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On bifurcation for semilinear elliptic Dirichlet problems and the Morse-Smale index theorem

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Abstract

We study bifurcation from a branch of trivial solutions of semilinear elliptic Dirichlet boundary value problems on star-shaped domains, where the bifurcation parameter is introduced by shrinking the domain. In the proof of our main theorem we obtain in addition a special case of an index theorem due to S. Smale.

1 Introduction

Let $\Omega \subset \mathbb{R}^N$ be a smooth domain which is star-shaped with respect to 0 and set

$$\Omega_r := \{r \cdot x \in \mathbb{R}^N : x \in \Omega\}, \quad 0 < r \leq 1.$$

We consider for $0 < r \leq 1$ the semilinear elliptic Dirichlet problems

$$\begin{cases} -\Delta u(x) + g(x, u(x)) = 0, & x \in \Omega_r \\ u(x) = 0, & x \in \partial\Omega_r, \end{cases} \tag{1}$$

where $g \in C^\infty(\overline{\Omega} \times \mathbb{R})$ and $g(x, 0) = 0$ for all $x \in \Omega$. Moreover, we assume that there exists $C > 0$ such that g satisfies the standard growth conditions

$$|g(x, \xi)| \leq C(1 + |\xi|^\alpha), \quad \left|\frac{\partial g}{\partial \xi}(x, \xi)\right| \leq C(1 + |\xi|^\beta), \quad (x, \xi) \in \Omega \times \mathbb{R},$$

for certain constants $\alpha, \beta \geq 0$ depending on the dimension N (cf. [AP93, §1.2]).

We call $r^* \in (0, 1]$ a *bifurcation point* for the equations (1) if there exists a sequence $r_n \to r^*$ and $u_n \in H_0^1(\Omega_{r_n})$ such that u_n is a non-trivial weak solution of the boundary value problem (1) on Ω_{r_n} and $\|u_n\|_{H_0^1(\Omega_{r_n})} \to 0$.

Now denote $f(x) = \frac{\partial g}{\partial \xi}(x, 0)$, $x \in \overline{\Omega}$, and consider the linearised equations

$$\begin{cases} -\Delta u(x) + f(x)u(x) = 0, & x \in \Omega_r \\ u(x) = 0, & x \in \partial\Omega_r. \end{cases} \tag{2}$$

We call $r^* \in (0, 1]$ a *conjugate point* for (2) if

$$m(r^*) := \dim\{u \in C^2(\Omega_{r^*}) : u \quad \text{solves} \quad (2)\} > 0$$

and henceforth we assume that $m(1) = 0$. Our main theorem reads as follows:

Theorem 1.1. *The bifurcation points of* (1) *are precisely the conjugate points of* (2).

Our proof of Theorem 1.1 uses crossing forms and the connection between bifurcation of branches of critical points and the spectral flow from [FPR99]. In particular, we do not use any domain monotonicity properties of eigenvalues of (2).

Interestingly, we moreover obtain from our computations a new and simple proof of a classical theorem due to Smale [Sm65] for the equations (2). In what follows we denote by M the Morse index of (2) on the full domain $\Omega_1 = \Omega$, i.e. the number of negative eigenvalues counted according to their multiplicities.

Corollary 1.2. $m(r) = 0$ *for almost all* $0 < r < 1$ *and*

$$M = \sum_{0 < r < 1} m(r).$$

We want to point out that Smale considered in [Sm65] the Dirichlet problem for general strongly elliptic differential operators on vector bundles over compact manifolds with boundary. It seems to us that a computation of the corresponding crossing forms, as we will do for the equation (2) in Section 2.2, is no longer possible in this generality. However, Smale's theorem has attracted some interest in recent years for scalar equations on star-shaped domains in Euclidean spaces (cf. [DJ11] and [DP12]).

Finally, from Theorem 1.1 and Corollary 1.2 we immediately obtain the following result:

Corollary 1.3. *If* $M \neq 0$, *then there exist at least*

$$\left\lfloor \frac{M}{\max_{0 < r < 1} m(r)} \right\rfloor$$

distinct bifurcation points in $(0, 1)$.

In the special case $N = 1$, i.e. if (1) is a semilinear ODE on a compact interval, we have $0 \leq m(r) \leq 1$ and hence the number of bifurcation points in Corollary 1.3 is precisely the Morse index M.

2 The proof

In this section we prove Theorem 1.1 and Corollary 1.2. At first, we recall in Section 2.1 the definitions of crossing forms and their application in bifurcation theory. Afterwards we compute the crossing forms for the equations (2) in Section 2.2.

