DETECTABLE SUBSPACES AND INVERSE PROBLEMS FOR HAIN-LÜST-TYPE OPERATORS

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ABSTRACT. We examine the extent to which a block operator matrix of Hain-Lüst type can be reconstructed from its Titchmarsh-Weyl coefficients. The detectable subspace of the operator is determined in a variety of cases and the question of unique determination of the coefficients is considered for both first and second order operators.

1. INTRODUCTION

In recent articles [6, 8, 9] the authors have considered forward and inverse problems for operators in the boundary triples setting. In particular, we have been interested in the *detectable subspaces* (see (12) below) related to the Titchmarsh-Weyl functions $M(\lambda)$, $\widetilde{M}(\lambda)$ associated with a formally adjoint pair, which determine upper bounds on the spaces in which the operators can be reconstructed, to some extent, from the information about boundary measurements contained in the Titchmarsh-Weyl functions. For instance, Derkach and Malamud [11] (see also Ryzhov [30]) show that in the formally symmetric case, if the detectable subspace is the whole Hilbert space, then the operator can be reconstructed up to unitary equivalence. In terms of the Q-function, this result was proved earlier by Kreĭn, Langer and Textorius [17, 22].

If the underlying operator is not symmetric, but the detectable subspace is the whole Hilbert space, then the Titchmarsh-Weyl function determines the operators of an adjoint pair up to weak equivalence [24]. However, weak equivalence does not preserve the spectral properties of the operators. Improving the result on weak equivalence in some special cases is the topic of [2–4, 13].

In an abstract setting these results are optimal: further information depends on having a priori knowledge of the operator. The fact that a Schrödinger operator in one dimension is uniquely determined (not just up to unitary equivalence) by the Titchmarsh-Weyl coefficient as a function of the spectral parameter has been known for more than sixty years [10, 12, 26] while in higher dimensions it suffices to know the Dirichlet to Neumann map for just one value of the spectral parameter [29]. Nevertheless an inverse-PDE application of the boundary triple approach may be found in [5].

To gain insight into what information may be determined from the Titchmarsh-Weyl functions in a general setting, it is instructive to look at particular examples. In this article we examine the extent to which a block operator matrix of Hain-Lüst type can be

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reconstructed from Titchmarsh-Weyl coefficients. We show that unique determination is generally impossible because the detectable subspace may have a non-trivial orthogonal complement, and we characterise the detectable subspace in various different cases. The fact that the results depend so much on the case under consideration shows that Hain-Lüst-type operators are very far from being a Schrödinger operator when questions of determination from boundary measurements are raised.

We also consider the case when the coefficients in the Hain-Lüst-type operator are analytic. In this case, some properties of the coefficients are uniquely determined by the Titchmarsh-Weyl coefficients (Theorem 4.1). One may expect that much more information should be contained in the Titchmarsh-Weyl coefficients in the analytic situation. However, our considerations of first order Hain-Lüst-type operators in Section 6 show that, in this simpler case, the operator is not uniquely determined by its Titchmarsh-Weyl coefficient. In terms of the detectable subspace, our results show that the first and second order results are very similar, so it seems plausible (Conjecture 4.3) that also in the second order case with analytic coefficients, the Titchmarsh-Weyl coefficient does not uniquely determine the coefficients. This remains an open problem.

In Section 5 we show that the *restricted resolvent*, which is closely related to the function $M(\lambda)$ [9], nevertheless contains just enough extra information to enable the coefficients in the second order Hain-Lüst-type operator to be largely reconstructed, with an explicit description of the exceptional sets on which two of the coefficients may be undetermined.

Regarding forward Hain-Lüst-type problems, there is now a substantial literature. As a very good starting point, we would recommend that the interested reader consult [1]. Further results can be found, e.g. in [7, 14, 16, 18–21, 27, 28].

2. Preliminaries

The Hain-Lüst-type operators we will study are given by

$$\widetilde{A}^* = \begin{pmatrix} -\frac{d^2}{dx^2} + q(x) & \tilde{w}(x) \\ w(x) & u(x) \end{pmatrix}, \quad A^* = \begin{pmatrix} -\frac{d^2}{dx^2} + \overline{q(x)} & \overline{w(x)} \\ \overline{\tilde{w}(x)} & \overline{u(x)} \end{pmatrix} \tag{1}$$

where q, u, $\tilde{w}$ and w are L^∞-functions, and the domains of the operators are given by

$$D(\widetilde{A}^*) = D(A^*) = H^2(0,1) \times L^2(0,1). \tag{2}$$

Integration by parts shows that

$$\begin{aligned} &\left\langle \widetilde{A}^* \begin{pmatrix} y \\ z \end{pmatrix}, \begin{pmatrix} f \\ g \end{pmatrix} \right\rangle - \left\langle \begin{pmatrix} y \\ z \end{pmatrix}, A^* \begin{pmatrix} f \\ g \end{pmatrix} \right\rangle \\ &\qquad = \left\langle \Gamma_1 \begin{pmatrix} y \\ z \end{pmatrix}, \Gamma_2 \begin{pmatrix} f \\ g \end{pmatrix} \right\rangle - \left\langle \Gamma_2 \begin{pmatrix} y \\ z \end{pmatrix}, \Gamma_1 \begin{pmatrix} f \\ g \end{pmatrix} \right\rangle, \end{aligned} \tag{3}$$

where the *boundary operators* Γ_j are given by

$$\Gamma_1 \begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} -y'(1) \\ y'(0) \end{pmatrix}, \quad \Gamma_2 \begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} y(1) \\ y(0) \end{pmatrix}$$

and the inner products on the right of (3) are in $\mathbb{C}^2$. The Titchmarsh-Weyl function or Neumann-to-Dirichlet operator $M(\lambda)$ is, in this context, the 2×2 matrix defined by

$$M(\lambda)\Gamma_1 \begin{pmatrix} y \\ z \end{pmatrix} = \Gamma_2 \begin{pmatrix} y \\ z \end{pmatrix}, \quad \text{for all } \begin{pmatrix} y \\ z \end{pmatrix} \in \ker(\widetilde{A}^* - \lambda I); \tag{4}$$

$\widetilde{M}(\lambda)$ is defined similarly but with $\widetilde{A}^*$ replaced by A^*. Given the definitions of Γ_1 and Γ_2 above, we have

$$M(\lambda)\begin{pmatrix} -y'(1) \\ y'(0) \end{pmatrix} = \begin{pmatrix} y(1) \\ y(0) \end{pmatrix}; \tag{5}$$

moreover the fact that $\begin{pmatrix} y \\ z \end{pmatrix} \in \ker(\widetilde{A}^* - \lambda I)$ yields two equations,

$$-y'' + (q-\lambda)y + \tilde{w}z = 0, \quad wy + (u-\lambda)z = 0,$$

from which z may be eliminated to give the Schur-complement equation for y, which is

$$\left(-\frac{d^2}{dx^2} + q(x) - \lambda - \frac{w(x)\tilde{w}(x)}{u(x)-\lambda}\right) y = 0. \tag{6}$$

Thus the Titchmarsh-Weyl function M for the Hain-Lüst operator $\widetilde{A}^*$ is determined by the formula (5) as applied to any basis of the set of solutions of (6) for λ outside the range of u. Explicit formulae, which we do not require here, are given in [6, eqn. (5.10-5.12)]. Similarly, $\widetilde{M}(\lambda)$ is determined by the formula (5) with M replaced by $\widetilde{M}$, and the function y now satisfying (6) with q, w, $\tilde{w}$ and u replaced by their complex conjugates. It follows that

$$\widetilde{M}(\lambda) = \overline{M(\overline{\lambda})}, \tag{7}$$

and so without loss of generality we can restrict our attention only to one M-function when considering the question of how much information on the operators is contained in the M-functions.

Hain-Lüst operator pairs therefore fall within the abstract setting of boundary triples for adjoint pairs [23]. In this setting we showed [8] that, given any bounded operator B (which in the Hain-Lüst context means that B is just a constant 2×2 matrix) one may define an operator A_B by imposing the *boundary condition* $(\Gamma_1 - B\Gamma_2)u = 0$ on the elements $u = \begin{pmatrix} y \\ z \end{pmatrix}$ of $D(\widetilde{A}^*)$,

$$A_B := \widetilde{A}^*\Big|_{\ker(\Gamma_1 - B\Gamma_2)}, \tag{8}$$

having the property that $M_B(\lambda) := M(\lambda)(I - BM(\lambda))^{-1}$, which maps according to the rule

$$M_B(\lambda)(\Gamma_1 - B\Gamma_2)u = \Gamma_2 u, \quad u \in \ker(\widetilde{A}^* - \lambda I), \tag{9}$$

is analytic in the resolvent set of A_B. We also show that the adjoint $(A_B)^*$ is obtained by imposing the boundary condition $(\Gamma_1 - B^*\Gamma_2)u = 0$ on the elements of $D(A^*)$:

$$(A_B)^* = \widetilde{A}_{B^*} := A^*|_{\ker(\Gamma_1 - B^*\Gamma_2)}, \tag{10}$$

and develop Kreĭn resolvent formulae which relate the resolvents of operators corresponding to different boundary conditions [25].

We need here the concept of *solution operator* or abstract Poisson operator. This is the operator defined by

$$u = S_{\lambda,B}h \ \text{ if and only if } \ (\widetilde{A}^* - \lambda)u = 0 \ \text{ and } \ (\Gamma_1 - B\Gamma_2)u = h. \tag{11}$$

Provided λ does not lie in the spectrum of A_B, the operator $S_{\lambda,B}$ is well defined on $\mathrm{Ran}(\Gamma_1 - B\Gamma_2)$, which is the whole boundary space $\mathbb{C}^2$ for our Hain-Lüst problem. The solution operator $\tilde{S}_{\lambda,B^*}$ is defined analogously, by solving the equation $(A^* - \lambda)u = 0$ subject to $(\Gamma_1 - B^*\Gamma_2)u = h$ and setting $\tilde{S}_{\lambda,B^*}h = u$.

In [6] we associated *detectable subspaces* with $\tilde{A}^*$ and A, in the abstract setting rather than the Hain-Lüst case. These detectable subspaces were written as the closures $\overline{\mathcal{S}}$ and $\overline{\widetilde{\mathcal{S}}}$ of some dense subsets $\mathcal{S}$ and $\widetilde{\mathcal{S}}$ respectively. We proved that the orthogonal complement of the detectable subspace $\overline{\mathcal{S}}$ is given by

$$\mathcal{S}^\perp = \bigcap_{B,\lambda\in\rho(A_B)} \ker(S^*_{\lambda,B}); \tag{12}$$

for the purposes of this article, the reader may take (12) as an implicit definition of the detectable subspace $\overline{\mathcal{S}}$. We proved that $\overline{\mathcal{S}}$ is a regular invariant subspace of the resolvent of A_B and, under some mild hypotheses, is independent of the boundary condition operator B. Under the assumption that $\mathrm{Ran}\,(\Gamma_1 - B\Gamma_2)$ is the whole boundary space (which is satisfied for Hain-Lüst), we also proved in [8, Proposition 3.9] that

$$\ker(S^*_{\lambda,B}) = \ker\left(\Gamma_2(\widetilde{A}_{B^*} - \overline{\lambda})^{-1}\right).$$

Similar results hold for $\overline{\widetilde{\mathcal{S}}}$.

Suppose now that $h \in \mathcal{S}^\perp$. Then we have $\Gamma_2(\widetilde{A}_{B^*} - \overline{\lambda})^{-1}h = 0$ for all suitable B and λ. Fixing B and λ and setting

$$y_B = (\widetilde{A}_{B^*} - \overline{\lambda})^{-1}h, \tag{13}$$

we get $\Gamma_2 y_B = 0$ and hence $\Gamma_1 y_B = B^*\Gamma_2 y_B = 0$, so y_B satisfies any homogeneous boundary condition and lies in the domain of the minimal operator. We have therefore proved the following.

Proposition 2.1. A vector h is orthogonal to the detectable subspace if and only if, for each boundary condition operator B and each λ in the resolvent set of $\widetilde{A}_{B^*}$, the vector $y_B := (\widetilde{A}_{B^*} - \lambda I)^{-1}h$ satisfies all homogeneous boundary conditions $\Gamma_1 y_B = 0 = \Gamma_2 y_B$. Equivalently, for our Hain-Lüst problem, $y_B(0)$, $y'_B(0)$, $y_B(1)$ and $y'_B(1)$ are all zero.

In Section 3 below, we will investigate the space $\overline{\mathcal{S}}$ for operators of Hain-Lüst type by asking whether the criteria in Proposition 2.1 imply that y_B and hence h is equal to zero.

3. SPECIAL CASES: SOME COEFFICIENTS ARE CONSTANT ON A SUB-INTERVAL

3.1. **The case $w = \tilde{w} \equiv 0$: the Sturm-Liouville problem.** When $w = \tilde{w} \equiv 0$ the Hain-Lüst operator decomposes into a direct sum of a Sturm-Liouville operator and the operator of multiplication by u. Since the boundary operators Γ_1 and Γ_2 contain only information about the first component y of an element $\begin{pmatrix} y \\ z \end{pmatrix}$ of the domain of A^* or $\widetilde{A}^*$, which is completely uncoupled from z, we cannot detect the coefficient u. The detectable subspace has the form $\overline{\mathcal{S}} = \overline{\mathcal{S}}_{SL} \oplus \mathbf{0}$, where $\overline{\mathcal{S}}_{SL}$ is the detectable subspace of the associated Sturm-Liouville problem. We have shown in [9] that $\overline{\mathcal{S}}_{SL} = L^2(0,1)$, but include the argument here for completeness.

