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Positivity of Continuous-Time Descriptor Systems With Time Delays

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***Abstract*—This paper is concerned with positivity characteristic of continuous-time descriptor systems with time delays. Firstly, a set of necessary and sufficient conditions is presented to check the property. Then, considering a descriptor time-delay system with two assumptions, a new time-delay system is established whose positivity is equivalent to that of the original system. Furthermore, a set of necessary and sufficient conditions is provided to check the positivity of the new system. Finally, a numerical example is given to illustrate the validity of the results obtained.**

***Index Terms*—Continuous-time systems, descriptor systems, positive systems, time-delay systems.**

NOMENCLATURE

$\mathbb{R}^n$	Set of all real vectors of n-dimension.
$\mathbb{R}^{n\times m}$	Set of all $n \times m$ real matrices.
A^T	Transpose of matrix A.
A^{-1}	Inverse of matrix A.
$A \geq (\leq) 0$	All elements of matrix A are nonnegative (nonpositive).
$A > 0$	$A \geq 0$ and at least one element of A is positive.
a_{ij} or $[A]_{ij}$	The ij^{th} element of matrix A.
$\text{rank}(A)$	Rank of matrix A.
$\det(A)$	Determinant of matrix A.

I. INTRODUCTION

Recently, positive systems which have formed a new branch play an important role in systems theory. Positive systems are dynamical systems whose state and output variables always take nonnegative values for any nonnegative initial states and any nonnegative inputs. Such systems are widespread in many areas such as population models, economics, biology and communications [1]–[4]. In these models, the variables may represent population levels, quantities of output of goods, densities of species and congestion window sizes of sources, which have no physical meaning with negative values. Since the nonnegative restriction on variables, the theory of nonnegative matrices is a basic mathematical tool for investigation of linear positive systems and there are many remarkable and straightforward results which are only available for such systems. Over the past several decades, the importance of positive systems has been highlighted by many researchers and a great many of topics such as controllability and reachability [5]–[7], positive stabilization [8]–[10], positive linear observers [11] have been investigated, and lots of theoretical contributions have been reported in the literature. It should be noted that [1] and [7] are two nice and complete monographs for positive standard systems.

However, a great many of systems in economics, chemical process and biology can be modeled more accurately using descriptor systems which are of more extensive applications than standard systems. In addition, the reaction of real world systems to exogenous signals is never instantaneous and always infected by certain time delays. The research on both standard systems with time delay [12] and descriptor systems with time delay [13] is very important. Naturally, positivity of the variables of descriptor systems in areas aforementioned with time delays should be guaranteed. Therefore, the study on positive descriptor systems with time delays is of profound and far reaching practical significance. Positive descriptor systems have only been investigated in recent years. Moreover, due to the singularity of derivative matrix and the positive restriction on variables, much of the developed theory for positive descriptor systems is still not up to a quantitative level. For details of the literature related to positive continuous-time descriptor systems, the reader is referred to [14]–[16], and the references cited therein. On the other hand, some recent developments on positive standard time-delay systems can be found in [17]–[20]. Weak positivity, weak external positivity, reachability and controllability for continuous-time descriptor systems with time delays were investigated in [21], [22]. Different from the work in [21], [22], positivity of the descriptor systems considered in this paper can be guaranteed.

It should be emphasized that the research on positivity of descriptor systems is very limited. Specifically, when descriptor systems involve time delays, the study of positivity property becomes very difficult and thus the corresponding results are very few. In this paper, positivity of continuous-time descriptor time-delay systems will be investigated by imposing limitation on the initial conditions relating to time delays. The main contributions of this paper lie in two aspects. Firstly, a set of novel necessary and sufficient conditions for continuous-time descriptor systems with time delays is presented such that the descriptor systems are positive even in the presence of time

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delays. Secondly, considering a descriptor time-delay system with two assumptions, a new time-delay system is established whose positivity is equivalent to that of the original system. Furthermore, a criterion for checking positivity of the new system is given. The obtained conditions can be directly checked which is convenient for application.

The remainder of this paper is organized as follows. Section II presents some necessary preliminaries, definitions and lemmas. Positivity of continuous-time descriptor time-delay systems is investigated in section III. Section IV is devoted to positivity of a continuous-time descriptor time-delay system with two assumptions. An illustrative example is given to show the effectiveness of the results obtained in section V. Finally, section VI concludes this paper.

