On bifurcation for semilinear elliptic Dirichlet problems on geodesic balls

Alessandro Portaluri and Nils Waterstraat

Abstract

We study bifurcation from a branch of trivial solutions of semilinear elliptic Dirichlet boundary value problems on a geodesic ball, whose radius is used as the bifurcation parameter. In the proof of our main theorem we obtain in addition a special case of an index theorem due to S. Smale.

1 Introduction

Let \((M, g)\) be an oriented Riemannian manifold of dimension \(n\) and let \(\Delta = \text{div grad} : C^\infty(M) \to C^\infty(M)\) denote the associated Laplace-Beltrami operator. Let \(V : M \times \mathbb{R} \to \mathbb{R}\) be a smooth function such that \(V(p, 0) = 0\) for all \(p \in M\) and

\[
|V(p, \xi)| \leq C(1 + |\xi|^\alpha), \quad \left| \frac{\partial V}{\partial \xi}(p, \xi) \right| \leq C(1 + |\xi|^\beta), \quad (p, \xi) \in M \times \mathbb{R},
\]

for some \(C > 0\) and constants \(\alpha, \beta \geq 0\) depending on the dimension \(n\) of \(M\) (cf. \cite{AP93}, §1.2). In this paper we deal with local solutions of the semilinear equation

\[
-\Delta u(p) + V(p, u(p)) = 0, \quad p \in M,
\]

under Dirichlet boundary conditions. Note that many equations from geometric analysis are of the type \((2)\). Let us refer to \cite{Au82}, \cite{Be87} and just mention as an example on compact manifolds of dimension \(n \geq 3\) the equation

\[
4 \frac{n-1}{n-2} \Delta u(p) + s(p)u(p) = \mu u(p)^{\frac{n+2}{n-2}}, \quad p \in M,
\]

where \(s : M \to \mathbb{R}\) denotes the scalar curvature function and \(\mu\) the Yamabe invariant of the metric \(g\) on \(M\). Positive solutions \(u \in C^\infty(M)\) of \((3)\) give rise to metrics \(\tilde{g}\) of constant scalar curvature on \(M\) by \(\tilde{g} = u^{\frac{4}{n-2}}g\).

We now fix a point \(p_0 \in M\) and assume that the unit ball \(B(0, 1) \subset T_{p_0}M\) is contained in the

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maximal domain on which the exponential map \( \exp_{p_0} \) at \( p_0 \) is an embedding. Let us denote by \( B(p_0, r) = \exp_{p_0}(B(0, r)) \) the geodesic ball of radius \( 0 < r \leq 1 \) around \( p_0 \) and consider the Dirichlet boundary value problems

\[
\begin{aligned}
-\Delta u(p) + V(p, u(p)) &= 0, & p &\in B(p_0, r) \\
u(p) &= 0, & p &\in \partial B(p_0, r).
\end{aligned}
\]  

We call \( r^* \in (0, 1] \) a bifurcation radius for the boundary value problems (4) if there exists a sequence of radii \( r_n \to r^* \) and functions \( u_n \in H^1_0(B(p_0, r_n)) \) such that \( u_n \) is a non-trivial weak solution of (4) on \( B(p_0, r_n) \) and \( \|u_n\|_{H^1_0(B(p_0, r_n))} \to 0 \). Note that we exclude from the definition the limiting case \( r^* = 0 \) in which the domain degenerates to a point. The reason is that \( \|u_n\|_{H^1_0(B(p_0, r_n))} \to 0 \) for \( r_n \to 0 \) holds, for example, for any sequence of functions \( u_n \in C^1(B(p_0, r_n)), n \in \mathbb{N} \), such that all \( u_n \) and their weak derivatives are bounded uniformly. Consequently, a bifurcation radius \( r^* = 0 \) would not imply the existence of non-trivial solutions of (4) for small \( r > 0 \) which are arbitrarily close to the trivial solution \( u \equiv 0 \) in a suitable sense.