Let $\theta(x,\lambda)$ and $\phi(x,\lambda)$ be solutions of $-y'' + qy = \lambda y$ which satisfy $\theta(0,\lambda) = 0$, $\theta'(0,\lambda) = 1$ and $\phi(0,\lambda) = 1$, $\phi'(0,\lambda) = 0$. Then $\overline{\theta}(x,\lambda)$ and $\overline{\phi}(x,\lambda)$ solve $-y'' + \overline{q}y = \overline{\lambda}y$ with the same initial conditions.

Let $h \in \overline{\mathcal{S}}_{SL}$ and y_B be as in (13). Then by the variation of constants formula, there exist $C, \tilde{C}$ such that

$$y_B(x,\lambda) \quad = \quad \int_0^x \overline{\phi}(t,\lambda)h(t)\;dt\;\overline{\theta}(x,\lambda) + \int_x^1 \overline{\theta}(t,\lambda)h(t)\;dt\;\overline{\phi}(x,\lambda) + C\overline{\theta}(x,\lambda) + \tilde{C}\overline{\phi}(x,\lambda).$$

y_B satisfies $\Gamma_1 y_B = 0 = \Gamma_2 y_B$. We choose λ so that it is not a Dirichlet eigenvalue. Then

$$y_B(0,\lambda) = \textstyle\int_0^1 \overline{\theta} h dt + \tilde{C} = 0, \quad y_B(1,\lambda) = \left(\int_0^1 \overline{\phi} h dt + C\right)\overline{\theta}(1,\lambda) + \tilde{C}\overline{\phi}(1,\lambda) = 0,$$
$$y_B'(0,\lambda) = C = 0, \quad y_B'(1,\lambda) = \left(\textstyle\int_0^1 \overline{\phi} h dt + C\right)\overline{\theta}'(1,\lambda) + \tilde{C}\overline{\phi}'(1,\lambda) = 0.$$

This simplifies to

$$\int_0^1 \overline{\phi} h dt\;\overline{\theta}(1,\lambda) - \int_0^1 \overline{\theta} h dt\;\overline{\phi}(1,\lambda) = 0, \qquad \int_0^1 \overline{\phi} h dt\;\overline{\theta}'(1,\lambda) - \int_0^1 \overline{\theta} h dt\;\overline{\phi}'(1,\lambda) = 0.$$

As the Wronskian of $\overline{\theta}$ and $\overline{\phi}$ is non-zero, we have

$$\int_0^1 \overline{\theta} h dt = \int_0^1 \overline{\phi} h dt = 0.$$

This holds for almost all λ. Analyticity in λ implies that these equations hold for all λ. Choosing λ to run through the Dirichlet eigenvalues shows that h is orthogonal to all Dirichlet eigenfunctions and also to any possible root vectors. Hence, $h \equiv 0$ proving the result.

3.2. **The Hain-Lüst problem when $w\tilde{w}$ is constant on an interval.** We now consider the full Hain-Lüst operator

$$\widetilde{A}^* = \begin{pmatrix} -\frac{d^2}{dx^2} + q(x) & \tilde{w}(x) \\ w(x) & u(x) \end{pmatrix}$$

and present three special results on detectability. The first, Theorem 3.1, covers the case where u is constant on an interval where $\tilde{w}$ vanishes. The second result, Theorem 3.4, covers the case when $w\tilde{w}$ is identically zero, without any special hypotheses on u. Finally, Theorem 3.10 deals with the case in which $w\tilde{w}$ is a non-zero constant and u is constant.

Theorem 3.1. Assume there exists an interval $I \subset (0,1)$ of positive measure such that

- $u|_I = u_0$ is constant,
- $|w| \geq \varepsilon > 0$ a.e. on I,
- $\tilde{w}|_I = 0$ a.e.

Then for all $\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \in \overline{\mathcal{S}}$ we have

(1) $f_2/w \in H^2_{\mathrm{loc}}(I)$ and
(2) $f_1 = -(f_2/w)'' + (q - u_0)f_2/w$ on I.

Remark 3.2. In particular, there is a restriction on the first component. We will see below that there are cases where the interplay between f_1 and f_2 given in (2) actually arises, i.e. there are elements of $\overline{\mathcal{S}}$ with non-zero $f_1|_I$.

Proof. Let $h \in C_0^\infty(I)$ be arbitrary and $\mu \in \mathbb{C} \setminus \{u_0\}$ such that $\overline{\mu} \in \rho(A_B)$ for some B. Set

$$y_\mu = \frac{h}{\overline{u_0} - \mu}, \quad g = \frac{h'' - (\overline{q} - \overline{u_0})h}{\overline{w}} \quad \text{and} \quad z_\mu = \frac{g}{\overline{u_0} - \mu}.$$

As all functions are supported on $\overline{I}$ and y_μ is smooth, we have that $\begin{pmatrix} y_\mu \\ z_\mu \end{pmatrix}$ lies in the domain of the minimal operator $\widetilde{A}$ and
(14)
$$(\widetilde{A}-\mu)\begin{pmatrix} y_\mu \\ z_\mu \end{pmatrix} = \begin{pmatrix} -y_\mu'' + (\overline{q}-\mu)y_\mu + \overline{w}z_\mu \\ \overline{\widetilde{w}}y_\mu + (\overline{u_0}-\mu)z_\mu \end{pmatrix} = \begin{pmatrix} \frac{-h''+(\overline{q}-\mu)h}{\overline{u_0}-\mu} + \frac{h''-(\overline{q}-\overline{u_0})h}{\overline{u_0}-\mu} \\ g \end{pmatrix} = \begin{pmatrix} h \\ g \end{pmatrix}.$$

Therefore, for any B
$$\Gamma_2(\widetilde{A}_{B^*}-\mu)^{-1}\begin{pmatrix} h \\ g \end{pmatrix} = \Gamma_2\begin{pmatrix} y_\mu \\ z_\mu \end{pmatrix} = 0.$$

This implies by Proposition 2.1 that
$$\begin{pmatrix} h \\ g \end{pmatrix} \in \bigcap_{B,\mu} \ker \Gamma_2(\widetilde{A}_{B^*}-\mu)^{-1} = \mathcal{S}^\perp,$$

i.e. for any $h \in C_0^\infty(I)$ we have
$$\begin{pmatrix} h \\ \frac{h''-(\overline{q}-\overline{u_0})h}{\overline{w}} \end{pmatrix} \in \mathcal{S}^\perp.$$

Now, let $\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \in \overline{\mathcal{S}}$. Then for all $h \in C_0^\infty(I)$ we have
$$\langle f_1, h\rangle + \left\langle f_2, \frac{h''}{\overline{w}}\right\rangle - \left\langle f_2, \frac{(\overline{q}-\overline{u_0})h}{\overline{w}}\right\rangle = 0.$$

This implies $f_2/w \in H^2(I)$ and $f_1 = -(f_2/w)'' + (q-u_0)f_2/w$ on I. □

We now show that the complicated interplay between the two components suggested in the theorem does occur, by proving that for more general Hain-Lüst-type operators there is always an element $\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \in \overline{\mathcal{S}}$ such that $f_1|_I \neq 0$:

Lemma 3.3. Let $q, u, w, \tilde{w}$ be bounded functions such that $w\tilde{w}|_I = 0$ on some interval $I \subset (0,1)$. Then there exists $\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \in \overline{\mathcal{S}}$ such that $f_1|_I \neq 0$.

Proof. Assume for a contradiction that $f_1|_I = 0$ for all $\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \in \overline{\mathcal{S}}$. Then $\begin{pmatrix} L^2(I) \\ 0 \end{pmatrix} \subset \overline{\mathcal{S}}^\perp$, where we view $L^2(I)$ as a subset of $L^2(0,1)$ by trivially extending functions to the whole interval. Let $0 \neq h \in L^2(I)$ and μ, B as in the previous proof. Let
$$\begin{pmatrix} y \\ z \end{pmatrix} = (\widetilde{A}_{B^*}-\mu)^{-1}\begin{pmatrix} h \\ 0 \end{pmatrix}.$$

As $\begin{pmatrix} h \\ 0 \end{pmatrix} \in \overline{\mathcal{S}}^\perp = \bigcap_{B,\mu} \ker \Gamma_2(\widetilde{A}_{B^*}-\mu)^{-1}$, we get $\Gamma_2\begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} y(1) \\ y(0) \end{pmatrix} = 0$. As $\begin{pmatrix} y \\ z \end{pmatrix} \in D(\widetilde{A}_{B^*})$, we also have $\Gamma_1\begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} -y'(1) \\ y'(0) \end{pmatrix} = 0$. Therefore,
$$-y'' + (\overline{q}-\mu)y + \overline{w}z = h, \quad \overline{\widetilde{w}}y + (\overline{u}-\mu)z = 0, \quad y(0) = y(1) = y'(0) = y'(1) = 0.$$

On $(0,1)\setminus I$, the function y therefore satisfies

$$-y'' + (\overline{q}-\mu)y - \frac{\overline{w}\overline{\tilde{w}}}{\overline{u}-\mu}y = 0.$$

Together with the boundary conditions at 0 and 1, this means $y \equiv 0$ on $(0,1)\setminus I$, so $\operatorname{supp}(y) \subset \overline{I}$ and thus $\operatorname{supp}(z) \subset \overline{I}$. We now use that $w\tilde{w} = 0$ on I. This means that in fact

$$-y'' + (\overline{q}-\mu)y = h$$

and y satisfies any boundary condition at 0 and 1. Since this is true for all $\mu \in \rho(\widetilde{A}_{B^*})$, it follows by the same argument as for the Sturm-Liouville problem in Section 3.1 that $h = 0$, giving a contradiction. ☐

The second case we consider characterises the detectable subspace $\overline{\mathcal{S}}$ in the case $w\tilde{w} = 0$. In the following, we denote by $L^2(w=0)$ the set of functions $f \in L^2(0,1)$ such that $w(x)f(x) = 0$ a.e. We denote its orthogonal complement in $L^2(0,1)$ by $L^2(w \neq 0)$.

Theorem 3.4. Assume $w\tilde{w} = 0$ and that $\theta(x,\lambda), \phi(x,\lambda)$ solve $(l-\lambda)y = -y'' + (q-\lambda)y = 0$ subject to the boundary conditions

$$\theta(0,\lambda) = 0, \quad \theta'(0,\lambda) = 1 \quad \text{and} \quad \phi(0,\lambda) = 1, \quad \phi'(0,\lambda) = 0. \tag{15}$$

Define

$$E_{u,w} := \operatorname{Span}_{n\in\mathbb{N}}\{w(x)\theta(x,u(x))u(x)^n\} + \operatorname{Span}_{n\in\mathbb{N}}\{w(x)\phi(x,u(x))u(x)^n\}.$$

Then

$$\mathcal{S}^\perp = \left\{ \begin{pmatrix} h \\ g \end{pmatrix} : g \perp E_{u,w},\ h(x) = \int_0^x (\overline{w}g)(t)[\overline{\phi}(t,u(t))\overline{\theta}(x,u(t)) - \overline{\theta}(t,u(t))\overline{\phi}(x,u(t))]dt \right\} \tag{16}$$

In particular,

$$\overline{\mathcal{S}} = \begin{pmatrix} L^2(0,1) \\ L^2(w\neq 0) \end{pmatrix} \tag{17}$$

iff $E_{u,w} = L^2(w \neq 0)$.

We defer the somewhat lengthy proof of the theorem to section 3.3, and first consider some special cases.

Corollary 3.5. If $w \equiv 0$, then $\overline{\mathcal{S}} = \begin{pmatrix} L^2(0,1) \\ 0 \end{pmatrix}$.

Proof. In this case, $E_{u,w} = \{0\}$, so from (16), $\mathcal{S}^\perp = \begin{pmatrix} 0 \\ L^2(0,1) \end{pmatrix}$. ☐

More generally, from Theorem 3.4, we immediately have the following inclusion:

Corollary 3.6. Let $w\tilde{w} \equiv 0$. Then $\overline{\mathcal{S}} \subseteq \begin{pmatrix} L^2(0,1) \\ L^2(w\neq 0) \end{pmatrix}$.

Proof. In fact this result is required in order to prove Theorem 3.4 and is re-stated as Lemma 3.12 below, where it is proved independently. However it also follows immediately from Theorem 3.4 since (16) implies that $\mathcal{S}^\perp \supseteq \begin{pmatrix} 0 \\ L^2(w=0) \end{pmatrix}$. ☐

For the reverse inclusion, we have the following results. We start with a lemma.

Lemma 3.7. Assume that

$$\operatorname{Span}_{n\in\mathbb{N}_0}\left\{\chi_{\{w\neq 0\}}u^n\right\} = L^2(w\neq 0). \tag{18}$$

Then $E_{u,w} = L^2(w\neq 0)$.

Remark 3.8. Equality (18) holds for $u(x) = x$. More generally, if u is continuous and strictly monotone then it also holds, as one may verify by using the Stone-Weierstrass Theorem.

Proof. Let Ψ be the set of solutions of $-y'' + (q-u)y = 0$. Then $v\in E_{u,w}^{\perp}$ if and only if

$$\int_0^1 \overline{v(x)}w(x)\psi(x)u(x)^n\,dx = 0$$

for any $\psi\in\Psi$. By our assumption (18), this is equivalent to $\overline{v(x)}w(x))\psi(x)|_{w\neq 0} = 0$ for any $\psi\in\Psi$. As not all $\psi\in\Psi$ can simultaneously vanish at a point x, this is equivalent to $v(x)|_{w\neq 0} = 0$, which concludes the proof of the lemma. $\square$

As an immediate consequence of Lemma 3.7 and Theorem 3.4 we have:

Corollary 3.9. Let $w\tilde{w}\equiv 0$. Assume that equality (18) holds. Then $\overline{\mathcal{S}} \supseteq \begin{pmatrix} L^2(0,1) \\ L^2(w\neq 0)\end{pmatrix}$.