II. Preliminaries

In this section, some preliminaries, basic definitions and lemmas are to be introduced which will be essentially used in this paper and helpful to understanding of subsequent results.

Consider the following continuous-time descriptor system with time delays in state and input [21]

$$E\dot{x}(t)=\sum_{j=0}^{q}A_jx(t-h_j)+\sum_{j=0}^{q'}B_ju(t-h_j'),\qquad y(t)=Cx(t), \tag{1}$$

where $x(t)\in\mathbb{R}^n$, $u(t)\in\mathbb{R}^m$, $y(t)\in\mathbb{R}^p$ are the state, input and output vectors, respectively. E, $A_j\in\mathbb{R}^{n\times n}$, $j=0,1,\ldots,q$, B_j, $j=0,1,\ldots,q'$ and C are real matrices of compatible dimensions and $0=h_0<h_j<h_{j+1}=h<\infty$, $j=1,2,\ldots,q-1$, $0=h_0'<h_j'<h_{j+1}'=h'<\infty$, $j=1,2,\ldots,q'-1$ are delays in state and input, respectively. $\mathrm{rank}(E)=r\le n$. When $E=I$, system (1) is called a continuous-time standard time-delay system. The initial condition of system (1) is given as

$$x(t)=\varphi(t),\ t\in[-h,0],$$

where $\varphi(t)$ is a given continuous vector function and $x(0)=\varphi(0)$.

A matrix pair (E,A_0) is said to be regular if there exists a constant scalar s such that $\det(sE-A_0)\ne0$.

Suppose that matrix pair (E,A_0) is regular, then there exist two nonsingular matrices $P,Q\in\mathbb{R}^{n\times n}$ such that

$$PEQ=\begin{bmatrix}I_r&0\\0&N\end{bmatrix},\ PA_0Q=\begin{bmatrix}J&0\\0&I_{n-r}\end{bmatrix},$$

where J and N are matrices in the Jordan canonical form, moreover, N is a nilpotent matrix. This transformation is called the Weierstrass canonical form transformation.

For a matrix $A\in\mathbb{R}^{n\times n}$, its index is defined as the smallest integer k such that $\mathrm{rank}(A^{k+1})=\mathrm{rank}(A^k)$ and denoted by $\mathrm{ind}(A)$. For a regular matrix pair (E,A_0), its index is defined as the nilpotent index of the sub-matrix N in the Weierstrass canonical form and denoted by $\mathrm{ind}(E,A_0)$. In particular, $\mathrm{ind}(E,A_0)=0$ if matrix E is nonsingular, and $\mathrm{ind}(E,A_0)=1$ if $\mathrm{rank}(E)=r<n$ and $N=0$.

Suppose that matrix pair (E,A_0) is regular. Then, according to Theorem 1 in [21], in the sense of differential equation, an explicit solution to the state equation in (1) is given as

$$\begin{aligned}x(t)&=e^{\Phi_0A_0t}\Phi_0Ex(0)+\sum_{j=1}^{q}\int_0^t e^{\Phi_0A_0(t-\tau)}\Phi_0A_jx(\tau-h_j)d\tau\\&+\sum_{j=0}^{q'}\int_0^t e^{\Phi_0A_0(t-\tau)}\Phi_0B_ju(\tau-h_j')d\tau\\&+\sum_{i=1}^{\mu}\sum_{j=1}^{q}\Phi_{-i}A_jx^{(i-1)}(t-h_j)+\sum_{i=1}^{\mu}\sum_{j=0}^{q'}\Phi_{-i}B_ju^{(i-1)}(t-h_j'),\end{aligned} \tag{2}$$

where $x^{(i)}$ and $u^{(i)}$ are the i^{th} time derivatives of vectors x and u, respectively. μ is the index of matrix pair (E,A_0). Φ_i are the fundamental matrices of matrix pair (E,A_0) which are solutions of the following equations [7]

$$E\Phi_i-A_0\Phi_{i-1}=\Phi_iE-\Phi_{i-1}A_0=\begin{cases}I, & \text{for } i=0\\0, & \text{for } i\ne0\end{cases}$$

and $\Phi_i=0$, $i<-\mu$, $E\Phi_{-\mu}=\Phi_{-\mu}E=0$. It is worth noting that Φ_i satisfy the following relations

$$\Phi_iE\Phi_j=\Phi_jE\Phi_i,\ \text{for all } i,j,$$
$$\Phi_iE\Phi_j=0,\ \text{for } i\ge0,\ j<0,$$

and

$$\Phi_iE\Phi_j=\begin{cases}-\Phi_{i+j}, & i<0,\ j<0\\\Phi_{i+j}, & i\ge0,\ j\ge0\\0, & \text{otherwise.}\end{cases}$$

