$ . To do this we use the combinatorial description of the log terminal condition on  $X = X(\mathcal{E})$  given in Theorem 5.6.

**Lemma 5.23.** *Let  $\tau \in \mathcal{E}_y$  and  $\rho \in \tau(1)$ . Then  $\omega_X(y, \rho) < 0$ .*

*Proof.* Let  $(\mathfrak{D}, \mathcal{F})$  be a colored polyhedral divisor in  $\mathcal{E}$  such that  $\rho$  is a primitive generator of a ray of  $C_y(\mathfrak{D})$ . Assume first that  $\rho \in \sigma(1)$ , where  $\sigma$  is the tail of  $\mathfrak{D}$ . If  $\rho \in \text{Ray}(\mathcal{E})$  or if a color of  $\mathcal{F}$  is mapped onto the ray  $\mathbb{Q}_{\geq 0}\rho$ , then by Proposition 5.11 the rational number  $\omega_X(y, \rho)$  is equal, respectively, to  $-1$  or to  $-\ell a_{\alpha} \in \mathbb{Q}_{<0}$  for some  $\ell \in \mathbb{Q}_{>0}$ .

Otherwise, by Theorem 5.6 the log terminal condition implies that the locus of  $\mathfrak{D}$  is the projective line  $\mathbb{P}^1$  and that  $\deg(\mathfrak{D}) \cap \mathbb{Q}_{\geq 0}\rho \neq \emptyset$ . By Lemma 5.22, there exists of a vertex  $v$  of  $\deg(\mathfrak{D})$  such that  $\rho = \mu v$  for some  $\mu \in \mathbb{Q}_{>0}$ .

Let us use a similar argument as in the proof of [LT16, Theorem 2.22]. Write  $\mathfrak{D} = \sum_{z \in \mathbb{P}^1} \Delta_z \cdot [z]$ . Then (see Definition 5.3), we can find  $e \in M$ ,  $\alpha \in \mathbb{Q}_{>0}$  and a rational function  $f \in \mathbb{C}(\mathbb{P}^1)^*$  such that for any  $z \in \mathbb{P}^1$  and any vertex  $v \in \Delta_z$  we have

$$(30) \quad \theta_X(z, \kappa(v)v, \kappa(v)) = \alpha \kappa(v) (\langle e, v \rangle + \text{ord}_z(f)) = \kappa(v) b_z + \kappa(v) - 1,$$



where  $K_{\mathbb{P}^1} = \sum_{z \in \mathbb{P}^1} b_z \cdot [z]$  is a canonical divisor. Let  $(v_z)_{z \in \mathbb{P}^1}$  be a sequence of elements of  $N_{\mathbb{Q}}$  such that for any  $z \in \mathbb{P}^1$ ,  $v_z$  is a vertex of  $\Delta_z$  and  $v = \sum_{z \in \mathbb{P}^1} v_z$ . From Equation (30) and Proposition 5.11 we obtain

$$(31) \quad \begin{aligned} \omega_X(C, v, 0) &= \theta_X(C, v, 0) = \langle e, v \rangle \\ &= \sum_{z \in \mathbb{P}^1} \frac{1}{\alpha \kappa(v_z)} (\kappa(v_z) \langle e, v_z \rangle + \text{ord}_z(f)) \\ &= \frac{1}{\alpha} \left( \deg K_{\mathbb{P}^1} + \sum_{z \in \mathbb{P}^1} \left( 1 - \frac{1}{\kappa(v_z)} \right) \right) < 0. \end{aligned}$$

The inequality is a consequence of the fact that  $\deg K_{\mathbb{P}^1} = -2$  and of the log terminal assumption on  $X = X(\mathcal{E})$  (see Theorem 5.6). If  $\rho$  does not belong to  $N_{\mathbb{Q}}$ , then  $\rho = (\kappa(w)w, \kappa(w))$  for some vertex  $w$  of  $\Delta_y$ , and hence by Proposition 5.11 we have

$$\omega_X(y, \rho) = \theta_X(y, \kappa(w)w, \kappa(w)) = -1.$$

This concludes the proof of the lemma.  $\square$

*Remark 5.24.* The argument of the proof of Lemma 5.23 shows as expected from Theorem 5.17 that the function  $\omega_X$  does not depend on the choice of a canonical divisor  $K_C$  of the curve  $C$ . Indeed, assume that there exists a ray  $\rho \in N_{\mathbb{Q}}$  of a Cayley cone which is not in  $\text{Ray}(\mathcal{E})$ . Then by Theorem 5.6 we have  $C = \mathbb{P}^1$ . Using Equation 31, we observe that the value of  $\omega_X$  at  $(C, \rho, 0)$  depends only on the canonical class of  $\mathbb{P}^1$ . The values of  $\omega_X$  at the other rays of the Cayley cones of  $\mathcal{E}$  are by definition equal to  $-1$ , which concludes the remark.

**Corollary 5.25.** *Let  $y \in \text{Sp}(\mathcal{E})$  be a special point and  $\tau$  be an element of  $\mathcal{E}$ . Denote by  $\mathbb{C}[[\tau \cap (N \oplus \mathbb{Z})]]$  the formal completion of the ring  $\mathbb{C}[\tau \cap (N \oplus \mathbb{Z})]$  with respect to the ideal  $I_\tau$  generated by  $\{\chi^v \mid v \in \tau \cap (N \oplus \mathbb{Z}) \setminus \{0\}\}$ . Elements of  $I_\tau$  are of the form*

$$\sum_{v \in \tau \cap (N \oplus \mathbb{Z})} a_v \chi^v \text{ with } a_v \in \mathbb{C}.$$

Let  $m := \min\{a \in \mathbb{Z}_{>0} \mid a\omega_X \text{ takes its values in } \mathbb{Z}\}$ . Then the map

$$\varphi_\tau : (\mathbb{C}[\tau \cap (N \oplus \mathbb{Z})], I_\tau) \rightarrow (\mathbb{C}[\mathbb{L}^{-\frac{1}{m}}], \mathbb{L}^{-\frac{1}{m}} \mathbb{C}[\mathbb{L}^{-\frac{1}{m}}]), \chi^v \mapsto \mathbb{L}^{\omega_X(y, v)}$$

is a well-defined continuous morphism, and therefore it extends to the formal completions

$$\bar{\varphi}_\tau : \mathbb{C}[[\tau \cap (N \oplus \mathbb{Z})]] \rightarrow \mathbb{C}[[\mathbb{L}^{-\frac{1}{m}}]], \chi^v \mapsto \mathbb{L}^{\omega_X(y, v)}.$$

*Proof.* By Lemma 5.23 we have  $\varphi_\tau(I_\tau) \subset \mathbb{L}^{-\frac{1}{m}} \mathbb{C}[\mathbb{L}^{-\frac{1}{m}}]$ . This implies the existence of the morphism  $\bar{\varphi}_\tau$  (see [Mat80, §(23.H)]).  $\square$

Inspired by the theory of Stanley–Reisner rings we introduce the following terminology (see also [BM13, Section 6]).

