Pricing and Hedging Basket Options with Exact Moment Matching

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Abstract

Theoretical models applied to option pricing should take into account the empirical characteristics of financial time series. In this paper, we show how to price basket options when the underlying asset prices follow a displaced log-normal process with jumps, capable of accommodating negative skewness and excess kurtosis. Our technique involves Hermite polynomial expansion that can match exactly the first m moments of the model-implied basket return. This method is shown to provide superior results for basket options not only with respect to pricing but also for hedging.

Keywords: Displaced log-normal jump-diffusion process, Hermite polynomials, moment matching, Quasi-analytical pricing, Basket options

JEL: C18, C63, G13, G19

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1. Introduction

Basket options are contingent claims on a group of assets such as equities, commodities, currencies and even other vanilla derivatives. They are a subclass of exotic options and commonly traded over-the-counter in order to hedge away exposure to correlation or contagion risk. Additionally, they are also employed by hedge-funds for investment purposes, to combine diversification with leverage.

Baskets consist of several assets and, consequently, any modelling ought to be multidimensional. Many pricing models that seem to work well for single assets cannot be easily extended to a multidimensional set-up, mainly due to computational difficulties. The major problem is that in many cases the probability density function of the basket values at expiration is not known. Hence, practitioners usually resort to classic multidimensional Geometric Brownian motion type models to keep the modelling framework as simple as possible. However, by doing so, the computational problems are not completely solved because the probability density function of the sum of log-normal variables is not known and additionally the empirical characteristics of the assets in the basket are simply overlooked. In particular, the negative skewness and excess kurtosis, which are well known to characterize equities, cannot be captured properly by these simple models because they can produce a limited range of values for these statistics.

Ideally, one would like the best of both worlds, realistic modelling and precise calculations. In this paper, we present a general computational solution to the problem of multidimensional models lacking closed-form formulae or requiring burdensome numerical procedures. The purpose of this paper is to provide a robust and precise methodology for pricing and hedging basket options when the price of each of the assets in the basket follows a model able to accommodate the empirical characteristics. One such model is the displaced jump-diffusion which will be used as test subject to show the superiority of the presented methodology. This model is very useful for the dynamics of one asset, but expanding the set-up to a basket of assets leads to computational problems related to the calculation of the probability distribution of the basket price. Therefore, we circumvent this problem by employing a Hermite polynomial expansion matching exactly the first m moments of the model-implied basket return.

The pricing and hedging methodology we propose consists of quasi–analytical formulae: they are Black and Scholes type formulae and some of their inputs are given as the solution of a system of m equations in m unknowns. The main advantages of the new methodology are: low computational cost compared to numerical methods, especially when one prices a portfolio of options written on the same basket with different strikes and/or payoffs, since the matching procedure needs to be carried out only once; precise calculations and the availability of formulae for the Greeks. Additionally, the only prerequisite of our method is the existence of the moments of the basket and, consequently, it is applicable to the situation when some assets in the basket follow one diffusion model and other assets follow a different diffusion model.

The remaining of the article is structured as follows. Section 2 reviews the existing literature on pricing and hedging basket options. Section 3 describes the continuous-time models employed here. The new methodology is discussed in Section 4 and a numerical comparison is presented in Section 5. The final section concludes.

2. Existing contributions

The number of papers covering basket options has increased considerably in the last three decades. The available methods can be classified into analytical, purely numerical and a hybrid quasi-analytical class which is based on various expansions and moment matching techniques. Our method belongs to the last category.

By analogy to early papers on pricing Asian options, Gentle (1993) proposed pricing basket options by approximating the arithmetic weighted average with its geometrical-average counterpart so that a Black-Scholes type formula could be applied. Korn and Zeytun (2013) improved this approximation using the fact that, if the spot prices of assets in the basket are shifted by a large scalar constant C, their arithmetic and geometric means converge asymptotically. They consider log-normally distributed assets and approximate the C-shifted distribution by standard log-normal distributions. Kirk (1995) developed a technique for pricing a spread option by coupling the asset with negative weight with the strike price, considering their combination as one asset having a shifted distribution and then employing the Margrabe (1978) formula for exchanging two assets. The methods in Li et al. (2008) and Li et al. (2010) extended the procedure proposed in Kirk (1995) to the case of multi-asset spread options. Curran (1994) priced basket options with only positive weights by conditioning on the geometric basket value: the resulting formula is given as an exact term plus an approximated term. Deelstra et al. (2004, 2010) extended on Curran (1994) and obtained lower and upper bounds for the prices of basket options and Asian basket options, respectively. Similarly, Xu and Zheng (2009) derived bounds for basket options on assets following a jump-diffusion model with idiosyncratic and systematic jumps. A completely different approach has been proposed by Laurence and Wang (2004, 2005), and Hobson et al. (2005b,a). They derived model-free upper and lower bounds for basket option prices based on the prices of the European options, each on a single-asset. While the literature on pricing basket options is large, there is sparse research on calculating the hedging parameters for basket options. A notable exception is Hurd and Zhou (2010) who priced spread options and derived the Greek parameters by using fast Fourier transform under different models.

When analytical formulae are difficult to be derived under a particular model, it is common, in the finance industry, to resort to Monte Carlo methods. Control variate techniques for pricing basket options are described in Pellizzari (2001) and Korn and Zeytun (2013). While Monte Carlo methods offer a feasible solution, the computational cost may be too high even for standard-size baskets commonly traded on the financial markets. Hence, the majority of the literature on basket option pricing gravitates around approximation methods that circumvent the numerical problems generated by the high-dimensionality of basket models. Levy (1992) approximated the distribution of a basket by matching its first two moments with the moments of a log-normal density function, and then derived a Black-Scholes type pricing formula. Other works modified the log-normal approximation allowing for improved skewness and kurtosis calibration. Borovkova et al. (2007) have proposed a new methodology that can incorporate negative skewness while still retaining analytical tractability, under a shifted log-normal distribution, by considering the entire basket as one single asset.[1] This strong assumption allows the derivation of closed-form formulae for basket option pricing. On the other hand, some other research has priced basket options whose asset dynamics are more appropriate to accommodate the empirical characteristics of the asset returns. Flamouris and Giamouridis (2007) priced basket options on assets following a Bernoulli jump-diffusion process using the Edgeworth expansion; Wu et al. (2009) assumed that asset prices follow the multivariate normal inverse Gaussian model (mNIG) and employed the fast Fourier transform together with the methodology outlined by Milevsky and Posner (1998) to approximate the sum of assets following the mNIGs model as a mNIG; Xu and Zheng (2009) priced correlated local volatility jump-diffusion model deriving the Partial Integro Differential Equation (PIDE) driving the basket and approximating it via the asymptotic expansion method. Bae et al. (2011) priced basket options (with positive weights) on assets following a jump-diffusion process by using the Taylor expansion method of Ju (2002).

The technique we propose in this paper approximates the basket return at the option maturity by an Hermite polynomial expansion of a standard normal variable. This aims to solve the problems encountered by existing pricing approaches that employ polynomial expansions to approximate the probability density function of the basket values (Dionne et al., 2006, among

[1]Brigo et al. (2004) proposed a similar method to that of Borovkova et al. (2007) but their method can cope only with positive-value baskets.

others). In particular, these methods provide valid approximations only for a limited set of skewness-kurtosis pairs. The main advantage of our new methodology over these previous approaches is that the matching of the moments is exact for a wider set of skewness-kurtosis set.

3. The Modeling Framework

From a modeling point of view, it would be more appropriate for the assets in the basket to follow models that are capable of generating negative skewness and excess kurtosis reflecting the empirical evidence in equity markets. One such flexible model is the displaced (or shifted) jump diffusion, that is a jump diffusion process for the displaced or shifted asset value, similar to the model discussed by Câmara et al. (2009). In the following, we define the modeling framework.

