Sliding Mode Observer Based-Controller Design for Nonlinear Systems with Time Varying Delay

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Abstract A class of nonlinear time varying delay systems in the presence of time delay uncertainties is considered in this chapter. The input distribution of the system is nonlinear. Under mild limitations on the uncertainty, an observer is synthesised using sliding mode techniques such that the error dynamics are ultimately uniformly bounded in the presence of uncertainties and time delay. Then, a nonlinear control scheme is developed based on the estimated states, and a set of sufficient conditions is presented such that the corresponding closed-loop systems are uniformly ultimately bounded using the well-known Lyapunov-Razumikhin approach. It is not required that the structure of the uncertainty is known. Finally, a numerical example is presented to demonstrate the approach and simulation results show the effectiveness of the developed paradigm.

1 Introduction

Theoretical studies often assume that all system states are available for control design. This assumption is not valid for many real systems. In order to implement such control schemes, a pertinent way forward is to construct an appropriate dynamic system which is called an observer, to estimate the state variables. Unfortunately, the traditional separation principle for linear control systems usually does not hold for the nonlinear counterpart, which implies that for nonlinear systems, the properties of a state feedback control law may not be achieved when the control law is imple-

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mented with the estimated states ([26]). Therefore, it is necessary to develop and formally analyse control strategies based upon observer state estimates in this case.

During recent decades, several approaches have been developed for observer design, such as the geometric approach, high-gain techniques and error linearisation ([16, 25]). The earliest work can be traced to the well-known Luenberger observer for linear systems. In Luenberger's approach, the observer dynamics are driven by the system input and the difference between the output of the system considered and the output of the observer designed. This output error should become zero in the ideal case. The need to achieve zero output error naturally suggests generating a sliding motion on the subspace for which the output is zero, which has motivated the development of sliding mode observers. Although sliding mode control has been widely studied due to its high robustness, observer design using sliding mode techniques is much less mature especially for nonlinear time delay systems (see survey paper [22]).

Time delay systems widely exist in the practical world. Such systems have been studied extensively (see [21] and the references therein) since Krasovskii extended the Lyapunov theory to time-delay systems and Razumikhin proposed a method to avoid the functional in Lyapunov stability analysis. Although the problem of observer design for time delay systems has been studied for a relatively long period ([3, 4, 24]), results concerning sliding mode observer design for time delay systems are very few and only a very limited literature is available ([2, 13, 18]). Two integral sliding mode control compensators were designed to suppress disturbances for stochastic systems with input and observation delays in [2]. Later, a sliding mode observer was proposed for a class of systems with parametric uncertainty in [18]. However, in both [2] and [18], the considered systems are linear. Higher order sliding mode techniques are employed in [5, 7] where time delay is not considered and the uncertainties are required to satisfy a linear growth condition.

[13] proposed a sliding mode observer for both delayed and non-delayed systems but only matched uncertainty and matched nonlinearities are considered. A sliding mode observer have been designed for nonlinear systems in [23] but time delay is not considered. More recently, [28] proposed a sliding mode observer for nonlinear time delay systems where the focus was on state and parameter estimation. Adaptive techniques were utilised but the control problem was not considered. Moreover, the error dynamics between the system considered and the observer designed in [28] are uniformly ultimately bounded instead of asymptotically stable.

Observer-based control for time delay systems has received much attention (see e.g., [11, 12, 15, 17, 29]). The backstepping approach is employed in [11] where it is required that the nominal system has a triangular structure. By choosing an appropriate Lyapunov-Krasovskii functional, a high gain linear controller is presented in [12]. In both [11] and [12], it is required that the systems considered have a particular structure. An observer-based sliding mode control is proposed in [17] where it is required that the nonlinear term is matched. [15] studied a class of time-delay systems using static and dynamic output feedback but it is required that the uncertainty is matched. Moreover, all the existing results require that the bounds on the mismatched uncertainties satisfy a linear growth condition (i.e. the bounds are linear functions of $\|x\|$ and/or $\|x(t-d)\|$). Since uncertainty bounds may have nonlinear forms in reality, it is pertinent to consider the case when the bounds on the uncertainties are nonlinear. Recently, a sliding mode control scheme has been proposed for a class of nonlinear systems in [29] where the bounds on uncertainties have been extended to nonlinear case but it is required that the input distribution matrix is constant and the designed observer is actually not a sliding mode observer. A finite-time stabilization scheme is proposed using observer based output feedback control in [1] and [20] where the considered systems are linear time-invariant with matched disturbances and delay is not considered.

This paper is focused on the observer-based output feedback control synthesis for a class of nonlinear time varying delay systems with uncertainties. The bounds on the uncertainties are nonlinear and time delayed. The accessible parts of the bounds and the nonlinear terms are fully employed in the observer and controller design to reduce the effects of the uncertainty and nonlinearity. Unlike the work of [28], a robust sliding mode observer is designed for the system by employing the system structure and the uncertainty distribution structure to ensure that the error dynamics are uniformly asymptotically stable. Then, based on the designed observer, a discontinuous control law is proposed to stabilise the system uniformly asymptotically even in the presence of the uncertainties and time delay. The well known Lyapunov-Razumikhin approach is employed to deal with the time delay in the stability analysis of the closed-loop system formed by the system, observer, and the proposed control law. It is not required that either the nonlinear term or the uncertainty acts on the input channel and thus they are mismatched. The input distribution matrix is a nonlinear function matrix. The only limitation on the time varying delay is that it is continuous and bounded. There is no limitation on the rate of change (time derivative) of the delay. Simulation results reflect the effectiveness of the approach proposed.

2 System Description and Preliminaries

Notation: The set of $n \times m$ matrices with elements defined in $\mathbb{R}$ will be denoted by $\mathbb{R}^{n\times m}$. For $A \in \mathbb{R}^{n\times n}$, $A > 0$ denotes a symmetric positive definite matrix, and $\lambda_{\min}(A)$ ($\lambda_{\max}(A)$) denotes the minimum (maximum) eigenvalue of A. The symbol I_n represents the nth order unit matrix and $\mathbb{R}^+$ represents the set of non-negative real numbers. A function $f(x_1, \ldots, x_{n_1}, y_1, \ldots, y_{n_2})$ is also written as $f(x, y)$ where $x = [x_1 \ \ldots \ x_{n_1}]^T \in \mathbb{R}^{n_1}$ and $y = [y_1 \ \ldots \ y_{n_2}]^T \in \mathbb{R}^{n_2}$. The Lipschitz constant or the generalised Lipschitz constant of a function f will be written as $\mathcal{L}_f$. Finally, $\|\cdot\|$ denotes the Euclidean norm or its induced norm.

Definition 1. A continuous function $\alpha : [0, a) \mapsto [0, \infty)$ is called a class K function if it is strictly increasing and $\alpha(0) = 0$ (see, [14]).