**Definition 5.26.** Let  $y \in \text{Sp}(\mathcal{E})$  be a special point and consider  $\tau \in \mathcal{E}_y$ . Let  $m \in \mathbb{Z}_{>0}$  as in 5.25. According to Corollary 5.25 we have  $\bar{\varphi}_\tau(Q(\tau)) \in \mathbb{Z}[t^{-1}]$ , where  $t = \mathbb{L}^{\frac{1}{m}}$  and  $Q(\tau) = Q(\tau, \Sigma_\tau)$  is the function of Lemma 5.20. The *Stanley–Reisner polynomial* associated with the pair  $(\tau, \omega_X)$  is the polynomial defined by the equality

$$P(\tau, \omega_X) := P(\tau, \omega_X)(t) = \mathbb{L}^{\eta(\tau, \omega_X)} \cdot \bar{\varphi}_\tau(Q(\tau)) \in \mathbb{Z}[t],$$

where  $\eta(\tau, \omega_X) \in \frac{1}{m}\mathbb{Z}$  is the degree of  $P(\tau, \omega_X)$  divided by  $m$ .

The following result gives a more precise description of the rational form of the stringy motivic volume  $\mathcal{E}_{st}(X)$  of  $X = X(\mathcal{E})$ . In particular, we obtain a finite set of candidate poles of  $\mathcal{E}_{st}(X)$  given in terms of the values of the stringy support function  $\omega_X$ .

**Theorem 5.27.** *For every  $y \in \mathrm{Sp}(\mathcal{E})$ , let us denote by  $\mathcal{E}_y^*$  the set of cones  $\tau$  of  $\mathcal{E}_y$  such that  $\tau \not\subset N_{\mathbb{Q}}$ . Let  $\Gamma = C \setminus \mathrm{Sp}(\mathcal{E})$  and consider the tail fan  $\Sigma(\mathcal{E})$  of  $\mathcal{E}$ . Then the stringy motivic volume of  $X = X(\mathcal{E})$  is expressed by the formula*

$$\begin{aligned} \mathcal{E}_{st}(X) = & [G/H][\Gamma] \sum_{\tau \in \Sigma(\mathcal{E})} \left( P(\tau, \omega_X) \mathbb{L}^{-\eta(\tau, \omega_X)} \prod_{\rho \in \tau(1)} \left( 1 - \mathbb{L}^{\omega_X(C, \rho, 0)} \right)^{-1} \right) \\ & + [G/H](\mathbb{L} - 1) \sum_{y \in \mathrm{Sp}(\mathcal{E})} \sum_{\tau \in \mathcal{E}_y^*} \left( P(\tau, \omega_X) \mathbb{L}^{-\eta(\tau, \omega_X)} \prod_{\rho \in \tau(1)} \left( 1 - \mathbb{L}^{\omega_X(y, \rho)} \right)^{-1} \right), \end{aligned}$$

where  $P(\tau, \omega_X)$  is the Stanley–Reisner polynomial associated with the pair  $(\tau, \omega_X)$ .

*Proof.* Using Theorem 5.17 and Corollary 5.25 we obtain

$$\mathcal{E}_{st}(X) = [G/H] \left( [\Gamma] \sum_{\tau \in \Sigma(\mathcal{E})} \bar{\varphi}_{\tau}(S(\tau)) + (\mathbb{L} - 1) \sum_{y \in \mathrm{Sp}(\mathcal{E})} \sum_{\tau \in \mathcal{E}_y^*} \bar{\varphi}_{\tau}(S(\tau)) \right),$$

where  $S(\tau)$  is defined in 5.20. We conclude by using the rational form of  $S(\tau)$  from Lemma 5.20.  $\square$

As a consequence of the theorem, under some assumptions on  $\mathcal{E}$ , we obtain a formula for the stringy Euler characteristic of the variety  $X$ .

**Corollary 5.28.** *Let  $(M, I)$  be the pair describing the horospherical homogeneous space  $G/H$ . Consider  $W := N_G(Q)/Q$  the Weyl group of  $(G, Q)$  and  $W_I \subset W$  the subset defined in Section 3.1. Let  $r := \dim(N_{\mathbb{Q}})$ . Assume that for every tail cone  $\tau \in \Sigma(\mathcal{E})$  and every  $\tau' \in \mathcal{E}_y$  for  $y$  in  $\mathrm{Sp}(\mathcal{E})$ , we have  $r \geq |\tau(1)|$  and  $r + 1 \geq |\tau'(1)|$ . Then*

$$\begin{aligned} \frac{|W_I|}{|W|} e_{st}(X) = & e(\Gamma) \sum_{\tau \in \Sigma(\mathcal{E}), \dim(\tau)=r} \left( P(\tau, \omega_X)(1) \prod_{\rho \in \tau(1)} \frac{1}{(-\omega_X(C, \rho, 0))} \right) \\ & + \sum_{y \in \mathrm{Sp}(\mathcal{E})} \sum_{\tau \in \mathcal{E}_y^*, \dim(\tau)=r+1} \left( P(\tau, \omega_X)(1) \prod_{\rho \in \tau(1)} \frac{1}{(-\omega_X(y, \rho))} \right), \end{aligned}$$

where  $P(\tau, \omega_X)$  is the Stanley–Reisner polynomial associated with the pair  $(\tau, \omega_X)$ .

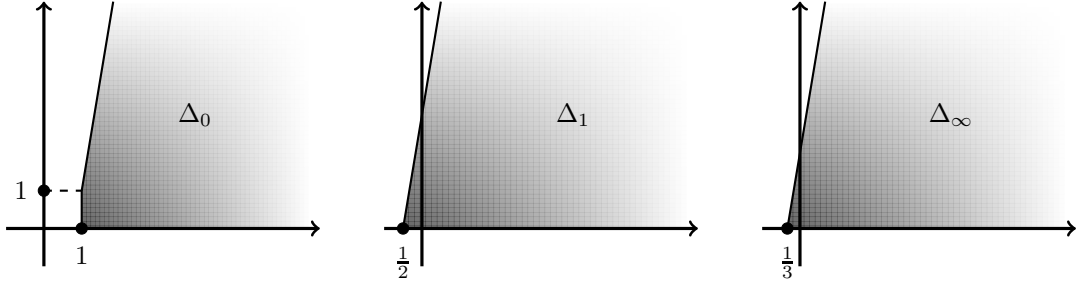
*Proof.* Let  $P \subset G$  be the closed subgroup containing  $B$  and associated with the set  $I$ . By Theorem 5.27 we have

$$\begin{aligned} E_{st}(X) = & E(G/P)E(\Gamma) \sum_{\tau \in \Sigma(\mathcal{E})} \left( P(\tau, \omega_X) \left( (uv)^{\frac{1}{m}} \right) (uv)^{-\eta(\tau, \omega_X)} \frac{(uv - 1)^r}{\prod_{\rho \in \tau(1)} (1 - (uv)^{\omega_X(C, \rho, 0)})} \right) \\ & + E(G/P) \sum_{y \in \mathrm{Sp}(\mathcal{E})} \sum_{\tau \in \mathcal{E}_y^*} \left( P(\tau, \omega_X) \left( (uv)^{\frac{1}{m}} \right) uv^{-\eta(\tau, \omega_X)} \frac{(uv - 1)^{r+1}}{\prod_{\rho \in \tau(1)} (1 - (uv)^{\omega_X(y, \rho)})} \right). \end{aligned}$$

Studying the fixed points of the action of maximal torus  $Q$  on  $G/P$  we see that that  $e(G/P) = \frac{|W|}{|W_I|}$ . We conclude by letting  $u, v$  tend to 1.  $\square$

## 6. EXAMPLES AND APPLICATIONS

In this section we start in 6.1 by illustrating Theorem 5.17 on the example of a hypersurface endowed with a  $(\mathbb{C}^*)^2$ -action which was initially studied by Liendo and Süß [LS13]. Then in 6.2 we deduce from Theorem 5.17 a smoothness criterion for locally factorial horospherical varieties of complexity one.