Consider the filtered probability space[2] $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{0\leq t\leq T}, \mathbb{P})$. Let us define, on this space, the financial market consisting of the asset price processes $S^{(i)}$, $i = 1, \cdots, \Upsilon$ and the bank account $M_t = \mathrm{e}^{rt}$ that can be used to borrow and deposit money with continuously compounded interest rate $r \geq 0$, assumed constant over time. The asset price processes $S^{(i)}$ are assumed to follow the correlated displaced jump diffusions, defined by their dynamics

$$\begin{aligned} d\left(S_t^{(i)} - \delta_t^{(i)}\right) &= (\alpha_i - \beta_i \lambda_i)\left(S_t^{(i)} - \delta_t^{(i)}\right) dt + \left(S_t^{(i)} - \delta_t^{(i)}\right) \sum_{j=1}^{n_w} \gamma_{ij}\, dW_t^{(j)} \\ &\quad + \left(S_{t^-}^{(i)} - \delta_t^{(i)}\right) d\mathrm{Q}_t^{(i)} \end{aligned} \tag{3.1}$$

[2]The results in this section are proved both in Câmara et al. (2009) and in Shreve (2004), chap. 11.5. In the latter, the standard multidimensional jump-diffusion model is described and the theory can be adapted to deal with shifted assets.

for $i = 1, \ldots, \Upsilon$, where α_i is the expected rate of return on the shifted asset i, $\left\{ W_t^{(j)} \right\}_{t \geq 0}$ are n_w mutually independent Wiener processes, so that $Z_t^{(i)} = \sum_{j=1}^{n_w} \gamma_{ij} W_t^{(j)}$ are dependent Wiener processes with $Var[Z_t^{(i)}] = t \sum_{j=1}^{n_w} \gamma_{ij}^2$ and $Cov[Z_t^{(i)}, Z_t^{(j)}] = t \sum_{k=1}^{n_w} \gamma_{ik} \gamma_{jk}$. $\left\{ \mathrm{Q}_t^{(i)} \right\}_{t \geq 0}$ are independent compound Poisson processes formed from some underlying Poisson processes $\left\{ N_t^{(i)} \right\}_{t \geq 0}$ with intensity $\lambda_i \geq 0$. In addition, $Y_j^{(i)}$ represents the amplitude of the j-th jump (of the shifted process) of $N_t^{(i)}$ for any $i = 1, \cdots, \Upsilon$, the jumps being i.i.d. random variables with probability density function $f^{(i)}(y) : [-1, +\infty) \to \mathbb{R}_+$ having the expected value[3] under the physical measure $\beta_i = \mathbb{E}[Y^{(i)}] = \int_{-1}^{\infty} y f^{(i)}(y) dy$. Moreover, jumps sizes for different assets are assumed to be independent. Finally, $\delta_t^{(i)} = \delta_0^{(i)} \mathrm{e}^{rt}$ is the shift applied to $S_t^{(i)}$ at time t with non-negative initial shift.

Câmara (1999) studied the relationship between the shift δ_0 and probability density function of the displaced log-normal process (without jumps, as in Rubinstein (1981)): a positive (negative) value of δ_0 is associated with a more positively (negatively) skewed and leptokurtic (mesokurtic) distribution. A drawback of this process is that negative values of δ_0 may imply negative stock prices with positive probability. However, introducing jumps as in (3.1), see also Câmara et al. (2009), allows to capture the empirical properties of stocks even for $\delta_0 \geq 0$. For this reason, in the following we assume $\delta_0^{(i)} \geq 0$, for any $i = 1, \ldots, \Upsilon$.

[3]Henceforth, $\mathbb{E}$ and $\tilde{\mathbb{E}}$ are used to indicate the expectation operator under the physical measure $\mathbb{P}$ and the risk-neutral measure $\tilde{\mathbb{P}}$, respectively.

The solution of SDE (3.1), under the risk-neutral pricing measure $\tilde{\mathbb{P}}$, is

$$S_t^{(i)} = \left(S_0^{(i)} - \delta_0^{(i)}\right) e^{\left(r - \tilde{\beta}_i \tilde{\lambda}_i - \frac{1}{2}\sum_{j=1}^{n_w} \gamma_{ij}^2\right)t + \sum_{j=1}^{n_w} \gamma_{ij} \tilde{W}_t^{(j)}} \prod_{l=1}^{N_t^{(i)}} \left(Y_l^{(i)} + 1\right) + \delta_0^{(i)} e^{rt} \tag{3.2}$$

where the intensity of the Poisson process $\left\{N_t^{(i)}\right\}_{t\geq 0}$ is $\tilde{\lambda}_i$, $\tilde{\beta}$ is the expected value of $Y^{(i)}$, and all the remaining quantities are defined in similar way as those under $\mathbb{P}$. For (3.2) not to introduce arbitrage, the parameters $\tilde{\beta}_1, \cdots, \tilde{\beta}_\Upsilon$, $\tilde{\lambda}_1, \cdots, \tilde{\lambda}_\Upsilon$, and $\theta_1, \cdots, \theta_{n_w}$ need to satisfy the system of equations

$$\alpha_i - \beta_i \lambda_i - r = \sum_{j=1}^{n_w} \gamma_{ij} \theta_j - \tilde{\beta}_i \tilde{\lambda}_i, \quad i = 1, \cdots, \Upsilon. \tag{3.3}$$

Solution to (3.3) is, in general, not unique, so we are in incomplete markets. Nevertheless, we assume that one solution of the system (3.3) is selected[4] and a pricing measure $\tilde{\mathbb{P}}$ is fixed.

For each asset, jumps are taken i.i.d. log-normally distributed such that $\tilde{\mathbb{E}}[\log(Y_j^{(i)} + 1)] = \eta_i$ and $\widetilde{Var}[\log(Y_j^{(i)} + 1)] = \upsilon_i^2$. This assumption cor-

[4]There is a large literature devoted to the issue of selecting a pricing measure. For a review, see Frittelli (2000) and references within. General principles about the martingale approach of pricing contingent claims are provided in Chapter 15 of Bjork (2009). More specific techniques for pricing and hedging in incomplete markets are described in Chapter 10 of the excellent book by Cont and Tankov (2004). Secondly, standard market practice is to calibrates the volatility parameters to volatility surfaces, in general using vanilla products for each asset in the basket, where models are more robust and market data is available. What are very difficult to calibrate are the correlations between assets, particularly in markets that operate mainly OTC. Some traders estimate their correlations from historical data, others use some copula functions and so on. Any approach has pros and cons, as documented in the literature.

responds to the displaced jump diffusion with unsystematic jump risk in Câmara et al. (2009). We remark that when in addition $\delta_0^{(i)} = 0$, the model corresponds to that of Merton (1976). In order to simplify the notation, we denote $V_t^{(i)} = \sum_{j=1}^{n_w} \frac{\gamma_{ij}}{\sigma_i} \tilde{W}_t^{(j)}$ where $\sigma_i^2 = \sum_{j=1}^{n_w} \gamma_{ij}^2$. Thus $\left\{V_t^{(i)}\right\}_{t\geq 0}$ are dependent standard Brownian motions with

$$\rho_{l_1 l_2} = \mathrm{corr}(V_t^{(l_1)}, V_t^{(l_2)}) = \frac{1}{\sigma_{l_1}\sigma_{l_2}} \sum_{j=1}^{n_w} \gamma_{l_1 j}\gamma_{l_2 j},$$

and, consequently, (3.2) can be rewritten as

$$S_t^{(i)} = \left(S_0^{(i)} - \delta_0^{(i)}\right) e^{\left(r - \tilde{\beta}_i \tilde{\lambda}_i - \frac{1}{2}\sigma_i^2\right)t + \sigma_i V_t^{(i)}} \prod_{l=1}^{N_t^{(i)}} \left(Y_l^{(i)} + 1\right) + \delta_0^{(i)} e^{rt}. \tag{3.4}$$

Finally, we point out that the shifted jump-diffusion model will encompass three sub-cases: the geometric Brownian motion (GBM) when $\delta_0^{(i)} = 0$ and $\tilde{\lambda}_i = 0$ for each asset i, the shifted GBM when $\tilde{\lambda}_i = 0$ for each asset i, and the standard jump-diffusion model when $\delta_0^{(i)} = 0$ for each asset i.

4. Pricing and hedging methodology

The aim of this section is to price European basket options. The method we introduce here is general and works for any choice of price models for the assets in the basket under the assumption that we can calculate the moments of the basket at maturity. However, in order to simplify the description of the methodology, we consider the modelling framework in the previous section[5]

[5]Any change in the price dynamics will impact exclusively on the calculation of the moments of the basket returns (i.e. the results in Proposition 4.1).

and price a basket call option expiring at time T, whose payoff at maturity is $(B_T^* - K^*)^+$. The variable underlying the option is the basket

$$B_t^* = \sum_{i=1}^{\Upsilon} a_i S_t^{(i)}, \tag{4.1}$$

where Υ is the number of assets in the basket, K^* is the strike price, and $a_i \in \mathbb{R}$ is the quantity invested in asset i, $i = 1, \ldots, \Upsilon$.

Under the majority of models applied in practice, including the shifted jump-diffusion model, the probability density of the basket B_t^*—required for pricing and hedging—cannot be obtained in closed-form. The methodology proposed here circumvents this problem by approximating the standardized return of the basket by a polynomial transformation of a standard normal random variable. The approximation derived in this paper is constructed in such a way to match up exactly[6] the first m moments of the model implied risk-neutral return. While the methodology may work for any required m, from an investment finance perspective only the first four moments have been identified in the asset pricing literature as having a clear significance.