It is assumed that matrix pair (E,A_0) is regular and $\mathrm{ind}(E,A_0)=\mu$. Then, the admissible initial state $x(0)$ of system (1) is given by

$$x(0)=\Phi_0Ew+\sum_{i=1}^{\mu}\sum_{j=1}^{q}\Phi_{-i}A_j\varphi^{(i-1)}(-h_j)+\sum_{i=1}^{\mu}\sum_{j=0}^{q'}\Phi_{-i}B_ju^{(i-1)}(-h_j'), \tag{3}$$

where $w\in\mathbb{R}^n$.

According to definitions of positive descriptor systems and positive standard time-delay systems, the following definition is introduced.

Definition 1: Suppose that matrix pair (E,A_0) is regular and $\mathrm{ind}(E,A_0)=\mu$. Then system (1) is said to be positive if $x(t)\ge0$, $y(t)\ge0$, $\forall t\ge0$ for every admissible initial conditions $\varphi^{(i)}(\tau)\ge0$, $-h\le\tau\le0$, $i=0,1,2,\ldots$, which is a continuously differentiable function, and every continuously differentiable input $u^{(i)}(\tau)\ge0$, $-h'\le\tau\le t$, $i=0,1,2,\ldots$.

Remark 1. It should be noted that in the definition of positivity for a standard system with time delays, in order to guarantee $x(t)\ge0$, it is required that all the initial conditions of states and inputs are positive (see [24]). This can be seen from the solution representation of a standard time-delay system (see [24]). However, for descriptor systems, the situation becomes more complicated. In order to guarantee $x(t)\ge0$, in the Definition 1 above, it is required from the equation (3) that not only the initial conditions for states and inputs but also their associated derivatives, are positive. Similar conditions have been employed in [14].

Definition 2 [7, Ch. 1]: A matrix $A \in \mathbb{R}^{n\times n}$ is called a Metzler matrix if all its off-diagonal entries are nonnegative.

Definition 3 [7, Ch. 1]: For a matrix $A \in \mathbb{R}^{n\times n}$ and $A > 0$, it is called a monomial matrix if its every row and its every column has only one positive element and the remaining elements equal zero.

Lemma 1 [7, Ch. 1]: For a matrix $A \in \mathbb{R}^{n\times n}$, $e^{At} \geq 0$, $\forall t \geq 0$ if and only if A is a Metzler matrix.

Lemma 2 [7, Ch. 1]: Let $A \in \mathbb{R}^{n\times n}$ be nonsingular and $A > 0$. Then $A^{-1} > 0$ if and only if A is a monomial matrix. Moreover, A^{-1} is equal to the transpose matrix A^T in which every nonzero entry is replaced by its inverse.

Lemma 3 [7, Ch. 1]: Let $A \in \mathbb{R}^{n\times n}$ and $A \geq 0$. If $M > 0$ is a monomial matrix, then $\bar{A} = MAM^{-1} \geq 0$.

III. Positivity for Descriptor Time-Delay Systems

This section is concerned with positivity of system (1). The results presented here provide a set of necessary and sufficient conditions to check positivity.

Theorem 1: Suppose that matrix pair (E, A_0) is regular and $\mathrm{ind}(E, A_0) = \mu$, $\Phi_0 E \geq 0$, $\Phi_0 A_0 \geq 0$. Then, system (1) is positive if and only if the following conditions hold:

1) $\Phi_{-i}A_j \geq 0$, $i = 1,2,\ldots,\mu$, $j = 1,2,\ldots,q$, $\Phi_{-i}B_j \geq 0$, $i = 1,2,\ldots,\mu$, $j = 0,1,\ldots,q'$;
2) $\Phi_0 A_j \geq 0$, $j = 1,2,\ldots,q$, $\Phi_0 B_j \geq 0$, $j = 0,1,\ldots,q'$;
3) $C\Phi_0 E \geq 0$, $C\Phi_{-i}A_j \geq 0$, $i = 1,2,\ldots,\mu$, $j = 1,2,\ldots,q$, $C\Phi_{-i}B_j \geq 0$, $i = 1,2,\ldots,\mu$, $j = 0,1,\ldots,q'$.