FIGURE 1. The  $\sigma$ -polyhedra  $\Delta_0$ ,  $\Delta_1$  and  $\Delta_\infty$ 

**6.1. An example where the acting group is a torus.** Here we borrow [LS13, Example 1.1]. Let  $N = \mathbb{Z}^2$  and  $\sigma = \text{Cone}((1,0), (1,6)) \subset N_{\mathbb{Q}} = \mathbb{Q}^2$ . Define a  $\sigma$ -polyhedral divisor on  $(\mathbb{P}^1, (\mathbb{C}^*)^2)$  by  $\mathfrak{D} = \sum_{y \in \mathbb{P}^1} \Delta_y \cdot [y]$ , where

$$\Delta_y = \begin{cases} \text{Conv}((1,0), (1,1)) + \sigma & \text{if } y = 0, \\ (-\frac{1}{2}, 0) + \sigma & \text{if } y = 1, \\ (-\frac{1}{3}, 0) + \sigma & \text{if } y = \infty, \\ \sigma & \text{otherwise.} \end{cases}$$

The variety  $X(\mathfrak{D})$  is a hypersurface in  $\mathbb{A}^4$ , given by the equation  $x_2^3 - x_3^2 + x_1x_4 = 0$ , where the  $(\mathbb{C}^*)^2$ -action is given by

$$(\lambda_1, \lambda_2) \cdot (x_1, x_2, x_3, x_4) = (\lambda_2x_1, \lambda_1^2x_2, \lambda_1^3x_3, \lambda_1^6\lambda_2^{-1}x_4).$$

We choose  $\Gamma = \mathbb{P}^1 \setminus \{0, 1, \infty\}$ . The (non-trivial) Cayley cones are

$$\begin{aligned} C_0(\mathfrak{D}) &= \text{Cone}((1,0,0), (1,6,0), (1,0,1), (1,1,1)) \\ C_1(\mathfrak{D}) &= \text{Cone}\left((1,0,0), (1,6,0), \left(-\frac{1}{2}, 0, 1\right)\right) \\ C_\infty(\mathfrak{D}) &= \text{Cone}\left((1,0,0), (1,6,0), \left(-\frac{1}{3}, 0, 1\right)\right). \end{aligned}$$

Since the locus of  $\mathfrak{D}$  is projective we need to compute the degree of  $\mathfrak{D}$ :

$$\deg \mathfrak{D} := \Delta_0 + \Delta_1 + \Delta_\infty = \text{Conv}\left(\left(\frac{1}{6}, 0\right), \left(\frac{1}{6}, 1\right)\right) + \sigma.$$

We notice that all rays of  $\sigma$  intersect  $\deg \mathfrak{D}$ . Thus to prove that  $X$  is  $\mathbb{Q}$ -Gorenstein we need to construct a function  $\vartheta_X$  satisfying the following conditions.

- (i) There exists rational numbers  $m_1, m_2$  and a principal divisor  $D = \sum_{y \in \mathbb{P}^1} c_y \cdot [y]$  such that

$$\vartheta_X(y; \nu_1, \nu_2, \ell) = m_1\nu_1 + m_2\nu_2 + \ell c_y$$

for any  $y \in \mathbb{P}^1$  and any  $(\nu, \ell) \in C_y(\mathfrak{D})$ ,

- (ii) There exists a canonical divisor  $K_{\mathbb{P}^1} = \sum_{y \in \mathbb{P}^1} b_y \cdot [y]$  such that

$$\vartheta_X(0; 1, 0, 1) = \vartheta_X(0; 1, 1, 1) = b_0$$

$$\vartheta_X(1; -1, 0, 2) = 2b_1 + 1$$

$$\vartheta_X(\infty; -1, 0, 3) = 3b_\infty + 2.$$

A simple computation shows that taking  $m_1 = -5$ ,  $m_2 = 0$ ,  $D = 0$ , and  $K_{\mathbb{P}^1} = -5 \cdot [0] + 2 \cdot [1] + [\infty]$  gives a solution. Hence by Proposition 5.4 it follows that  $X$  is  $\mathbb{Q}$ -Gorenstein. It is moreover log terminal.

Indeed, with the same notation as in Theorem 5.6 we have

$$\kappa_y = \begin{cases} 1 & \text{if } y = 0, \\ 2 & \text{if } y = 1, \\ 3 & \text{if } y = \infty, \\ 0 & \text{otherwise.} \end{cases}$$

We may now compute the function  $\omega_X$ :

$$\omega_X(y; a, b, c) = \begin{cases} -5a + 4c & \text{if } y = 0, \\ -5a - 3c & \text{if } y = 1, \\ -5a - 2c & \text{if } y = \infty, \\ -5a - c & \text{otherwise.} \end{cases}$$

To compute the stringy motivic volume of  $X(\mathfrak{D})$  we compute the sum of the four following series:

$$S_\sigma := \sum_{\nu \in \sigma \cap \mathbb{Z}^2} \mathbb{L}^{\omega_X(\mathbb{P}^1; \nu, 0)},$$

$$S_y := \sum_{(\nu, \ell) \in C_y(\mathfrak{D}) \setminus (\sigma \times \{0\})} \mathbb{L}^{\omega_X(y; \nu, \ell)} \text{ for } y \in \{0, 1, \infty\}.$$

Let us first describe the inequations defining the corresponding cones:

$$\begin{aligned} \sigma &= \{(a, b, 0) \in \mathbb{Z}^3 \mid b \geq 0 \text{ and } 6a - b \geq 0\}, \\ C_0(\mathfrak{D}) &= \{(a, b, c) \in \mathbb{Z}^3 \mid c \geq 0, a - c \geq 0, b - c \geq 0 \text{ and } 6a - b - 5c \geq 0\}, \\ C_1(\mathfrak{D}) &= \{(a, b, c) \in \mathbb{Z}^3 \mid c \geq 0, b \geq 0 \text{ and } 6a - b + 3c \geq 0\}, \\ C_\infty(\mathfrak{D}) &= \{(a, b, c) \in \mathbb{Z}^3 \mid c \geq 0, b \geq 0 \text{ and } 6a - b + 2c \geq 0\}. \end{aligned}$$

Since the cones  $\sigma$ ,  $C_1(\mathfrak{D})$ , and  $C_\infty(\mathfrak{D})$  are simplicial, we may compute the series  $S_\sigma$ ,  $S_1$  and  $S_\infty$  by using Theorem 5.19.