For practical purposes, since the assets follow a shifted process, we shall work with shifted quantities defined as: 'shifted strike price'

$$K = K^* - \sum_{i=1}^{\Upsilon} a_i \delta_0^{(i)} \mathrm{e}^{rT} \tag{4.2}$$

[6]A different approach using different Hermite polynomials, called the "physicists" Hermite polynomials, has been exploited elegantly for option pricing by Necula et al. (2015). The main advantage of their approach is that convergence for fat-tailed distributions is guaranteed when using the full infinite polynomial Hermite expansion. Our approach is focused on matching the first four moments exactly, using the "probabilists" Hermite polynomials.

and 'shifted basket'

$$B_T \;=\; B_T^* - \sum_{i=1}^{\Upsilon} a_i \delta_0^{(i)} \mathrm{e}^{rT}. \tag{4.3}$$

Our methodology uses the random variable

$$J(Z) = \sum_{k=0}^{m-1} \varphi_k H_k(Z), \tag{4.4}$$

to approximate the standardized basket return quantity[7]

$$X_T = \frac{B_T}{B_0 \mathrm{e}^{rT}} - h_1. \tag{4.5}$$

To explain how the approximation works, denote by $\Phi_t = B_t \mathrm{e}^{-rt}$ be the discounted basket payoff at time t. We use $J(Z)$ to approximate Φ_T/Φ_0 when $h_1 = 0$ and to approximate the gross return of Φ over the interval $[0, T]$ when $h_1 = 1$. There are only two values for the coefficient h_1, 0 and 1 respectively. Furthermore, $H_k(x)$ denotes the kth-order Hermite polynomial $H_k(x) = \frac{(-1)^k}{\phi(x)} \frac{\partial^k \phi(x)}{\partial x^k}$, $\phi(\cdot)$ is the standard normal density function and Z is a standard normal random variable.

The coefficients φ_k are calculated by matching the first m moments of (4.5), i.e. as the solution of the system of equations

$$\begin{cases} \tilde{\mathbb{E}}[J] = \tilde{\mathbb{E}}[X_T] \\ \tilde{\mathbb{E}}[J^2] = \tilde{\mathbb{E}}[X_T^2] \\ \qquad \cdots \\ \tilde{\mathbb{E}}[J^m] = \tilde{\mathbb{E}}[X_T^m] \end{cases} \tag{4.6}$$

[7] B_0 is assumed to be different from 0 and h_1 can take only the values 0 and 1 to indicate what type of returns are used for calculations.

In order to solve this system we need to calculate the first m moments of J, i.e. we need $\tilde{\mathbb{E}}[J^k]$ for $k = 1, \ldots, m$. In particular, the k-th moment of m-degree polynomial J is given by

$$\tilde{\mathbb{E}}[J^k] = \tilde{\mathbb{E}}\left\{\left[\sum_{i=0}^{m-1} \varphi_i H_i(Z)\right]^k\right\} =$$

$$\tilde{\mathbb{E}}\left\{\left[\sum_{i_1=0}^{m-1} \varphi_{i_1} H_{i_1}(Z)\right] \times \left[\sum_{i_2=0}^{m-1} \varphi_{i_2} H_{i_2}(Z)\right] \cdots \times \left[\sum_{i_k=0}^{m-1} \varphi_{i_k} H_{i_k}(Z)\right]\right\} =$$

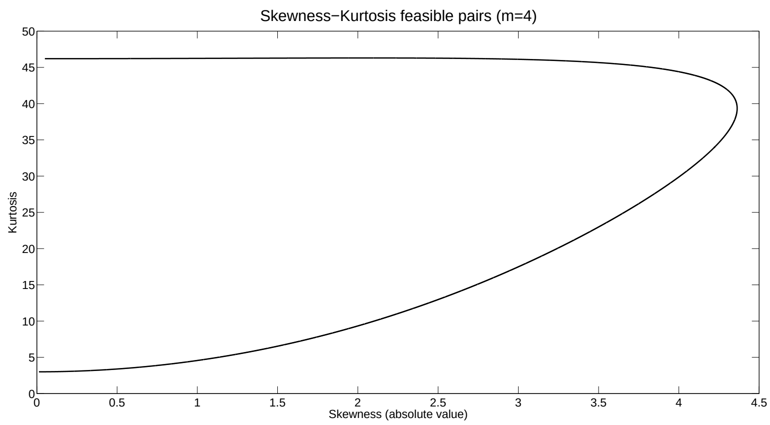
$$\sum_{i_1=0}^{m-1} \cdots \sum_{i_k=0}^{m-1} \varphi_{i_1} \times \cdots \times \varphi_{i_k} \tilde{\mathbb{E}}[H_{i_1}(Z) \times \cdots \times H_{i_k}(Z)]$$

In Appendix A.1, we provide the analytic formulae for the first 4 moments of $J(Z)$. Following Leccadito et al. (2014), p. 79-80, it is possible to determine all the possible values of skewness (in absolute value) and kurtosis of X_T for which the proposed method can be employed when $m = 4$, see Figure 1. We refer the reader to Headrick (2009), p. 23, for a discussion regarding the feasible skewness-kurtosis pairs for larger values of m, where the author states that the region associated to $m = 6$ is larger than the one represented in Figure 1.

The three variants of our moment-matching method that will be analyzed in Section 5 are:

1. mGA indicates a moment matching procedure that matches the first m moments of X_T with $h_1 = 0$;
2. mGB indicates a moment matching procedure that matches the first m moments of X_T with $h_1 = 1$;
3. $mGAB$ is a hybrid methodology spanned by the two methods mGA and mGB. It returns the solution of the method that correctly matches

Figure 1: The locus of skewness-kurtosis pairs of X_T, eq. (4.5), for which the proposed approximation (4.4) is feasible when $m = 4$.



the moments if one of mGA and mGB works properly and takes into account the worst error between the two variants if both correctly match the moments.

For the three variants above, the mnemonics driven by m stands for the number of moments matched and G highlights that a transformation of the Gaussian distribution is considered.

This moment-matching procedure is an extension of the method presented in Leccadito et al. (2012). They proposed the Hermite tree method for pricing financial derivatives and, in a nutshell, the idea is to match the moments of the log-returns of the underlying asset with the moments of a discrete random variable. Our methodology extends Leccadito et al. (2012) to deal with baskets that may take on negative values and replaces the binomial distribution they employed with the asymptotically equivalent Gaussian distribution. Consequently, our new methodology consists of quasi-analytic pricing and hedging formula, which do not employ a tree or lattice method.

Proposition 4.1 shows how to calculate the moments of the standardized basket return quantity for assets that follow the shifted jump-diffusion process (SDE (3.4)).

Proposition 4.1. *The k-moment of the standardized return X_T in formula (4.5), under $\tilde{\mathbb{P}}$, is given by*

$$\tilde{\mathbb{E}}[X_T^k] = \tilde{\mathbb{E}}\left[\left(\frac{B_T}{B_0 \mathrm{e}^{rT}} - h_1\right)^k\right] = \sum_{i=0}^{k} \binom{k}{i} \frac{(-h_1)^i}{(B_0\mathrm{e}^{rT})^{k-i}} \tilde{\mathbb{E}}[B_T^{k-i}]. \tag{4.7}$$

where

$$\tilde{\mathbb{E}}[B_T^k] = \sum_{i_1=1}^{\Upsilon} \cdots \sum_{i_k=1}^{\Upsilon} a_{i_1}\left(S_0^{(i_1)} - \delta_0^{(i_1)}\right)\mathrm{e}^{(r+\omega_{i_1})T} \cdots a_{i_k}\left(S_0^{(i_k)} - \delta_0^{(i_k)}\right)\mathrm{e}^{(r+\omega_{i_k})T}\,\mathrm{mgf}(\boldsymbol{e}_{i_1}+\ldots+\boldsymbol{e}_{i_k}), \tag{4.8}$$

$\omega_j = -\tilde{\beta}_j\tilde{\lambda}_j - \frac{1}{2}\sigma_j^2$, $\boldsymbol{e}_j \in \Re^{\Upsilon}$ is the vector having 1 in position j and 0 elsewhere. Furthermore, the moment generation function of $\sigma_i V_T^{(i)} + \sum_{l=1}^{N_T^{(i)}} \log\left(Y_l^{(i)} + 1\right)$ is given by

$$\mathrm{mgf}(\boldsymbol{u}) = \exp\left\{T\boldsymbol{u}'\boldsymbol{\Sigma}\boldsymbol{u}/2\right\} \prod_{i=1}^{\Upsilon} \mathrm{mgf}_{N_T^{(i)}}\left(\eta_i u_i + \upsilon_i^2 u_i^2/2\right) \tag{4.9}$$

where $\boldsymbol{\Sigma}$ denotes the covariance matrix of $\boldsymbol{V} = \left(V_T^{(1)}, \cdots, V_T^{(\Upsilon)}\right)'$, and

$$\mathrm{mgf}_{N_T^{(i)}}(u) = \exp(T\tilde{\lambda}_i(\mathrm{e}^u - 1)). \tag{4.10}$$

Proof. Formulae (4.7) and (4.8) are derived by exponentiation of formulae (4.5) and (4.3), respectively and the linear property of the expectation operator. Additionally, the moment generation function of $\sigma_i V_T^{(i)} + \sum_{l=1}^{N_T^{(i)}} \log\left(Y_l^{(i)} + 1\right)$ in (4.9) is calculated by conditioning with respect to $N_T^{(i)}$. □

Once the parameters φ are calculated by solving the moment-matching system (4.6), the two propositions in the next sections are our main results for pricing and hedging basket options. The solution of this system of equations is done numerically and the moment matching requires little computational effort.