Some further consequences of Theorem 3.4 are given in Remark 3.14 below.

We conclude this section on special cases with the following result, whose proof uses a result in the proof of Theorem 3.4 and is therefore deferred to section 3.4.

Theorem 3.10. Assume $w\tilde{w} = c_0$, a non-zero constant, and that $u = u_0$, a constant. Then $\overline{\mathcal{S}} = L^2(0,1)\oplus L^2(0,1)$.

3.3. **Proof of Theorem 3.4.** Throughout this subsection, we assume the hypotheses of Theorem 3.4 hold.

Using Proposition 2.1, a simple calculation using $w\tilde{w} = 0$ shows that $\begin{pmatrix} h \\ g\end{pmatrix}\in\mathcal{S}^{\perp}$ if and only if

$$-y_B'' + (\overline{q}-\mu)y_B = h - \frac{\overline{w}g}{\overline{u}-\mu}$$

for all suitable B, μ and y_B as in the proposition. By the arguments for the Sturm-Liouville problem, we therefore have $\begin{pmatrix} h \\ g\end{pmatrix}\in\mathcal{S}^{\perp}$ if and only if

$$\int_0^1 \left(h - \frac{\overline{w}g}{\overline{u}-\mu}\right)\overline{\psi}(x,\overline{\mu})dx = 0 \text{ for both } \psi = \theta \text{ and } \psi = \phi. \tag{19}$$

For the first part of the proof of Theorem 3.4 we need a lemma:

Lemma 3.11. Let $w\tilde{w} = 0$. If $\begin{pmatrix} h \\ g\end{pmatrix}\in\mathcal{S}^{\perp}$, then $g\perp E_{u,w}$.

Proof. Choose a contour Γ around essran$(\overline{u})$, multiply (19) by a power of μ and integrate with respect to μ. Then for any $n\in\mathbb{N}_0$,

$$0 \;=\; -\int_0^1 \overline{w(x)}g(x)\frac{1}{2\pi i}\int_\Gamma \frac{\overline{\psi}(x,\bar{\mu})}{\overline{u(x)}-\mu}\mu^n d\mu dx \;=\; -\int_0^1 \overline{w(x)}g(x)\overline{\psi(x,u(x))u(x)}^n dx.$$

This proves the lemma. $\square$

This establishes one of the inclusions in Theorem 3.4 in the case when $E_{u,w} = L^2(w \neq 0)$: for in this case, Lemma 3.11 implies that $g|_{\{w\neq 0\}} = 0$. From (19), we then get that $\int_0^1 h(x)\psi(x,\mu)\ dx \equiv 0$ a.e. As in Section 3.1, this implies $h \equiv 0$, so we get $\begin{pmatrix} 0 \\ L^2(w=0) \end{pmatrix} \supseteq \mathcal{S}^\perp$. The reverse inclusion is established in the following lemma.

Lemma 3.12. Let $w\tilde{w} = 0$. We have $\overline{\mathcal{S}} \subseteq \begin{pmatrix} L^2(0,1) \\ L^2(w \neq 0) \end{pmatrix}$.

Proof. Let $\begin{pmatrix} y \\ z \end{pmatrix} = S_{\mu,B}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$, for some $c_1, c_2 \in \mathbb{C}$. Then $wy + (u-\mu)z = 0$, so $z = \frac{wy}{\mu - u}$ is supported only on the support of w. Moreover, $-y'' + (q-\mu)y + \tilde{w}z = -y'' + (q-\mu)y = 0$, so $y \in L^2(0,1)$ and $z \in L^2(w \neq 0)$. Since from [6, Lemma 3.1], we have that $\overline{\mathcal{S}} = \overline{\operatorname{Span}_{\mu\in\rho(A_B)} \operatorname{Ran} S_{\mu,B}}$, this concludes the proof. □

Therefore, we have shown that if $E_{u,w} = L_2(w \neq 0)$, then $\mathcal{S}^\perp = \begin{pmatrix} 0 \\ L_2(w=0) \end{pmatrix}$.

We now consider the case when $E_{u,w} \neq L_2(w \neq 0)$. Then there exists $g \neq 0$ such that $g \perp E_{u,w}$. Put

$$h(x) = \int_0^x (\overline{w}g)(t)[\overline{\phi}(t,u(t))\overline{\theta}(x,u(t)) - \overline{\theta}(t,u(t))\overline{\phi}(x,u(t))]\ dt. \tag{20}$$

We need to check that h defined in this way satisfies the condition

$$0 = \int_0^1 \left(h - \frac{\overline{w}g}{\overline{u}-\mu}\right)\overline{\phi}(x,\overline{\mu})dx.$$

We first note the following: Let $(\overline{l_0} - \mu)y = f$ mean that

$$\begin{cases} -y'' + \overline{q}y - \mu y = f, \\ y(0) = y'(0) = 0. \end{cases}$$

An explicit calculation gives

$$y(x) = \int_0^x [\overline{\phi}(x,\overline{\mu})\overline{\theta}(t,\overline{\mu}) - \overline{\theta}(x,\overline{\mu})\overline{\phi}(t,\overline{\mu})]f(t)dt.$$

We let ψ stand for either θ or ϕ. Then we consider the function

$$\Delta(x,\mu,\rho) := \frac{\overline{\psi}(x,\overline{\mu})}{\rho - \mu} + (\overline{l_0} - \rho)^{-1}\overline{\psi}(x,\overline{\mu})$$

for $\rho \in \mathbb{C}\backslash\{\mu\}$. Since $(\bar{l} - \rho)\Delta = 0$, we have that $\Delta(x,\mu,\rho) = C_1\overline{\theta}(x,\overline{\rho}) + C_2\overline{\phi}(x,\overline{\rho})$ and with $\overline{\psi}(x,\overline{\mu}) = \overline{\psi}'(0,\overline{\mu})\overline{\theta}(x,\overline{\mu}) + \overline{\psi}(0,\overline{\mu})\overline{\phi}(x,\overline{\mu})$, consideration of the domain of $\overline{l_0}$ leads to

$$\Delta(x,\mu,\rho) = \frac{\overline{\psi}'(0,\overline{\mu})}{\rho-\mu}\overline{\theta}(x,\overline{\rho}) + \frac{\overline{\psi}(0,\overline{\mu})}{\rho-\mu}\overline{\phi}(x,\overline{\rho}).$$

This implies

$$\frac{\overline{\psi}(x,\overline{\mu})}{\rho-\mu} \equiv -\int_0^x [\overline{\phi}(x,\overline{\rho})\overline{\theta}(t,\overline{\rho}) - \overline{\theta}(x,\overline{\rho})\overline{\phi}(t,\overline{\rho})]\overline{\psi}(t,\overline{\mu})dt + \frac{\overline{\psi}'(0,\overline{\mu})}{\rho-\mu}\overline{\theta}(x,\overline{\rho}) + \frac{\overline{\psi}(0,\overline{\mu})}{\rho-\mu}\overline{\phi}(x,\overline{\rho}). \tag{21}$$

We now fix $x \in [0,1]$ and put $\rho = \overline{u(x)}$. As $\mu \in \rho(\widetilde{A}_{B^*})$ for some B, we have $\mu \notin \overline{\mathrm{essran}\overline{u}}$ and this choice is always possible. This gives

$$\frac{\overline{\psi}(x,\overline{\mu})}{\overline{u(x)}-\mu} = -\int_0^x [\overline{\phi}(x,u(x))\overline{\theta}(t,u(x)) - \overline{\theta}(x,u(x))\overline{\phi}(t,u(x))]\overline{\psi}(t,\overline{\mu})dt + \frac{\overline{\psi}'(0,\overline{\mu})}{\overline{u(x)}-\mu}\overline{\theta}(x,u(x)) + \frac{\overline{\psi}(0,\overline{\mu})}{\overline{u(x)}-\mu}\overline{\phi}(x,u(x))$$

for both choices θ, ϕ of ψ. Using this expression for $\frac{\overline{\psi}(x,\overline{\mu})}{\overline{u(x)}-\mu}$, a calculation gives

$$\begin{aligned}
&\int_0^1 \overline{w}g(x)\frac{\overline{\psi}(x,\overline{\mu})}{\overline{u(x)}-\mu}\,dx \\
&= -\int_0^1 (\overline{w}g)(x)\int_0^x [\overline{\phi}(x,u(x))\overline{\theta}(t,u(x)) - \overline{\theta}(x,u(x))\overline{\phi}(t,u(x))]\overline{\psi}(t,\overline{\mu})\,dt\,dx \qquad (22)\\
&\quad + \int_0^1 \frac{(\overline{w}g)(x)\overline{\theta}(x,u(x))}{\overline{u(x)}-\mu}\overline{\psi}'(0,\overline{\mu})\,dx + \int_0^1 (\overline{w}g)(x)\frac{\overline{\phi}(x,u(x))}{\overline{u(x)}-\mu}\overline{\psi}(0,\overline{\mu})\,dx.
\end{aligned}$$

As $g \perp E_{u,w}$, we have that

$$\int_0^1 (\overline{w}g)(x)\frac{\overline{\psi}(x,u(x))}{\overline{u(x)}-\mu}\,dx = 0,$$

so the last two terms on the right of (22) cancel. Exchanging the order of integration, we get

$$\begin{aligned}
&\int_0^1 \overline{w}g(x)\frac{\overline{\psi}(x,\overline{\mu})}{\overline{u(x)}-\mu}\,dx \\
&= -\int_0^1 \overline{\psi}(t,\overline{\mu})\int_t^1 (\overline{w}g)(x)[\overline{\phi}(x,u(x))\overline{\theta}(t,u(x)) - \overline{\theta}(x,u(x))\overline{\phi}(t,u(x))]\,dx\,dt \\
&= \int_0^1 \overline{\psi}(t,\overline{\mu})\int_0^t (\overline{w}g)(x)\big(\overline{\phi}(x,u(x))\overline{\theta}(t,u(x)) - \overline{\theta}(x,u(x))\overline{\phi}(t,u(x))\big)\,dx\,dt,
\end{aligned}$$

where for the second equality, we use that since θ and ϕ depend analytically on the second variable, they can be developed into series of the form

$$\theta(t,u(x)) = \sum_{k=0}^{\infty} c_k(t)u(x)^k, \quad \phi(t,u(x)) = \sum_{k=0}^{\infty} \tilde{c}_k(t)u(x)^k \qquad (23)$$

and make use of the orthogonality condition on g. Therefore, by (20),

$$\int_0^1 \overline{w}g(x)\frac{\overline{\psi}(x,\overline{\mu})}{\overline{u(x)}-\mu}\,dx = \int_0^1 \overline{\psi}(t,\overline{\mu})h(t)\,dt.$$

Hence, as (19) is satisfied, we have $\begin{pmatrix} h \\ g \end{pmatrix} \in \mathcal{S}^\perp$.

Bearing Lemma 3.11 in mind, to complete the proof it is now enough to check that h is uniquely determined by g whenever $\begin{pmatrix} h \\ g \end{pmatrix} \in \mathcal{S}^\perp$. We know that

$$\int_0^1 h(x)\overline{\psi}(x,\overline{\mu})dx = \int_0^1 \frac{(\overline{w}g)(x)}{\overline{u(x)}-\mu}\overline{\psi}(x,\overline{\mu})dx$$

for $\mu \notin \overline{\mathrm{essran}(\overline{u})}$ and therefore for all μ since the left integral is analytic in μ. Letting μ_n run through the spectrum of the operator $-\dfrac{d^2}{dx^2} + q$ with Dirichlet boundary conditions, the functions $\{\overline{\theta}(x, \overline{\mu}_n)\}$ form a basis. This implies that h is uniquely determined by g and completes the proof of Theorem 3.4. $\square$.

Remark 3.13. If $\mathrm{Span}_{k=0}^{\infty} wu^k = L_2(w \neq 0)$, then $E_{u,w} = L^2(w \neq 0)$. Indeed, assume $E_{u,w} \neq L^2(w \neq 0)$. Then there exists $g \in L^2(w \neq 0)$, $g \neq 0$ such that $\overline{\phi}(x, u(x))g \perp wu^k$ for $k = 0, 1, ...$ and $\overline{\theta}(x, u(x))g \perp wu^k$ for $k = 0, 1, ...$ This implies $\overline{\phi}(x, u(x))g(x) = 0, \quad \overline{\theta}(x, u(x))g(x) = 0$ a.e. on $\{w \neq 0\}$. Since $\overline{\phi}(x, u(x))$ and $\overline{\theta}(x, u(x))$ have no common zeroes we get $g(x) = 0$ a.e.