2.1 Crossing forms and bifurcation

We follow in this section [FPR99]. Let H be an infinite dimensional separable real Hilbert space and let $I = [0, 1]$ denote the unit interval. We consider C^2-functions $\psi : I \times H \to \mathbb{R}$ and assume throughout that $0 \in H$ is a critical point of each functional $\psi_\lambda = \psi(\lambda, \cdot) : H \to \mathbb{R}$. We call λ_0 a bifurcation point if any neighbourhood of $(\lambda_0, 0) \in I \times H$ contains elements (λ, u) such that $u \neq 0$ is a critical point of ψ_λ.

The Hessians $h_\lambda = D_0^2 \psi_\lambda : H \to \mathbb{R}$ of ψ_λ at $0 \in H$ define a path of quadratic forms which we require henceforth to be continuously differentiable. Moreover, we assume that the Riesz representations of h_λ, $\lambda \in I$, with respect to the scalar product of H are Fredholm operators. We call $\lambda_0 \in (0, 1)$ a *crossing* if $\ker h_{\lambda_0} \neq 0$. The *crossing form* at λ_0 is defined by

$$\Gamma(h, \lambda_0) : \ker h_{\lambda_0} \to \mathbb{R}, \quad \Gamma(h, \lambda_0)[u] = \left(\frac{d}{d\lambda} \mid_{\lambda = \lambda_0} h \right) [u] \tag{3}$$

and a crossing λ_0 is said to be regular if $\Gamma(h, \lambda_0)$ is non-degenerate. It is easily seen that regular crossings are isolated and hence finite in number.

The proof of Theorem 1.1 will be based on the theorem below which follows from Theorem 1 and Theorem 4.1 in [FPR99] (cf. also [Ra89]).

Theorem 2.1. *If $\lambda_0 \in (0, 1)$ is a regular crossing and*

$$\mathrm{sgn}\,\Gamma(h, \lambda_0) \neq 0,$$

then λ_0 is a bifurcation point.

For the proof of Corollary 1.2 we need a further result from [FPR99]. In what follows we assume that the quadratic forms h_λ, $\lambda \in I$, have finite Morse indices $M(h_\lambda)$. From Proposition 3.9 and Theorem 4.1 of [FPR99] we obtain:

Theorem 2.2. *If all crossings of the path h_λ, $\lambda \in I$, are regular and h_0, h_1 are non-degenerate, then*

$$M(h_0) - M(h_1) = \sum_{\lambda \in (0,1)} \mathrm{sgn}\,\Gamma(h, \lambda).$$

2.2 Computation of the crossing forms

We consider the family of boundary value problems (1) for $0 < r \leq 1$. After rescaling, equation (1) on Ω_r is equivalent to

$$\begin{cases} -\Delta u(x) + r^2\, g(r \cdot x, u(x)) = 0, & x \in \Omega \\ \qquad\qquad\qquad\qquad u(x) = 0, & x \in \partial\Omega, \end{cases} \tag{4}$$

and equation (2) reads as

$$\begin{cases} -\Delta u(x) + r^2\, f(r \cdot x) u(x) = 0, & x \in \Omega \\ \qquad\qquad\qquad\qquad u(x) = 0, & x \in \partial\Omega. \end{cases} \tag{5}$$

Now consider the function $\psi : I \times H_0^1(\Omega) \to \mathbb{R}$ defined by

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$$\psi(r, u) = \frac{1}{2} \int_\Omega \langle \nabla u, \nabla u \rangle \, dx + r^2 \int_\Omega G(r \cdot x, u(x)) \, dx,$$

where

$$G(x, t) = \int_0^t g(x, \xi) \, d\xi.$$

Then ψ is C^2 and the critical points of ψ_r, $r \in I$, are precisely the weak solutions of equation (4). Moreover, $0 \in H_0^1(\Omega)$ is a critical point for all r and $r^* \in [0, 1]$ is a bifurcation point of ψ in the sense of Section 2.1 if and only if it is a bifurcation point for (1).

The Hessian of ψ_r at the critical point 0 is given by

$$h_r(u) = \int_\Omega \langle \nabla u, \nabla u \rangle \, dx + r^2 \int_\Omega f(r \cdot x) u^2 \, dx, \quad u \in H_0^1(\Omega).$$

Note that the kernel of h_r consists of all classical solutions of the corresponding equation (5). From the compactness of the embedding $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$ it is easily seen that the Riesz representation of h_r with respect to the usual scalar product of $H_0^1(\Omega)$ is a compact perturbation of the identity and hence a Fredholm operator. Moreover, the Morse index $M(h_r)$ is finite and so the function ψ satisfies all assumptions of Section 2.1.