Let us now consider the linearised boundary value problems

\[
\begin{aligned}
-\Delta u(p) + f(p)u(p) &= 0, & p &\in B(p_0, r) \\
u(p) &= 0, & p &\in \partial B(p_0, r),
\end{aligned}
\]  

where \( f(p) = \frac{\partial V}{\partial \xi}(p, 0), p \in M \). We call \( r^* \in (0, 1] \) a conjugate radius for (5) if

\[
m(r^*) := \dim\{u \in C^2(B(p_0, r^*)) : u \text{ solves (5)}\} > 0,
\]

and from now on we assume that \( m(1) = 0 \). Our main result reads as follows:

**Theorem 1.1.** The bifurcation radii of (4) are precisely the conjugate radii of (5).

We explain below in the proof of Theorem 1.1 that we obtain from our methods a new proof of the Morse-Smale index theorem \([\text{Sm65}]\) (cf. also \([\text{Sm67}]\)) for the linearised equations (5). As a consequence, we conclude that \( m(r) = 0 \) for almost all radii \( r \in (0, 1) \), and moreover, we derive from Theorem 1.1 the following corollary:

**Corollary 1.2.** Let \( \mu \) denote the Morse index of (5) on \( B(p_0, 1) \), i.e. the number of negative eigenvalues counted according to their multiplicities. If \( \mu \neq 0 \), then there exist at least

\[
\left\lfloor \frac{\mu}{\max_{0 < r < 1} m(r)} \right\rfloor
\]

distinct bifurcation radii in \( (0, 1) \), where \( \lfloor \cdot \rfloor \) denotes the integral part of a real number.

Let us point out that a proof of Theorem 1.1 and Corollary 1.2 for the special case that \( M \) is a star-shaped domain in \( \mathbb{R}^n \) can be found in \([\text{PW13}]\). The following section is devoted to the more general setting which we consider here.

## 2 The proof

Our main reference for the Laplace-Beltrami operator on manifolds with boundary is \([\text{La96}] \S2.4\). Let us recall at first that in local coordinates...
\[ \Delta u = \sum_{j,k=1}^{n} |g|^{-\frac{1}{2}} \frac{\partial}{\partial x_j} \left( g^{jk} \frac{\partial u}{\partial x_k} \right), \]

where \( g^{jk} \), \( 1 \leq j, k \leq n \), are the components of the inverse of the metric tensor \( g = \{g_{jk}\} \) and \( |g| := |\det \{g_{jk}\}| \) is the absolute value of its determinant. Denoting by \( \text{dvol}_g \) the volume form of \( g \), we find for \( u \in H^1_0(B(p_0, r)) \), \( 0 < r \leq 1 \),

\[
- \int_{B(p_0, r)} (\Delta u)(p)v(p) \text{ dvol}_g + \int_{B(p_0, r)} V(p, u(p)) v(p) \text{ dvol}_g \\
= - \int_{B(0, r)} v(x) \sum_{j,k=1}^{n} \frac{\partial}{\partial x_j} \left( g^{jk}(x) |g(x)|^{-\frac{1}{2}} \frac{\partial u}{\partial x_k}(x) \right) dx + \int_{B(0, r)} |g(x)|^{-\frac{1}{2}} V(x, u(x)) v(x) dx \\
= r \int_{B(0, 1)} \sum_{j,k=1}^{n} g^{jk}(r \cdot x) |g(r \cdot x)|^{-\frac{1}{2}} \frac{\partial u}{\partial x_k}(r \cdot x) \frac{\partial v}{\partial x_j}(r \cdot x) dx \\
+ r \int_{B(0, 1)} |g(r \cdot x)|^{-\frac{1}{2}} V(r \cdot x, u(r \cdot x)) v(r \cdot x) dx,
\]

and analogously

\[
- \int_{B(p_0, r)} (\Delta u)(p) v(p) \text{ dvol}_g + \int_{B(p_0, r)} f(p) u(p) v(p) \text{ dvol}_g \\
= r \int_{B(0, 1)} \sum_{j,k=1}^{n} g^{jk}(r \cdot x) |g(r \cdot x)|^{-\frac{1}{2}} \frac{\partial u}{\partial x_k}(r \cdot x) \frac{\partial v}{\partial x_j}(r \cdot x) dx \\
+ r \int_{B(0, 1)} |g(r \cdot x)|^{-\frac{1}{2}} f(r \cdot x) u(r \cdot x) v(r \cdot x) dx.
\]

We now set \( B := B(0, 1) \) and define for \( 0 \leq r \leq 1 \) a functional \( q_r : H^1_0(B) \times H^1_0(B) \to \mathbb{R} \) by

\[
q_r(u, v) = \int_{B} \sum_{j,k=1}^{n} g^{jk}(r \cdot x) |g(r \cdot x)|^{-\frac{1}{2}} \frac{\partial u}{\partial x_k}(r \cdot x) \frac{\partial v}{\partial x_j}(r \cdot x) dx + r^2 \int_{B} |g(r \cdot x)|^{-\frac{1}{2}} V(r \cdot x, u(x)) v(x) dx
\]

as well as a quadratic form \( h_r : H^1_0(B) \to \mathbb{R} \) by

\[
h_r(u) = \int_{B} \sum_{j,k=1}^{n} g^{jk}(r \cdot x) |g(r \cdot x)|^{-\frac{1}{2}} \frac{\partial u}{\partial x_k}(r \cdot x) \frac{\partial u}{\partial x_j}(r \cdot x) dx + r^2 \int_{B} |g(r \cdot x)|^{-\frac{1}{2}} f(r \cdot x) u(x)^2 dx.
\]