The fundamental sets are

$$\begin{aligned} P_\sigma &= \{(0, 0, 0), (1, 1, 0), (1, 2, 0), (1, 3, 0), (1, 4, 0), (1, 5, 0)\}, \\ P_{C_1(\mathfrak{D})} &= \{(0, 0, 0), (0, 0, 1), (0, 1, 1), (0, 2, 1), (0, 3, 1), (1, 1, 0), (1, 2, 0), (1, 3, 0), (1, 4, 0), (1, 4, 1), (1, 5, 0), (1, 5, 1)\}, \\ P_{C_\infty(\mathfrak{D})} &= \{(0, 0, 0), (0, 0, 1), (0, 0, 2), (0, 1, 1), (0, 1, 2), (0, 2, 1), (0, 2, 2), (0, 3, 2), (0, 4, 2), (1, 1, 0), (1, 2, 0), \\ &\quad (1, 3, 0), (1, 3, 1), (1, 4, 0), (1, 4, 1), (1, 5, 0), (1, 5, 1), (1, 5, 2)\}. \end{aligned}$$

Using Theorem 5.19, we find

$$\begin{aligned} S_\sigma &= \frac{1 + 5\mathbb{L}^{-5}}{(1 - \mathbb{L}^{-5})^2}, \\ S_1 &= \frac{\mathbb{L}^{-1} + 4\mathbb{L}^{-3} + 5\mathbb{L}^{-6} + 2\mathbb{L}^{-8}}{(1 - \mathbb{L}^{-5})^2(1 - \mathbb{L}^{-1})}, \\ S_\infty &= \frac{\mathbb{L}^{-1} + 3\mathbb{L}^{-2} + 5\mathbb{L}^{-4} + 5\mathbb{L}^{-6} + 3\mathbb{L}^{-7} + \mathbb{L}^{-9}}{(1 - \mathbb{L}^{-5})^2(1 - \mathbb{L}^{-1})}. \end{aligned}$$

By a direct computation we obtain

$$S_0 = \frac{\mathbb{L}^{-1} + 5\mathbb{L}^{-6}}{(1 - \mathbb{L}^{-5})^2(1 - \mathbb{L}^{-1})}.$$

We may now compute the stringy motivic volume of  $X(\mathfrak{D})$ :

$$\begin{aligned} \mathcal{E}_{st}(X) &= [(\mathbb{C}^*)^2] ([\mathbb{P}^1 \setminus \{0, 1, \infty\}] S_\sigma + (\mathbb{L} - 1)(S_0 + S_1 + S_\infty)) \\ &= \frac{\mathbb{L}(\mathbb{L} - 1)^2}{(1 - \mathbb{L}^{-5})^2} (1 - 2\mathbb{L}^{-1} + 3\mathbb{L}^{-2} + 3\mathbb{L}^{-3} + 4\mathbb{L}^{-4} + 10\mathbb{L}^{-5} - 10\mathbb{L}^{-6} + 15\mathbb{L}^{-7} + 3\mathbb{L}^{-8} + 2\mathbb{L}^{-9} + \mathbb{L}^{-10}) \end{aligned}$$

Thus the stringy Euler characteristic of  $X$  is

$$e_{st}(X) = \frac{6}{5}.$$

Let us compare this with the usual Euler characteristic of  $X$ . We may embed  $(\mathbb{C}^*)^3$  into  $X(\mathfrak{D})$  by  $(t_1, t_2, t_3) \mapsto (t_1, t_2, t_3, \frac{t_2^2 - t_3^2}{t_1})$ . The complement can be decomposed as follows

$$\begin{aligned} X(\mathfrak{D}) \setminus (\mathbb{C}^*)^3 = & \{(0, x_2, x_3, x_4) \mid x_2^3 = x_3^2, x_2 \neq 0\} \sqcup \{(0, 0, 0, x_4)\} \sqcup \{(x_1, 0, x_3, x_4) \mid x_2^3 = x_1 x_4, x_1 \neq 0, x_3 \neq 0\} \\ & \sqcup \{(x_1, 0, 0, 0) \mid x_1 \neq 0\} \sqcup \{(x_1, x_2, 0, x_4) \mid x_2^3 + x_1 x_4 = 0, x_1 \neq 0, x_2 \neq 0\}. \end{aligned}$$

Hence

$$e(X) = e((\mathbb{C}^*)^3) + e(\mathbb{A}^1)e(\mathbb{C}^*) + e(\mathbb{A}^1) + e((\mathbb{C}^*)^2) + e(\mathbb{C}^*) + e((\mathbb{C}^*)^2) = 1.$$

Indeed, the Euler characteristic of an algebraic torus is zero. We observe that  $e_{st}(X) > e(X)$ .

**6.2. Stringy Euler characteristic and a smoothness criterion.** Here we prove a smoothness criterion based on the computation of the stringy Euler characteristic. It is analogous to [BM13, Theorem 5.3].

**Lemma 6.1.** *Let  $(\mathfrak{D}, \mathcal{F})$  be a colored  $\sigma$ -polyhedral divisor on  $(C, G/H)$  with affine locus  $C_0$ . Assume that  $X(\mathfrak{D})$  is locally factorial (that is, that any Weil divisor is Cartier), and that  $\sigma$  has dimension  $\dim X - 1$ . Then the stringy Euler characteristic is equal to*

$$e_{st}(X) = e(C_0) \frac{|W|}{|W_I| \prod_{D_\alpha \in \mathcal{F}} a_\alpha},$$

where, as usual,  $W$  denotes the Weyl group of  $(G, Q)$  and  $W_I$  denotes the Weyl group of  $(N_G(H), Q)$ .

*Proof.* Let  $\Gamma \subset C_0$  be an open subset of  $C_0$  which does not contain any special point of  $\mathfrak{D}$ . By Theorem 5.17 we have

$$\begin{aligned} \mathcal{E}_{st}(X) &= [\Gamma][G/H] \left( \sum_{\nu \in \sigma \cap N} \mathbb{L}^{\omega_X(C, \nu, 0)} \right) + (\mathbb{L} - 1)[G/H] \left( \sum_{y \in C_0 \setminus \Gamma} \sum_{(\nu, \ell) \in C_y(\mathfrak{D}) \setminus (\sigma \times \{0\})} \mathbb{L}^{\omega_X(y, \nu, \ell)} \right) \\ &= [\Gamma] \mathcal{E}_{st}(X(\sigma, \mathcal{F})) + \sum_{y \in C_0 \setminus \Gamma} (\mathcal{E}_{st}(X(C_y(\mathfrak{D}), \mathcal{F})) - (\mathbb{L} - 1) \mathcal{E}_{st}(X(\sigma, \mathcal{F}))), \end{aligned}$$

where  $X(\sigma, \mathcal{F})$  (respectively  $X(C_y(\mathfrak{D}), \mathcal{F})$ ) is the  $G$ -equivariant (respectively  $G \times \mathbb{C}^*$ -equivariant) embedding of the horospherical homogeneous space  $G/H$  (respectively  $G/H \times \mathbb{C}^*$ ) associated with the colored cone  $(\sigma, \mathcal{F})$  (respectively  $(C_y(\mathfrak{D}), \mathcal{F})$ ). Here the last equality comes from [BM13, Theorem 4.3]

Passing to the stringy  $E$ -polynomial and evaluating at  $u = v = 1$ , we obtain

$$e_{st}(X) = e(\Gamma) e_{st}(X(\sigma, \mathcal{F})) + \sum_{y \in C_0 \setminus \Gamma} e_{st}(X(C_y(\mathfrak{D}), \mathcal{F})).$$

Using [BM13, Proposition 5.11] we see that  $e_{st}(X(C_y(\mathfrak{D}), \mathcal{F})) = e_{st}(X(\sigma, \mathcal{F}))$  since the stringy Euler characteristic only depends on the Weyl groups and the colors, which are unchanged. We deduce

$$e_{st}(X) = e(C_0) e_{st}(X(\sigma, \mathcal{F})).$$

Finally, using again [BM13, Proposition 5.11], we obtain

$$e_{st}(X(\sigma, \mathcal{F})) = \frac{|W|}{|W_I| \prod_{D_\alpha \in \mathcal{F}} a_\alpha},$$

which concludes the proof.  $\square$

The next proposition gives a full description of the  $G$ -orbits of a simple  $G$ -model of  $C \times G/H$  corresponding to a colored  $\sigma$ -polyhedral divisor  $(\mathfrak{D}, \mathcal{F})$  with affine locus  $C_0$ . For the complexity zero case we refer to [BM13, Proposition 2.4] and [GH15, Theorem 1.1].