Definition 2. A function vector/matrix $f(x_1, x_2)$ ($x_i \in \Omega_i \subset \mathbb{R}^{n_i}$ for $i = 1, 2$) is said to satisfy the generalised Lipschitz condition with respect to (w.r.t.) x_2 in Ω_2 for $x_1 \in \Omega_1$ if there exists a function $\mathcal{L}_f(\cdot)$ defined in $x_1 \in \Omega_1$ such that for any

$x_2, \hat{x}_2 \in \Omega_2$

$$\|f(x_1,x_2) - f(x_1,\hat{x}_2)\| \leq \mathcal{L}_f(x_1)\|x_2 - \hat{x}_2\|, \qquad x_1 \in \Omega_1$$

where the function $\mathcal{L}_f(\cdot)$ is called the generalised Lipschitz constant.
Remark 1. It should be noted that the generalised Lipschitz condition defined in Definition 2 is for partial variables. It can be considered as an extension of the normal Lipschitz condition. The generalised Lipschitz constant $\mathcal{L}_f(\cdot)$ is usually a function instead of a constant. However for simplicity, the symbol $\mathcal{L}_f$ is used instead of $\mathcal{L}_f(\cdot)$ throughout the paper unless it is necessary.

Consider nonlinear systems described by

$$\dot{x} = Ax + G(t,y)u + \Phi(t,x,x_d) + \Psi(t,x,x_d) \tag{1}$$

$$y = Cx, \tag{2}$$

where $x \in \Omega \subset \mathbb{R}^n$, $u, y \in \mathbb{R}^m$ $(m < n)$ are the system states, inputs and outputs respectively; A and C are constant matrices with appropriate dimensions; the nonlinear function matrix $G(\cdot) \in \mathbb{R}^{n\times m}$ is assumed to be known and full rank; the nonlinear term $\Phi(\cdot)$ is known and satisfies generalised Lipschitz condition w.r.t. the variables x and x_d for $t \in \mathbb{R}^+$; the term $\Psi(\cdot)$ includes all the uncertainties. The symbol $x_d := x(t-d)$ represents the delayed state where $d := d(t)$ is the time varying delay which is assumed to be known, continuous, nonnegative and bounded in $\mathbb{R}^+ := \{t \mid t \geq 0\}$, that is

$$\overline{d} := \sup_{t\in\mathbb{R}^+}\{d(t)\} < \infty$$

The initial condition related to the delay is given by

$$x(t) = \phi(t), \qquad t \in [-\overline{d}, 0] \tag{3}$$

where $\phi(\cdot)$ is continuous in $[-\overline{d}, 0]$. It is assumed that all the nonlinear functions are smooth enough for the subsequent analysis, which guarantees that the unforced system has a unique continuous solution.

Firstly, the following Assumptions are imposed on the system (1)–(2).
Assumption 1. The matrix pair (A, C) is observable with C being of full rank.

Under Assumption 1, there exists a matrix L such that the inequality

$$(A - LC)^T P + P(A - LC) < 0 \tag{4}$$

is solvable for $P > 0$.
Remark 2. Assumption 1 is a limitation on the triple (A, E, C). The solvability of the Lyapunov equation (4) with limitation (5) is called the Constrained Lyapunov Problem (CLP). A similar condition has been imposed by many authors (see e.g, [**?**, 8, 13, 17]). Necessary and sufficient conditions for solving the CLP can be found in [8] and [6].
Assumption 2. The uncertainty $\Psi(\cdot)$ satisfies

$$\|\Psi(t,x,x_d)\| \leq \xi_1(t,y)\,\xi_2(t,x,x_d) \tag{5}$$

where $\xi_1(\cdot)$ is a known C^1 function with $\xi_1(t,0)=0$ and $\xi_2(t,x,x_d)$ is a known generalised Lipschitz function w.r.t. x and x_d for $t\in\mathbb{R}^+$.

Remark 3. Assumption 2 is the limitation on the uncertainty $\Psi(\cdot)$. It requires that the bounds on the uncertainty $\Psi(\cdot)$ is known which is to be employed in both observer and controller design to reduce/reject the effects of the uncertainty.

Assumption 3. There exist a continuous function $u^a(\cdot):\mathbb{R}^+\times\mathbb{R}^n\mapsto\mathbb{R}^m$ which is generalised Lipshitz w.r.t x for $t\in\mathbb{R}^+$, and a C^1 function $V_0(t,x):\mathbb{R}^+\times\mathbb{R}^n\mapsto\mathbb{R}^+$ such that

i). $\alpha_1\|x\|^2\leq V_0(t,x)\leq\alpha_2\|x\|^2$

ii). $\frac{\partial V_0}{\partial t}+\left(\frac{\partial V_0}{\partial x}\right)^T\big(Ax+G(\cdot)u^a(t,x)\big)\leq -\alpha_3\|x\|^2$

iii). $\left\|\frac{\partial V_0}{\partial x}\right\|\leq\alpha_4\|x\|$

for positive constants α_i for $i=1,\ldots,4$, where $x:=\text{col}\,(x_1,\cdots,x_n)$ and $\frac{\partial V_0}{\partial x}:=\left[\frac{\partial V_0}{\partial x_1}\cdots\frac{\partial V_0}{\partial x_n}\right]^T$.

Remark 4. Assumption 3 has been used in the converse Lyapunov theorems (see, pages 162-163 in [14]). Due to the complex of nonlinear input channel $G(t,y)$, Assumption 3 is introduced to guarantee that the system $\dot{x}=Ax+G(t,y)u$ is stabilisable using state feedback $u=u^a(t,x)$.

Assumption 4. There exist continuous function matrices $N(\cdot)$ and $M(\cdot)$ where $M(\cdot)$ is nonsingular such that

$$G^T(t,y)\frac{\partial V_0}{\partial x}=M(t,y)y \tag{6}$$

$$\Phi^T(t,x,x_d)\frac{\partial V_0}{\partial x}=N(t,x,x_d)y \tag{7}$$

where $V_0(\cdot)$ is given in Assumption 3 and $N(\cdot)\in\mathbb{R}^{n\times m}$ is generalised Lipschitz w.r.t. x and x_d for $t\in\mathbb{R}^+$.

Remark 5. Assumptions 4 and 3 together can be considered as an extension of CLP for nonlinear case. It is straightforward to see that the equation (8) will be satisfied if (7) holds and $\Phi(\cdot)$ is matched (i.e. $\Phi(\cdot)=G(\cdot)\bar{\Phi}(\cdot)$ for some continuous $\bar{\Phi}(\cdot)$). However, condition (8) does not imply that the nonlinear term $\Phi(\cdot)$ is matched (see, e.g. the simulation example in Section 5).