4.1. Pricing and hedging methods

Following standard non arbitrage principles, the price of a European basket call option is calculated by discounting the expected value of the option payoff at maturity. The mechanism of shifting the basket and strike price, in equations (4.2) and (4.3), allows rewriting the pricing formula in two equivalent ways:

$$c_0(B_0^*, T, K^*) = \mathrm{e}^{-rT}\tilde{\mathbb{E}}[(B_T^* - K^*)^+] = \mathrm{e}^{-rT}\tilde{\mathbb{E}}[(B_T - K)^+] = c_0(B_0, T, K). \tag{4.11}$$

The next proposition provides a formula for the European call basket option price under the Hermite polynomial approximations considered in this paper.

Proposition 4.2. *The price of a European call basket option with the Hermite expansion variant mGA or mGB is given by:*

$$c_0(B_0, T, K) = B_0\left[(\varphi_0 + h_1)\Phi(-h_2\tilde{z}) + h_2 g(\tilde{z})\right] - K\mathrm{e}^{-rT}\Phi(-h_2\tilde{z}) \tag{4.12}$$

where

$$g(\tilde{z}) = \phi(\tilde{z})\sum_{k=0}^{m-2}\varphi_{k+1}H_k(\tilde{z}), \tag{4.13}$$

K is the shifted strike price, $h_1 = 0$ for the variant mGA and $h_1 = 1$ for the variants mGB, $h_2 = \mathrm{sgn}(B_0)$, $\tilde{z}$ is the solution of $[J(\tilde{z}) + h_1]B_0\mathrm{e}^{rT} = K$,

$\phi(\cdot)$ is the standard normal density function, $\Phi(\cdot)$ is the standard normal cumulative distribution function and $\varphi_1, \ldots, \varphi_{m-1}$ are calculated by matching the first m moments of the standardized return quantity X_T.

Proof. Let us consider the approximation of X_T by the random variable $J(Z)$ via the solution of the moment-matching procedure. Consequently,

$$B_T \approx B_0 \mathrm{e}^{rT}(J(Z) + h_1) \tag{4.14}$$

and substituting it into the equality (4.11) leads to:

$$\begin{aligned} c_0(B_0, T, K) &= \mathrm{e}^{-rT}\tilde{\mathbb{E}}[(B_T - K)^+] \approx \mathrm{e}^{-rT} \int_{l_1}^{l_2} \left[B_0 \mathrm{e}^{rT}(J(z) + h_1) - K\right] \phi(z) dz \\ &= B_0 \int_{l_1}^{l_2} J(z)\phi(z)dz + (h_1 B_0 - K\mathrm{e}^{-rT})\Phi(-h_2 \tilde{z}) \end{aligned} \tag{4.15}$$

where, for $B_0 > 0$, $l_1 = \tilde{z}$ and $l_2 = +\infty$ and, for $B_0 < 0$, $l_1 = -\infty$ and $l_2 = \tilde{z}$.

For the calculation of the integral $\int_{l_1}^{l_2} J(z)\phi(z)dz$, the results in formulae (A.1) and (A.2) (see Appendix A.2) are employed for $B_0 > 0$ and $B_0 < 0$, respectively. Formula (4.12) is then proved by rearranging the terms. □

The next proposition reports the formula for the hedging parameter[8] with respect to the variable u, which can be any of the quantities $S_0^{(i)}$, B_0^*, σ_i, r, T, a_i, $\tilde{\lambda}_i$, $\delta_0^{(i)}$, $\tilde{\beta}_i$, η_i or υ_i.

[8]One may remark that this formula is an approximation of the theoretical Greek parameter, that is not analytically available for our model. On their own, the hedging values for delta parameter for example will indeed not give the exact analytical option price under the assumed model but it will help with faster hedging calculations. We thank an anonymous referee for indicating this point.

Proposition 4.3. *For* c_0, h_1, h_2, $\tilde{z}$, $g(\cdot)$, $\phi(\cdot)$ *and* $\Phi(\cdot)$ *defined in Proposition 4.2, the hedging parameter of a European call basket option, with respect to the variable* u*, under the Hermite expansion variant mGA or mGB, is given by*

$$\begin{aligned}\frac{\partial c_0}{\partial u} &= c_0 \mathrm{e}^{rT}\frac{\partial \mathrm{e}^{-rT}}{\partial u} + B_0\left[h_2 g'(\tilde{z}) + \frac{\partial \varphi_0}{\partial u}\phi(-h_2\tilde{z})\right] + h_2\mathrm{e}^{-rT}\Phi(\tilde{z})\frac{\partial K}{\partial u} \\ &+ \mathrm{e}^{-rT}\frac{\partial(B_0\mathrm{e}^{rT})}{\partial u}\left[h_2 g(\tilde{z}) + \varphi_0\phi(-h_2\tilde{z}) + h_1\left(-h_2\Phi(\tilde{z}) + \frac{h_2+1}{2}\right)\right]\end{aligned} \tag{4.16}$$

where

$$g'(\tilde{z}) = \phi(\tilde{z})\sum_{k=0}^{m-2}\frac{\partial\varphi_{k+1}}{\partial u}H_k(\tilde{z}) \tag{4.17}$$

and c_0 *is the short for* $c_0(B_0, T, K)$.

Proof. The calculation of the hedging parameter can be achieved by direct differentiation using Leibniz' rule of the pricing formula (4.11) considered together with approximation (4.14), as follows:

$$\begin{aligned}\frac{\partial c_0}{\partial u} &= \frac{\partial \mathrm{e}^{-rT}}{\partial u}\int_{l_1}^{l_2}\left[B_0\mathrm{e}^{rT}(J(z)+h_1) - K\right]\phi(z)dz \\ &\quad + \mathrm{e}^{-rT}\int_{l_1}^{l_2}\frac{\partial\left[B_0\mathrm{e}^{rT}(J(z)+h_1)-K\right]}{\partial u}\phi(z)dz \\ &= c_0\mathrm{e}^{rT}\frac{\partial \mathrm{e}^{-rT}}{\partial u} + \mathrm{e}^{-rT}\int_{l_1}^{l_2}\frac{\partial\left[B_0\mathrm{e}^{rT}(J(z)+h_1)-K\right]}{\partial u}\phi(z)dz.\end{aligned}$$

Additionally, since the Hermite polynomials do not depend on u, $\frac{\partial J(z)}{\partial u} = \sum_{k=0}^{m-1}\frac{\partial\varphi_k}{\partial u}H_k(z)$, and, consequently, formulae (A.1) and (A.2) in Appendix A.2 can also be used for the integral $\int_{l_1}^{l_2}\frac{\partial J(z)}{\partial u}\phi(z)dz$ where l_1 and l_2 are as defined for function (4.15). Formula (4.16) is then proved by rearranging the terms. The calculation of $\frac{\partial\varphi_k}{\partial u}$ is shown in Appendix A.3. □

In Section 5.3, a comparison of our method with other methods in the literature is carried out using the Delta-hedging performances as a yardstick. For that exercise, we calculate the delta parameter as:

$$\frac{\partial c_0}{\partial B_0^*} = \frac{\partial c_0}{\partial S_0^{(i)}}\frac{\partial S_0^{(i)}}{\partial B_0^*} = \frac{\partial c_0}{\partial S_0^{(i)}}\frac{1}{a_1}, \tag{4.18}$$

where $\frac{\partial c_0}{\partial S_0^{(i)}}$ is calculated by using formula (4.16) for $u = S_0^{(i)}$:

$$\begin{aligned}\frac{\partial c_0}{\partial S_0^{(i)}} &= B_0\left[h_2 g'(\tilde{z}) + \frac{\partial \varphi_0}{\partial S_0^{(i)}}\phi(-h_2\tilde{z})\right] + \\ &\quad + a_1\left[h_2 g(\tilde{z}) + \varphi_0\phi(-h_2\tilde{z}) + h_1\left(-h_2\Phi(\tilde{z}) + \frac{h_2+1}{2}\right)\right]\end{aligned} \tag{4.19}$$

The calculation of $\frac{\partial \varphi_k}{\partial S_0^{(i)}}$ is shown in Appendix A.3.