From the computations above we conclude that:

i) \( r^* \in (0, 1] \) is a bifurcation radius for \( f \), if and only if there exist a sequence \( \{r_n\}_{n \in \mathbb{N}} \subset (0, 1] \), \( r_n \to r^* \), and a sequence of non-trivial functions \( \{u_n\}_{n \in \mathbb{N}} \subset H^1_0(B) \), \( u_n \to 0 \), such that \( q_{r_n}(u_n, \cdot) = 0 \in (H^1_0(B))^* \) for all \( n \in \mathbb{N} \).


ii) \( r^* \in (0, 1) \) is a conjugate radius for \( (\mathbb{I}) \) if and only if \( h_r \) is degenerate.

We now define a function \( \psi : [0, 1] \times H_0^1(B) \to \mathbb{R} \) by

\[
\psi(r, u) = \int_B \sum_{j,k=1}^n g^{jk}(r \cdot x)|g(r \cdot x)|^2 \frac{\partial u}{\partial x} \left( \frac{\partial u}{\partial x} \right) dx + 2 \int_B \frac{1}{2} g(r \cdot x) \nabla G(r \cdot x, u(x)) + r^2. \]

where

\[
G(x, t) = \int_0^t V(x, \xi) d\xi.
\]

It is a standard result that \( \psi \) is \( C^2 \)-smooth under the growth conditions \( (\mathbb{I}) \), and \( D_u \psi_r = q_r(u, \cdot) \), \( u \in H_0^1(B) \). Moreover, \( 0 \in H_0^1(B) \) is a critical point of all functionals \( \psi \), and the corresponding Hessians are given by \( D_r^2 \psi_r = h_r \). From the compactness of the inclusion \( H_0^1(B) \hookrightarrow L^2(B) \), we see at once that the Riesz representation of the quadratic form \( h_r \) is a self-adjoint Fredholm operator. In particular, it is invertible if \( h_r \) is non-degenerate.

Let us now assume at first that \( r^* \in (0, 1) \) is not a conjugate radius. Then \( h_{r^*} \) is non-degenerate and we conclude by the implicit function theorem \( [\text{AP93}, \S 2.2] \) that the equation \( q_r(u, \cdot) = 0 \) has no other solutions than \( (r, 0) \in [0, 1] \times H_0^1(B) \) in a neighbourhood of \( (r^*, 0) \). Consequently, \( (r^*, 0) \) is not a bifurcation radius, and we have shown that every bifurcation radius in \( (0, 1) \) is a conjugate radius.

In order to prove the remaining implication of Theorem \( (\mathbb{I}) \), we make use of the bifurcation theory for critical points of smooth functionals developed in \( [\text{FPR99}] \). Accordingly, we consider for \( r_0 \in (0, 1) \) the quadratic forms

\[
\Gamma(h, r_0) : \ker h_{r_0} \to \mathbb{R}, \quad \Gamma(h, r_0)[u] = \left( \frac{d}{dr} |_{r=r_0} h_r \right) u.
\]

By \( [\text{FPR99}] \) Thm. 1 & Thm. 4.1, \( r_0 \) is a bifurcation radius if \( \Gamma(h, r_0) \) is non-degenerate and has a non-vanishing signature (cf. also Section 2.1 in \( [\text{PW13}] \)). Consequently, we now assume that \( r_0 \in (0, 1) \) is a conjugate radius and our aim is to compute \( \Gamma(h, r_0) \). Let us write for simplicity of notation

\[
a^{jk}(x) = g^{jk}(x)|g(x)|^2, \quad x \in B, \quad 1 \leq j, k \leq n,
\]

\[
\hat{f}(x) = \frac{g(x)}{|g(x)|^2} f(x), \quad x \in B.
\]

For \( u \in \ker h_{r_0} \) we have by definition

\[
\Gamma(h, r_0)[u] = \int_B \sum_{j,k=1}^n \langle \nabla a^{jk}(r_0 \cdot x), x \rangle \frac{\partial u}{\partial x} \left( \frac{\partial u}{\partial x} \right) dx + \int_B \left( \frac{d}{dr} |_{r=r_0} (r^2 \hat{f}(r \cdot x)) u(x) \right)^2 dx. \quad (6)
\]