**Proposition 6.2.** *Let  $(\mathfrak{D}, \mathcal{F})$  be a colored  $\sigma$ -polyhedral divisor on  $(C, G/H)$  with affine locus  $C_0$ , and  $\text{Sp}(\mathfrak{D})$  be the set of special points of  $\mathfrak{D}$ . Then  $X := X(\mathfrak{D}, \mathcal{F})$  has two types of  $G$ -orbits:*

- (i) horizontal orbits, which are contained in the  $G$ -invariant open subset  $\Gamma \times X(\sigma, \mathcal{F})$  of  $X$ , where  $\Gamma = C_0 \setminus \text{Sp}(\mathfrak{D})$  and  $X(\sigma, \mathcal{F})$  is the  $G/H$ -embedding associated with the colored cone  $(\sigma, \mathcal{F})$ ,
- (ii) vertical orbits, which are the remaining  $G$ -orbits.

Moreover, the horizontal  $G$ -orbits of  $X$  are parametrized by triples  $(y, \tau, \mathcal{F}_\tau)$ , where  $y \in \Gamma$ ,  $\tau$  is a face of  $\sigma$ , and  $\mathcal{F}_\tau = \{D \in \mathcal{F} \mid \varrho(D) \in \tau\}$ . The stabilizer  $H_{y, \tau, \mathcal{F}_\tau} =: H_\tau$  of such a triple is given by

$$H_\tau = \{g \in P_{I_{\mathcal{F}_\tau}} \mid \chi^m(g) = 1 \text{ for any } m \in \tau^\perp \cap M\},$$

where  $I_{\mathcal{F}_\tau} \subset \Phi$  is the reunion with  $I$  of the set of simple roots indexing the colors in  $\mathcal{F}_\tau$ , and  $\chi^m$  is the character of  $P_{I_{\mathcal{F}_\tau}}$  associated with  $m$ .

Finally, the vertical  $G$ -orbits of  $X$  are parametrized by triples  $(y, F, \mathcal{F}_F)$ , where  $y \in \text{Sp}(\mathfrak{D})$ ,  $F$  is a face of the  $\sigma$ -polyhedron  $\Delta_y$  of  $\mathfrak{D}$  associated with  $y$ , and

$$\mathcal{F}_F = \{D_\alpha \in \mathcal{F} \mid \varrho(D_\alpha) \in \lambda(F)\}.$$

Here

$$\lambda(F) = \{m \in M_{\mathbb{Q}} \mid \langle m, v - v' \rangle \geq 0 \text{ for any } v \in \Delta_y, v' \in F\}.$$

The stabilizer  $H_{y, F, \mathcal{F}_F} =: H_{y, F}$  of a triple  $(y, F, \mathcal{F}_F)$  is given by

$$H_{y, F} = \{g \in P_{I_{\mathcal{F}_F}} \mid \chi^m(g) = 1 \text{ for any } m \in M_y\},$$

where

$$M_y = \left\{ m \in M \cap \lambda(F) \cap (-\lambda(F)) \mid \min_{v \in \Delta_y} \langle m, v \rangle \in \mathbb{Z} \right\}.$$

*Proof.* Using parabolic induction, we may assume that  $X(\mathfrak{D}) = \text{Spec } A(C_0, \mathfrak{D}, \mathcal{F})$  is affine, where  $A(C_0, \mathfrak{D}, \mathcal{F})$  is defined in Equation 11. Now the description of horizontal  $G$ -orbits is an immediate consequence of [BM13, Proposition 2.4]. For vertical  $G$ -orbits, we adapt [AH06, Section 7] to study the quotient map  $\pi : X(\mathfrak{D}) \rightarrow X(\mathfrak{D}) // G = C_0$  induced by the inclusion of algebras  $\mathbb{C}[C_0] \subset A(C_0, \mathfrak{D}, \mathcal{F})$ . In this case, the result follows from the description of the algebra of regular global functions on the irreducible components of each reduced fiber of  $\pi$  and from [Tim11, Section 28.2].  $\square$

We may now compute the (usual) Euler characteristic, which we then compare to the stringy Euler characteristic of Lemma 6.1.

**Lemma 6.3.** *Under the same assumptions as in Lemma 6.1, the Euler characteristic of  $X$  is given by*

$$e(X) = e(C_0) \frac{|W|}{|W_{I_{\mathcal{F}}}|}.$$

*Proof.* By Proposition 6.2 we have a decomposition

$$X = X(\mathfrak{D}) = (\Gamma \times X(\sigma, \mathcal{F})) \sqcup \bigsqcup_{O \text{ vertical } G\text{-orbit}} O,$$

hence

$$e(X) = e(\Gamma \times X(\sigma, \mathcal{F})) + \sum_{O \text{ vertical } G\text{-orbit}} e(O).$$

Using [BM13, Proposition 5.11] and the parametrization of vertical orbits in Proposition 6.2, we obtain

$$e(X) = e(\Gamma) \frac{|W|}{|W_{I_{\mathcal{F}}}|} + \sum_{y \in \text{Sp}(\mathfrak{D})} \sum_{v \in \Delta_y} \frac{|W|}{|W_{I_{\mathcal{F}\{v\}}}|}.$$

Thus it only remains to prove that

$$(32) \quad \frac{|W|}{|W_{I_{\mathcal{F}}}|} = \sum_{v \in \Delta_y} \frac{|W|}{|W_{I_{\mathcal{F}\{v\}}}|}.$$

Fix a point  $y_0 \in C_0$  and define a colored  $\sigma$ -polyhedral divisor  $(\hat{\mathfrak{D}}, \mathcal{F})$  on  $(\mathbb{P}^1, G/H)$  by setting  $\hat{\mathfrak{D}} := \sum_{z \in \mathbb{A}^1} \hat{\Delta}_z \cdot [z]$ , where

$$\hat{\Delta}_z = \begin{cases} \Delta_{y_0} & \text{if } z = 0, \\ \sigma & \text{otherwise.} \end{cases}$$

*Claim.* The variety  $X(\hat{\mathfrak{D}})$  is locally factorial.

*Proof of the claim.* It is enough to prove that any  $B$ -divisor of  $X(\hat{\mathfrak{D}})$  is Cartier. There are five types of  $B$ -divisors in  $X(\hat{\mathfrak{D}})$ :

- (1) divisors  $\hat{D}_\tau$  for any uncolored ray of  $\sigma$ ,
- (2) divisors  $\hat{D}_{(0,p)}$  for any vertex  $p \in \hat{\Delta}_0(0) = \Delta_{y_0}(0)$ ,
- (3) divisors  $\hat{D}_{(z,0)}$  for any  $z \in \mathbb{A}^1 \setminus \{0\}$  (since  $\mathbb{A}^1 \setminus \{0\}$  contains only non-special points),
- (4) colors  $\hat{D}_\alpha \in \mathcal{F}$ ,
- (5) colors  $\hat{D}_\alpha \in \mathcal{F}_{G/H} \setminus \mathcal{F}$ .

We will prove that the divisors in each family are Cartier by constructing suitable support functions.