Remark 3.14. If $\begin{pmatrix} h \\ g \end{pmatrix} \in \mathcal{S}^{\perp}$, then h has the following properties:

(1) $h(0) = 0$;
(2) $h(1) = 0$ by the orthogonality condition on g and developing $\phi(t, u(x))$ and $\theta(t, u(x))$ into power series as in (23);
(3) $h \in H^1$ and by explicit calculations
$$h'(x) = \int_0^x (\overline{w}g)(t)[\overline{\phi}(t, u(t))\overline{\theta}'(x, u(t)) - \overline{\theta}(t, u(t))\overline{\phi}'(x, u(t))]\, dt;$$
(4) $h'(0) = 0$;
(5) $h'(1) = 0$ (as for $h(1)$);
(6) $h \in H^2$ and using that the Wronskian of θ and ϕ is 1,
$$\begin{aligned} h''(x) &= (\overline{w}g)(x) + \int_0^x (\overline{w}g)(t)[\overline{\phi}(t, u(t))\overline{\theta}(x, u(t)) - \overline{\theta}(t, u(t))\overline{\phi}(t, u(t))](\overline{q}(x) - \overline{u}(t))\, dt \\ &= (\overline{w}g)(x) - \int_0^x (\overline{w}g)(t)\overline{u}(t)[\overline{\phi}(t, u(t))\overline{\theta}(x, u(t)) - \overline{\theta}(t, u(x))\overline{\phi}(x, u(t))\, dt + \overline{q}(x)h(x), \end{aligned}$$
i.e.
$$-h''(x) + \overline{q}(x)h(x) = -(\overline{w}g)(x) + \int_0^x (\overline{w}g)(t)\overline{u}(t)[\overline{\phi}(t, u(t))\overline{\theta}(x, u(x)) - \overline{\theta}(t, u(t))\overline{\phi}(x, u(t))]\, dt.$$
(7) In the special case when $\tilde{w} \equiv 0$ and u is constant, say $u \equiv u_0$, then $\begin{pmatrix} h \\ g \end{pmatrix} \in \mathcal{S}^{\perp}$ if and only if $h \in H_0^2(0, 1)$ and $-h'' + (\overline{q} - \overline{u}_0)h = -\overline{w}g$.

Remark 3.15. We see from these results that in the one-dimensional case, the description of the detectable subspace is complicated. However, in the multi-dimensional case, the description is much easier. This is due to the fact that in higher dimensions the operator-valued function $M(\lambda)$ at one point λ contains much more information than the scalar function $M(\lambda)$ in the one-dimensional case. Using the now-classical results about recovery of potentials in Schrödinger PDEs, e.g. [29], one sees that knowing $M(\lambda)$ for just one λ uniquely determines $q - \lambda + w^2/(\lambda - u)$. If one knows this quantity for three different values of λ then reduction to a 3×3 linear system with essentially a van der Monde determinant shows that one knows q, w and the values of u on the set where w is non-zero.

3.4. **Proof of Theorem 3.10.** We shall show that if $\begin{pmatrix} h \\ g \end{pmatrix} \in \mathcal{S}^{\perp}$ then $h = wg = 0$. Since $w\tilde{w} \neq 0$ this implies $h = g = 0$.

Let $\theta(x,\lambda)$ and $\phi(x,\lambda)$ solve the Schrödinger equation $-y'' + (q-\lambda)y = 0$ subject to the boundary conditions (15). A similar calculation as in (19) shows that $\begin{pmatrix} h \\ g \end{pmatrix} \in \mathcal{S}^\perp$ if and only if

$$\int_0^1 \left(h - \frac{\overline{w} g}{\overline{u}_0 - \mu} \right) \overline{\psi}(x, \overline{\rho}) dx = 0 \text{ for both } \psi = \theta \text{ and } \psi = \phi, \tag{24}$$

where $\rho := \mu + \frac{\bar{c}_0}{\overline{u}_0 - \mu}$. Observe that the mapping from μ to ρ is not injective; indeed two different values of μ,

$$\mu_\pm := \frac{\rho + \overline{u}_0}{2} \pm \sqrt{\left(\frac{\rho - \overline{u}_0}{2}\right)^2 + \bar{c}_0}$$

yield the same value of ρ. With this notation, we have the two equations

$$\int_0^1 (h(\overline{u}_0 - \mu_\pm) - \overline{w} g) \overline{\theta}(x, \overline{\rho}) dt = 0 \quad \text{for a.e. } \rho.$$

Subtracting yields

$$\int_0^1 h \overline{\theta}(x, \overline{\rho}) dt = 0 \quad \text{for a.e. } \rho \in \mathbb{C},$$

and hence for all $\rho \in \mathbb{C}$. Choosing ρ to lie in the spectrum of the Schrödinger operator with Dirichlet boundary conditions gives the result. □

4. Analytic coefficients: partial uniqueness results

The previous section shows that the Hain-Lüst operator generally cannot be reconstructed from a knowledge of its Titchmarsh-Weyl M-function; even worse, its detectable space $\overline{\mathcal{S}}$ is generally not the whole Hilbert space. The vanishing of the coefficients w and $\tilde{w}$ in some part of the interval $[0,1]$ is very important in constructing these non-uniqueness and non-detectability results.

In this section we investigate some uniqueness results for the case of real-valued analytic coefficients.

Theorem 4.1. Consider two Hain-Lüst problems with coefficients $q_1, q_2, u_1, u_2, w_1 = \tilde{w}_1$ and $w_2 = \tilde{w}_2$, satisfying the following properties:

(1) all the coefficients mentioned are analytic in a neighbourhood $\mathcal{N}$ of the line-segment $[0,1]$ in $\mathbb{C}$;
(2) w_1, w_2 are bounded away from zero in $\mathcal{N}$;
(3) The u_j are invertible as functions on $\mathcal{N}$ and are real-valued on $[0,1]$, with either $u_1(0) \neq u_2(0)$ or $u_1'(0) \neq u_2'(0)$, or similar inequalities at $x = 1$.

Then the two Hain-Lüst problems must have distinct Titchmarsh-Weyl M-functions.

Proof. Following the discussion around (5) and (6) in the Introduction, to prove our result we need some basic information about analyticity properties of solutions of the Schur complement equation (6), which here has the form

$$\left(-\frac{d^2}{dx^2} + q_j - \lambda - \frac{w_j^2}{u_j - \lambda} \right) y = 0 \tag{25}$$

for $j = 1, 2$. Specifically, we have solutions of the form

$$y_j(x,\lambda) = \sum_{n=1}^{\infty} c_{n,j}(\lambda)(x - u_j^{-1}(\lambda))^n, \quad c_1(\lambda) \equiv 1, \tag{26}$$

$$\tilde{y}_j(x,\lambda) = \log(x - u_j^{-1}(\lambda)) y_j(x,\lambda) + z_j(x,\lambda), \tag{27}$$

in which y_j and z_j can be continued analytically in x to the whole of $\mathcal{N}$ and in λ to the whole of $u_j(\mathcal{N})$ for all $x \in \mathcal{N}$. The choice of the branch of logarithm is obviously important, but observe that different choices of branch only add multiples of the analytic solution y_j to the logarithmically singular solution $\tilde{y}_j$. The presence of the logarithmic singularity depends on the fact that w_j does not vanish. These formulae are easily proved using Frobenius expansion formulae, see, e.g. [15].

We assume for a contradiction that $u_1(0) \neq u_2(0)$ or $u_1'(0) \neq u_2'(0)$ but that the M-functions coincide. The proof is similar in the case when the inequalities hold at $x = 1$.

Suppose that λ is non-real. In this case all the singularities of solutions lie off the real axis. Since $\tilde{y}_1$ and y_2 solve (25) for $j = 1, 2$ respectively and there are no singularities in the interval $[0, 1]$, a standard integration by parts yields

$$\Big[-\tilde{y}_1' y_2 + \tilde{y}_1 y_2'\Big]_0^1 + \int_0^1 \left[(q_1 - q_2)(x) + \frac{w_1(x)^2}{\lambda - u_1(x)} - \frac{w_2(x)^2}{\lambda - u_2(x)}\right] \tilde{y}_1(x,\lambda) y_2(x,\lambda) dx = 0. \tag{28}$$

Using the coincidence of the M-functions we can write

$$\begin{pmatrix} \tilde{y}_1(1) \\ \tilde{y}_1(0) \end{pmatrix} = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} \begin{pmatrix} -\tilde{y}_1'(1) \\ \tilde{y}_1'(0) \end{pmatrix}, \quad \begin{pmatrix} y_2(1) \\ y_2(0) \end{pmatrix} = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} \begin{pmatrix} -y_2'(1) \\ y_2'(0) \end{pmatrix},$$

which we use to replace the function values at $x = 0$ and $x = 1$ in (28) by derivative values, leaving

$$(\tilde{y}_1'(0) y_2'(1) - \tilde{y}_1'(1) y_2'(0))(m_{12} - m_{21}) + \int_0^1 \left[(q_1 - q_2)(x) + \frac{w_1(x)^2}{\lambda - u_1(x)} - \frac{w_2(x)^2}{\lambda - u_2(x)}\right] \tilde{y}_1(x,\lambda) y_2(x,\lambda) dx = 0.$$

However the M-function is symmetric: $m_{12}(\lambda) = m_{21}(\lambda)$, a fact which follows from the constancy of the Wronskian of solutions of the equations (25). Thus, for all non-real λ,

$$\int_0^1 \left[(q_1 - q_2)(x) + \frac{w_1(x)^2}{\lambda - u_1(x)} - \frac{w_2(x)^2}{\lambda - u_2(x)}\right] \tilde{y}_1(x,\lambda) y_2(x,\lambda) dx = 0. \tag{29}$$

Fix any point $t \in (0, 1)$. We shall consider the limits of the integral (29) as $\lambda \to u_1(t)$ from above and from below in the complex plane. We need to avoid the singularity which will appear in the term $1/(\lambda - u_1(x))$ at $x = t$ when $\lambda = u_1(t)$; there may also be a singularity in the term $1/(\lambda - u_2(x))$, generally not at $x = t$ but at some other point; however this is cancelled by the factor $y_2(x, \lambda)$ which vanishes at precisely such a point thanks to (26). Thus it suffices to avoid the singularity which will appear at $x = t$. Assuming without loss of generality that $\Im(u_1(t + i\epsilon)) > 0$ and $\Im(u_1(t - i\epsilon)) < 0$ for small $\epsilon > 0$, the avoidance is achieved by (a) deforming the contour $[0, 1]$ into the lower-half of the x-plane when $\Im\lambda > 0$, taking a detour around a semi-circle of small radius $r > 0$ passing below $x = t$, and (b) deforming above $x = t$ on a semi-circle for $\Im\lambda < 0$. We denote the deformed contours (including the segments $[0, t - r] \cup [t + r, 1]$) by C_{+r}

and C_{-r} respectively, and we have

$$\int_{C_{\pm r}} \left[(q_1 - q_2)(x) + \frac{w_1(x)^2}{\lambda - u_1(x)} - \frac{w_2(x)^2}{\lambda - u_2(x)} \right] \tilde{y}_1(x,\lambda) y_2(x,\lambda) dx = 0$$

where $\mp\Im\lambda > 0$ on $C_{\pm r}$. For the solution $\tilde{y}_1$, when $\Im\lambda > 0$, we can cut the x-plane along a curve $(-\infty, u_1^{-1}(\lambda)]$ in the upper half-plane while integrating with respect to x on contour C_{-r} in the lower half plane; and when $\Im\lambda < 0$ we can cut along a curve $(-\infty, u_1^{-1}(\lambda)]$ in the lower half-plane while integrating on contour C_{+r} in the upper half plane. With these choices of cuts we have the limits

$$\lim_{\lambda \to u_1(t)+i0} \tilde{y}_1(x,\lambda) = \begin{cases} \log|x-t| y_1(x,u_1(t)) + z_1(x,u_1(t)), & x > t, \\ \log|x-t| y_1(x,u_1(t)) + z_1(x,u_1(t)) - i\pi y_1(x,u_1(t)), & x < t, \end{cases} \tag{30}$$

$$\lim_{\lambda \to u_1(t)-i0} \tilde{y}_1(x,\lambda) = \begin{cases} \log|x-t| y_1(x,u_1(t)) + z_1(x,u_1(t)), & x > t, \\ \log|x-t| y_1(x,u_1(t)) + z_1(x,u_1(t)) + i\pi y_1(x,u_1(t)), & x < t. \end{cases} \tag{31}$$

Taking the difference between the two contour integrals and letting $\lambda \to u_1(t)$ from the appropriate half-plane in each, and using the information about the solution $\tilde{y}_1$ in (30,31) we therefore obtain

$$\begin{aligned} 0 = -2\pi i \int_0^{t-r} & \left[(q_1 - q_2)(x) + \frac{w_1(x)^2}{u_1(t) - u_1(x)} - \frac{w_2(x)^2}{u_1(t) - u_2(x)} \right] y_1(x,u_1(t)) y_2(x,u_1(t)) dx \\ + \int_{-\pi}^{\pi} & \left[(q_1 - q_2)(t + r\mathrm{e}^{i\theta}) + \frac{w_1(t+r\mathrm{e}^{i\theta})^2}{u_1(t) - u_1(t+r\mathrm{e}^{i\theta})} - \frac{w_2(t+r\mathrm{e}^{i\theta})^2}{u_1(t) - u_2(t+r\mathrm{e}^{i\theta})} \right] \\ & \tilde{y}_1(t + r\mathrm{e}^{i\theta}, u_1(t)) y_2(t + r\mathrm{e}^{i\theta}, u_1(t)) ir\mathrm{e}^{i\theta} d\theta, \end{aligned}$$

in which we observe that any zeros in the denominator $u_1(t) - u_2(\cdot)$ will be cancelled by zeros of $y_2(\cdot, u_1(t))$. For small r we have $\tilde{y}_1(t + r\mathrm{e}^{i\theta}, u_1(t)) \sim z_1(t, u_1(t))$ so that letting $r \searrow 0$ gives

$$\begin{aligned} & \frac{w_1(t)^2}{u_1'(t)} z_1(t,u_1(t)) y_2(t,u_1(t)) \\ = & \int_0^t \left[q_1(x) - q_2(x) + \frac{w_1(x)^2}{u_1(t) - u_1(x)} - \frac{w_2(x)^2}{u_1(t) - u_2(x)} \right] y_1(x,u_1(t)) y_2(x,u_1(t)) dx. \end{aligned} \tag{32}$$

Our strategy now is to consider the limit $t \searrow 0$ and prove that $w_1(0) = 0$, contradicting the hypothesis w_1 and w_2 are bounded away from zero. There are different cases depending on whether or not $u_1(0) = u_2(0)$.