3 Sliding Mode Observer Design

In this section, a sliding mode observer will be proposed. Without loss of generality, it is assumed that the output matrix C in equation (2) has the form

$$C=\begin{bmatrix}0 & I_m\end{bmatrix} \tag{8}$$

Otherwise there exists a nonsingular transformation matrix T_c such that $CT_c = [0 \quad I_m]$ because C is of full rank. Then, the transformed system will have the output matrix in (9). Therefore, system (1)–(2) can be rewritten as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \underbrace{\begin{bmatrix} G_1(t,y) \\ G_2(t,y) \end{bmatrix}}_{G(\cdot)} u + \underbrace{\begin{bmatrix} \Phi_1(t,x_1,x_2,x_{1d},x_{2d}) \\ \Phi_2(t,x_1,x_2,x_{1d},x_{2d}) \end{bmatrix}}_{\Phi(\cdot)} + \underbrace{\begin{bmatrix} E_1 \\ E_2 \end{bmatrix}}_{E} \Psi(\cdot) \tag{9}$$

$$y = \begin{bmatrix} 0 & I_m \end{bmatrix} x \tag{10}$$

where $x = \text{col}\,(x_1, x_2)$ with $x_1 \in \mathbb{R}^{n-m}$, $A_1 \in \mathbb{R}^{(n-m)\times(n-m)}$ and $E_1 \in \mathbb{R}^{(n-m)\times p}$. The terms $G_1(\cdot)$ and $\Phi_1(\cdot)$ are the first $n-m$ components of $G(\cdot)$ and $\Phi(\cdot)$ respectively. Introduce partitions of P and Q which are conformable with the decomposition in (10)–(11):

$$P = \begin{bmatrix} P_1 & P_2 \\ P_2^T & P_3 \end{bmatrix}, \qquad Q = \begin{bmatrix} Q_1 & Q_2 \\ Q_2^T & Q_3 \end{bmatrix} \tag{11}$$

It is clear from $P > 0$ and $Q > 0$ that $P_1 > 0$, $P_3 > 0$, $Q_1 > 0$ and $Q_3 > 0$. Using the matrix partitions in (12), it follows from (5) and (9) that

$$\begin{aligned} [0 \quad F] = FC &= \begin{bmatrix} E_1^T & E_2^T \end{bmatrix} P = \begin{bmatrix} E_1^T P_1 + E_2^T P_2^T & E_1^T P_2 + E_2^T P_3 \end{bmatrix} \\ &= \begin{bmatrix} \left(P_1(E_1 + P_1^{-1}P_2E_2)\right)^T & E_1^T P_2 + E_2^T P_3 \end{bmatrix} \end{aligned}$$

which implies that

$$P_1(E_1 + P_1^{-1}P_2E_2) = 0 \tag{12}$$

Now, introduce a coordinate transformation:

$$z = \underbrace{\begin{bmatrix} I_{n-m} & P_1^{-1}P_2 \\ 0 & I_m \end{bmatrix}}_{T} x \tag{13}$$

From (13), system (10)–(11) in the new coordinate system z, can be described by

$$\begin{aligned} \dot{z}_1 &= (A_1 + P_1^{-1}P_2A_3)z_1 + \left(A_2 - A_1P_1^{-1}P_2 + P_1^{-1}P_2(A_4 - A_3P_1^{-1}P_2)\right)z_2 + [I_{n-m} \quad P_1^{-1}P_2]G(\cdot)u \\ &\quad + [I_{n-m} \quad P_1^{-1}P_2]\Phi(t, T^{-1}z, T^{-1}z_d) \end{aligned} \tag{14}$$

$$\dot{z}_2 = A_3z_1 + \left(A_4 - A_3P_1^{-1}P_2\right) z_2 + G_2(t,y)u + \Phi_2(t, T^{-1}z, T^{-1}z_d) + E_2\Psi(t, T^{-1}z, T^{-1}z_d) \tag{15}$$

$$y = z_2 \tag{16}$$

where $z = \text{col}\,(z_1, z_2)$ with $z_1 \in \mathbb{R}^{n-m}$. From (3), the initial condition related to the delay is given by

$$z(t) = T\phi(t) := \rho_1(t), \qquad t \in [-\bar{d}, 0] \tag{17}$$

For system (15)–(17), consider a dynamical system

$$\dot{\hat{z}}_1 = (A_1 + P_1^{-1}P_2A_3)\hat{z}_1 + \big(A_2 - A_1P_1^{-1}P_2 + P_1^{-1}P_2(A_4 - A_3P_1^{-1}P_2)\big)y$$
$$+[I_{n-m} \quad P_1^{-1}P_2]G(\cdot)u + [I_{n-m} \quad P_1^{-1}P_2]\Phi(t, T^{-1}\hat{z}_y, T^{-1}\hat{z}_{yd}) \tag{18}$$
$$\dot{\hat{z}}_2 = A_3\hat{z}_1 + \big(A_4 - A_3P_1^{-1}P_2\big)\hat{z}_2 + D(y - \hat{z}_2) + G_2(t, y)u + \Phi_2(t, T^{-1}\hat{z}_y, T^{-1}\hat{z}_{yd}) + \nu(\cdot) \tag{19}$$

where

$$\hat{z}_y := \begin{bmatrix} \hat{z}_1 \\ y \end{bmatrix}, \qquad \hat{z}_{yd} := \begin{bmatrix} \hat{z}_{1d} \\ y_d \end{bmatrix} \tag{20}$$

the matrix D is chosen such that $A_4 - A_3P_1^{-1}P_2 - D$ is Hurwitz stable, and the term $\nu(\cdot)$ is defined by

$$\nu(\cdot) = \big(A_4 - A_3P_1^{-1}P_2 - D\big)(y - \hat{z}_2) + \Big(\|E_2\|\xi_1(t, y)\xi_2(t, T^{-1}\hat{z}_y, T^{-1}\hat{z}_{yd})$$
$$+k(\cdot)\Big)\mathrm{sgn}(y - \hat{z}_2) \tag{21}$$

where sgn denotes the usual sign vector function and $k(\cdot)$ is to be determined later. The initial condition related to the delay is given by

$$\hat{z}(t) = \rho_2(t), \qquad t \in [-\bar{d}, 0] \tag{22}$$

where $\rho_2(\cdot)$ can be chosen as any continuous function such that

$$\|\rho_1(t) - \rho_2(t)\| \leq b_0 \tag{23}$$

for some constant b_0, where $\rho_1(\cdot)$ is given in (18).

Let $e_{z_1} = z_1 - \hat{z}_1$ and $e_{z_2} = z_2 - \hat{z}_2$. Then by comparing (15)–(17) with (19)–(20), the error dynamical equation is described by

$$\dot{e}_{z_1} = (A_1 + P_1^{-1}P_2A_3)e_{z_1} + [I_{n-m} \ \ P_1^{-1}P_2]\delta(\Phi) \tag{24}$$
$$\dot{e}_{z_2} = A_3e_{z_1} + \big(A_4 - A_3P_1^{-1}P_2 - D\big)e_{z_2} + \delta(\Phi_2) + E_2\Psi(t, T^{-1}z, T^{-1}z_d) - \nu(\cdot) \tag{25}$$

where $\nu(\cdot)$ is defined by (22), and the functional operator $\delta(\cdot)$ is defined by

$$\delta(\Theta) := \Theta(t, T^{-1}z, T^{-1}z_d) - \Theta(t, T^{-1}\hat{z}_y, T^{-1}\hat{z}_{yd}) \tag{26}$$

where $\Theta(\cdot)$ is a function of z, z_d and t, T is defined in (14), and $\hat{z}_y$ and $\hat{z}_{yd}$ are defined by (21).

For system (25)–(26), consider a sliding surface

$$S := \{(e_{z_1}, e_{z_2}) \mid e_{z_2} = 0\} \tag{27}$$

In order to study the stability of the associate sliding motion, it is necessary to prove the following result at first.