Proof: (Necessity). Suppose that system (1) is positive. Then it follows that $x(t) \geq 0$, $y(t) \geq 0$, $\forall t \geq 0$ for every admissible initial condition $\varphi^{(i)}(\tau) \geq 0$, $-h \leq \tau \leq 0$, $i = 0,1,2,\ldots$ and every continuously differentiable input $u^{(i)}(\tau) \geq 0$, $-h' \leq \tau \leq t$, $i = 0,1,2,\ldots$.

1) According to Definition 1, we have admissible initial state $x(0) \geq 0$, that is,

$$x(0) = \Phi_0 Ew + \sum_{i=1}^{\mu}\sum_{j=1}^{q}\Phi_{-i}A_j\varphi^{(i-1)}(-h_j) + \sum_{i=1}^{\mu}\sum_{j=0}^{q'}\Phi_{-i}B_j u^{(i-1)}(-h'_j) \geq 0.$$

Take $u \equiv 0$, $\Phi_0 Ew = 0$ and

$$\varphi(\tau) = \begin{cases} \varepsilon e_v, & -h_{j-1} < \tau \leq -h_j,\ j = 1,2,\ldots,q \\ 0, & \text{otherwise,} \end{cases}$$

where $\varepsilon > 0$ is small enough. Then,

$$x(0) = \varepsilon\Phi_{-1}A_j e_v \geq 0,\ j = 1,2,\ldots,q,$$

which implies $\Phi_{-1}A_j \geq 0$, $j = 1,2,\ldots,q$.

Choose $u \equiv 0$, $\Phi_0 Ew = 0$ and

$$\varphi(\tau) = \begin{cases} \varepsilon(\tau + h_j)e_v, & -h_{j-1} < \tau \leq -h_j,\ j = 1,2,\ldots,q \\ 0, & \text{otherwise,} \end{cases}$$

where $\varepsilon > 0$ is small enough. It follows that

$$x(0) = \varepsilon\Phi_{-2}A_j e_v \geq 0,\ j = 1,2,\ldots,q.$$

Then, $\Phi_{-2}A_j \geq 0$, $j = 1,2,\ldots,q$.

Following the same analysis aforementioned, it is concluded that $\Phi_{-i}A_j \geq 0$ for $i = 3,4,\ldots,\mu$, $j = 1,2,\ldots,q$ and $\Phi_{-i}B_j \geq 0$, $j = 0,1,\ldots,q'$, $i = 1,2,\ldots,\mu$.

2) According to the property of the fundamental matrix Φ_0, $\Phi_0 E\Phi_0 = \Phi_0$ [7],

$$\begin{aligned}\Phi_0 Ee^{\Phi_0 A_0 t}\Phi_0 &= \Phi_0 E(I + \Phi_0 A_0 t + \frac{(\Phi_0 A_0 t)^2}{2} + \cdots)\Phi_0 \\ &= \Phi_0 + \Phi_0 A_0 \Phi_0 t + \frac{(\Phi_0 A_0 t)^2\Phi_0}{2} + \cdots \\ &= e^{\Phi_0 A_0 t}\Phi_0.\end{aligned}$$

Choose $\Phi_0 Ew = 0$, $u \equiv 0$ and

$$\varphi(\tau) = \begin{cases} \varepsilon e_v, & -h_j < \tau < -h_{j-1},\ j = 1,2,\ldots,q \\ 0, & \text{otherwise,} \end{cases}$$

where $\varepsilon > 0$ is small enough. Choosing small enough $t > 0$ satisfying $-h_j < t - h_j < -h_{j-1}$, it follows from (2) that

$$\Phi_0 Ex(t) = \int_0^t \varepsilon e^{\Phi_0 A_0(t-\tau)}\Phi_0 A_j e_v d\tau \geq 0,$$

which is due to the fact that $\Phi_0 E \geq 0$, $\Phi_0 E\Phi_{-i} = 0$, $i > 0$. The objective next is to show $\Phi_0 A_j \geq 0$. Suppose that $[\Phi_0 A_j]_{lv} < 0$, $l, v = 1,2,\ldots,n$, then,

$$\begin{aligned}[\Phi_0 Ex(t)]_l &= \int_0^t [(I + \Phi_0 A_0(t-\tau) + o((t-\tau)^2))\Phi_0 A_j\varepsilon e_v]_l d\tau \\ &= \int_0^t \{\varepsilon[\Phi_0 A_j]_{lv} + o(\varepsilon(t-\tau))\}d\tau < 0,\end{aligned}$$

which is a contradiction. Therefore, $\Phi_0 A_j \geq 0$, $j = 1,2,\ldots,q$.