Case 1: $u_1(0) \neq u_2(0)$. Then for all sufficiently small t the term $1/(u_1(t) - u_2(x))$ is bounded independently of x and t. We first assume that

$$\text{the function } t \mapsto y_2(t, u_1(t)) \text{ is not identically zero.} \tag{33}$$

Under this assumption the dominant term in the integral on the right hand side of (32) is

$$\int_0^t \frac{w_1(x)^2}{u_1(t) - u_1(x)} y_1(x,u_1(t)) y_2(x,u_1(t)) dx \sim \left[w_1(0)^2 \, \partial_\lambda y_1(0,\lambda)|_{\lambda = u_1(0)} \, y_2(t,u_1(t)) \right] t$$

Combining this with (32) and cancelling the common factor $y_2(t, u_1(t))$ shows that

$$\frac{w_1(t)^2}{u_1'(t)} z_1(t,u_1(t)) \sim t w_1(0)^2 \, \partial_\lambda y_1(0,\lambda)|_{\lambda = u_1(0)} \leq O(t),$$

which in particular implies that $w_1(0) = 0$. This contradicts the hypothesis that w_1 is bounded away from zero.

The case $u_1(0) \neq u_2(0)$ is therefore complete if we can show that the assumption (33) always holds. Assume for a contradiction that (33) does not hold. Then (32) becomes

$$\int_0^t \left[q_1(x) - q_2(x) + \frac{w_1(x)^2}{u_1(t) - u_1(x)} - \frac{w_2(x)^2}{u_1(t) - u_2(x)} \right] y_1(x, u_1(t)) y_2(x, u_1(t)) dx \equiv 0. \tag{34}$$

We now know that $y_1(t, u_1(t)) = 0$ for all t and we have assumed that $y_2(t, u_1(t)) = 0$ for all t, so

$$y_1(x, u_1(t)) = (x-t) y_1'(t, u_1(t)) + O((x-t)^2); \quad y_2(x, u_1(t)) = (x-t) y_2'(t, u_1(t)) + O((x-t)^2),$$

where dash denotes differentiation with respect to the first argument. From (26) we know that $y_1'(t, u_1(t)) \neq 0$. Also we know that $y_2'(t, u_1(t)) \neq 0$ since t is a regular point for the equation (25) with $j = 2$ and $\lambda = u_1(t) \neq u_2(t)$ (by the assumption $u_1(0) \neq u_2(0)$ and the fact that t is small) and so it is impossible for both $y_2(t, u_1(t))$ and $y_2'(t, u_1(t))$ to be zero. Thus the leading order term in the small-t expansion of the integral in (34) is

$$\frac{t^2 w_1(0)}{2 u_1'(0)} y_1'(0, u_1(0)) y_2'(0, u_1(0)).$$

This term must be identically zero and, in view of the fact that neither $y_1'(0, u_1(0))$ nor $y_2'(0, u_1(0))$ may vanish, we deduce that $w_1(0) = 0$, and arrive again at a contradiction.

Case 2: $u_1(0) = u_2(0)$ but $u_1'(0) \neq u_2'(0)$. Since $u_1 \not\equiv u_2$ we may assume that t is sufficiently small to ensure $u_1(x) \neq u_2(x)$ for all $x \in (0, t]$. In (32) we shall use first-order Taylor expansions, for which purpose we note that by virtue of (26),

$$(\partial y_2(t, \lambda)/\partial \lambda)|_{\lambda = u_2(t)} = -1/u_2'(t),$$

and hence, for small $t > 0$,

$$y_2(t, u_1(t)) = y_2(t, u_1(t)) - y_2(t, u_2(t)) = -t(u_1'(0) - u_2'(0))/u_2'(0) + O(t^2),$$

$$\begin{aligned} \frac{y_1(x, u_1(t)) y_2(x, u_1(t))}{u_1(t) - u_2(x)} &= y_1(x, u_1(t)) \left[\frac{y_2(x, u_1(t)) - y_2(x, u_2(x))}{u_1(t) - u_2(x)} \right] \\ &= -y_1(x, u_1(t)) \frac{1}{u_2'(x)} (1 + o(1)) = O(t - x); \end{aligned}$$

$$\begin{aligned} \frac{y_1(x, u_1(t)) y_2(x, u_1(t))}{u_1(t) - u_1(x)} &= \left[\frac{y_1(x, u_1(t)) - y_1(x, u_1(x))}{u_1(t) - u_1(x)} \right] y_2(x, u_1(t)) \\ &= -y_2(x, u_1(t)) \frac{1}{u_1'(x)} (1 + o(1)) = O(t - x). \end{aligned}$$

It follows that the right hand side of (32) is $O(t^2)$ or smaller. The dominant term of the left hand side is

$$\begin{aligned} \frac{w_1(t)^2}{u_1'(t)} z_1(t, u_1(t)) y_2(t, u_1(t)) &= \frac{w_1(t)^2}{u_1'(t)} z_1(t, u_1(t)) (y_2(t, u_1(t)) - y_2(t, u_2(t))) \\ &= \frac{w_1(t)^2}{u_1'(t)} z_1(t, u_1(t)) \left(t \frac{u_1'(0) - u_2'(0)}{u_2'(t)} + O(t^2) \right). \end{aligned}$$

Bearing in mind the assumption $u_1'(0) \neq u_2'(0)$ we see that comparing the left and right hand sides of (32) has given us, for small t,

$$\frac{w_1(t)^2}{u_1'(t)} z_1(t, u_1(t)) = O(t).$$

Since $z_1(t, u_1(t)) = O(1)$ we deduce that $w_1(0) = 0$, which is again a contradiction. $\square$

Remark 4.2. The asymptotic behaviours of the solutions which we have used in this theorem can be seen explicitly in the case $u(x) = x$, $q(x) \equiv 0$, $w(x) \equiv 1$, for instance, when the analytic solutions (y as opposed to $\tilde{y}$) are all scalar multiples of

$$y(x, \lambda) = \sum_{n=1}^{\infty} c_n (x - \lambda)^n,$$

in which $c_n = -c_{n-1}/(n(n+1))$. Clearly in this case y is an entire function of both of its arguments. The second solution $\tilde{y}$ can be found by the method of D'Alembert.

Conjecture 4.3. Despite Theorem 4.1, we conjecture that in the general case of analytic coefficients, $M_B(\lambda)$ does not determine the coefficients uniquely. In the first order case this non-uniqueness is established below, see Remark 6.2.

5. Reconstruction of the operator from one restricted resolvent $(A_B - \lambda)^{-1}|_{\overline{\mathcal{S}}}$

The detectable subspace $\overline{\mathcal{S}}$ is the largest space on which we may recover information about an operator from its M-functions. We now consider, if the resolvent of A_B is known on this space, for one unknown B, how much information about A_B can be recovered.

Theorem 5.1. For the Hain-Lüst operator

$$\widetilde{A}^* = \begin{pmatrix} -\frac{d^2}{dx^2} + q(x) & \widetilde{w}(x) \\ w(x) & u(x) \end{pmatrix}$$

with coefficients q, w, $\widetilde{w}$ and u all in $L^\infty(0,1)$, the restricted resolvent $(A_B - \lambda)^{-1}|_{\overline{\mathcal{S}}}$ determines q, $u|_{w \neq 0}$, w and $\widetilde{w}|_{w\neq 0}$, as well as the boundary condition matrix B.

The proof of this result is distributed over the following subsections.

5.1. Preliminaries.

For any fixed $\lambda_0 \notin \mathrm{Ran}(u)$ a straightforward calculation shows that

$$\mathrm{Ran}(S_{\lambda_0,B}) = \ker(\widetilde{A}^* - \lambda_0) = \left\{ \begin{pmatrix} 1 \\ -\frac{w}{u-\lambda_0} \end{pmatrix} (c_1 y_1 + c_2 y_2) \right\}$$

where y_i, $i = 1, 2$, are the solutions of the Schur complement equation (6), namely

$$-y'' + (q - \lambda_0) y - \frac{w\widetilde{w}}{u - \lambda_0} y = 0, \tag{35}$$

with initial conditions

$$\begin{cases} y_1(0) = 0, & y_1'(0) = 1; \\ y_2(0) = 1, & y_2'(0) = 0. \end{cases} \tag{36}$$

Recall eqn. (9), namely

$$M_B(\lambda)(\Gamma_1 - B\Gamma_2) \begin{pmatrix} y \\ z \end{pmatrix} = \Gamma_2 \begin{pmatrix} y \\ z \end{pmatrix} \quad \text{for all } \begin{pmatrix} y \\ z \end{pmatrix} \in \ker(\widetilde{A}^* - \lambda).$$

In particular, then,

$$M_B(\lambda)\left[\begin{pmatrix} -y_1'(1) \\ 1 \end{pmatrix} - B\begin{pmatrix} y_1(1) \\ 0 \end{pmatrix}\right] = \begin{pmatrix} y_1(1) \\ 0 \end{pmatrix}; \tag{37}$$

$$M_B(\lambda)\left[\begin{pmatrix} -y_2'(1) \\ 0 \end{pmatrix} - B\begin{pmatrix} y_2(1) \\ 1 \end{pmatrix}\right] = \begin{pmatrix} y_2(1) \\ 1 \end{pmatrix}. \tag{38}$$

It follows immediately from these expressions that

$$M_B(\lambda)^{-1}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \left[\begin{pmatrix} -y_1'(1) \\ 1 \end{pmatrix} - B\begin{pmatrix} y_1(1) \\ 0 \end{pmatrix}\right]/y_1(1);$$

$$M_B(\lambda)^{-1}\begin{pmatrix} y_2(1) \\ 1 \end{pmatrix} = \left[\begin{pmatrix} -y_2'(1) \\ 0 \end{pmatrix} - B\begin{pmatrix} y_2(1) \\ 1 \end{pmatrix}\right].$$

We note that $y_1(1) \neq 0$ for almost all $\lambda \in \mathbb{C}$ – in particular, $y_1(1) \neq 0$ if $\lambda \notin \mathrm{Ran}(u)$ is not an eigenvalue of $\widetilde{A}^*|_{\ker(\Gamma_2)}$.

5.2. **Useful vectors.** We know that in $\overline{\mathcal{S}}$ there are vectors of the form

$$\begin{pmatrix} 1 \\ \frac{-w}{u-\lambda_i} \end{pmatrix} y_j, \quad i = 1,2;\ j = 1,2;\ \lambda_1 \neq \lambda_2.$$

At this stage we cannot construct any such vectors explicitly. However we can certainly assert the existence of a pair of vectors $\mathbf{u}_i = \begin{pmatrix} f_i \\ g_i \end{pmatrix}$, $i = 1,2$, such that

(1) $f_i(x) \neq 0$ for a.e. $x \in (0,1)$;
(2) $g_i(x) \neq 0$ for a.e. s such that $w(x) \neq 0$;
(3) $g_i(x) = 0$ for all x such that $w(x) = 0$;
(4) $g_1(x)/f_1(x) \neq g_2(x)/f_2(x)$ for a.e. x such that $w(x) \neq 0$.
(5) $f_i \in C^1[0,1]$, $i = 1,2$.

To see that vectors satisfying these properties exist, choose

$$\begin{cases} f_i = y_1, \\ g_i = \frac{-w}{u-\lambda_i} y_1, \quad \lambda_1 \neq \lambda_2, \end{cases}$$

and observe that

$$\frac{g_1}{f_1} = \frac{-w}{u-\lambda_1} \neq \frac{-w}{u-\lambda_2} = \frac{g_2}{f_2}.$$

Note that, at this stage, we do not know w; however, since we know $\overline{\mathcal{S}}$, we certainly know the set $\{x : w(x) \neq 0\}$ as the union of supports of second components of vectors in $\overline{\mathcal{S}}$.