Lemma 1 Assume that the function $\Theta(t, z, z_d)$ is Lipschitz w.r.t. z and z_d in their definition domain, and the operator $\delta(\cdot)$ is defined in (27). Then

$$\|\delta(\Theta)\| \leq \|T^{-1}\|\mathcal{L}_{\Theta}(\|e_{z_1}\| + \|e_{z_1 d}\|) \tag{28}$$
$$\|\delta(\Theta)\|^2 \leq \|T^{-1}\|^2\mathcal{L}_{\Theta}^2 \left(\|e_{z_1}\|^2 + \|e_{z_1 d}\|^2\right) \tag{29}$$

where T is defined in (14), $e_{z_1} := z_1 - \hat{z}_1$ and $e_{z_1 d} := \hat{z}_{1d} - \hat{z}_{1d}$.
Proof: Since $\Theta(t, z, z_d)$ is Lipschitz w.r.t. the variables z and z_d in their definition domain, it follows from the structure of T in (14) that

$$\begin{aligned}
\|\delta(\Theta)\| &= \left\|\Theta(t, T^{-1}z, T^{-1}z_d) - \Theta(t, T^{-1}\hat{z}_y, T^{-1}\hat{z}_{yd})\right\| \\
&\leq \mathcal{L}_{\Theta} \left\| diag\{T^{-1}, T^{-1}\} \begin{bmatrix} z - \hat{z}_y \\ z_d - \hat{z}_{yd} \end{bmatrix} \right\| \\
&\leq \mathcal{L}_{\Theta} \|T^{-1}\| \left\| \underbrace{\begin{bmatrix} z_1 - \hat{z}_1 \\ 0 \\ z_{1d} - \hat{z}_{1d} \\ 0 \end{bmatrix}}_{Y} \right\|
\end{aligned} \tag{30}$$

It is clear that

$$\|Y\| \leq \|z_1 - \hat{z}_1\| + \|z_{1d} - \hat{z}_{1d}\| = \|e_{z_1}\| + \|e_{z_1 d}\|$$
$$\|Y\|^2 = \|z_1 - \hat{z}_1\|^2 + \|z_{1d} - \hat{z}_{1d}\|^2 = \|e_{z_1}\|^2 + \|e_{z_1 d}\|^2$$

Hence the conclusion follows. ∇

Note the inequality (30) cannot be obtained directly from (29). The following result is ready to be presented:

Theorem 1. Under Assumptions 1 and 2, the sliding motion of system (25)–(26) associated with the sliding surface (28) is uniformly asymptotically stable if there exists a constant $q_0 > 1$ such that

$$q := \lambda_{\min}(Q_1) - 2\|[P_1 \;\; P_2]\| \, \|T^{-1}\|\mathcal{L}_{\Phi}\left(1 + \sqrt{q_0 \frac{\lambda_{\max}(P_1)}{\lambda_{\min}(P_1)}}\,\right) > 0$$

where P_1, P_2 and Q_1 are given in (12).
Proof: From the definition of the sliding surface in (28), it is clear that system (25) is the sliding mode dynamics which govern the sliding motion, and thus it is only necessary to prove that (25) is uniformly asymptotically stable.

Applying matrix block multiplication to equation (4), it follows from the partition (12) that

$$A_1^T P_1 + A_3^T P_2^T + P_1 A_1 + P_2 A_3 = -Q_1$$

This implies

$$(A_1 + P_1^{-1} P_2 A_3)^T P_1 + P_1(A_1 + P_1^{-1} P_2 A_3) = -Q_1 \tag{31}$$

From (29) in Lemma 1,

$$\|\delta(\Phi)\| \leq \|T^{-1}\| \, \mathcal{L}_{\Phi} \left(\|e_{z_1}(t)\| + \|e_{z_1 d}(t)\|\right) \tag{32}$$

For system (25), consider the Lyapunov function candidate $V_e = e_{z_1}^T P_1 e_{z_1}$. If there is a constant $q_0 > 1$ such that $V_e(e_{z_1 d}) \le q_0 V_e(e_{z_1})$, then,

$$\lambda_{\min}(P_1)\|e_{z_1 d}\| \le e_{z_1 d} P_1 e_{z_1 d} \le q_0 e_{z_1}^T P_1 e_{z_1} \le \lambda_{\max}(P_1)\|e_{z_1}\|$$

and thus

$$\|e_{z_1 d}\| \le \sqrt{q_0 \tfrac{\lambda_{\max}(P_1)}{\lambda_{\min}(P_1)}}\; \|e_{z_1}\| \tag{33}$$

Using (32), (33) and (34), the derivative of V_e along the trajectories of the system (25) is described by

$$\begin{aligned}
\dot V_e &= -e_{z_1}^T Q_1 e_{z_1} + 2e_{z_1}^T P_1 [I_{n-m} \;\; P_1^{-1} P_2]\delta(\Phi)\\
&\le \lambda_{\min}(Q_1)\|e_{z_1}\|^2 + 2\|e_{z_1}\|\,\|[P_1 \;\; P_2]\|\,\|T^{-1}\|\,\mathcal{L}_{\Phi}\left(\|e_{z_1}\| + \|e_{z_1 d}\|\right)\\
&\le \lambda_{\min}(Q_1)\|e_{z_1}\|^2 + 2\|e_{z_1}\|\,\|[P_1 \;\; P_2]\|\,\|T^{-1}\|\,\mathcal{L}_{\Phi}\left(\|e_{z_1}\| + \sqrt{q_0 \tfrac{\lambda_{\max}(P_1)}{\lambda_{\min}(P_1)}}\;\|e_{z_1}\|\right)\\
&= -q\|e_{z_1}\|^2
\end{aligned} \tag{34}$$

Hence the conclusion follows from $q > 0$. ∇

Remark 6. Theorem 1 has shown that $e_{z_1}(t)$ is uniform asymptotic stable. From (35) and the definition of V_e, it follows that

$$\dot V_e \le -q\|e_{z_1}\|^2 \le -\frac{q}{\lambda_{\max}(P_1)} V_e \quad \Longrightarrow \quad V_e \le V_{e_0}\exp\{-\frac{q}{\lambda_{\max}(P_1)}t\}$$

and thus

$$\|e_{z_1}(t)\| \le \sqrt{\frac{V_{e_0}}{\lambda_{\min}(P_1)}}\;\exp\{-\frac{q}{2\lambda_{\max}(P_1)}t\} =: b_1(t), \qquad t \ge 0 \tag{35}$$

From (24) and (36),

$$\|e_{z_1 d}(t)\| \le \max\{b_1(t), b_0\} =: b_2(t) \tag{36}$$

where b_0 is given in (24).

Theorem 2. Under Assumptions 1 and 2, the error dynamical system (25)–(26) is driven to the sliding surface (28) in finite time and remains on it thereafter if $k(\cdot)$ is chosen as

$$k = \|A_3\| b_1(t) + \left(\mathcal{L}_{\Phi_2} + \|E_2\|\,\xi_1(t,y)\mathcal{L}_{\xi_2}\right)\|T^{-1}\|\left(b_1(t) + b_2(t)\right) + \eta \tag{37}$$

where the functions $b_1(\cdot)$ and $b_2(\cdot)$ are determined by (36) and (37) respectively, $\xi_1(\cdot)$ and $\xi_2(\cdot)$ are defined in (6), and η is any positive constant.