The same methodology can be applied to other payoff structures. Additionally, we note that for pricing basket put options, defining p_0 as the put price, one can employ the put-call parity relationship:

$$p_0(B_0^*, T, K^*) = c_0(B_0^*, T, K^*) + K^*\mathrm{e}^{-rT} - B_0^* \tag{4.20}$$

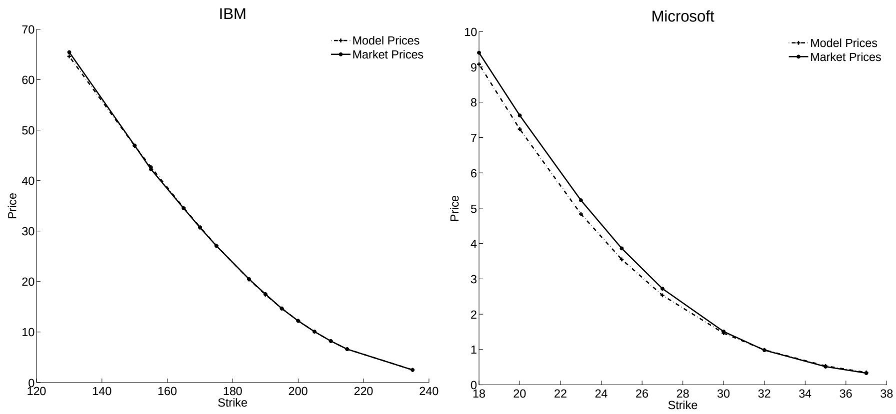
that can equivalently be written for any hedging parameter w.r.t. u as:

$$\frac{\partial p_0}{\partial u} = \frac{\partial c_0}{\partial u} + \frac{\partial (K^*\mathrm{e}^{-rT} - B_0^*)}{\partial u}. \tag{4.21}$$

5. Numerical Comparisons

In this section we first provide a short example of how the model is calibrated and show how the resulting approximated densities compare with the true ones. We then investigate the performance of the proposed approximation both in a pricing and in a delta-hedging context.

Figure 2: Calibration of single-stock European options.

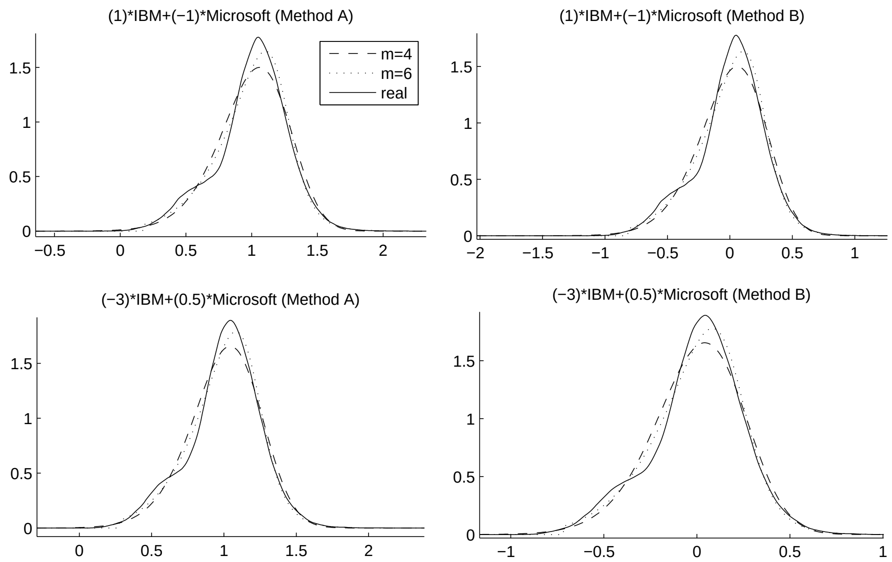


5.1. Model Calibration

All parameters (including δ_0) excluding correlations are calibrated from single-stock European options. All the correlations are estimated from historical data. As an example, we consider a basket with two assets, IBM and Microsoft. The calibration exercise is performed on European options quotes available on 14/12/2012. Among the options available, we select the ones with maturity closes to 1 year, the maturity of the basket options considered. The calibrated parameters are $\sigma = 0.1552$, $\tilde{\lambda} = 0.3151$, $\eta = -0.3541$, $\upsilon = 0.2403$ and $\delta_0 = 38.3615$ for IBM and $\sigma = 0.2785$, $\tilde{\lambda} = 0.0955$, $\eta = 0.0186$, $\upsilon = 0.03$ and $\delta_0 = 5.3491$ for Microsoft. Figure 2 reports, for the two companies, market option prices vs. model prices for various strike levels.

The correlation estimated using the time series of daily returns equals 0.4274. We consider two baskets comprising these two stocks ($a_{IBM} = 1, a_{Microsoft} = -1$ and $a_{IBM} = -3, a_{Microsoft} = 0.5$) with maturity 1 year

Figure 3: Approximated vs. true densities.



and using the calibrated parameters we compare in Figure 3 the densities resulting from the approximations 4GA, 4GB, 6GA, and 6GB with the true ones (obtained by simulation). Moreover, the risk-free rate is $r = 0.0526$ and the spot prices are \$191.76 and \$26.81 for IBM and for Microsoft, respectively.

5.2. Pricing performances

The usefulness of a newly proposed method can be gauged by comparing it with other established methods in the literature. The variants mGA, mGB and $mGAB$—introduced in Section 4—of our Hermite approximation approach are compared on a large set of simulated option-scenarios with the method in Borovkova et al. (2007), (BPW) from now on, which is capable of matching quite large ranges of skewness and kurtosis, is also supported by a Black-and-Scholes type pricing formula, is shown to be one of the best available method, has a similar running time as our methodology and is, consequently, our main competitor. In addition, the benchmark option price is taken as the Monte Carlo with control variate methodology outlined in Pellizzari (2001), henceforth MC, adapted to deal with assets having the dynamics specified by equation (3.4). The pricing performance of each method is determined considering two measures of error: C1 and C2. C1 is the *percentage of 'good prices'*, defined as number of times the absolute percentage error under the specified method is lower than 2% over total number of options:

$$C1_j = \frac{1}{|O|}\sum_{i\in O} 1_{APE_{i,j}<2\%}$$

where O is the set of the option scenarios, $|O|$ its cardinality, $1_{\{\cdot\}}$ the indicator function, $APE_{i,j} = \left|\frac{P_{i,j}-MC_i}{MC_i}\right|$ and $P_{i,j}$ and MC_i are the price of option scenario i under method j and the benchmark price, respectively. C2 is the *mean absolute percentage error*, calculated only relative to the options for which the method was able to find a numerical solution, i.e.,

$$C2_j = \frac{1}{|O_j|}\sum_{i\in O_j} APE_{i,j}$$

where $O_j \subseteq O$ represents the only option in O for which method j could find a numerical solution for the moment matching procedure. In particular, a numerical solution is not found whenever the system of m equations for the moment matching procedure does not admit a solution, i.e. the moments of the basket return are outside the moments' domain of the Hermite polynomial expansions (for mGA or mGB) and/or the log-normal density (for the method of Borovkova et al. (2007)). For more details see

Jondeau and Rockinger (2001). For the considered scenarios, the percentage of numerical solution found by the BPW method , mGA and mGB are above 90%. We perform two separate pricing performance studies. The first is based on the option scenarios described in Borovkova et al. (2007) and the second based on 2,000 simulated options scenarios.

5.2.1. Comparison under the scenarios in Borovkova et al. (2007)

This section is a direct comparison with the method in Borovkova et al. (2007) on the six basket options they considered. It is assumed that the i-th asset in the basket follows the process described by SDE (3.4) with $\tilde{\lambda}_i = 0$ and $\delta_0^{(i)} = 0$ and the other parameters as in Table 1 (i.e. the asset prices are assumed to follow a geometric Brownian motion). The results are depicted in Table 2: the BPW prices in the table had to be adjusted from the ones in their paper because, to be consistent with the other models considered in this paper, we are pricing basket options on equities and not on forward contracts.

The numerical results indicate that $4GA$ and $4GB$ give, for these six basket options, exactly the same prices, and the two methods appear to be as good as the BPW method according to the C1 criterion and to outperform it according to the C2 criterion. The methods $6GA$ and $6GB$ underperform the other three methods and, consequently, for the baskets analysed here, there is very little advantage in matching all six moments, the Hermite approximation method working overall better when only the first four moments are matched. To further stress the superiority of our approximation over competing methods, we remark that there are baskets for which the BPW method cannot be used (because moments are not matched) while our method is still valid. One such case is easily obtained by changing a_1 to -4 in the first basket of Table 1.

[Table 1 about here.]

[Table 2 about here.]