Lemma 5.2. Assume that $q, w, \widetilde{w}, u \in L^\infty(0,1)$, that $f \in C^1[0,1]$, $g \in L^2(0,1)$, and that y is the solution of

$$\begin{cases} -y'' + (q-\lambda)y - \dfrac{w\widetilde{w}}{u-\lambda} y = f - \dfrac{\widetilde{w} g}{u-\lambda}; \\ \\ (\Gamma_1 - B\Gamma_2)\begin{pmatrix} y \\ 0 \end{pmatrix} = \mathbf{0}. \end{cases}$$

Then, with $\|\cdot\|$ denoting the norm in $L^2(0,1)$,

$$\left\| y + \frac{f}{\lambda} \right\| = o(\lambda^{-1}), \quad \lambda \to -\infty;\ \ \lambda \in \mathbb{R}.$$

Proof. Let L_0 denote the operator defined by $L_0 y = -y'' + qy$ with boundary condition

$$\begin{pmatrix} -y'(1) \\ y'(0) \end{pmatrix} = B \begin{pmatrix} y(1) \\ y(0) \end{pmatrix}.$$

Then

$$\langle L_0 y, y\rangle = (y(1), y(0))B(y(1), y(0))^* + \int_0^1 [|y'|^2 + q|y|^2]dx,$$

and since the trace operator $y \mapsto (y(1), y(0))$ is bounded with respect to the norm in $H^1(0,1)$ with relative bound zero, it follows that for any $\epsilon > 0$ there exists $c_\epsilon, d_\epsilon \in \mathbb{R}$ such that the numerical range of L_0 is contained in a set

$$\{\lambda \in \mathbb{C} \mid \Re(\lambda) \geq c_\epsilon, \ \ |\Im(\lambda)| \leq \epsilon|\Re(\lambda)| + d_\epsilon\}.$$

It follows that when $\lambda \to -\infty$, one has a uniform bound

$$\|L_0(L_0 - \lambda)^{-1}\| \leq \text{const.}$$

Together with the fact that the domain of L_0 is dense in $L^2(0,1)$ this implies that for any u in $L^2(0,1)$,

$$L_0(L_0 - \lambda)^{-1}u \to 0, \quad \lambda \to -\infty.$$

Now the equation in the Lemma may be written as $(L_0 - \lambda)y = f + A(\lambda)y + G(\lambda)$ in which $\|A(\lambda)\| = O(\lambda^{-1})$ and $\|G(\lambda)\| = O(\lambda^{-1})$, $\lambda \to -\infty$. This can be written as

$$\left[I - (L_0 - \lambda)^{-1}A(\lambda)\right] y = \frac{-1}{\lambda} f + \frac{L_0(L_0 - \lambda)^{-1} f}{\lambda} + (L_0 - \lambda)^{-1}G(\lambda),$$

and the result follows by using the Neumann series for the resolvent of the operator on the left hand side. □

Remark 5.3. One may prove that the result holds when $\lambda \to \infty$ in any sector $|\arg(-\lambda)| < \pi/2 - \epsilon$, where $\epsilon > 0$ is fixed.

5.3. **Reconstruction of** $\widetilde{w}|_{w\neq 0}$ **and** q**.** Let $\begin{pmatrix} f_i \\ g_i \end{pmatrix}$, $i = 1, 2$, be two vectors from $\overline{\mathcal{S}}$ for which the conditions of our previous sub-section are satisfied. Define

$$\begin{pmatrix} Y_i \\ Z_i \end{pmatrix} = (A_B - \lambda)^{-1}|_{\overline{\mathcal{S}}} \begin{pmatrix} f_i \\ g_i \end{pmatrix}, \quad i = 1, 2, \quad \lambda \in \rho(A_B).$$

These vectors are known since they require only the restricted resolvent for their computation. Performing this computation explicitly, we have

$$\begin{cases} -Y_i'' + (q - \lambda)Y_i - \dfrac{w\widetilde{w}}{u - \lambda} Y_i = f_i - \dfrac{\widetilde{w} g_i}{u - \lambda}, \\[2ex] Z_i = \dfrac{g_i}{u - \lambda} - \dfrac{wY_i}{u - \lambda} \\[2ex] (\Gamma_1 - B\Gamma_2) \begin{pmatrix} Y_i \\ Z_i \end{pmatrix} = \mathbf{0}. \end{cases} \tag{39}$$

Rearranging the first equation slightly we obtain

$$-Y_i'' - \lambda Y_i - f_i = -qY_i + \frac{w\widetilde{w}}{u - \lambda} Y_i - \frac{\widetilde{w} g_i}{u - \lambda},$$

in which the left hand side is known, and hence the right hand side is known. However by Lemma 5.2, we have

$$-qY_i + \frac{w\widetilde{w}}{u-\lambda}Y_i - \frac{\widetilde{w}g_i}{u-\lambda} = \frac{qf_i}{\lambda} + \frac{\widetilde{w}g_i}{\lambda} + o(\lambda^{-1}), \quad \lambda \to -\infty.$$

It follows that $qf_i+\widetilde{w}g_i$ are known, for $i = 1, 2$; hence that $q+\widetilde{w}\frac{g_i}{f_i}$ are known. Subtracting, we deduce that

$$\widetilde{w}\left(\frac{g_1}{f_1} - \frac{g_2}{f_2}\right)$$

is known. However g_1/f_1 and g_2/f_2 are known on the set of x such that $w(x) \neq 0$ (and are zero outside this set). Hence we deduce that

$$\widetilde{w}|_{w\neq 0} \text{ is known.}$$

Since (say) $q + wg_1/f_1$ is now fully known, it follows that

$$q \text{ is known.}$$

5.4. **Reconstruction of w and $u|_{w\neq 0}$.** From the second equation in (39) we know the functions

$$Z_i = \frac{g_i}{u-\lambda} - \frac{wY_i}{u-\lambda}, \quad i = 1, 2.$$

Since, from Lemma 5.2, we have $Y_i = -f_i/\lambda + o(\lambda^{-1})$, we obtain

$$Z_i = -\frac{g_i}{\lambda} - \frac{g_iu + wf_i}{\lambda^2} + o(\lambda^{-2}).$$

From this expansion it follows that $g_iu + wf_i$ are known, $i = 1, 2$, and hence that $w + \frac{g_i}{f_i}u$ are known, $i = 1, 2$. Subtracting, we find that $u\left(\frac{g_1}{f_1} - \frac{g_2}{f_2}\right)$ are known, $i = 1, 2$; moreover $\frac{g_1}{f_1} - \frac{g_2}{f_2}$ is non-zero a.e. on the set of x such that $w(x) \neq 0$. It follows that

$$u|_{w\neq 0} \text{ is known.}$$

Repeating the argument at the end of the previous section we conclude that

$$w \text{ is known.}$$

5.5. **Reconstruction of the boundary condition matrix B.** We now know the coefficients q, $w\widetilde{w}$ and $u|_{w\neq 0}$ and so the solutions y_1 and y_2 of (35,36) appearing in (37,38) are completely determined. To reconstruct B we first re-write (37,38) as

$$(M_B(\lambda)^{-1} + B)\begin{pmatrix} y_1(1) \\ 0 \end{pmatrix} = \begin{pmatrix} -y_1'(1) \\ 1 \end{pmatrix}, \tag{40}$$

$$(M_B(\lambda)^{-1} + B)\begin{pmatrix} y_2(1) \\ 1 \end{pmatrix} = \begin{pmatrix} -y_2'(1) \\ 0 \end{pmatrix}. \tag{41}$$

In order for $M_B(\lambda)^{-1} + B$ to be completely determined for any fixed λ it suffices that the vectors

$$\begin{pmatrix} y_1(1) \\ 0 \end{pmatrix}, \quad \begin{pmatrix} y_2(1) \\ 1 \end{pmatrix}$$

be linearly independent, which is true provided $y_1(1) \neq 0$.

For $\lambda \notin \mathrm{Ran}(u)$, the requirement $y_1(1) \neq 0$ is equivalent to the requirement that λ not be an eigenvalue of $\widetilde{A}^*|_{\ker(\Gamma_2)}$. Under our hypotheses of L^∞ coefficients, the numerical range of $\widetilde{A}^*|_{\ker(\Gamma_2)}$ is confined to a semi-infinite strip of the form

$$\Re(\lambda) \geq \alpha, \quad |\Im(\lambda)| \leq \beta.$$

In particular, $y_1(1)$ is non-zero for any λ outside this semi-infinite strip. Thus $M_B(\lambda)^{-1} + B$ is determined outside the numerical range of $\widetilde{A}^*|_{\ker(\Gamma_2)}$.

In order to recover B it is therefore sufficient to know that $M_B(\lambda)^{-1}$ is determined. However by [9, Theorem 4.1], the resolvent $(A_B - \lambda)^{-1}|_{\overline{\mathcal{S}}}$ uniquely determines $M_B(\lambda)$. Thus B is uniquely determined, and Theorem 5.1 is proved. □

6. The first order Hain-Lüst operator

In this section we consider a first order toy model replacement of the Hain-Lüst equation and show that, even for this simple case in which many quantities are explicitly computable by quadrature, many results remain non-trivial. In particular we show that the M-function does not determine the coefficients in the operator uniquely, even when the coefficients are analytic.

We consider on the domain $D(A) = D(\tilde{A}) = H_0^1(0,1) \times L^2(0,1)$ the first order operators

$$A = \begin{pmatrix} i\dfrac{d}{dx} + q & \tilde{w} \\ w & u \end{pmatrix}, \quad \tilde{A} = \begin{pmatrix} i\dfrac{d}{dx} + \overline{q} & \overline{w} \\ \overline{\widetilde{w}} & \overline{u} \end{pmatrix},$$

with coefficients q, u, w and $\widetilde{w}$ all $L^\infty(0,1)$ functions. The adjoints A^* and $\widetilde{A}^*$ have domain $H^1(0,1) \oplus L^2(0,1)$.

Definition 6.1. The boundary operators $\Gamma_1, \Gamma_2, \tilde{\Gamma}_1, \tilde{\Gamma}_2$ are defined by

$$\Gamma_1 \begin{pmatrix} y \\ z \end{pmatrix} = iy(1), \quad \tilde{\Gamma}_1 \begin{pmatrix} y \\ z \end{pmatrix} = -iy(0),$$
$$\Gamma_2 \begin{pmatrix} y \\ z \end{pmatrix} = y(0), \quad \tilde{\Gamma}_2 \begin{pmatrix} y \\ z \end{pmatrix} = y(1).$$

Then, the Lagrange identity is

$$\left\langle \tilde{A}^\star \begin{pmatrix} y \\ z \end{pmatrix}, \begin{pmatrix} f \\ g \end{pmatrix} \right\rangle - \left\langle \begin{pmatrix} y \\ z \end{pmatrix}, A^\star \begin{pmatrix} f \\ g \end{pmatrix} \right\rangle = \langle iy', f\rangle - \langle y, if'\rangle$$
$$= iy\overline{f}|_0^1 = i(y(1)\overline{f(1)} - y(0)\overline{f(0)}) = \left\langle \Gamma_1 \begin{pmatrix} y \\ z \end{pmatrix}, \tilde{\Gamma}_2 \begin{pmatrix} f \\ g \end{pmatrix} \right\rangle - \left\langle \Gamma_2 \begin{pmatrix} y \\ z \end{pmatrix}, \tilde{\Gamma}_1 \begin{pmatrix} f \\ g \end{pmatrix} \right\rangle.$$

6.1. Calculation of the M-function.

In line with our review in Section 2 the M-function is defined by the equation

$$M_0(\lambda)\Gamma_1 \begin{pmatrix} y \\ z \end{pmatrix} = \Gamma_2 \begin{pmatrix} y \\ z \end{pmatrix} \text{ for } \begin{pmatrix} y \\ z \end{pmatrix} \in \ker(\tilde{A}^* - \lambda).$$

This gives $iM_0(\lambda)y(1) = y(0)$ and thus

$$M_0(\lambda) = -i\frac{y(0)}{y(1)}.$$

Now $\begin{pmatrix} y \\ z \end{pmatrix} \in \ker(\tilde{A}^* - \lambda)$ holds if and only if we have

$$iy' + qy + \tilde{w}z = \lambda y, \qquad wy + uz = \lambda z.$$

Solving these equations yields $z = \dfrac{wy}{\lambda - u}$ and $iy' + (q-\lambda)y + \dfrac{w\tilde{w}}{\lambda - u}y = 0$, so we have $\dfrac{y'}{y} = i(q - \lambda + \dfrac{w\tilde{w}}{\lambda - u})$ giving

$$y(x) = y(0)\exp\left(i\int_0^x [q(t) - \lambda + \frac{w(t)\tilde{w}(t)}{\lambda - u(t)}]dt\right). \tag{42}$$

Thus we have an explicit expression for the M-function in terms of the coefficients in the operator:

$$M_0(\lambda) = -i\exp\left(i\lambda - i\int_0^1 q(t)dt + i\int_0^1 \frac{w\tilde{w}(t)}{u(t)-\lambda}dt\right).$$

Remark 6.2.

(1) Observe that the only information on q from M_0 is its mean value, $\int_0^1 q(t)dt$. In the Hermitian case $i\dfrac{d}{dx} + q$ is unitarily equivalent to $i\dfrac{d}{dx} + \int_0^1 q(t)dt$ by a gauge transformation; as these operators have the same form, the fact that M_0 can only determine the mean value of q also follows from abstract results (e.g. [30]).

(2) This also shows that for the scalar equation $i\dfrac{d}{dx}+q$ we can only recover $\int_0^1 q(t)dt$. This is despite $\overline{\mathcal{S}} = \overline{\mathrm{Span}\,\{e^{i\int^x q(t)dt}, e^{i\lambda x}\}} = L^2(0,1)$.

(3) From $\int_0^1 \frac{w\tilde{w}(t)}{u(t)-\lambda}dt$ we can reconstruct $\mathrm{Ran}\, u$ but not u and $w\tilde{w}$. To see this let $\phi : [0,1] \to [0,1]$ be an analytic change of coordinates. Then $\int_0^1 \frac{w\tilde{w}}{u-\lambda}dt = \int_0^1 \frac{(w\tilde{w}\circ\phi)}{u\circ\phi - \lambda}\phi' dt$. Thus the analytic change of coordinates $w\tilde{w} \to (w\tilde{w}\circ\phi)\phi'$ and $u \to u\circ\phi$ gives non uniqueness.

In view of these remarks, and to provide a comparison with our results for the second-order Hain-Lüst case, it is interesting to consider the calculation of the detectable subspace for the first order Hain-Lüst operator. Despite the availability of explicit expressions such as (42) the computations are only tractable in some special cases.