Proof: From equation (26)

$$e_{z_2}^T \dot e_{z_2} = e_{z_2}^T (A_4 - A_3 P_1^{-1} P_2 - D) e_{z_2} + e_{z_2}^T\left(A_3 e_{z_1} + \delta(\Phi_2) + E_2 \Psi(t, T^{-1}z, T^{-1}z_d)\right) - e_{z_2}^T \nu(\cdot) \tag{38}$$

It is clear that for any vector e_{z_2},

$$e_{z_2}^T \mathrm{sgn}(y - \hat{z}_2) = e_{z_2}^T \mathrm{sgn}(e_{z_2}) \geq \|e_{z_2}\| \tag{39}$$

Then, by applying (6), (40) and (22) to (39),

$$\begin{aligned} e_{z_2}^T(t)\dot{e}_{z_2}(t) \leq & \|e_{z_2}\| \Big(\|A_3\| \, \|e_{z_1}\| + \|\delta(\Phi_2)\| + \|E_2\| \xi_1(t,y) \xi_2(t, T^{-1}z, T^{-1}z_d) \Big) \\ & - \Big(\|E_2\| \xi_1(t,y) \xi_2(t, T^{-1}\hat{z}_y, T^{-1}\hat{z}_{yd}) + k(\cdot) \Big) e_{z_2}^T \mathrm{sgn}(y - \hat{z}_2) \\ \leq & \|A_3\| \, \|e_{z_1}\| \, \|e_{z_2}\| + \Big(\|E_2\| \xi_1(t,y) \delta(\xi_2) + \delta(\Phi_2) \Big) \|e_{z_2}\| - k(\cdot) \|e_{z_2}\| \end{aligned} \tag{40}$$

where $\delta(\cdot)$ is a functional operator defined in (27). From (29) in Lemma 1, (36) and (37),

$$\|\delta(\xi_2)\| \leq \mathcal{L}_{\xi_2} \|T^{-1}\| (b_1(t) + b_2(t)) \tag{41}$$

$$\|\delta(\Phi_2)\| \leq \mathcal{L}_{\Phi_2} \|T^{-1}\| (b_1(t) + b_2(t)) \tag{42}$$

Applying (38), (42) and (43) to (41) yields

$$\begin{aligned} e_{z_2}^T \dot{e}_{z_2} \leq & b_1(t) \|A_3\| \|e_{z_2}\| + (\mathcal{L}_{\Phi_2} + \|E_2\| \, \xi_1(t,y) \, \mathcal{L}_{\xi_2}) \, \|T^{-1}\| (b_1(t) + b_2(t)) \|e_{z_2}\| - k(\cdot) \|e_{z_2}\| \\ = & -\eta \|e_{z_2}\| \end{aligned} \tag{43}$$

which shows that the reachability condition is satisfied. Hence the conclusion follows. ∇

By combining Theorem 1 with Theorem 2, it follows from sliding mode theory that the system (25)–(26) is uniformly asymptotically stable. Therefore, (19)–(20) is a sliding mode observer for the system (15)–(17). Clearly , the formula

$$\hat{x} = T^{-1} \hat{z}_y \tag{44}$$

provides an estimate for the states x of the dynamical system (1), where T is defined in (14) and $\hat{z}_y$ is defined in (21) with $\hat{z}_1$ given by (19)–(20). In fact, from $z = Tx$,

$$\|x - \hat{x}\| = \left\| T^{-1} z - T^{-1} \hat{z}_y \right\| \leq \|T^{-1}\| \, \|e_{z_1}\| \tag{45}$$

and thus $\hat{x}$ defined in (45) gives an estimate for the state x. From (45), it is clear to see that the operator $\delta(\cdot)$ defined in (27) can be expressed by

$$\delta(\Theta) = \Theta(t, x, x_d) - \Theta(t, \hat{x}, \hat{x}_d) \tag{46}$$

and both (29) and (30) hold.

4 Stabilising Controller Synthesis

In this section, it is assumed that the observer (19)–(20) has been well designed. A discontinuous control law based on the associated state estimates will be proposed to stabilise the system (1)–(2) uniformly asymptotically.

For system (1)–(2), consider the control law

$$u := u^a(t,\hat{x}) + u^b(t,y,\hat{x},\hat{x}_d) + u^c(t,y,\hat{x},\hat{x}_d) \tag{47}$$

where $u^a(\cdot)$ satisfies Assumption 3, and $u^b(\cdot)$ and $u^c(\cdot)$ are, respectively, defined by

$$u^b(\cdot) := \begin{cases} -\frac{\varepsilon_1 M^{-T}(t,y)y}{2\|y\|^2}\,\alpha_4^2\,\|E\|^2\,\xi_1^2(t,y)\,\xi_2^2(t,\hat{x},\hat{x}_d), & y \neq 0 \\ 0 & y = 0 \end{cases} \tag{48}$$

$$u^c(\cdot) := \begin{cases} -M^{-T}(t,y)y\left(\frac{\varepsilon_2}{2} + \frac{\|y^T N(t,\hat{x},\hat{x}_d)\|}{\|y\|^2}\right), & y \neq 0 \\ 0 & y = 0 \end{cases} \tag{49}$$

where $\hat{x}$ is given by (45), ε_1 and ε_2 are positive constants, and $M(\cdot)$ satisfies (7).

Remark 7. Consider the control (48). From the condition that $\xi_1(\cdot)$ is of class C^1 with $\xi_1(t,0)=0$ for $t \in \mathbb{R}^+$, it is straightforward to see that $\lim_{y\to 0} u^b(t,y,\hat{x},\hat{x}_d) = 0$ which implies that the control component $u^b(t,y,\hat{x},\hat{x}_d)$ defined in (49) is continuous. The value of the control component $u^c(t,y,\hat{x},\hat{x}_d) = 0$ at $y=0$ has been pre-specified in (50) according to the equivalent control method. The extension of this method to time delay systems has been justified in [19].

Theorem 3. Under Assumptions 1-4, system (1)–(2) is stabilised uniformly asymptotically by the controller (48) if the matrix $W(\cdot) := [w_{ij}(\cdot)]_{3\times 3}$ is positive definite with $\gamma_0 := \inf\{\lambda_{\min}(W(\cdot))\} > 0$ where

$$\begin{aligned}
w_{11} &:= \alpha_3 - \frac{1}{2\varepsilon_1} - \alpha_2\gamma \\
w_{22} &:= \lambda_{\min}(Q_1) - \frac{\mathcal{L}_N^2}{2\varepsilon_2}\|T^{-1}\|^2 - 2\mathcal{L}_\Phi\,\|[P_1\ \ P_2]\|\,\|T^{-1}\| - \lambda_{\max}(P_1)\gamma \\
w_{33} &:= \lambda_{\min}(P_1) - \frac{\mathcal{L}_N^2}{2\varepsilon_2}\|T^{-1}\|^2 \\
w_{12} &= w_{21} := -\frac{1}{2}\alpha_4\|T^{-1}\|\left(\|G(t,y)\|\mathcal{L}_{u^a} + \mathcal{L}_{\xi_2}\,\xi_1(\cdot)\|E\|\right) \\
w_{23} &= w_{32} := -\mathcal{L}_\Phi\,\|[P_1\ \ P_2]\|\,\|T^{-1}\| \\
w_{13} &= w_{31} := -\frac{1}{2}\,\alpha_4\,\mathcal{L}_{\xi_2}\,\xi_1(\cdot)\|E\|\,\|T^{-1}\|
\end{aligned}$$

for some $\varepsilon_1 > 0$, $\varepsilon_2 > 0$ and $\gamma > 1$, where P_1, P_2 and Q_1 are given in (12).