5.2.2. Comparison under a set of simulated scenarios

A general comparison is performed considering 2,000 generated options scenarios. In the first 1,000 scenarios (henceforth, the first 1,000 scenarios will be called 'Set 1') each asset in the baskets follows the shifted jump-diffusion model with dynamics given by SDE (3.4) where the parameters are drawn based on the following specifications: all σ_i are independently uniformly distributed between 0.1 and 0.6; $S_0^{(i)}$ are uniformly distributed between 70 and 130; the shifts $\delta_0^{(i)}$ range uniformly between 0 and 20; the intensities of the Poisson processes $\tilde{\lambda}_i$ are uniformly distributed between 0 and 0.2; the average jump size (η_i) is uniformly distributed between -0.3 and 0; and the volatility (υ_i) is uniformly distributed between 0 and 0.3. Furthermore, the number of assets in the basket in each scenario is uniformly distributed between 2 and 15, r is uniformly distributed between 0.0 and 0.1, T is uniformly distributed between 0.1 and 1 years, the weights a_i of the assets in the basket are uniformly distributed between -1 and 1, the ratios K^*/B_0^* are uniformly distributed between 0.8 and 1.2, and the correlation matrix among assets is randomly generated satisfying the semi-positiveness condition. The second 1,000 scenarios (henceforth 'Set 2') are identical to the scenarios in Set 1 except for the average jump size (η_i) which is uniformly distributed between -0.3 and 0.3.

The number of simulations used when applying the Monte Carlo method are between 10^5 and 10^6, depending on the number of assets in the baskets. The methods we compare are $4GA$, $4GB$, $4GAB$, $6GA$, $6GB$ and BPW methodology.

The results in relation to Set 1 and Set 2 are summarized in Tables 3 and 4, respectively. The two tables show similar results. Overall methods $4GA$ and $4GB$ have analogous performance in terms of C1 and C2 criteria, with $4GB$ slightly better than $4GA$. $4GA$ outperforms $4GB$ only for longer maturities (greater than 0.5 years) scenarios under C1 measure and for near-the-money scenarios under C2. Both $4GA$ and $4GB$ are robust to a change in the risk-free rate. However, the performances of both methods improve for longer maturities under C1 and worsen under C2. Comparing our two Hermite approximations with the BPW method, it is clear that the latter is not as good as the former at matching the model-implied characteristics and that the fourth moment is necessary for pricing basket options. Both $4GA$ and $4GB$ show greater improvements on the BPW method the greater the basket size. Finally, the method $4GAB$ outperforms the two methods under C1, performing almost as well under C2. Consequently, one can use this hybrid method for practical purposes.

A cross analysis of Table 3 and Table 4 shows that changes in the expected jump intensity impact on the performances of the Hermite-approximate methods that are slightly better for $\eta_i \in [-0.3, 0]$. Additionally, the two tables show the pricing performances when the methods mGA and mGB are used for $m = 6$ moments. The method $6GB$ outperforms all the other while $6GA$ also outperforms the other methods under C1 but underperforms $4GA$ and

$4GB$ under C2. While results may improve for this exercise when using more moments matching, it is difficult to interpret moments larger than four.

[Table 3 about here.]

[Table 4 about here.]

Our numerical results reveal that our methodology improves the performance of the approach described in Borovkova et al. (2007). Furthermore, under our method, models that match the first six moments seem to produce some marginal performance improvement over models matching four moments. In theory, one could use any number of moments m, a higher m being associated with improved performance. However, since only the first four moments have a clear association with known features of empirical series– the mean, variance, skewness and kurtosis–, we recommend working with Hermite polynomials determined by matching only the first four moments.

5.3. Delta-hedging performances

A comparison of Delta-hedging performance between our formula (4.19) and the formula proposed in Borovkova et al. (2007) is illustrated in this section. A sample of $n_S = 1,000$ simulated paths with a 1-week-interval hedging rolling frequency is generated for six basket option scenarios. All the scenarios have: $\Upsilon = 2$, $\sigma_1 = 0.3$, $\sigma_2 = 0.2$, $T = 0.5$ years, $S_0^{(1)} = 110$, $S_0^{(2)} = 90$, $a_1 = 0.7$, $a_2 = 0.3$, $\delta_0^{(1)} = \delta_0^{(2)} = 20$, $\tilde{\lambda}_1 = \tilde{\lambda}_2 = 0.2$, and $\upsilon_1 = \upsilon_2 = 0.2$. Additionally, we consider for three of the scenarios $r = 2\%$ and $\eta_1 = \eta_2 = -0.3$ and for the other three $r = 5\%$ and $\eta_1 = \eta_2 = 0.3$. The strikes considered are $K^* = \{100, 104, 110\}$.

For each path, the option delta are calculated at each time step by the three methods $4GA$, $4GB$ and the BPW method. The evaluation of the performances for the Delta-hedged portfolios is performed via two measures: C3, i.e. the *average hedging error* among all the simulations

$$C3 = \frac{1}{n_S} \sum_{i \in n_S} HE_i$$

and C4, i.e. the *average quadratic hedging error*, where the hedging error is defined as the difference in values between hedging portfolio at the maturity date and option's payoff,

$$C4 = \frac{1}{n_S} \sum_{i \in n_S} HE_i^2.$$

The results for the hedging performances are reported in Table 5. The methods $4GA$ and $4GB$ produce good results and their performances are almost identical on the six scenarios considered. For $\eta_i = -0.3$, the two Hermite expansion methods tend to super-hedge, as the measure C3 indicates, although the average errors are almost negligible. However, when $\eta_i = 0.3$ on average the hedging error is negative showing a sub-hedge and this is caused by the high average jump size.

The BPW method also performs fairly well for the three scenarios with $\eta_i = -0.3$ with virtually the same performances of $4GA$ and $4GB$ under the measure C4. However, under these three scenarios, the new methods have much better performances than the BPW method under the C3 measures (a remarkable reduction of more than 25% is reached). When one considers the positive average jump size ($\eta_i = 0.3$), also the method of Borovkova et al. (2007) sub-hedges under each scenarios and both its C3 and C4 measures of error are worse than the $4GA$ and $4GB$ ones.

[Table 5 about here.]

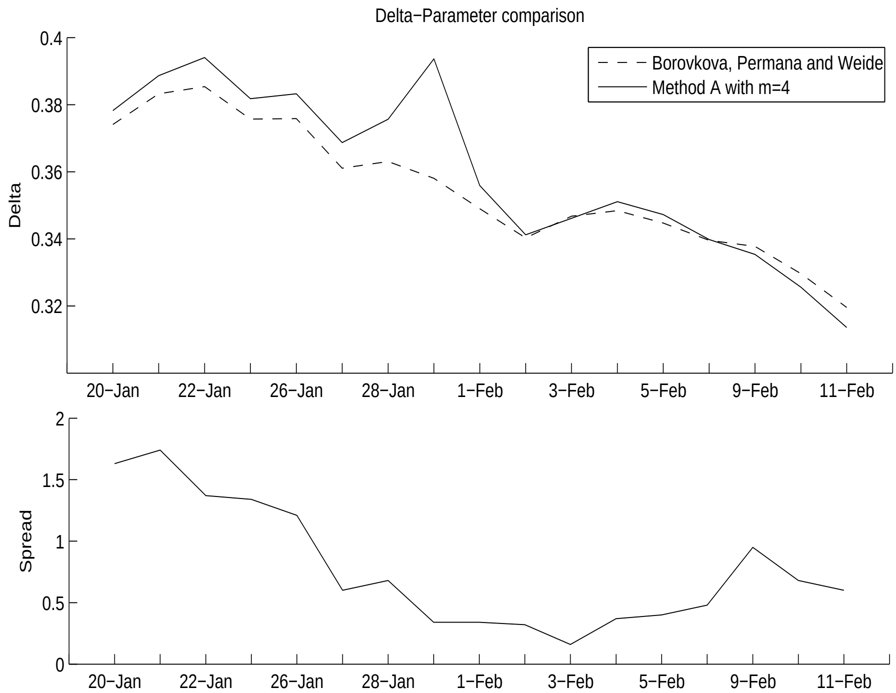
5.4. A real-world example

Here we apply the delta-hedging formula to hedge a position in a Jun-expire WTI-Brent futures spread option with strike price 6.5$ (Bloomberg ticker BYM6P).[9]

The underlying assets are consider to follow two correlated displaced jump diffusion. These are calibrated to single asset vanilla options on the futures. The correlation between the two assets was estimated as the historical correlation of log returns based on daily observations. The plot 4 illustrates the comparison of the delta parameters calculated via our method A with 4 moments and the BPW method. We run the exercise from 20th January 2016 when the option price is 0.81$ to 11th February 2016 when the option price is 0.95$.

[9]Please note that the WTI-Brent futures spread is directly quoted in the market but for the sake of this exercise we consider the option driven by the two underlyings: WTI crude futures (CLM6) and Brent futures (BZAM6). The weights of the the two futures are 1 and -1, respectively

Figure 4: Delta-Hedging comparison for a June-expire WTI-Brent futures spread option from 20th January 2016 when the option price is 0.81$ to 11th February 2016 when the option price is 0.95$. Market data from Bloomberg. The underlying spread price is described in the lower graph.



6. Conclusions

One can account for the empirical characteristics of historical prices by considering a shift into the jump-diffusion process underlying the assets of a basket. However, recent techniques imposed strong assumptions on the overall evolution dynamics of the basket, searching for closed-form solution and repackaging of log-normal Black-Scholes type pricing formulae.