- (1) Let  $\tau_0$  be an uncolored ray of  $\sigma$ . Since  $X$  is locally factorial, there exists a support function  $\varpi_{\tau_0} = (\omega_{\tau_0}, (r_\alpha)_\alpha)$  in  $\text{PL}(\mathfrak{D})$  such that  $D_{\tau_0} = D_{\varpi_{\tau_0}}$ . Define

$$\hat{\omega}_{\tau_0}(z, \nu, \ell) = \begin{cases} \omega_{\tau_0}(C, \nu, 0) & \text{if } \ell = 0, \\ \omega_{\tau_0}(y_0, \nu, \ell) & \text{if } z = 0, \\ \omega_{\tau_0}(y_1, \nu, \ell) & \text{otherwise,} \end{cases}$$

where  $y_1 \in C_0$  is a fixed non-special point. Clearly the function  $\hat{\omega}_{\tau_0} := (\hat{\omega}_{\tau_0}, (r_\alpha)_\alpha)$  is well-defined and linear on each Cayley cone, so that  $\hat{\omega}_{\tau_0} \in \text{PL}(\hat{\mathfrak{D}})$ . Moreover

$$\begin{aligned} \hat{D}_{\hat{\omega}_{\tau_0}} &= \sum_{\tau \text{ uncolored ray of } \sigma} \hat{\omega}_{\tau_0}(\mathbb{P}^1, \tau, 0) \hat{D}_\tau + \sum_{p \in \hat{\Delta}_0(0)} \hat{\omega}_{\tau_0}(0, p) \hat{D}_{(0,p)} + \sum_{\hat{D}_\alpha \in \mathcal{F}} \hat{\omega}_{\tau_0}(\mathbb{P}^1, \varrho(\hat{D}_\alpha), 0) \hat{D}_\alpha \\ &\quad + \sum_{\hat{D}_\alpha \notin \mathcal{F}} r_\alpha \hat{D}_\alpha + \sum_{z \in \mathbb{A}^1 \setminus \{0\}} \hat{\omega}_{\tau_0}(z, 0, 1) \hat{D}_{(z,0)} \\ &= \sum_{\tau \text{ uncolored ray of } \sigma} \omega_{\tau_0}(C, \tau, 0) \hat{D}_\tau + \sum_{p \in \Delta_{y_0}(0)} \omega_{\tau_0}(y_0, p) \hat{D}_{(0,p)} + \sum_{D_\alpha \in \mathcal{F}} \omega_{\tau_0}(C, \varrho(D_\alpha), 0) \hat{D}_\alpha \\ &\quad + \sum_{D_\alpha \notin \mathcal{F}} r_\alpha \hat{D}_\alpha + \sum_{z \in \mathbb{A}^1 \setminus \{0\}} \omega_{\tau_0}(y_1, 0, 1) \hat{D}_{(z,0)}. \end{aligned}$$

We also have

$$\begin{aligned} D_{\varpi_{\tau_0}} &= \sum_{\tau \text{ uncolored ray of } \sigma} \omega_{\tau_0}(C, \tau, 0) D_\tau + \sum_{p \in \Delta_{y_0}(0)} \omega_{\tau_0}(y_0, p) D_{(y_0,p)} + \sum_{D_\alpha \in \mathcal{F}} \omega_{\tau_0}(C, \varrho(D_\alpha), 0) D_\alpha \\ &\quad + \sum_{D_\alpha \notin \mathcal{F}} r_\alpha D_\alpha + \sum_{y \in C_0 \setminus \{y_0, y_1\}, p \in \Delta_y(0)} \omega_{\tau_0}(y, p) D_{(y,p)} + \omega_{\tau_0}(y_1, 0, 1) D_{(y_1,0)}. \end{aligned}$$

Since  $D_{\varpi_{\tau_0}} = D_{\tau_0}$ , it follows that all the coefficients of the divisors in the previous identity are 0, except for the coefficient of  $D_{\tau_0}$  which is equal to one. Replacing in the expression of  $\hat{D}_{\hat{\omega}_{\tau_0}}$ , we see that  $\hat{D}_{\hat{\omega}_{\tau_0}} = \hat{D}_{\tau_0}$ , thus  $\hat{D}_{\tau_0}$  is Cartier.

- (2) Let  $p_0$  be a vertex in  $\hat{\Delta}_0(0) = \Delta_{y_0}(0)$ . Let  $\varpi_{p_0} = (\omega_{p_0}, (r_\alpha)_\alpha)$  in  $\text{PL}(\mathfrak{D})$  be a support function such that  $D_{(0,p_0)} = D_{\varpi_{p_0}}$ , and define

$$\hat{\omega}_{p_0}(z, \nu, \ell) = \begin{cases} \omega_{p_0}(C, \nu, 0) & \text{if } \ell = 0, \\ \omega_{p_0}(y_0, \nu, \ell) & \text{if } z = 0, \\ \omega_{p_0}(y_1, \nu, \ell) & \text{otherwise,} \end{cases}$$

where  $y_1 \in C_0$  is a fixed non-special point. As in the previous case  $\hat{\omega}_{p_0} := (\hat{\omega}_{p_0}, (r_\alpha)_\alpha) \in \text{PL}(\hat{\mathfrak{D}})$  and  $\hat{D}_{\hat{\omega}_{p_0}} = \hat{D}_{(0,p_0)}$ , which proves that  $\hat{D}_{\tau_0}$  is Cartier.

- (3) Consider  $z_0 \in \mathbb{A}^1 \setminus \{0\}$ . If  $y_1$  is any non-special point of  $C_0$ , then  $(y_1, 0) \in \text{Vert}(\mathfrak{D})$ . Thus there exists a support function  $\varpi_1 = (\omega_1, (r_\alpha)_\alpha)$  in  $\text{PL}(\mathfrak{D})$  such that  $D_{\varpi_1} = D_{(y_1,0)}$ . To check that  $\hat{D}_{(z_0,0)}$  is Cartier, we define a function  $\hat{\omega}_1 := (\hat{\omega}_1, (r_\alpha)_\alpha) \in \text{PL}(\hat{\mathfrak{D}})$ , where

$$\hat{\omega}_1(z, \nu, \ell) = \begin{cases} \omega_1(C, \nu, 0) & \text{if } \ell = 0, \\ \omega_1(y_0, \nu, \ell) & \text{if } z = 0, \\ \omega_1(y_1, \nu, \ell) & \text{if } z = z_0, \\ \omega_1(y_2, \nu, \ell) & \text{otherwise.} \end{cases}$$

Here  $y_2$  is a fixed non-special point in  $C_0 \setminus \{y_1\}$ . Clearly  $\hat{D}_{\hat{\omega}_1} = \hat{D}_{(z_0,0)}$ , which proves that  $\hat{D}_{(z_0,0)}$  is Cartier.

(4) For a color  $\hat{D}_{\alpha_0} \in \mathcal{F}$ , the proof is extremely similar to that of the previous cases.

(5) If  $\hat{D}_{\alpha_0} \notin \mathcal{F}$ , the function  $\hat{\omega}_{\alpha_0} = (0, (\delta_{\alpha_0, \alpha})_\alpha)$  is a suitable support function.