Proof: By applying the control law in (48) to system (1)–(2), the closed loop system is described by system (19), (20) and the system

$$\dot{x} = Ax + G(t,y)(u^a(t,\hat{x}) + u^b(t,y,\hat{x},\hat{x}_d) + u^c(t,y,\hat{x},\hat{x}_d)) + \Phi(t,x,x_d) + E\Psi(t,x,x_d) \tag{50}$$

where $\hat{x}$ is determined by (45). Let $e_{z_1} = z_1 - \hat{z}_1$ and $e_{z_2} = z_2 - \hat{z}_2$. Based on the analysis in Section 3, the closed-loop system in $\mathrm{col}(x, e_{z_1}, e_{z_2})$ coordinates can be described by (51), (25) and (26). For the closed-loop system, consider the sliding surface

$$S := \{(x, e_{z_1}, e_{z_2}) \mid e_{z_2} = 0\} \tag{51}$$

It follows that the sliding mode dynamics are described by (25) and (51). Theorem 2 has provided a reachability condition. It remains to prove that the sliding mode dynamics (25) and (51) which govern the sliding motion, are uniformly asymptotically stable.

consider the Lyapunov candidate function

$$V(t,x,e_{z_1},e_{z_2}) = V_0(t,x) + e_{z_1}^T P_1 e_{z_1} \tag{52}$$

where $V_0(\cdot)$ satisfies Assumption 3, and P_1 is given in (12). Then, the time derivative of $V(\cdot)$ along the trajectories of the closed-loop system is described by

$$\begin{aligned}\dot{V} =& \tfrac{\partial V_0}{\partial t} + \Big(\tfrac{\partial V_0}{\partial x}\Big)^T \big(Ax + G(t,y)u^a(t,\hat{x})\big) + \Big(\tfrac{\partial V_0}{\partial x}\Big)^T G(t,y)u^b(\cdot) + \Big(\tfrac{\partial V_0}{\partial x}\Big)^T E\Psi(t,x,x_d)\\ &+ \Big(\tfrac{\partial V_0}{\partial x}\Big)^T G(t,y)u^c(\cdot) + \Big(\tfrac{\partial V_0}{\partial x}\Big)^T \Phi(t,x,x_d) + \dot{e}_{z_1}^T(t)P_1 e_{z_1}(t) + e_{z_1}^T P_1 \dot{e}_{z_1}\end{aligned} \tag{53}$$

From Assumption 3 and equation (7),

$$\begin{aligned}&\tfrac{\partial V_0}{\partial t} + \Big(\tfrac{\partial V_0}{\partial x}\Big)^T \big(Ax + G(t,y)u^a(t,\hat{x})\big)\\ =& \tfrac{\partial V_0}{\partial t} + \Big(\tfrac{\partial V_0}{\partial x}\Big)^T \big(Ax + G(t,y)u^a(t,x)\big) + \Big(\tfrac{\partial V_0}{\partial x}\Big)^T G(t,y)\Big(u^a(t,\hat{x}) - u^a(t,x)\Big)\\ \le& -\alpha_3\|x\|^2 + \alpha_4\,\|x\|\,\mathcal{L}_{u^a}\,\|G(t,y)\|\,\|x-\hat{x}\|\\ \le& -\alpha_3\|x\|^2 + \alpha_4\mathcal{L}_{u^a}\|G(t,y)\|\,\|T^{-1}\|\,\|e_{z_1}\|\,\|x\|\end{aligned} \tag{54}$$

where (46) is employed above. From (7), Assumptions 2 and 3, and Young's inequality $ab \le \frac{1}{2\varepsilon}a^2 + \frac{\varepsilon}{2}b^2$ for any $\varepsilon > 0$,

$$\begin{aligned}&\Big(\tfrac{\partial V_0}{\partial x}\Big)^T G(t,y)u^b(\cdot) + \Big(\tfrac{\partial V_0}{\partial x}\Big)^T E\Psi(t,x,x_d)\\ \le& y^T M^T(t,y)u^b(\cdot) + \alpha_4\,\|x\|\,\|E\|\,\xi_1(t,y)\,\xi_2(t,x,x_d)\\ =& y^T M^T(t,y)u^b(\cdot) + \alpha_4\|x\|\,\|E\|\xi_1(t,y)\xi_2(t,\hat{x},\hat{x}_d) + \alpha_4\|x\|\,\|E\|\xi_1(t,y)\delta(\xi_2)\\ \le& y^T M^T(t,y)u^b(\cdot) + \frac{\varepsilon_1}{2}\,\alpha_4^2\,\|E\|^2\xi_1^2(t,y)\xi_2^2(t,\hat{x},\hat{x}_d) + \frac{1}{2\varepsilon_1}\|x\|^2 + \alpha_4\|x\|\,\|E\|\xi_1(t,y)\delta(\xi_2)\end{aligned} \tag{55}$$

where the operator $\delta(\cdot)$ is defined in (47). From the definition of $u^b(\cdot)$ in (49), if $y \neq 0$, then,

$$
\begin{aligned}
& y^T M^T(t,y) u^b(\cdot) + \frac{\varepsilon_1}{2}\,\alpha_4^2\,\|E\|^2 \xi_1^2(t,y)\xi_2^2(t,\hat{x},\hat{x}_d) \\
&= -y^T M^T(t,y)\frac{\varepsilon_1 M^{-T}(\cdot) y}{2\|y\|^2}\|E\|^2\,\alpha_4^2\,\xi_1^2(t,y)\xi_2^2(t,\hat{x},\hat{x}_d) + \frac{\varepsilon_1}{2}\,\alpha_4^2\,\|E\|^2\xi_1^2(t,y)\xi_2^2(t,\hat{x},\hat{x}_d) \\
&= 0
\end{aligned} \tag{56}
$$

and if $y = 0$, it is easy to see from $\xi_1(t,0) = 0$ that (57) holds. Then, From (29)

$$
\|x\|\,\|E\|\xi_1(t,y)\delta(\xi_2) \le \xi_1(t,y)\|E\|\,\|T^{-1}\|\mathcal{L}_{\xi_2}\left(\|e_{z_1}\| + \|e_{z_1 d}\|\right)\|x\| \tag{57}
$$

Substituting (57) and (58) into (56) yields

$$
\left(\tfrac{\partial V_0}{\partial x}\right)^T G(t,y)u^b(\cdot) + \left(\tfrac{\partial V_0}{\partial x}\right)^T E\Psi(t,x,x_d) \le \tfrac{\|x\|^2}{2\varepsilon_1} + \alpha_4\,\xi_1(t,y)\|E\|\,\|T^{-1}\|\mathcal{L}_{\xi_2}\left(\|e_{z_1}\| + \|e_{z_1 d}\|\right)\|x\| \tag{58}
$$