Let us now introduce a new function by \( v_r(x) := u(\frac{x}{r_0}, x), r \in (0, r_0], x \in B \), and denote

\[
\dot{u}(x) := \frac{d}{dr} |_{r=r_0} v_r(x) = \frac{1}{r_0} (\nabla u(x), x).
\]

It is readily seen that \( v_r \) satisfies

\[
\dot{u}(x) := \frac{d}{dr} |_{r=r_0} v_r(x) = \frac{1}{r_0} (\nabla u(x), x).
\]
\[- \sum_{j,k=1}^{n} \frac{\partial}{\partial x^j} \left( a^{jk}(r \cdot x) \frac{\partial \nu_j}{\partial x^k} \right) + r^2 \bar{f}(r \cdot x) v_r(x) = 0, \]

and by differentiating with respect to \( r \) at \( r = r_0 \) we have

\[
0 = - \sum_{j,k=1}^{n} \frac{\partial}{\partial x^j} \left( \langle \nabla a^{jk}(r_0 \cdot x), x \rangle \frac{\partial \bar{u}}{\partial x^k} \right) - \sum_{j,k=1}^{n} \frac{\partial}{\partial x^j} \left( a^{jk}(r_0 \cdot x) \frac{\partial \bar{u}}{\partial x^k} \right) + \frac{d}{dr} |_{r=r_0} \left( r^2 \bar{f}(r \cdot x) \right) \bar{u}(x) 
\]

We multiply (8) by \( u \) and integrate over \( B \):

\[
0 = - \int_B \sum_{j,k=1}^{n} \frac{\partial}{\partial x^j} \left( \langle \nabla a^{jk}(r_0 \cdot x), x \rangle \frac{\partial \bar{u}}{\partial x^k} \right) u(x) \, dx - \int_B \sum_{j,k=1}^{n} \frac{\partial}{\partial x^j} \left( a^{jk}(r_0 \cdot x) \frac{\partial \bar{u}}{\partial x^k} \right) u(x) \, dx 
+ \int_B \frac{d}{dr} |_{r=r_0} \left( r^2 \bar{f}(r \cdot x) \right) u(x)^2 \, dx + \int_B r_0^2 \bar{f}(r_0 \cdot x) \bar{u}(x) u(x) \, dx.
\]

Let \( \nu(x) = (\nu_1(x), \ldots, \nu_n(x)) \), \( x \in \partial B \), denote the outward pointing unit normal to the boundary of \( B \). Using \( u \big|_{\partial B} = 0 \), we obtain from integration by parts

\[
0 = \int_B \sum_{j,k=1}^{n} \langle \nabla a^{jk}(r_0 \cdot x), x \rangle \frac{\partial \bar{u}}{\partial x^j} \frac{\partial u}{\partial x^k} \, dx - \int_{\partial B} \left( \sum_{j,k=1}^{n} \langle \nabla a^{jk}(r_0 \cdot x), x \rangle \nu_j(x) \frac{\partial \bar{u}}{\partial x^k} \right) u(x) \, dS 
+ \int_B \sum_{j,k=1}^{n} a^{jk}(r_0 \cdot x) \frac{\partial \bar{u}}{\partial x^j} \frac{\partial u}{\partial x^k} \, dx - \int_{\partial B} \left( \sum_{j,k=1}^{n} a^{jk}(r_0 \cdot x) \nu_j(x) \frac{\partial \bar{u}}{\partial x^k} \right) u(x) \, dS 
+ \int_B \frac{d}{dr} |_{r=r_0} \left( r^2 \bar{f}(r \cdot x) \right) u(x)^2 \, dx + \int_B r_0^2 \bar{f}(r_0 \cdot x) \bar{u}(x) u(x) \, dx
\]

Since \( u \in \ker h_{r_0} \),

\[
- \sum_{j,k=1}^{n} \frac{\partial}{\partial x^j} \left( a^{jk}(r_0 \cdot x) \frac{\partial u}{\partial x^k} \right) + r_0^2 \bar{f}(r_0 \cdot x) u(x) = 0, \quad x \in B, \tag{9}
\]

and it follows from (8) and (9) that
\begin{equation}
\Gamma(h, r_0)[u] = -\frac{1}{r_0} \int_{\partial B} \left( \sum_{j,k=1}^n a^{jk}(r_0 \cdot x) \nu_j(x) \frac{\partial u}{\partial x^k} \right) \langle \nabla u(x), x \rangle \, dS. \tag{10}
\end{equation}