In this paper, we have highlighted a methodology that can handle baskets of assets following correlated shifted log-normal diffusions with jumps and that is applicable whenever the spot price of the basket is not zero. We demonstrated with numerical comparisons that our Hermite expansion approach provides pricing and hedging results for basket options that are as good as competing methods, and in many cases superior.

The improved results emphasized in the paper are not surprising since the technique is fundamentally based on matching the first four moments under model specification. Thus, we allow granular specification of dynamics for each asset but then we only need to determine the moments of the basket. While our paper was focused on equity baskets, it is clear that the same methodology can be applied for mixtures of assets and models, as long as moments can be calculated easily.

Disclaimer: The views expressed are those of the authors and do not necessarily reflect those of ING Bank.

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Appendix A. Computational Tools

Appendix A.1. Moments of the approximating variable $J(Z)$

The k-th moment of $J(Z) = \sum_{k=0}^{m-1} \varphi_k H_k(Z)$ (formula (4.4)) can be calculated in closed form as a weighted sum of the moments of the standard normal variable Z. For $m = 4$, the moments of J are

$$
\begin{aligned}
\tilde{\mathbb{E}}[J] &= \varphi_0 \\
\tilde{\mathbb{E}}[J^2] &= \sum_{i=0}^{m-1} i!\varphi_i^2 \\
\tilde{\mathbb{E}}[J^3] &= \varphi_0^3 + (3\varphi_1^2 + 6\varphi_2^2 + 18\varphi_3^2)\varphi_0 + 6\varphi_1^2\varphi_2 + 36\varphi_1\varphi_2\varphi_3 + 8\varphi_2^3 + 108\varphi_2\varphi_3^2 \\
\tilde{\mathbb{E}}[J^4] &= \varphi_0^4 + (6\varphi_1^2 + 12\varphi_2^2 + 36\varphi_3^2)\varphi_0^2 + (24\varphi_1^2\varphi_2 + 144\varphi_1\varphi_2\varphi_3 + 32\varphi_2^3 + 432\varphi_2 \\
&\quad \varphi_3^2)\varphi_0 + 3\varphi_1^4 + 24\varphi_1^3\varphi_3 + 60\varphi_1^2\varphi_2^2 + 252\varphi_1^2\varphi_3^2 + 576\varphi_1\varphi_2^2\varphi_3 + 1296\varphi_1\varphi_3^3 \\
&\quad +60\varphi_2^4 + 2232\varphi_2^2\varphi_3^2 + 3348\varphi_3^4.
\end{aligned}
$$

Appendix A.2. Tools for the pricing formula (Proposition 4.2)

The Hermite polynomials satisfy the recursive relation

$$H_k(z) = zH_{k-1}(z) - H'_{k-1}(z) \quad k = 1, 2, \ldots$$

with $H_0(z) = 1$ and where $H'_k(\cdot)$ is the first derivative of $H_k(\cdot)$ with respect to z. Since $\int_{\tilde{z}}^{+\infty} H_0(z)\phi(z)dz = \Phi(-\tilde{z})$, for $k \geq 1$

$$\int_{\tilde{z}}^{+\infty} H_k(z)\phi(z)dz = \int_{\tilde{z}}^{+\infty} zH_{k-1}(z)\phi(z)dz - \int_{\tilde{z}}^{+\infty} H'_{k-1}(z)\phi(z)dz.$$

Solving the second integral by parts and using $\phi'(z) = -z\phi(z)$,

$$
\begin{aligned}
\int_{\tilde{z}}^{+\infty} H_k(z)\phi(z)dz &= \int_{\tilde{z}}^{+\infty} zH_{k-1}(z)\phi(z)dz - H_{k-1}(z)\phi(z) \mid_{\tilde{z}}^{+\infty} - \int_{\tilde{z}}^{+\infty} zH_{k-1}(z)\phi(z)dz \\
&= H_{k-1}(\tilde{z})\phi(\tilde{z})
\end{aligned}
$$

$$\int_{\tilde{z}}^{+\infty} J(z)\phi(z)dz = g(\tilde{z}) + \varphi_0\Phi(-\tilde{z}). \tag{A.1}$$

where $g(\cdot)$ is defined in formula (4.13).

Given the orthogonality feature of the Hermite polynomials,

$$\begin{aligned}\int_{-\infty}^{\tilde{z}} H_k(z)\phi(z)dz &= -H_{k-1}(\tilde{z})\phi(\tilde{z}) \\ \int_{-\infty}^{\tilde{z}} J(z)\phi(z)dz &= -g(\tilde{z}) + \varphi_0\Phi(\tilde{z}).\end{aligned} \tag{A.2}$$

In the proof of Proposition 4.2, formula (A.1) and formula (A.2) are used for $B_0 > 0$ and $B_0 < 0$, respectively.

Appendix A.3. Tools for the Hedging formula (Proposition 4.3)

In the following, we calculate $\frac{\partial \varphi_k}{\partial u}$ using a similar technique to that in Borovkova et al. (2007). Consider the 'moment-matching' system of equations in (4.6), where the formulae for the expectations are as in Appendix A.1, and differentiate both sides of each equation with respect to u. The quantities $\frac{\partial \varphi_k}{\partial u}$ are given by the solution of this new system of equations when the coefficients of the Hermite polynomials φ_k are the ones used for the pricing (solution of the first system of equations).

As an exemplification, we explicitly write the formula of the first derivative of $\mathbb{E}[X_T^k]$ wrt $S_0^{(1)}$, being useful for the delta parameter in function (4.19):

$$\frac{\partial \tilde{\mathbb{E}}[X_T^k]}{\partial S_0^{(1)}} = \sum_{i=0}^{k} \binom{k}{i} \frac{(-h_1)^i}{(B_0 \mathrm{e}^{rT})^{k-i}} \left(\frac{\partial \tilde{\mathbb{E}}[{B_T}^{k-i}]}{\partial S_0^{(1)}} - a_1 \frac{(k-i)\tilde{\mathbb{E}}[{B_T}^{k-i}]}{B_0} \right), \tag{A.3}$$

where $\frac{\partial \tilde{\mathbb{E}}[B_t]}{\partial S_0^{(1)}} = a_1$ and for $k > 1$

$$\begin{aligned}\frac{\partial \tilde{\mathbb{E}}[B_t^k]}{\partial S_0^{(1)}} &= \frac{\partial \tilde{\mathbb{E}}[B_t^k]}{\partial S_0^{(1)}} = k a_1 \mathrm{e}^{(r+\omega_1)t} \sum_{i_1=1}^{\Upsilon} \cdots \sum_{i_{k-1}=1}^{\Upsilon} \left(a_{i_1} S_0^{(i_1)} \mathrm{e}^{(r+\omega_{i_1})t} \right) \times \cdots \\ &\cdots \times \left(a_{i_{k-1}} S_0^{(i_{k-1})} \mathrm{e}^{(r+\omega_{i_{k-1}})t} \right) \mathrm{mgf}(\boldsymbol{e}_1 + \boldsymbol{e}_{i_1} + \ldots + \boldsymbol{e}_{i_{k-1}}),\end{aligned}$$

and mgf$(\cdot)$ is defined in (4.9). Derivatives with respect to other variables u are calculated in a similar way.

Table 1: Specification of the basket option scenarios in Borovkova et al. (2007)

	Basket 1	Basket 2	Basket 3	Basket 4	Basket 5	Basket 6
Stock Prices	[100,120]	[150,100]	[110,90]	[200,50]	[95,90,105]	[100,90,95]
Volatilities	[0.2,0.3]	[0.3,0.2]	[0.3,0.2]	[0.1,0.15]	[0.2,0.3,0.25]	[0.25,0.3,0.2]
a_i	[-1,1]	[-1,1]	[0.7,0.3]	[-1,1]	[1,-0.8,-0.5]	[0.6,0.8,-1]
Correlations	$\rho_{1,2} = 0.9$	$\rho_{1,2} = 0.3$	$\rho_{1,2} = 0.9$	$\rho_{1,2} = 0.8$	$\rho_{1,2} = 0.9$, $\rho_{2,3} = 0.9$ $\rho_{1,3} = 0.8$	$\rho_{1,2} = 0.9$, $\rho_{2,3} = 0.9$ $\rho_{1,3} = 0.8$
Strike price	20	-50	104	-140	-30	35

Notes: Other relevant parameters are $r = 3\%$, 1-year maturity, $\tilde{\lambda}_i = 0$ and $\delta_0^{(i)} = 0$. The first row indicates the stock prices $S_0^{(i)}$, the second the volatilities σ_i, the third the weights a_i of the assets in the basket, the forth the correlation $\rho_{i,j}$ for each couple (i, j) of assets and the fifth the strike K^*. The only difference compared with the scenarios in Borovkova et al. (2007) is that they price options on basket of forward contracts, while we price options on basket of equities.