This concludes the proof of the Claim.  $\square$

Let us now finish the proof of Lemma 6.3. According to the claim we have that  $X(\hat{\mathfrak{D}})$  is locally factorial. Moreover it is also horospherical, and it identifies with the  $(G \times \mathbb{C}^*)$ -equivariant embedding of  $G/H \times \mathbb{C}^*$  associated with the colored cone  $(C_0(\hat{\mathfrak{D}}), \mathcal{F}) = (C_y(\mathfrak{D}), \mathcal{F})$ . Thus by [BM13, Proposition 5.11], we obtain

$$\frac{|W|}{|W_{I_{\mathcal{F}}}|} = e(X(C_0(\hat{\mathfrak{D}}), \mathcal{F})) = e(X(\hat{\mathfrak{D}})) = e(\mathbb{C}^*)e(X(\sigma, \mathcal{F})) + \sum_{v \in \Delta_y} \frac{|W|}{|W_{I_{\mathcal{F}\{v\}}}|}.$$

Since  $e(\mathbb{C}^*) = 0$ , this proves Equation (32) and concludes the proof.  $\square$

Combining Lemmas 6.1 and 6.3 with the smoothness criterion of [LT16, Theorem 2.5], we obtain the following result.

**Theorem 6.4.** *Let  $(\mathfrak{D}, \mathcal{F})$  be a colored  $\sigma$ -polyhedral divisor on  $(C, G/H)$  with affine locus  $C_0 \subset C$ . Assume that  $X = X(\mathfrak{D})$  is locally factorial and that for any  $y \in C_0$ , the Cayley cone  $C_y(\mathfrak{D})$  has dimension  $d = \dim X$ . Then we have  $e_{st}(X) \geq e(X)$ . Moreover, if  $2 - 2g - |C \setminus C_0| \neq 0$ , then  $X$  is smooth if and only if  $e_{st}(X) = e(X)$ .*

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#### REFERENCES

- [AH06] Klaus Altmann and Jürgen Hausen. Polyhedral divisors and algebraic torus actions. *Math. Ann.*, 334(3):557–607, 2006.
- [AHS08] Klaus Altmann, Jürgen Hausen, and Hendrik Süß. Gluing affine torus actions via divisorial fans. *Transform. Groups*, 13(2):215–242, 2008.
- [AIP<sup>+</sup>12] Klaus Altmann, Nathan Owen Ilten, Lars Petersen, Hendrik Süß, and Robert Vollmert. The geometry of  $T$ -varieties. In *Contributions to algebraic geometry*, EMS Ser. Congr. Rep., pages 17–69. Eur. Math. Soc., Zürich, 2012.
- [Bat98] Victor V. Batyrev. Stringy Hodge numbers of varieties with Gorenstein canonical singularities. In *Integrable systems and algebraic geometry (Kobe/Kyoto, 1997)*, pages 1–32. World Sci. Publ., River Edge, NJ, 1998.
- [Bat99] Victor V. Batyrev. Non-Archimedean integrals and stringy Euler numbers of log-terminal pairs. *J. Eur. Math. Soc. (JEMS)*, 1(1):5–33, 1999.
- [Bat00] Victor V. Batyrev. Stringy Hodge numbers and Virasoro algebra. *Math. Res. Lett.*, 7(2-3):155–164, 2000.
- [Bit04] Franziska Bittner. The universal Euler characteristic for varieties of characteristic zero. *Compos. Math.*, 140(4):1011–1032, 2004.
- [Bli11] Manuel Blickle. A short course on geometric motivic integration. In *Motivic integration and its interactions with model theory and non-Archimedean geometry. Volume I*, volume 383 of *London Math. Soc. Lecture Note Ser.*, pages 189–243. Cambridge Univ. Press, Cambridge, 2011.
- [BLR90] Siegfried Bosch, Werner Lütkebohmert, and Michel Raynaud. *Néron models*, volume 21 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]*. Springer-Verlag, Berlin, 1990.
- [BM13] Victor Batyrev and Anne Moreau. The arc space of horospherical varieties and motivic integration. *Compos. Math.*, 149(8):1327–1352, 2013.
- [Bor14] Lev Borisov. Class of the affine line is a zero divisor in the Grothendieck ring. arXiv:1412.6194, 2014.
- [Bri88] Michel Brion. Points entiers dans les polyèdres convexes. *Ann. Sci. École Norm. Sup. (4)*, 21(4):653–663, 1988.
- [Bri91] Michel Brion. Sur la géométrie des variétés sphériques. *Comment. Math. Helv.*, 66(2):237–262, 1991.