Theorem 6.3. In the special case $w\tilde{w} \equiv 0$ the orthogonal complement $\mathcal{S}^\perp$ of the detectable subspace is given by

$$\mathcal{S}^\perp = \left\{ \begin{pmatrix} f \\ g \end{pmatrix} : g \perp E_{u,w}, \quad f = \mathcal{J}(g) \right\}.$$

Here $E_{u,w}$ is the space

$$E_{u,w} = \bigvee_{n=0}^{\infty} w(x)\psi(x,u(x))u^n(x),$$

where $\psi(x,\lambda)$ is the unique-up-to-scalar-multiples solution of the differential equation

$$i\psi' + (q-\lambda)\psi = 0; \tag{43}$$

and $\mathcal{J}$ is the functional defined by $f = \mathcal{J}(g)$ precisely when

$$f(x) = i\exp\left(i\int_0^x \overline{q}\right)\int_0^x \exp\left(-i\int_0^t \overline{q}\right)\exp(i(t-x)\overline{u}(t))\overline{w}(t)g(t)dt. \tag{44}$$

The proof of this result will use Proposition 2.1 from Section 2 and follows closely the methods in Section 3.

Proof. Following the ideas which lead to Proposition 2.1 we know that $\begin{pmatrix} f \\ g \end{pmatrix} \in \mathcal{S}^\perp$ if and only if, for a.e. $\lambda \in \mathbb{C}$ there exist y, z such that $y(0) = 0 = y(1)$,

$$iy' + (\overline{q} - \lambda)y + \overline{w}z = f, \quad \text{and} \quad \overline{\widetilde{w}}y + (\overline{u} - \lambda)z = g.$$

We assume without loss of generality that λ does not lie in $\overline{\mathrm{essran}(\overline{u})}$. The second equation can then be used to determine z, giving

$$z = \frac{g}{\overline{u} - \lambda} - \frac{\overline{\widetilde{w}}y}{\overline{u} - \lambda};$$

then, since $w\widetilde{w} \equiv 0$, the equation for y becomes

$$iy' + (\overline{q} - \lambda)y = f - \frac{\overline{w}g}{\overline{u} - \lambda}, \tag{45}$$

equipped with boundary conditions $y(0) = 0 = y(1)$. It is easy to see that given any non-trivial function $\psi(\cdot, \lambda)$ solving (43), $\overline{\psi}(\cdot, \overline{\lambda})$ provides an integrating factor for (45) and hence that $y(0) = 0 = y(1)$ if and only if

$$\int_0^1 \left(f(x) - \frac{\overline{w}(x)g(x)}{\overline{u}(x) - \lambda} \right) \overline{\psi}(x, \overline{\lambda}) dx = 0. \tag{46}$$

This condition must hold for a.e. λ in $\mathbb{C}$. To obtain the conditions on g in the theorem we multiply by λ^n and integrate with respect to λ round a contour enclosing $\overline{\mathrm{essran}(\overline{u})}$, exactly as in the proof of Lemma 3.11, yielding

$$\int_0^1 \overline{w}(x)\overline{\psi}(x, u(x))\overline{u}(x)^n g(x) dx = 0. \tag{47}$$

This shows that $g \perp E_{u,w}$ is a necessary condition. It is not difficult to verify that it is also sufficient.

Knowing now that $\int_0^1 \overline{w}(x)\overline{\psi}(x, u(x))\overline{u}(x)^n g(x) dx = 0$ we can use the Taylor expansion

$$(\lambda - \overline{u}(x))^{-1} = \lambda^{-1} \sum_{n=0}^{\infty} \left(\frac{\overline{u}(x)}{\lambda} \right)^n,$$

valid for large λ, in conjunction with (47) to deduce that

$$\int_0^1 \overline{w}(x) \frac{\overline{\psi}(x, u(x))}{\overline{u}(x) - \lambda} g(x) dx = 0,$$

and hence obtain from (46) that

$$\int_0^1 f(x)\overline{\psi}(x, \overline{\lambda}) dx = \int_0^1 \overline{w}(x) \frac{\overline{\psi}(x, \overline{\lambda}) - \overline{\psi}(x, u(x))}{\overline{u}(x) - \lambda} g(x) dx. \tag{48}$$

From (43) we have an explicit formula for ψ, correct up to scalar multiples,

$$\psi(x, \lambda) = \exp(-i\lambda x) \exp\left(i \int_0^x q \right), \quad \text{so} \quad \overline{\psi}(x, \overline{\lambda}) = \exp(i\lambda x) \exp\left(-i \int_0^x \overline{q} \right)$$

and if we define the compactly supported function $\tilde{f}(x) := \chi_{(0,1)}(x) f(x) \exp\left(-i \int_0^x \overline{q}\right)$ then (48) yields

$$\mathcal{F}(\tilde{f})(\lambda) = \int_0^1 \overline{w}(x) g(x) \frac{\overline{\psi}(x, \overline{\lambda}) - \overline{\psi}(x, u(x))}{\overline{u}(x) - \lambda} dx,$$

where $\mathcal{F}$ denotes Fourier transform. The function on the right hand side is an entire function; it is of the appropriate exponential type to be, by the Paley-Wiener theorem, the

Fourier transform of an L^2 function supported on $(0,1)$. The equation therefore has a unique solution for $\tilde{f}$ and hence f is uniquely determined from (48). The fact that the expression (44) gives the solution of this equation is a calculation which we omit here. $\square$

As an immediate consequence of Theorem 6.3 we have the following.

Corollary 6.4. Under the hypotheses of Theorem 6.3 the detectable subspace satisfies

$$\mathcal{S}^\perp = \begin{pmatrix} 0 \\ L^2(w=0) \end{pmatrix}$$

if and only if

$$\bigvee_{n=0}^{\infty} w(x)u^n(x) = L^2(w \neq 0). \tag{49}$$

One may ask whether the characterisation of $\mathcal{S}^\perp$ in Theorem 6.3 holds in the case when $w\widetilde{w}$ is nonzero. The following example shows that it does not.

Example 6.5. We show that any example with $q \equiv 0$, $u(x) = x$, $w\widetilde{w} \equiv i$, has the property that $\mathcal{S}^\perp \neq \begin{pmatrix} 0 \\ L^2(w=0) \end{pmatrix}$ even though (49) holds.

To this end we observe that in a general situation the function ψ appearing in the proof of Theorem 6.3 is replaced by the solution of the differential equation

$$\left(i\frac{d}{dx} + q - \frac{w\widetilde{w}}{u(x)-\lambda}\right)\psi(x,\lambda) = \lambda\psi(x,\lambda),$$

which, for the coefficients chosen here, means that

$$\psi(x,\lambda) = \exp\left(-i\lambda x + \int_0^x \frac{dt}{t-\lambda}\right) = \left(1 - \frac{x}{\lambda}\right)\exp(-i\lambda x) \quad \text{and} \quad \overline{\psi}(x,\overline{\lambda}) = \frac{\lambda - x}{\lambda}\exp(i\lambda x).$$

The condition that $\begin{pmatrix} f \\ g \end{pmatrix} \in \mathcal{S}^\perp$ is still (46), which here is equivalent to

$$\int_0^1 (xf(x) - \overline{w}(x)g(x) - \lambda f(x))\exp(i\lambda x)dx = 0.$$

We may write this as

$$\int_0^1 (xf(x) - \overline{w}(x)g(x))\exp(i\lambda x)dx = -i\int_0^1 f(x)\left(\frac{d}{dx}\exp(i\lambda x)\right)dx. \tag{50}$$

Take any smooth $\phi \in C_0^\infty(0,1)$, multiply (50) by $\mathcal{F}(\phi)(\lambda)$ and invert the Fourier transforms to obtain

$$\int_0^1 (xf(x) - \overline{w}(x)g(x))\phi(x)dx = \int_0^1 (-if)(x)\phi'(x)dx.$$

Since this holds for all $\phi \in C_0^\infty(0,1)$ we deduce that $f \in H^1(0,1)$ and

$$if'(x) = xf(x) - \overline{w}(x)g(x).$$

Replacing $xf(x) - \overline{w}(x)g(x)$ by $if'(x)$ on the left hand side of (50) we deduce that for all λ,

$$0 = [-if(x)\exp(i\lambda x)]_{x=0}^{x=1} = -i(f(1)\exp(i\lambda) - f(0)),$$

which implies that $f(0) = 0 = f(1)$. Thus our complete characterisation of the arbitrary element $\begin{pmatrix} f \\ g \end{pmatrix}$ of $\mathcal{S}^\perp$ is that

$$g \in L^2(0,1); \quad f \in H^1(0,1); \quad -if' + xf = \overline{w}g; \quad f(0) = 0 = f(1).$$

In particular, given any $f \in H_0^1(0,1)$, we may simply choose $g = (-if' + xf)/\overline{w}$ and, provided $1/w \in L^\infty$, which can be arranged within our hypotheses, we shall have an element of $\mathcal{S}^\perp$ for which f is not identically zero.

Our final result on the first order Hain-Lüst model concerns the reconstruction of the operator from the resolvent restricted to the detectable subspace. Before stating the theorem, we prove a lemma.

Lemma 6.6. Suppose that $y(1) = C_B y(0)$ and that $iy' + (q(x) - \lambda)y + A(\lambda)y = h + O(1/\lambda)$, where the last term is $O(1/\lambda)$ in $L^2(0,1)$ and $|A(\lambda)| = O(1/\lambda)$. Then as $\Im\lambda \to \pm\infty$ with λ in a cone $\arg(\lambda) \mp \frac{\pi}{2} < \frac{\pi}{2} - \epsilon_0$, for any fixed, small $\epsilon_0 > 0$, one has, in $L^2(0,1)$,

$$y = -\frac{h}{\lambda} + o(1/\lambda). \tag{51}$$

Proof. Define an operator $L_0 = i\frac{d}{dx}$ with domain $D(L_0) = \{u \in H^1(0,1) \,|\, u(1) = C_B u(0)\}$, where $C_B = (i + B)/(i - B)$. A direct calculation shows that

$$2\Im\langle L_0 u, u\rangle = 2\Im\langle iu', u\rangle = \int_0^1 (u'\overline{u} + \overline{u'}u) = |u(1)|^2 - |u(0)|^2 = (|C_B|^2 - 1)|u(0)|^2.$$

Thus L_0 is either dissipative ($|C_B| \geq 1$) or anti-dissipative ($|C_B| \leq 1$). It follows from basic numerical range estimates that in the operator norm,

$$\|(L_0 - \lambda)^{-1}\| \leq \frac{1}{|\Im(\lambda)|}, \quad |C_B| \geq 1 \text{ and } \lambda \text{ in } \mathbb{C}^-, \text{ or } |C_B| \leq 1 \text{ and } \lambda \text{ in } \mathbb{C}^+. \tag{52}$$

Combining these with the identity $L_0(L_0 - \lambda)^{-1} = I + \lambda(L_0 - \lambda)^{-1}$ it follows that one has bounds

$$\|L_0(L_0 - \lambda)^{-1}\| \leq \text{const.}, \quad \pm(1 - |C_B|) \geq 0 \text{ and } \lambda \text{ tends to infinity on a non-real ray in } \mathbb{C}^\pm.$$

Since the domain of L_0 is dense in $L^2(0,1)$ it then follows that one has the following strong limits for any $u \in L^2(0,1)$:

$$L_0(L_0 - \lambda)^{-1}u \to 0, \quad \pm(1 - |C_B|) \geq 0 \text{ and } \lambda \text{ tends to infinity on a non-real ray in } \mathbb{C}^\pm. \tag{53}$$

Under the hypotheses in Lemma 6.6 we know that y satisfies the equation

$$(L_0 - \lambda)y = (-q(x) - A(\lambda))y + h + O(1/\lambda)$$

and so, if λ tends to infinity along a non-real ray, in the upper half-plane for $|C_B| \leq 1$ or in the lower half-plane for $|C_B| \geq 1$, we have

$$y - (L_0 - \lambda)^{-1}(q + A(\lambda))y = (L_0 - \lambda)^{-1}h + O(1/\lambda^2).$$

This means that under these conditions,

$$y = \left[I - (L_0 - \lambda)^{-1}(q + A(\lambda))\right]^{-1}\left[-\frac{h}{\lambda} + \frac{1}{\lambda}L_0(L_0 - \lambda)^{-1}h + O(1/\lambda^2)\right].$$

From (53) we know that $L_0(L_0 - \lambda)^{-1}h = o(1/\lambda)$, and from (52) we know $\|(L_0 - \lambda)^{-1}(q + A(\lambda))\| = O(1/\lambda)$ when λ tends to infinity on an appropriate non-real ray. The result follows immediately. $\square$

Theorem 6.7. Let A_B denote the restriction of $\widetilde{A}^*$ to the space of functions satisfying the boundary condition $\Gamma_1 u = B\Gamma_2 u$, where $B \in \mathbb{C}$. Then the coefficients q, w, $\widetilde{w}|_{w\neq 0}$ and $u|_{w\neq 0}$, as well as the boundary condition parameter B, are all uniquely determined by a knowledge of the detectable subspace itself, together with a knowledge of $(A_B - \lambda)^{-1}$ on the detectable subspace.