In a similar way, it is easy to check the fact that $\Phi_0 B_j \geq 0$, $j = 1,2,\ldots,q'$.

3) For $y(0) = Cx(0) \geq 0$, along the same line for the analysis of admissible initial condition $x(0) \geq 0$, we can conclude that $C\Phi_0 E \geq 0$, $C\Phi_{-i}A_j \geq 0$, $i = 1,2,\ldots,\mu$, $j = 1,2,\ldots,q$, and $C\Phi_{-i}B_j \geq 0$, $i = 1,2,\ldots,\mu$, $j = 0,1,\ldots,q'$.

(Sufficiency). Suppose that conditions 1)–3) hold. It is easy to check that $x(t) \geq 0$, $t \in [0, h_1]$ for every admissible initial condition $\varphi^{(i)}(\tau) \geq 0$, $-h \leq \tau \leq 0$, $i = 0,1,2,\ldots$ and every input $u^{(i)}(\tau) \geq 0$, $-h' \leq \tau \leq t$, $i = 0,1,2,\ldots$.

Now, we will show $y(t) \geq 0$, $t \in [0, h_1]$. Substituting (2) into output equation in (1) leads to

$$\begin{aligned}y(t) &= Ce^{\Phi_0 A_0 t}\Phi_0 Ex(0) + \sum_{j=1}^{q}\int_0^t Ce^{\Phi_0 A_0(t-\tau)}\Phi_0 A_j x(\tau - h_j)d\tau \\ &+ \sum_{j=0}^{q'}\int_0^t Ce^{\Phi_0 A_0(t-\tau)}\Phi_0 B_j u(\tau - h'_j)d\tau \\ &+ \sum_{i=1}^{\mu}\sum_{j=1}^{q}C\Phi_{-i}A_j x^{(i-1)}(t - h_j) + \sum_{i=1}^{\mu}\sum_{j=0}^{q'}C\Phi_{-i}B_j u^{(i-1)}(t - h'_j).\end{aligned}$$

From $e^{\Phi_0 A_0 t}\Phi_0 = \Phi_0 Ee^{\Phi_0 A_0 t}\Phi_0$, it follows that

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$$Ce^{\Phi_0 A_0 t}\Phi_0 E = C\Phi_0 E e^{\Phi_0 A_0 t}\Phi_0 E \geq 0,$$
$$Ce^{\Phi_0 A_0 (t-\tau)}\Phi_0 A_j = C\Phi_0 E e^{\Phi_0 A_0 (t-\tau)}\Phi_0 A_j \geq 0,$$
$$Ce^{\Phi_0 A_0 (t-\tau)}\Phi_0 B_j = C\Phi_0 E e^{\Phi_0 A_0 (t-\tau)}\Phi_0 B_j \geq 0.$$

Therefore, we have $y(t) \geq 0$, $t \in [0, h_1]$.

Using the step method we can extend the considerations to the intervals $[h_1, 2h_1], [2h_1, 3h_1], \ldots$. ■

It should be pointed out that $\operatorname{ind}(E, A_0) = 1$ if and only if $N = 0$, where N is the sub-matrix in the Weierstrass canonical form. Then the following corollary can be obtained.

Corollary 1: Suppose that matrix pair (E, A_0) is regular, and $\operatorname{ind}(E, A_0) = 1$, $\Phi_0 E \geq 0$. Then, system (1) is positive if and only if the following conditions hold:

1) $\Phi_{-1} A_j \geq 0$, $j = 1, 2, \ldots, q$, $\Phi_{-1} B_j \geq 0$, $j = 0, 1, \ldots, q'$;
2) There exists a scalar $\alpha \geq 0$ such that the matrix $\Phi = \Phi_0 A_0 + \alpha \Phi_0 E - \alpha I$ is a Metzler matrix;
3) $\Phi_0 A_j \geq 0$, $j = 1, 2, \ldots, q$, $\Phi_0 B_j \geq 0$, $j = 0, 1, \ldots, q'$;
4) $C\Phi_0 E \geq 0$, $C\Phi_{-1} A_j \geq 0$, $j = 1, 2, \ldots, q$, $C\Phi_{-1} B_j \geq 0$, $j = 0, 1, \ldots, q'$.