- [Bri93] Michel Brion. Variétés sphériques et théorie de Mori. *Duke Math. J.*, 72(2):369–404, 1993.
- [Bri95] Michel Brion. Spherical varieties. In *Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Zürich, 1994)*, pages 753–760. Birkhäuser, Basel, 1995.
- [CL08] Raf Cluckers and François Loeser. Constructible motivic functions and motivic integration. *Invent. Math.*, 173(1):23–121, 2008.
- [CLS11] David A. Cox, John B. Little, and Henry K. Schenck. *Toric varieties*, volume 124 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2011.
- [Cra04] Alastair Craw. An introduction to motivic integration. In *Strings and geometry*, volume 3 of *Clay Math. Proc.*, pages 203–225. Amer. Math. Soc., Providence, RI, 2004.
- [CS01] Pierre Colmez and Jean-Pierre Serre, editors. *Correspondance Grothendieck-Serre*. Documents Mathématiques (Paris) [Mathematical Documents (Paris)], 2. Société Mathématique de France, Paris, 2001.
- [Del71] Pierre Deligne. Théorie de Hodge. II. *Inst. Hautes Études Sci. Publ. Math.*, (40):5–57, 1971.
- [Del74] Pierre Deligne. Théorie de Hodge. III. *Inst. Hautes Études Sci. Publ. Math.*, (44):5–77, 1974.
- [DK86] V. I. Danilov and A. G. Khovanskiĭ. Newton polyhedra and an algorithm for calculating Hodge-Deligne numbers. *Izv. Akad. Nauk SSSR Ser. Mat.*, 50(5):925–945, 1986.
- [DL99] Jan Denef and François Loeser. Germs of arcs on singular algebraic varieties and motivic integration. *Invent. Math.*, 135(1):201–232, 1999.
- [DL01a] Jan Denef and François Loeser. Definable sets, motives and  $p$ -adic integrals. *J. Amer. Math. Soc.*, 14(2):429–469 (electronic), 2001.
- [DL01b] Jan Denef and François Loeser. Geometry on arc spaces of algebraic varieties. In *European Congress of Mathematics, Vol. I (Barcelona, 2000)*, volume 201 of *Progr. Math.*, pages 327–348. Birkhäuser, Basel, 2001.
- [DL02] Jan Denef and François Loeser. Motivic integration, quotient singularities and the McKay correspondence. *Compositio Math.*, 131(3):267–290, 2002.
- [Eis95] David Eisenbud. *Commutative algebra*, volume 150 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1995. With a view toward algebraic geometry.
- [Ewa96] Günter Ewald. *Combinatorial convexity and algebraic geometry*, volume 168 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1996.
- [FZ03] Hubert Flenner and Mikhail Zaidenberg. Normal affine surfaces with  $C^*$ -actions. *Osaka J. Math.*, 40(4):981–1009, 2003.
- [Gag15] Giuliano Gagliardi. A combinatorial smoothness criterion for spherical varieties. *Manuscripta Math.*, 146(3-4):445–461, 2015.
- [GH15] Giuliano Gagliardi and Johannes Hofscheier. Homogeneous spherical data of orbits in spherical embeddings. *Transform. Groups*, 20(1):83–98, 2015.
- [GN10] Dennis Gaiety and David Nadler. Spherical varieties and Langlands duality. *Mosc. Math. J.*, 10(1):65–137, 271, 2010.
- [Gre66] Marvin J. Greenberg. Rational points in Henselian discrete valuation rings. *Inst. Hautes Études Sci. Publ. Math.*, (31):59–64, 1966.
- [Har77] Robin Hartshorne. *Algebraic geometry*. Springer-Verlag, New York-Heidelberg, 1977. Graduate Texts in Mathematics, No. 52.
- [Hir64] Heisuke Hironaka. Resolution of singularities of an algebraic variety over a field of characteristic zero. I, II. *Ann. of Math. (2)* 79 (1964), 109–203; *ibid.* (2), 79:205–326, 1964.
- [HK06] Ehud Hrushovski and David Kazhdan. Integration in valued fields. In *Algebraic geometry and number theory*, volume 253 of *Progr. Math.*, pages 261–405. Birkhäuser Boston, Boston, MA, 2006.
- [Ish04] Shihoko Ishii. The arc space of a toric variety. *J. Algebra*, 278(2):666–683, 2004.
- [Jan87] Jens Carsten Jantzen. *Representations of algebraic groups*, volume 131 of *Pure and Applied Mathematics*. Academic Press, Inc., Boston, MA, 1987.
- [KKMSD73] G. Kempf, Finn Faye Knudsen, D. Mumford, and B. Saint-Donat. *Toroidal embeddings. I*. Lecture Notes in Mathematics, Vol. 339. Springer-Verlag, Berlin-New York, 1973.
- [KM98] János Kollár and Shigefumi Mori. *Birational geometry of algebraic varieties*, volume 134 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 1998. With the collaboration of C. H. Clemens and A. Corti, Translated from the 1998 Japanese original.
- [Kno90] Friedrich Knop. Weylgruppe und Momentabbildung. *Invent. Math.*, 99(1):1–23, 1990.
- [Kno91] Friedrich Knop. The Luna-Vust theory of spherical embeddings. In *Proceedings of the Hyderabad Conference on Algebraic Groups (Hyderabad, 1989)*, pages 225–249. Manoj Prakashan, Madras, 1991.
- [Kon95] Maxim Kontsevich. Lecture at Orsay, december 7, 1995.
- [Lan15] Kevin Langlois. Polyhedral divisors and torus actions of complexity one over arbitrary fields. *J. Pure Appl. Algebra*, 219(6):2015–2045, 2015.
- [Loe09] François Loeser. Seattle lectures on motivic integration. In *Algebraic geometry—Seattle 2005. Part 2*, volume 80 of *Proc. Sympos. Pure Math.*, pages 745–784. Amer. Math. Soc., Providence, RI, 2009.
- [LS13] Alvaro Liendo and Hendrik Süß. Normal singularities with torus actions. *Tohoku Math. J. (2)*, 65(1):105–130, 2013.
- [LT16] Kevin Langlois and Ronan Terpereau. On the geometry of normal horospherical  $G$ -varieties of complexity one. *J. Lie Theory*, 26(1):49–78, 2016.
- [LV83] D. Luna and Th. Vust. Plongements d’espaces homogènes. *Comment. Math. Helv.*, 58(2):186–245, 1983.

- [Mat80] Hideyuki Matsumura. *Commutative algebra*, volume 56 of *Mathematics Lecture Note Series*. Benjamin/Cummings Publishing Co., Inc., Reading, Mass., second edition, 1980.
- [MT11] Gunter Malle and Donna Testerman. *Linear algebraic groups and finite groups of Lie type*, volume 133 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2011.
- [Nau07] N. Naumann. Algebraic independence in the Grothendieck ring of varieties. *Trans. Amer. Math. Soc.*, 359(4):1653–1683 (electronic), 2007.
- [NS11] Johannes Nicaise and Julien Sebag. The Grothendieck ring of varieties. In *Motivic integration and its interactions with model theory and non-Archimedean geometry. Volume I*, volume 383 of *London Math. Soc. Lecture Note Ser.*, pages 145–188. Cambridge Univ. Press, Cambridge, 2011.
- [Pas08] Boris Pasquier. Variétés horosphériques de Fano. *Bull. Soc. Math. France*, 136(2):195–225, 2008.
- [Per14] Nicolas Perrin. On the geometry of spherical varieties. *Transform. Groups*, 19(1):171–223, 2014.
- [Poo02] Bjorn Poonen. The Grothendieck ring of varieties is not a domain. *Math. Res. Lett.*, 9(4):493–497, 2002.
- [PS11] Lars Petersen and Hendrik Süß. Torus invariant divisors. *Israel J. Math.*, 182:481–504, 2011.
- [Ros63] Maxwell Rosenlicht. A remark on quotient spaces. *An. Acad. Brasil. Ci.*, 35:487–489, 1963.
- [Seb04] Julien Sebag. Intégration motivique sur les schémas formels. *Bull. Soc. Math. France*, 132(1):1–54, 2004.
- [Sum74] Hideyasu Sumihiro. Equivariant completion. *J. Math. Kyoto Univ.*, 14:1–28, 1974.
- [SV07] Jan Schepers and Willem Veys. Stringy Hodge numbers for a class of isolated singularities and for threefolds. *Int. Math. Res. Not. IMRN*, (2):Art. ID rnm016, 14, 2007.
- [SV09] Jan Schepers and Willem Veys. Stringy  $E$ -functions of hypersurfaces and of Brieskorn singularities. *Adv. Geom.*, 9(2):199–217, 2009.
- [Tim97] D. A. Timashev. Classification of  $G$ -manifolds of complexity 1. *Izv. Ross. Akad. Nauk Ser. Mat.*, 61(2):127–162, 1997.
- [Tim00] D. A. Timashev. Cartier divisors and geometry of normal  $G$ -varieties. *Transform. Groups*, 5(2):181–204, 2000.
- [Tim08] Dmitri Timashev. Torus actions of complexity one. In *Toric topology*, volume 460 of *Contemp. Math.*, pages 349–364. Amer. Math. Soc., Providence, RI, 2008.
- [Tim11] Dmitry A. Timashev. *Homogeneous spaces and equivariant embeddings*, volume 138 of *Encyclopaedia of Mathematical Sciences*. Springer, Heidelberg, 2011. Invariant Theory and Algebraic Transformation Groups, 8.
- [Vey03] Willem Veys. Stringy zeta functions for  $\mathbb{Q}$ -Gorenstein varieties. *Duke Math. J.*, 120(3):469–514, 2003.
- [Vey04] Willem Veys. Stringy invariants of normal surfaces. *J. Algebraic Geom.*, 13(1):115–141, 2004.
- [Vey06] Willem Veys. Arc spaces, motivic integration and stringy invariants. In *Singularity theory and its applications*, volume 43 of *Adv. Stud. Pure Math.*, pages 529–572. Math. Soc. Japan, Tokyo, 2006.
- [Vin86] È. B. Vinberg. Complexity of actions of reductive groups. *Funktsional. Anal. i Prilozhen.*, 20(1):1–13, 96, 1986.
- [Yas04] Takehiko Yasuda. Twisted jets, motivic measures and orbifold cohomology. *Compos. Math.*, 140(2):396–422, 2004.
- [Yas06] Takehiko Yasuda. Motivic integration over Deligne-Mumford stacks. *Adv. Math.*, 207(2):707–761, 2006.

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