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Rigidity for the stable homotopy category
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If two model categories $\mathcal{C}$ and $\mathcal{D}$ are Quillen equivalent, then their homotopy categories $Ho(\mathcal{C})$ and $Ho(\mathcal{D})$ are equivalent. But if $Ho(\mathcal{C})$ and $Ho(\mathcal{D})$ are equivalent categories, can anything be said about the underlying model structures? For the stable homotopy category $Ho(S)$ (i.e. the homotopy category of spectra) there is the following result:

**Rigidity Theorem** [1]
Let $\mathcal{C}$ be a stable model category and
$$\Phi : Ho(S) \longrightarrow Ho(\mathcal{C})$$
be an equivalence of triangulated categories. Then $S$ and $\mathcal{C}$ are Quillen equivalent.

This means that all higher homotopy information in $S$ such as Toda brackets, $K$-theory or mapping spaces is already encoded in the triangulated structure of the stable homotopy category.

In this talk we will construct the most important category-theoretic tool used in the proof of the Rigidity Theorem:

**Universal Property of Spectra** [2]
Let $\mathcal{C}$ be a stable model category, $X \in \mathcal{C}$ a fibrant and cofibrant object. Then there is a Quillen adjoint functor pair
$$X \wedge : S \rightleftarrows \mathcal{C} : Hom(X, -)$$
with $X \wedge S^0 \simeq X$.

In the set-up of the Rigidity Theorem this Quillen pair will provide the desired Quillen equivalence. More precisely, for $X = \Phi(S^0)$, the composition
$$Ho(S) \xrightarrow{L(X \wedge -)} Ho(\mathcal{C}) \xrightarrow{\Phi^{-1}} Ho(S)$$
of the left derived Quillen functor and the given equivalence $\Phi$ is an endofunctor of the stable homotopy category sending the sphere to itself. Hence, as the following talks will show, it must be a self-equivalence of $Ho(S)$. Consequently, the derived functor of the Quillen functor $X \wedge -$ is an equivalence of categories which means that $S$ and $\mathcal{C}$ are Quillen equivalent.

**References**
[1] [2]