Remark 2: As known, system (1) is said to be impulse-free if $N = 0$. In general, whether a descriptor system is in normal operation largely depends on the impulse. When a descriptor system is concerned, impulse-free is usually desired. However, impulse frequently exists in descriptor systems in really, and thus the study on removing impulse is of great importance. It should be emphasized that only the analysis for positivity of descriptor systems is concerned. The problem of removing impulse for positive descriptor systems with positivity preserved is the subject of ongoing work.

IV. Model Reduction for Special Case

In this section, positivity of system (1) will be investigated under two assumptions. In this case, system (1) can be transformed into a new time-delay system whose positivity is equivalent to that of the original system. Furthermore, a set of necessary and sufficient conditions, which does not involve computation of fundamental matrices, is to be presented.

For the matrix E in system (1), it is known that there exist two nonsingular matrices $\bar{P}, \bar{Q} \in \mathbb{R}^{n\times n}$ such that

$$\bar{P} E \bar{Q} = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}. \tag{4}$$

Then A_j, $j = 0, 1, \ldots, q$, B_j, $j = 0, 1, \ldots, q'$, and C are partitioned accordingly

$$\bar{P} A_j \bar{Q} = \begin{bmatrix} A_{j1} & A_{j2} \\ A_{j3} & A_{j4} \end{bmatrix}, \ j = 0, 1, \ldots, q,$$

$$\bar{P} B_j = \begin{bmatrix} B_{j1} \\ B_{j2} \end{bmatrix}, \ j = 0, 1, \ldots, q',$$

$$C\bar{Q} = \begin{bmatrix} C_1 & C_2 \end{bmatrix}.$$

Define $\bar{Q}^{-1} x(t) = \begin{bmatrix} x_1^T(t) & x_2^T(t) \end{bmatrix}^T$. Suppose that $\det(A_{04}) \neq 0$. The following new system is obtained

$$\begin{aligned} \dot{x}_1(t) &= \bar{A}_{01} x_1(t) + \sum_{j=1}^{q} \bar{A}_{j1} x_1(t - h_j) \\ &\quad + \sum_{j=1}^{q} \bar{A}_{j2} x_2(t - h_j) + \sum_{j=0}^{q'} \bar{B}_{j1} u(t - h'_j), \\ x_2(t) &= -A_{04}^{-1} \sum_{j=0}^{q} A_{j3} x_1(t - h_j) - A_{04}^{-1} \sum_{j=1}^{q} A_{j4} x_2(t - h_j) \\ &\quad - A_{04}^{-1} \sum_{j=0}^{q'} B_{j2} u(t - h'_j), \\ y(t) &= C_1 x_1(t) + C_2 x_2(t), \end{aligned} \tag{5}$$

where

$$\bar{A}_{01} = A_{01} - A_{02} A_{04}^{-1} A_{03}, \ \bar{A}_{j1} = A_{j1} - A_{02} A_{04}^{-1} A_{j3},$$
$$\bar{A}_{j2} = A_{j2} - A_{02} A_{04}^{-1} A_{j4}, \ j = 1, 2, \ldots, q,$$
$$\bar{B}_{j1} = B_{j1} - A_{02} A_{04}^{-1} B_{j2}, \ j = 1, 2, \ldots, q'.$$

The solution of the first equation in (5) is given by

$$\begin{aligned} x_1(t) &= e^{\bar{A}_{01} t} x_1(0) + \int_0^t e^{\bar{A}_{01}(t-\tau)} \{ \sum_{j=1}^{q} \bar{A}_{j1} x_1(\tau - h_j) \\ &\quad + \sum_{j=1}^{q} \bar{A}_{j2} x_2(\tau - h_j) + \sum_{j=0}^{q'} \bar{B}_{j1} u(\tau - h'_j) \} d\tau. \end{aligned}$$

Lemma 4: Let $A \in \mathbb{R}^{n\times n}$ be a Metzler matrix and $M > 0$ be a monomial matrix, then $\bar{A} = MAM^{-1}$ is also a Metzler matrix.

Proof: For some $s \geq 0$, $A + sI \geq 0$ since matrix A is a Metzler matrix. Then it follows from Lemma 3 that $M(A + sI)M^{-1} \geq 0$. It is straightforward to see that MAM^{-1} is also a Metzler matrix. ■

Remark 3: It is important to mention that positivity of system (5) is equivalent to that of system (1) if and only if the matrix $\bar{Q}$ in (4) is a monomial matrix.