From the implicit function theorem applied to $\nabla \psi_r$, $r \in I$, we note at first that each bifurcation point of (1) is a conjugate point of (2). In order to show the remaining assertion of Theorem 1.1 we now investigate the crossing form $\Gamma(h, r)$ at some crossing $r \in (0, 1)$. We have by definition

$$\Gamma(h, r)[u] = \int_\Omega \frac{d}{ds} |_{s=r} \left(s^2 f(s \cdot x) \right) u^2(x) \, dx, \quad u \in \ker h_r. \tag{6}$$

Let now $0 \neq u \in \ker h_r$ be given. Then u is a classical solution of the equation (5). We define for $s \in \mathbb{R}$ sufficiently small $u_r^s(x) := u(\frac{s}{r} x)$ and set

$$\dot{u}(x) := \frac{d}{ds} |_{s=r} u_r^s(x) = \frac{1}{r} \langle \nabla u(x), x \rangle. \tag{7}$$

Note that

$$-\Delta u_r^s(x) + s^2 f(s \cdot x) u_r^s(x) = 0$$

and, by differentiating this equation with respect to s and evaluating at $s = r$, we get

$$-\Delta \dot{u}(x) + \frac{d}{ds} |_{s=r} \left(s^2 f(s \cdot x) \right) u(x) + r^2 f(r \cdot x) \dot{u}(x) = 0.$$

We multiply by u, integrate over Ω and conclude

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$$0 = -\int_{\Omega} \Delta \dot{u}(x)u(x)\,dx + \int_{\Omega} \frac{d}{ds}\,|_{s=r}\left(s^2 f(s \cdot x)\right)u(x)^2\,dx$$
$$+ r^2 \int_{\Omega} f(r \cdot x)\dot{u}(x)u(x)\,dx.$$

Denoting by $\partial_n u(x) = \langle \nabla u(x), n(x)\rangle$, $x \in \partial\Omega$, the normal derivative to the boundary of Ω, we obtain by applying Green's identity twice

$$0 = -\int_{\Omega} \dot{u}(x)\Delta u(x)\,dx + \int_{\partial\Omega}(\partial_n u)\dot{u}\,dS - \int_{\partial\Omega}(\partial_n\dot{u})u\,dS$$
$$+ \int_{\Omega}\frac{d}{ds}\,|_{s=r}\left(s^2 f(s \cdot x)\right)u(x)^2\,dx + r^2\int_{\Omega}f(r \cdot x)\dot{u}(x)u(x)\,dx.$$

Now, since u solves (5) and vanishes on $\partial\Omega$, we deduce by (6) and (7) that

$$\Gamma(h,r)[u] = -\int_{\partial\Omega}(\partial_n u)\dot{u}\,dS = -\frac{1}{r}\int_{\partial\Omega}\langle\nabla u(x),n(x)\rangle\,\langle\nabla u(x),x\rangle\,dS.$$

Denoting by x^T the tangential component of the vector $x \in \partial\Omega$, we have

$$\langle\nabla u(x),x\rangle = \langle\nabla u(x),n(x)\rangle\langle x,n(x)\rangle + \langle\nabla u(x),x^T\rangle$$

and hence

$$\Gamma(h,r)[u] = -\frac{1}{r}\int_{\partial\Omega}\langle\nabla u(x),n(x)\rangle^2\,\langle x,n(x)\rangle\,dS - \frac{1}{r}\int_{\partial\Omega}\langle\nabla u(x),n(x)\rangle\,\langle\nabla u(x),x^T\rangle\,dS.$$

It is easily seen that

$$\langle\nabla u(x),n(x)\rangle\,\langle\nabla u(x),x^T\rangle = \operatorname{div}(u(x)(\partial_n u(x))x^T), \quad x \in \partial\Omega,$$

and now we finally obtain by using Stokes' theorem on $\partial\Omega$

$$\Gamma(h,r)[u] = -\frac{1}{r}\int_{\partial\Omega}(\partial_n u(x))^2\langle x,n(x)\rangle\,dS \le 0,$$

where we use that $\langle x,n(x)\rangle > 0$ for all $x \in \partial\Omega$.

Moreover, if $\Gamma(h,r)[u] = 0$, then $\partial_n u(x) = 0$, $x \in \partial\Omega$, and we conclude that $u \equiv 0$ on Ω since u solves the boundary value problem (5).

Hence we have shown that $\Gamma(h,r)$ is negative definite. In particular, the crossing r of h is regular and

$$\operatorname{sgn}\Gamma(h,r) = -\dim\ker h_r = -m(r).$$

Now Theorem 1.1 follows from Theorem 2.1, and Corollary 1.2 follows from Theorem 2.2.

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