From (8), (46) and Young's inequality, it follows that for any $\varepsilon_2 > 0$

$$
\begin{aligned}
\left(\tfrac{\partial V_0}{\partial x}\right)^T \Phi(t,x,x_d) &= y^T\left(N(t,x,x_d) - N(t,\hat{x},\hat{x}_d)\right) + y^T N(t,\hat{x},\hat{x}_d) \\
&\le \|y\|\,\|\delta(N)\| + \|y^T N(t,\hat{x},\hat{x}_d)\| \\
&\le \frac{1}{2\varepsilon_2}\|\delta(N)\|^2 + \frac{\varepsilon_2}{2}\|y\|^2 + \|y^T N(t,\hat{x},\hat{x}_d)\| \\
&\le \frac{1}{2\varepsilon_2}\|T^{-1}\|^2\mathcal{L}_N^2\left(\|e_{z_1}\|^2 + \|e_{z_1 d}\|^2\right) + \frac{\varepsilon_2}{2}\|y\|^2 + \|y^T N(t,\hat{x},\hat{x}_d)\|
\end{aligned} \tag{59}
$$

where (30) in Lemma 1 is employed above. From (7), it follows that

i) if $y = 0$, then

$$
\left(\frac{\partial V_0}{\partial x}\right)^T G(t,y)u^c(\cdot) + \left(\frac{\partial V_0}{\partial x}\right)^T \Phi(t,x,x_d) = y^T M^T(\cdot)u^c(\cdot) + y^T N(t,x,x_d) = 0 \tag{60}
$$

ii) if $y \neq 0$, then From (60), the definition of $u^c(\cdot)$ in (50), and by the similar reasoning as for (59), it follows that

$$
\left(\frac{\partial V_0}{\partial x}\right)^T G(t,y)u^c(\cdot) + \left(\frac{\partial V_0}{\partial x}\right)^T \Phi(t,x,x_d) \le \frac{1}{2\varepsilon_2}\|T^{-1}\|^2\mathcal{L}_N^2\left(\|e_{z_1}\|^2 + \|e_{z_1 d}\|^2\right)
$$

The analysis in i) and ii) implies that the inequality

$$
\left(\tfrac{\partial V_0}{\partial x}\right)^T G(t,y)u^c(\cdot) + \left(\tfrac{\partial V_0}{\partial x}\right)^T \Phi(t,x,x_d) \le \tfrac{1}{2\varepsilon_2}\|T^{-1}\|^2\mathcal{L}_N^2\left(\|e_{z_1}\|^2 + \|e_{z_1 d}\|^2\right) \tag{61}
$$

holds. From (25), (32) and (33),

$$
\begin{aligned}
& \dot{e}_{z_1}^T(t)P_1 e_{z_1}(t) + e_{z_1}^T(t)P_1\dot{e}_{z_1}(t) \\
&= -e_{z_1}^T Q e_{z_1} + 2e_{z_1}^T P_1\left[I_{n-m}\;\; P_1^{-1}P_2\right]\sigma(\Phi) \\
&\le -\lambda_{\min}(Q_1)\|e_{z_1}\|^2 + 2\|e_{z_1}\|\,\|[P_1\;\; P_2]\|\,\|T^{-1}\|\mathcal{L}_{\Phi}(\|e_{z_1}\| + \|e_{z_1 d}\|)
\end{aligned} \tag{62}
$$

Substituting (55), (59), (62) and (63) into (54) yields

$$
\begin{aligned}
\dot{V} \leq & -\alpha_3\|x\|^2 + \alpha_4\,\mathcal{L}_{u^a}\|G(t,y)\|\,\|T^{-1}\|\,\|e_{z_1}\|\,\|x\| + \frac{\|x\|^2}{2\varepsilon_1} \\
& +\alpha_4\,\xi_1(t,y)\|E\|\,\|T^{-1}\|\mathcal{L}_{\xi_2}\left(\|e_{z_1}\| + \|e_{z_1 d}\|\right)\|x\| + \frac{1}{2\varepsilon_2}\|T^{-1}\|^2\mathcal{L}_N^2\left(\|e_{z_1}\|^2 + \|e_{z_1 d}\|^2\right) \\
& -\lambda_{\min}(Q_1)\|e_{z_1}\|^2 + 2\|e_{z_1}\|\,\|[P_1 \;\; P_2]\|\,\|T^{-1}\|\mathcal{L}_{\Phi}(\|e_{z_1}\| + \|e_{z_1 d}\|) \qquad (63)
\end{aligned}
$$

In order to apply Lyapunov-Razumikhin approach, it is assumed that for any $d \in [0,\ \bar{d}]$

$$V(t-d, x_d, e_{z_1 d}) \leq \gamma V(t, x, e_{z_1})$$

for $\gamma > 1$. Then, from Assumption 3 and the definition of $V(\cdot)$ in (53)

$$
\begin{aligned}
0 & \leq \gamma V(t,x,e_{z_1}) - V(t-d,x_d,e_{z_1 d}) \\
& = \gamma V_0(t,x) + \gamma e_{z_1}^T P_1 e_{z_1} - V_0(t-d,x_d) - e_{z_1 d}^T P_1 e_{z_1 d} \\
& \leq \alpha_2\gamma\|x\|^2 - \alpha_1\|x_d\|^2 + \gamma\lambda_{\max}(P_1)\|e_{z_1}\|^2 - \lambda_{\min}(P_1)\|e_{z_1 d}\|^2 \qquad (64)
\end{aligned}
$$

From (65) and (64),

$$
\begin{aligned}
\dot{V} \leq & -\alpha_3\|x\|^2 + \alpha_4\,\mathcal{L}_{u^a}\|G(t,y)\|\,\|T^{-1}\|\,\|e_{z_1}\|\,\|x\| + \frac{\|x\|^2}{2\varepsilon_1} + \alpha_4\mathcal{L}_{\xi_2}\,\xi_1(t,y)\|E\|\,\|T^{-1}\|\,\|e_{z_1}\|\,\|x\| \\
& +\alpha_4\mathcal{L}_{\xi_2}\,\xi_1(t,y)\|E\|\,\|T^{-1}\|\,\|e_{z_1 d}\|\,\|x\| + \frac{\mathcal{L}_N^2}{2\varepsilon_2}\|T^{-1}\|^2\|e_{z_1}\|^2 + \frac{\mathcal{L}_N^2}{2\varepsilon_2}\|T^{-1}\|^2\|e_{z_1 d}\|^2 \\
& -\lambda_{\min}(Q_1)\|e_{z_1}\|^2 + 2\mathcal{L}_{\Phi}\,\|[P_1 \;\; P_2]\|\,\|T^{-1}\|\,\|e_{z_1}\|^2 + 2\mathcal{L}_{\Phi}\,\|[P_1 \;\; P_2]\|\,\|T^{-1}\|\,\|e_{z_1}\|\,\|e_{z_1 d}\| \\
& +\alpha_2\,\gamma\|x\|^2 - \alpha_1\|x_d\|^2 + \gamma\lambda_{\max}(P_1)\|e_{z_1}\|^2 - \lambda_{\min}(P_1)\|e_{z_1 d}\|^2
\end{aligned}
$$

$$
\begin{aligned}
& \leq -\Big[\|x\| \;\; \|e_{z_1}\| \;\; \|e_{z_1 d}\|\Big] W(\cdot) \begin{bmatrix} \|x\| \\ \|e_{z_1}\| \\ \|e_{z_1 d}\| \end{bmatrix} - \alpha_1\|x_d\|^2 \\
& \leq -\gamma_0\left(\|x\|^2 + \|e_{z_1}\|^2 + \|e_{z_1 d}\|^2\right) - \alpha_1\|x_d\|^2 \\
& \leq -\gamma_0\left(\|x\|^2 + \|e_{z_1}\|^2\right)
\end{aligned}
$$

where $\gamma_2 > 0$ is used to obtain the last two inequalities. From Razumikhin Theorem (see, e.g. [9] and [10]), the conclusion follows from $\gamma_0 > 0$. ∇