It is easy to verify that the transformation of system (1) from the left does not change the solution according to the computation of fundamental matrices [7], [14]. Then the following two assumptions are imposed on system (1).

Assumption 1: There exist two nonsingular matrices $\bar{P}$, $\bar{Q} \in \mathbb{R}^{n\times n}$ such that E has the transformation of the form (4), where $\bar{Q}$ is a monomial matrix.

Assumption 2: $\det(A_{04}) \neq 0$, namely, $\operatorname{ind}(E, A_0) = 1$.

Theorem 2: Suppose that Assumptions 1 and 2 hold. Then system (5) is positive if and only if

1) $\bar{A}_{01}$ is a Metzler matrix;
2) $\bar{A}_{j1} \geq 0, \bar{A}_{j2} \geq 0, j = 1, 2, \ldots, q, \bar{B}_{j1} \geq 0, j = 0, 1, \ldots, q'$;
3) $A_{04}^{-1} A_{j3} \leq 0$, $j = 0, 1, \ldots, q$, $A_{04}^{-1} A_{j4} \leq 0$, $j = 1, 2, \ldots, q$, $A_{04}^{-1} B_{j2} \leq 0$, $j = 0, 1, \ldots, q'$;
4) $C_1 - C_2 A_{04}^{-1} A_{03} \geq 0$, $C_2 A_{04}^{-1} A_{j3} \leq 0$, $C_2 A_{04}^{-1} A_{j4} \leq 0$, $j = 1, 2, \ldots, q$, $C_2 A_{04}^{-1} B_{j2} \leq 0$, $j = 0, 1, \ldots, q$.

Proof: The proof follows exactly the same line of that of Theorem 1. ■

Remark 4: If Assumptions 1 and 2 hold, then by calculation, one can observe that the conditions for positivity in Corollary 1 and in Theorem 2 are equivalent.

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V. EXAMPLE

In this section, a numerical example is provided to demonstrate the effectiveness of the results obtained.

Example: Consider the system (1) with

$$E=\begin{bmatrix}0&0&0\\0&0&2\\1&0&0\end{bmatrix},\ A_0=\begin{bmatrix}0&5&0\\2&0&-8\\-1&0&3\end{bmatrix},\ A_1=\begin{bmatrix}-2&-20&0\\0&10&6\\2&20&4\end{bmatrix},$$

$$B_0=\begin{bmatrix}-1\\2\\1\end{bmatrix},\ B_1=\begin{bmatrix}0\\2\\5\end{bmatrix},\ C=\begin{bmatrix}3&5&2\end{bmatrix}.$$

Direct calculation shows that $\mu=1$ and

$$\Phi_0E=\begin{bmatrix}1&0&0\\0&0&0\\0&0&1\end{bmatrix}\geq 0.$$

By calculating directly, it follows that

1) $\Phi_{-1}B_0=\begin{bmatrix}0\\0.2\\0\end{bmatrix}\geq 0$, $\Phi_{-1}A_1=\begin{bmatrix}0&0&0\\0.2&4&0\\0&0&0\end{bmatrix}\geq 0$, $\Phi_{-1}B_1=0$;

2) $\Phi_0A_0=\begin{bmatrix}-1&0&3\\0&0&0\\1&0&-4\end{bmatrix}$ is a Metzler matrix;

3) $\Phi_0A_1=\begin{bmatrix}2&0&4\\0&0&0\\0&5&3\end{bmatrix}\geq 0$, $\Phi_0B_0=\begin{bmatrix}1\\0\\1\end{bmatrix}\geq 0$, $\Phi_0B_1=\begin{bmatrix}5\\0\\1\end{bmatrix}\geq 0$;

4) $C\Phi_0E=\begin{bmatrix}3&0&2\end{bmatrix}\geq 0$, $C\Phi_{-1}A_1=\begin{bmatrix}1&20&0\end{bmatrix}\geq 0$, $C\Phi_{-1}B_0=1\geq 0$, $C\Phi_{-1}B_1=0$.

Therefore, according to Corollary 1, the continuous-time descriptor time-delay system is positive.