Proof. We first identify some useful vectors in the detectable subspace. From (12) it follows that for all μ in the resolvent set $\rho(A_B)$, vectors in $\mathrm{Ran}(S_{\mu,B})$ lie in the detectable subspace. By solving the equation $(\widetilde{A}^* - \mu)\begin{pmatrix} f \\ g \end{pmatrix} = 0$, we see that $\mathrm{Ran}(S_{\mu,B})$ is a one-dimensional space given by

$$\mathrm{Ran}(S_{\mu,B}) = \mathrm{span}\left\{ \begin{pmatrix} 1 \\ -\frac{w(x)}{u(x)-\mu} \end{pmatrix} \exp(-i\mu x) \exp\left(i \int_0^x \left(q - \frac{w\widetilde{w}}{u-\mu} \right) \right) \right\}. \tag{54}$$

From this it follows that the detectable subspace should contain elements $\begin{pmatrix} f \\ g \end{pmatrix}$ such that

(1) $f, g \in L^\infty(0,1)$;
(2) $|f|$ is bounded below with a strictly positive lower bound;
(3) $g(x) \neq 0$ for all x such that $w(x) \neq 0$.

By doing this for a pair of different values of μ, say μ_1 and μ_2, one can generate two different vectors $U_1 = \begin{pmatrix} f_1 \\ g_1 \end{pmatrix}$ and $U_2 = \begin{pmatrix} f_2 \\ g_2 \end{pmatrix}$ in the detectable subspace with these three properties, with the additional feature that

$$\frac{g_1(x)}{f_1(x)} \neq \frac{g_2(x)}{f_2(x)} \quad \text{for all } x \text{ such that} \quad w(x) \neq 0. \tag{55}$$

This follows from the fact that $w(x)/(u(x)-\mu_1) \neq w(x)/(u(x)-\mu_2)$ for all x such that $w(x) \neq 0$. Observe that we do not claim to know what U_1 and U_2 are, because we do not know the operator $\widetilde{A}^*$ a priori; all we claim is that two such vectors exist in the detectable subspace with these properties.

Our first step is to reconstruct q and $\widetilde{w}|_{w\neq 0}$. We first pick two pairs $\begin{pmatrix} f_1 \\ g_1 \end{pmatrix}$ and $\begin{pmatrix} f_2 \\ g_2 \end{pmatrix}$ of vectors in $\overline{\mathcal{S}}$ satisfying the properties (1)-(3) above and (55). For any fixed λ in the resolvent set $\rho(A_B)$, define

$$\begin{pmatrix} y_j \\ z_j \end{pmatrix} = (A_B - \lambda)^{-1} U_j, \quad j = 1, 2.$$

This means that for $j = 1, 2$,

$$iy_j'(x) + (q(x) - \lambda)y_j(x) - \frac{w\widetilde{w}(x)}{u(x)-\lambda} y_j(x) = f_j - \frac{\widetilde{w}(x)}{u(x)-\lambda} g_j(x),$$

$$z_j(x) = \frac{g_j(x)}{u(x)-\lambda} - \frac{w(x)y_j(x)}{u(x)-\lambda}, \tag{56}$$

$$y_j(1) = C_B y_j(0),$$

where $C_B = (i+B)/(i-B)$. Since we know $(A_B-\lambda)^{-1}$ on the detectable subspace, we know f_j, g_j, y_j and z_j for $j=1,2$, for any $\lambda \in \rho(A_B)$. We observe that

$$iy_j'(x) - \lambda y_j(x) - f_j(x) = -q(x)y_j(x) - \frac{\widetilde{w}(x)g_j(x)}{u(x)-\lambda} + \frac{w\widetilde{w}(x)}{u(x)-\lambda}y_j(x). \tag{57}$$

From (57) and Lemma 6.6 we see that,

$$iy_j'(x) - \lambda y_j(x) - f_j(x) = \frac{q(x)f_j(x)}{\lambda} + \frac{\widetilde{w}(x)g_j(x)}{\lambda} + o(1/\lambda).$$

The left hand side of this equation is known since y_j and f_j are known. This means that $q(x)f_j(x) + \widetilde{w}(x)g_j(x)$ is known for $j=1,2$. Dividing by f_j, which is known and is bounded away from zero, we see that $q(x) + \widetilde{w}(x)\frac{g_j(x)}{f_j(x)}$ is known, for $j=1,2$. By taking differences between the $j=1$ and $j=2$ cases it follows from the property (55) that $\widetilde{w}(x)$ is known for all x such that $w(x)\neq 0$.

Observe that the set of x such that $w(x)\neq 0$ is known. This follows from the fact that we know the detectable subspace by hypothesis, and from the fact that we know vectors $U_j = \begin{pmatrix} f_j \\ g_j \end{pmatrix}$ with $g_j(x)$ non-vanishing at all x such that $w(x)\neq 0$.

Observe too that since the detectable subspace is the closure of the linear spans of all vectors of the form (54), one always has $g_j(x)=0$ whenever $w(x)=0$. It follows that $\widetilde{w}\frac{g_j}{f_j} = \widetilde{w}|_{w\neq 0}\frac{g_j}{f_j}$, and so $\widetilde{w}\frac{g_j}{f_j}$ are known for $j=1,2$. This implies that q is known.

From (56) and Lemma 6.6 we now have

$$z_j = \frac{g_j}{u-\lambda} - \frac{wy_j}{u-\lambda} = \frac{g_j}{u-\lambda} - \frac{wf_j}{\lambda^2} + o\left(\frac{1}{\lambda^2}\right) = -\frac{g_j}{\lambda} - \frac{g_ju + wf_j}{\lambda^2} + o\left(\frac{1}{\lambda^2}\right).$$

Since z_j is known this implies that $g_ju + wf_j$ is known for $j=1,2$. Thus $w + \frac{g_j}{f_j}u$ is known for $j=1,2$, and taking differences we see that $\left(\frac{g_1}{f_1} - \frac{g_2}{f_2}\right)u$ is known. Following our earlier reasoning for $\widetilde{w}$, we deduce that $u|_{w\neq 0}$ is known. But now since $w + \frac{g_1}{f_1}u$ is known on the set where w is nonzero, and since the set where w is nonzero is known, it follows that w also is known.

Finally we outline how the constant B in the boundary conditions can be reconstructed. Fix $\lambda_0 \in \mathbb{C}$. Given the information about q, w, $\widetilde{w}_{w\neq 0}$ and $u|_{w\neq 0}$ found above, the vector U in the detectable subspace given by

$$U := \begin{pmatrix} f \\ g \end{pmatrix} := \begin{pmatrix} 1 \\ -\frac{w}{u-\lambda_0} \end{pmatrix} \exp(-i\lambda_0 x)\exp\left(i\int_0^x \left(q - \frac{w\widetilde{w}}{u-\lambda_0}\right)dt\right) \tag{58}$$

is known. Since the resolvent is known on the detectable subspace, it follows that

$$\begin{pmatrix} y \\ z \end{pmatrix} = (A_B-\lambda)^{-1}U$$

is known, and moreover satisfies the boundary condition associated with A_B, namely $(i-B)y(1) = (i+B)y(0)$. In fact, since the differential equation satisfied by y uses only the coefficients q, $w\widetilde{w}$ and $u_{w\neq 0}$, y can be found in terms of quadratures by solving the equation with elementary methods. The only way that B can fail to be determined is if we have, for all λ_0 and λ, both $y(0)=0$ and $y(1)=0$. This turns out to be equivalent to

$$\int_0^1 \left(f(x) - \frac{\widetilde{w}(x)g(x)}{u(x)-\lambda}\right)\exp\left(-i\int_0^x\left(q-\lambda-\frac{w\widetilde{w}}{u-\lambda}\right)dt\right)dx = 0.$$

Substituting in the explicit expressions for f and g in (58), putting $\lambda = -i\tau$ and letting $\tau \to +\infty$ shows that this is impossible. $\square$

REFERENCES

[1] Adamjan, V., Langer, H. and Langer, M., A spectral theory for a λ-rational Sturm-Liouville problem. J. Diff. Eqns. **171** (2001), 315–345.

[2] Arlinskii, Yu.M., Hassi, S. and de Snoo, H., Q-functions of quasi-self-adjoint contractions. Oper. Theory Adv. Appl. **163** (2005), 23-54.

[3] Arlinskii, Yu. and Klotz, L., Weyl functions of bounded quasi-selfadjoint operators and operator Jacobi matrices. Acta Sci. Math. (Szeged) **76** (2010), 585-626.

[4] Arov, D.Z. and Nudelman, M.A., Tests for the similarity of all minimal passive realizations of a fixed transfer function (scattering or resistance matrix). Mat. Sb. **193** (6) (2002), 3-24.

[5] Behrndt, J. and Rohleder, J., An inverse problem of Calderón type with partial data. Comm. Partial Differential Equations **37** (2012), no. 6, 1141-1159.

[6] Brown, B.M., Hinchcliffe, J., Marletta, M., Naboko, S. and Wood, I., The abstract Titchmarsh-Weyl M-function for adjoint operator pairs and its relation to the spectrum, Int. Eq. Oper. Th, **63** (2009), 297-320.

[7] Brown, B.M., Langer, M. and Marletta, M., Spectral concentrations and resonances of a second-order block operator matrix and an associated λ-rational Sturm-Liouville problem. Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci. **460** (2004), 3403–3420.

[8] Brown, B.M., Marletta, M., Naboko, S. and Wood, I., Boundary triplets and M-functions for non-selfadjoint operators, with applications to elliptic PDEs and block operator matrices. J. Lond. Math. Soc. (2) **77** (2008), no. 3, 700-718.

[9] Brown, B.M., Marletta, M., Naboko, S. and Wood, I., An abstract inverse problem for boundary triples with an application to the Friedrichs model. `http://arxiv.org/abs/1404.6820`.

[10] Borg, G., Uniqueness theorems in the spectral theory of $y'' + (\lambda - q(x))y = 0$. Den 11th Skandinaviske Matematikerkongress, Trondheim, 1949; pp. 276-287. John Grundt Tanums Forlag, Oslo, 1952.

[11] Derkach, V. A. and Malamud, M. M., Generalized resolvents and the boundary value problems for Hermitian operators with gaps. J. Funct. Anal. **95** (1991), no. 1, 1-95.

[12] Gelfand, I.M. and Levitan, B.M., On the determination of a differential equation from its spectral function. Am. Math. Soc. Trans. **1** (1951), 253-304.

[13] Hassi, S., Malamud, M. and Mogileskij, V., Unitary equivalence of proper extensions of a symmetric operator and the Weyl function. Integr. Eq. Oper. Theory **77** (2013), 449-487.

[14] Hassi, S., Möller, M. and de Snoo, H., Singular Sturm-Liouville problems whose coefficients depend rationally on the eigenvalue parameter. J. Math. Anal. Appl. **295**, (2004), no. 1, 258–275.

[15] Ince, E.L., Ordinary Differential Equations. Dover, New York, 1956.

[16] Jonas, P. and Trunk, C., A Sturm-Liouville problem depending rationally on the eigenvalue parameter. Math. Nachr. **280**, (2007), no. 15, 1709–1726.

[17] Kreĭn, M.G. and Langer, H., Über die Q-Funktion eines π-hermiteschen Operators im Raume Π_κ. Acta Sci. Math. (Szeged) **34** (1973), 191–230.

[18] Kurasov, P., Lelyavin,I. and Naboko, S., On the essential spectrum of a class of singular matrix differential operators. II. Weyl's limit circles for the Hain-Lüst operator whenever quasi-regularity conditions are not satisfied. Proc. Roy. Soc. Edinburgh Sect. A **138** (2008), no. 1, 109–138.

[19] Kurasov, P. and Naboko, S., On the essential spectrum of a system of singular matrix-differential operators. I. Quasiregularity condition and essential selfadjointness. Math. Phys, Analysis and Geometry, **5** (2002), no. 3, 243-286.

[20] Langer, M., Eigenvalues of a λ-rational Sturm-Liouville problem. Math. Nachr. **210**, (2000), 163–176.

[21] Langer, M., Resonances of a λ-rational Sturm–Liouville problem. Proc. Roy. Soc. Edinburgh Sect. A bf 131 (2001) 709–720.

[22] Langer, H. and Textorius, B., On generalized resolvents and Q-functions of symmetric linear relations (subspaces) in Hilbert space. Pacific J. Math. **72**, 1 (1977), 135–165.

[23] Malamud, M.M. and Mogilevskii, V.I., On extensions of dual pairs of operators. Dopovidi Nation. Akad. Nauk Ukrainy **1** (1997) 30–37.

[24] Malamud, M.M., and Mogilevskii, V.I., On Weyl functions and Q-function of dual pairs of linear relations. Dopovidi Nation. Akad. Nauk Ukrainy **4** (1999) 32-37.

[25] Malamud, M.M., and Mogilevskii, V.I., Kreĭn type formula for canonical resolvents of dual pairs of linear relations. Methods Funct. Anal. Topology (4) **8** (2002) 72-100.

[26] Marchenko, V.A., Concerning the theory of a differential operator of the second order. (Russian) Doklady Akad. Nauk SSSR. (N.S.) **72** (1950) 457-460.

[27] Mennicken, R., Naboko, S. and Tretter, Ch., Essential spectrum of a system of singular differential operators and the asymptotic Hain-Lust operator. Proc.AMS. **130** (2002), no.6, 1699–1710.

[28] Muhammad, A. and Marletta, M., A numerical investigation of the quadratic numerical range of Hain-Lüst operators, Int. J. Comput. Math., **90**,(2013), no. 11, 2431–2451.

[29] Nachman, A.I., Sylvester, J. and Uhlmann, G., An n-dimensional Borg-Levinson theorem. Commun. Math. Phys. **115** (1988), 595-605.

[30] Ryzhov, V., A general boundary value problem and its Weyl function. Opuscula Math. (2) **27** (2007), 305-331.