Rigidity for the stable homotopy category

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If two model categories $\mathcal{C}$ and $\mathcal{D}$ are Quillen equivalent, then their homotopy categories $Ho(\mathcal{C})$ and $Ho(\mathcal{D})$ are equivalent. But if $Ho(\mathcal{C})$ and $Ho(\mathcal{D})$ are equivalent categories, can anything be said about the underlying model structures? For the stable homotopy category $Ho(\mathcal{S})$ (i.e. the homotopy category of spectra) there is the following result:

Rigidity Theorem [1]
Let $\mathcal{C}$ be a stable model category and

$$\Phi : Ho(\mathcal{S}) \longrightarrow Ho(\mathcal{C})$$

be an equivalence of triangulated categories. Then $\mathcal{S}$ and $\mathcal{C}$ are Quillen equivalent.

This means that all higher homotopy information in $\mathcal{S}$ such as Toda brackets, K-theory or mapping spaces is already encoded in the triangulated structure of the stable homotopy category.

In this talk we will construct the most important category-theoretic tool used in the proof of the Rigidity Theorem:

Universal Property of Spectra [2]
Let $\mathcal{C}$ be a stable model category, $X \in \mathcal{C}$ a fibrant and cofibrant object. Then there is a Quillen adjoint functor pair

$$X \wedge \_ : \mathcal{S} \leftrightarrows \mathcal{C} : \mathrm{Hom}(X, \_)$$

with $X \wedge S^0 \simeq X$.

In the set-up of the Rigidity Theorem this Quillen pair will provide the desired Quillen equivalence. More precisely, for $X = \Phi(S^0)$, the composition

$$Ho(\mathcal{S}) \xrightarrow{L(X \wedge \_)} Ho(\mathcal{C}) \xrightarrow{\Phi^{-1}} Ho(\mathcal{S})$$

of the left derived Quillen functor and the given equivalence Φ is an endofunctor of the stable homotopy category sending the sphere to itself. Hence, as the following talks will show, it must be a self-equivalence of $Ho(\mathcal{S})$. Consequently, the derived functor of the Quillen functor $X \wedge \_$ is an equivalence of categories which means that $\mathcal{S}$ and $\mathcal{C}$ are Quillen equivalent.

References

[1]
[2]