Table 2: Comparison over different option scenarios

# Basket	MC (SD)	BPW	4GA	4GB	6GA	6GB
1	8.2263 (0.0031)	8.2442	8.1977	8.1977	8.2222	8.2222
2	16.47 (0.0052)	16.6215	16.4424	16.4424	16.4631	16.3654
3	12.5887 (0.0005)	12.5911	12.5695	12.5695	12.5888	12.5888
4	1.1459 (0.0008)	1.1456	1.1453	1.1453	1.0938	1.1162
5	7.4681 (0.0027)	7.4951	7.4563	7.4563	7.4555	7.4555
6	9.7767 (0.0030)	9.7989	9.7628	9.7628	9.7856	9.7856
C1		100.00%	100.00%	100.00%	83.33%	83.33%
C2		0.30%	0.17%	0.17%	0.82%	0.59%

Notes: This table reports the comparison on the six basket option scenarios in Borovkova et al. (2007) (see Table 1). The second column shows the prices (standard deviation in bracket) calculated using the Monte Carlo method with control variate in Pellizzari (2001) with 10^6 simulations that are considered as benchmark. The third column shows the prices calculated by the method in Borovkova et al. (2007), BPW in the table. The last four columns contain the prices under the methods mGA and mGB when $m = 4$ and $m = 6$. The last two rows show the pricing performances: C1 is the percentage of absolute percentage errors smaller than 2% ('good price'), and C2 is the mean absolute percentage error.

Table 3: Pricing performance comparison: Set 1 (negative average jump-sizes)

		r		T		$\frac{K^*}{B_0^*}$			
		≤ 0.05	> 0.05	≤ 0.5	> 0.5	≤ 0.98	$(0.98, 1.02]$	> 1.02	Total
C1	BPW	58.86%	61.38%	69.72%	50.79%	64.82%	66.98%	53.62%	60.10%
	$4GA$	76.77%	74.39%	69.51%	81.50%	74.12%	74.53%	77.38%	75.60%
	$4GB$	77.36%	75.20%	72.97%	79.53%	74.34%	76.42%	78.28%	76.30%
	$4GAB$	82.48%	80.89%	80.08%	83.27%	79.42%	81.13%	84.16%	81.70%
	$6GA$	85.46%	84.75%	85.68%	84.58%	81.93%	88.30%	87.56%	85.11%
	$6GB$	88.99%	90.58%	86.37%	92.93%	89.60%	86.17%	90.80%	89.78%
C2	BPW	1.44%	1.29%	1.00%	1.73%	1.15%	1.35%	1.59%	1.37%
	$4GA$	0.42%	0.39%	0.29%	0.50%	0.46%	0.43%	0.34%	0.40%
	$4GB$	0.41%	0.39%	0.29%	0.50%	0.44%	0.45%	0.35%	0.40%
	$4GAB$	0.43%	0.38%	0.30%	0.50%	0.44%	0.44%	0.36%	0.41%
	$6GA$	0.58%	0.58%	0.54%	0.61%	0.56%	0.55%	0.60%	0.58%
	$6GB$	0.36%	0.38%	0.42%	0.32%	0.34%	0.64%	0.33%	0.37%
Scenarios		508	492	492	508	452	106	442	1,000

Notes: This table contains the summary of the performances of several methods for pricing options in Set 1. The assets follow equation (3.4) where the parameters are randomly generated and uniformly distributed in the following ranges: $\Upsilon \in [2, 15]$, $r \in (0; 0.1]$, $\sigma_i \in [0.1; 0.6]$, $T \in [0.1; 1]$, $S_0^{(i)} = [70; 130]$, $a_i \in [-1; 1]$, $\frac{K^*}{B^*} \in [0.8; 1.2]$, $\delta_0^{(i)} \in [0; 20]$, $\tilde{\lambda}_i \in [0; 0.2]$, $\eta_i \in [-0.3; 0]$ and $\upsilon_i \in [0; 0.3]$ for all $i = 2, \cdots, \Upsilon$. In each row the results per method are shown: BPW stands for the method in Borovkova et al. (2007), mGA and mGB are the Hermite approximation methods matching the first m moments of X_T with $m \in \{4, 6\}$. Furthermore, $4GAB$ is a mixture of $4GA$ and $4GB$ and returns the solution of the method that correctly matches the moments if only one of $4GA$ and $4GB$ works properly, or the solution of the method that is the worst out of the two. The Monte Carlo with control variate in Pellizzari (2001) is the benchmark price.

Table 4: Pricing performance comparison: Set 2 (positive and negative average jump-sizes)

		r		T		$\frac{K^*}{B_0^*}$			
		≤ 0.05	> 0.05	≤ 0.5	> 0.5	≤ 0.98	$(0.98, 1.02]$	> 1.02	Total
C1	BPW	52.95%	57.72%	65.04%	45.87%	59.73%	57.55%	50.23%	55.30%
	$4GA$	72.24%	72.56%	65.65%	78.94%	70.80%	68.87%	74.89%	72.40%
	$4GB$	74.02%	71.75%	68.70%	76.97%	71.46%	70.75%	74.89%	72.90%
	$4GAB$	78.15%	77.24%	74.80%	80.51%	75.88%	73.58%	80.54%	77.70%
	$6GA$	82.68%	84.96%	80.08%	87.40%	82.74%	83.96%	84.84%	83.80%
	$6GB$	88.19%	90.04%	83.33%	94.69%	87.83%	88.68%	90.50%	89.10%
C2	BPW	1.59%	1.42%	1.18%	1.84%	1.26%	1.55%	1.74%	1.51%
	$4GA$	0.59%	0.54%	0.46%	0.65%	0.64%	0.61%	0.49%	0.57%
	$4GB$	0.57%	0.55%	0.46%	0.65%	0.62%	0.66%	0.48%	0.56%
	$4GAB$	0.59%	0.55%	0.48%	0.66%	0.62%	0.68%	0.49%	0.57%
	$6GA$	0.76%	0.70%	0.71%	0.75%	0.67%	0.82%	0.77%	0.73%
	$6GB$	0.50%	0.52%	0.53%	0.49%	0.46%	0.62%	0.53%	0.51%
Scenarios		508	492	492	508	452	106	442	1,000

Notes: This table contains the summary of the performances of several methods for pricing options in Set 2. The assets follow equation (3.4) where the parameters are randomly generated and uniformly distributed in the following ranges: $\Upsilon \in [2, 15]$, $r \in (0; 0.1]$, $\sigma_i \in [0.1; 0.6]$, $T \in [0.1; 1]$, $S_0^{(i)} = [70; 130]$, $a_i \in [-1; 1]$, $\frac{K^*}{B^*} \in [0.8; 1.2]$, $\delta_0^{(i)} \in [0; 20]$, $\tilde{\lambda}_i \in [0; 0.2]$, $\eta_i \in [-0.3; 0.3]$ and $\upsilon_i \in [0; 0.3]$ for all $i = 2, \cdots, \Upsilon$. For other information see Table 3.

Table 5: Delta-hedging performance comparison

Scenario		C3	C4	Scenario		C3	C4	Scenario		C3	C4	
	$r = 2\%$	0.02	0.10		$r = 2\%$	0.04	0.11		$r = 2\%$	0.07	0.12	BPW
1	$\eta_i = -0.3$	0.01	0.10	2	$\eta_i = -0.3$	0.03	0.11	3	$\eta_i = -0.3$	0.06	0.12	4GA
	$K^* = 100$	0.01	0.10		$K^* = 104$	0.03	0.11		$K^* = 110$	0.059	0.12	4GB
	$r = 5\%$	-0.66	0.59		$r = 5\%$	-0.64	0.56		$r = 5\%$	-0.62	0.52	BPW
4	$\eta_i = 0.3$	-0.34	0.46	5	$\eta_i = 0.3$	-0.32	0.45	6	$\eta_i = 0.3$	-0.29	0.45	4GA
	$K^* = 100$	-0.34	0.46		$K^* = 104$	-0.32	0.45		$K^* = 110$	-0.29	0.45	4GB

Notes: This table contains the summary of the Delta-hedging performances of three methods: BPW stands for the method in Borovkova et al. (2007) and $4GA$ and $4GB$ are the Hermite approximation methods matching the first 4 moments of X_T. The measures of error considered are: C3– average error, C4– the average quadratic hedging error. The six scenarios considered are: $\varUpsilon = 2$, $\sigma_1 = 0.3$, $\sigma_2 = 0.2$, $T = 0.5$ years, $S_0^{(1)} = 110$, $S_0^{(2)} = 90$, $a_1 = 0.7$, $a_2 = 0.3$, $\delta_0^{(1)} = \delta_0^{(2)} = 20$, $\tilde{\lambda}_i = 0.2$, and $\upsilon_1 = \upsilon_2 = 0.2$ and the other parameter values are under the 'Scenario' columns.