Moreover, there exist two nonsingular matrices

$$\bar{P}=\begin{bmatrix}0&0&1\\0&0.5&0\\1&0&0\end{bmatrix},\ \bar{Q}=\begin{bmatrix}1&0&0\\0&0&0.2\\0&1&0\end{bmatrix}$$

such that

$$\bar{P}E\bar{Q}=\begin{bmatrix}1&0&\vdots&0\\0&1&\vdots&0\\\cdots&\cdots&\cdots&\cdots\\0&0&\vdots&0\end{bmatrix}.$$

Then, A_0, A_1, B_0, B_1 and C are partitioned accordingly

$$\bar{P}A_0\bar{Q}=\begin{bmatrix}-1&3&\vdots&0\\1&-4&\vdots&0\\\cdots&\cdots&\cdots&\cdots\\0&0&\vdots&1\end{bmatrix},\ \bar{P}A_1\bar{Q}=\begin{bmatrix}2&4&\vdots&0\\0&3&\vdots&1\\\cdots&\cdots&\cdots&\cdots\\-2&0&\vdots&-4\end{bmatrix},$$

$$\bar{P}B_0=\begin{bmatrix}1\\1\\\cdots\\-1\end{bmatrix},\ \bar{P}B_1=\begin{bmatrix}5\\1\\\cdots\\0\end{bmatrix},\ C\bar{Q}=\begin{bmatrix}3&2&\vdots&1\end{bmatrix}.$$

Therefore, it is easy to check by calculating that

1) $\bar{A}_{01}=\begin{bmatrix}-1&3\\1&-4\end{bmatrix}$ is a Metzler matrix;

2) $\bar{A}_{11}=\begin{bmatrix}2&4\\0&3\end{bmatrix}\geq 0$, $\bar{A}_{12}=\begin{bmatrix}0\\1\end{bmatrix}\geq 0$, $\bar{B}_{01}=\begin{bmatrix}1\\1\end{bmatrix}\geq 0$, $\bar{B}_{11}=\begin{bmatrix}5\\1\end{bmatrix}\geq 0$;

3) $A_{04}^{-1}A_{03}=0$, $A_{04}^{-1}A_{13}=\begin{bmatrix}-2&0\end{bmatrix}\leq 0$, $A_{04}^{-1}A_{14}=-4\leq 0$, $A_{04}^{-1}B_{02}=-1$, $A_{04}^{-1}B_{12}=0$;

4) $C_1-C_2A_{04}^{-1}A_{03}=\begin{bmatrix}3&2\end{bmatrix}\geq 0$, $C_2A_{04}^{-1}A_{13}=\begin{bmatrix}-2&0\end{bmatrix}\leq 0$, $C_2A_{04}^{-1}A_{14}=-4\leq 0$, $C_2A_{04}^{-1}B_{02}=-1\leq 0$, $C_2A_{04}^{-1}B_{12}=0$.

Hence, according to the analysis aforementioned and Theorem 2, the following new time-delay system

$$\dot{x}_1(t)=\begin{bmatrix}-1&3\\1&-4\end{bmatrix}x_1(t)+\begin{bmatrix}2&4\\0&3\end{bmatrix}x_1(t-h_1)+\begin{bmatrix}0\\1\end{bmatrix}x_2(t-h_1)+\begin{bmatrix}1\\1\end{bmatrix}u(t)+\begin{bmatrix}5\\1\end{bmatrix}u(t-h_1'),$$

$$x_2(t)=\begin{bmatrix}2&0\end{bmatrix}x_1(t-h_1)+4x_2(t-h_1)+u(t),$$

$$y=\begin{bmatrix}3&2\end{bmatrix}x_1(t)+x_2(t),$$

is positive. It is straightforward to see that the positivity of the new system can be checked by the system's matrices directly because there is no computation of fundamental matrices involved. It should be noted that the positivity of the new system is equivalent to that of the original descriptor system. Therefore, Theorem 2 provides a convenient way to check the positivity of such class of descriptor systems.

VI. CONCLUSION

In this paper, positivity of continuous-time descriptor time-delay systems has been investigated. A set of necessary and sufficient conditions has been established to check positivity of continuous-time descriptor time-delay systems. Also, under two assumptions, a new time-delay system is established and a set of necessary and sufficient conditions, which does not involve the computation of fundamental matrices and gives a simple way to check the positivity of the new system, is presented. Finally, the results obtained here have been illustrated by a numerical example. It should be pointed out that many practical systems can be modeled by discrete-time descriptor systems [2], [23]. Further study will focus on extending the results obtained in this paper to discrete-time descriptor systems with time delays. Moreover, some challenging problems, such as removing impulse and stabilization for positive descriptor systems with positivity preserved, are left for future research.

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