If we set \( A(x) := \{ a_{jk}(x) \}, x \in B \), and use that \( \nu(x) = x \) for all \( x \in \partial B \), we can rewrite (10) as

\[
\Gamma(h, r_0)[u] = -\frac{1}{r_0} \int_{\partial B} \langle A(r_0 \cdot x)x, \nabla u(x) \rangle \langle \nabla u(x), x \rangle \, dS.
\]

Denoting by \((A(r_0 \cdot x)x)^T, x \in \partial B\), the tangential component of the vector \( A(r_0 \cdot x)x \), we have

\[
\langle A(r_0 \cdot x)x, \nabla u(x) \rangle = \langle \nabla u(x), x \rangle \langle A(r_0 \cdot x)x, x \rangle + \langle \nabla u(x), (A(r_0 \cdot x)x)^T \rangle
\]
and hence

\[
\Gamma(h, r_0)[u] = -\frac{1}{r_0} \int_{\partial B} \langle \nabla u(x), x \rangle^2 \langle A(r_0 \cdot x)x, x \rangle \, dS
- \frac{1}{r_0} \int_{\partial B} \langle \nabla u(x), x \rangle \langle \nabla u(x), (A(r_0 \cdot x)x)^T \rangle \, dS.
\]

Since

\[
\langle \nabla u(x), x \rangle \langle \nabla u(x), (A(r_0 \cdot x)x)^T \rangle = \text{div}(u(x)\langle x, \nabla u(x) \rangle(A(r_0 \cdot x)x)^T), \quad x \in \partial B,
\]
we finally get by Stokes’ theorem

\[
\Gamma(h, r_0)[u] = -\frac{1}{r_0} \int_{\partial B} \langle \nabla u(x), x \rangle^2 \langle A(r_0 \cdot x)x, x \rangle \, dS \leq 0, \tag{11}
\]

where we use that \( A(x) \) is positive definite for all \( x \in B \).

Moreover, we obtain from (11) that if \( \Gamma(h, r_0)[u] = 0 \) for some \( u \in \ker h_{r_0} \), then

\[
\langle \nabla u(x), x \rangle = \langle \nabla u(x), \nu(x) \rangle = \frac{\partial u}{\partial \nu}(x) = 0
\]
for all \( x \in \partial B \) which implies \( u \equiv 0 \) by the unique continuation property.

In summary, we have shown that \( \Gamma(h, r_0) \) is negative definite, and so in particular non-degenerate with the non-vanishing signature

\[
\text{sgn} \, \Gamma(h, r_0) = m(r_0).
\tag{12}
\]

Consequently, \( r_0 \) is a bifurcation radius and Theorem 1.1 is proven.

Let us now prove Corollary 1.2. We note at first that the Morse index \( \mu \) of (5) on the full domain \( B(p_0, 1) \) is given by the Morse index \( \mu(h_1) \) of the quadratic form \( h_1 \). Moreover, since \( h_0 \) is positive, we see that \( \mu(h_0) = 0 \). It is shown in [FPR99, Prop. 3.9& Thm. 4.1] that if \( \Gamma(h, r) \) is non-degenerate for all \( r \in (0, 1) \), then \( \ker h_r = 0 \) for almost all \( r \in (0, 1) \) and
\[ \mu(h_1) - \mu(h_0) = \sum_{0 < r < 1} \text{sgn} \Gamma(h, r). \]

Consequently, we conclude from (12) that \( m(r) = \dim \ker h_r = 0 \) for almost all \( r \in (0, 1) \) and

\[ \mu = \sum_{0 < r < 1} m(r). \quad (13) \]

Let us point out that (13) was obtained by Smale in [Sm65] by studying the monotonicity of eigenvalues under shrinking of domains. Hence we have obtained a new proof of Smale’s theorem for the boundary value problem (9), and moreover, Corollary 1.2 is now an immediate consequence of (13) and Theorem 1.1.

References


Alessandro Portaluri
Department of Agriculture, Forest and Food Sciences
Università degli studi di Torino
Via Leonardo da Vinci, 44
10095 Grugliasco (TO)
Italy
E-mail: alessandro.portaluri@unito.it
Nils Waterstraat
Dipartimento di Scienze Matematiche
Politecnico di Torino
Corso Duca degli Abruzzi, 24
10129 Torino
Italy
E-mail: waterstraat@daad-alumni.de