Remark 8. It should be emphasised that if $\Phi(t,x,x_d) = \Phi(t,x_1,x_2,x_{1d},x_{2d})$ where $y = x_2$, then the condition that $\Phi(\cdot)$ is generalised Lipschitz w.r.t. x and x_d for t can be relaxed to the condition that $\Phi(t,x_1,x_2,x_{1d},x_{2d})$ is generalised Lipschitz w.r.t. the variables x_1 and x_{1d} for the variables t, x_2 and x_{2d}. This is applicable to all nonlinear functions which is required to satisfy the generalised Lipschitz condition throughout the paper.

5 Illustrative Example

Consider a nonlinear time varying delay system

$$\dot{x} = \underbrace{\begin{bmatrix} -5 & 0 \\ 1 & 1 \end{bmatrix}}_{A} x + \underbrace{\begin{bmatrix} 0 \\ \frac{1}{1+\sin^2(t+x_2)} \end{bmatrix}}_{G(\cdot)} u(t) + \underbrace{\begin{bmatrix} 0.2x_2 \\ 0.2x_{1d}x_2 \exp\{-t\} \end{bmatrix}}_{\Phi(\cdot)} + \underbrace{\begin{bmatrix} 1 \\ -5 \end{bmatrix}}_{E} \Psi(\cdot) \quad (65)$$

$$y = \underbrace{\begin{bmatrix} 0 & 1 \end{bmatrix}}_{C} x \quad (66)$$

where $x = \mathrm{col}(x_1, x_2) \in \mathbb{R}^2$, $u \in \mathbb{R}$ and $y \in \mathbb{R}$ are respectively the states, input and output of the system. The term $\Psi(\cdot)$ includes all uncertainties which satisfy

$$\|\Psi(\cdot)\| \le \frac{1}{4}(|x_{1d}| + |x_{2d}|) \exp\{-2-t\} \sin^2 x_2$$

The domain considered here is

$$\Omega = \{(x_1, x_2) \mid x_1 \in \mathbb{R}, |x_2| < 9.15\}$$

Clearly system (66)–(67) has the form in (10)–(11). It is easy to see that (A, C) is observable, and $\Phi(\cdot)$ is generalised Lipschitz w.r.t. x_{1d} for t and x_2 with $\mathcal{L}_\Phi = 0.2|y| \exp\{-t\}$. Let

$$\begin{aligned} &E = \begin{bmatrix} 1 & -5 \end{bmatrix}^T, && L = \begin{bmatrix} -1 & 6 \end{bmatrix}^T, \\ &\xi_1 = \sin^2 y, && \xi_2 = \tfrac{1}{4}(|x_{1d}| + |x_{2d}|) \exp\{-2-t\}, \\ &F = -5, && Q = 10I_2. \end{aligned}$$

Then,

$$P = \left[\begin{array}{c|c} 1.041667 & 0.208333 \\ \hline 0.208333 & 1.041667 \end{array}\right] = \begin{bmatrix} P_1 & P_2 \\ P_3 & P_4 \end{bmatrix}, \qquad T = \begin{bmatrix} 1 & 0.2 \\ 0 & 1 \end{bmatrix}$$

and Assumption 2 is satisfied. Let

$$u^a = -(1 + \sin^2(t+y))(x_1 + 6y), \qquad V_0 = 0.1(x_1^2 + x_2^2)$$

It follows that Assumption 3 holds with $\alpha_1 = \alpha_2 = 0.1$, $\alpha_3 = 1$ and $\alpha_4 = 0.2$. Let

$$M(\cdot) = \frac{0.2}{1+\sin^2(t+y)}, \qquad N(\cdot) = 0.04\,(x_1 + x_{1d}x_2 \exp\{-t\})$$

It is straightforward to check that Assumptions 4 holds. Choose $q_0 = 1.01$. By computation directly,

$$\begin{aligned} &\mathcal{L}_N = 0.04\sqrt{1 + x_2^2 \exp\{-2t\}}, && \mathcal{L}_{u^a} = 1 + \sin^2(t+y), && \mathcal{L}_\Phi = 0.2|y| \exp\{-t\}, \\ &\mathcal{L}_{\xi_2} = 0.0677 \exp\{-t\}, && \gamma = 1.01, && q > 10 - 0.9414|y| \exp\{-t\} \end{aligned}$$

and the entries of the matrix W is given by

$$
\begin{aligned}
w_{11} &= 1 - \tfrac{1}{2\varepsilon_1} - 0.1\gamma \\
w_{22} &= 10 - \tfrac{9.768\times 10^{-4}}{\varepsilon_2}\left(1 + x_2^2\exp\{-2t\}\right) - 0.4695|y|\exp\{-t\} - 1.0417\gamma \\
w_{33} &= 1.0417 - \tfrac{9.768\times 10^{-4}}{\varepsilon_2}\left(1 + 0.5x_2^2\exp\{-2t\}\right) \\
w_{12} &= w_{21} = -(0.1105 + 0.0763\sin^2 y\,\exp\{-t\}) \\
w_{23} &= w_{32} = -0.2348|y|\,\exp\{-t\} \\
w_{13} &= w_{31} = 0.0382\sin^2 y\,\exp\{-t\}
\end{aligned}
$$

By direct computation, all the conditions in Theorems 1–3 are satisfied in the domain Ω with $\gamma = 1.01$. Both the observer (19)-(20) and the controller (48) are well defined. According to (48), (49) and (50), the designed control is given by

$$
u = -(1 + \sin^2(t+y))(x_1 + 6y) + u^b(t, y, \hat{x}, \hat{x}_d) + u^c(t, y, \hat{x}, \hat{x}_d) \qquad (67)
$$

where u^b and u^c are defined by

$$
u^b = \begin{cases} -\frac{0.5099\varepsilon_1 y}{y^2}(1 + \sin^2(t+y))y_d^2(\sin y)^4(|\hat{x}_{1d}| + |\hat{x}_1|)^2\exp\{-4-2t\}, & y \neq 0 \\ 0, & y = 0 \end{cases}
$$

$$
u^c = \begin{cases} -\frac{1+\sin^2(t+y)}{0.2}\left(\frac{\varepsilon_2}{2} + \frac{0.04|\hat{x}_1 y_d + \hat{x}_{1d} y\exp\{-t\}|}{|y|}\right)y, & y \neq 0 \\ 0, & y = 0 \end{cases}
$$

For implementation purposes, choose $\eta = 5$ and $b_0 = 5$. The time-varying delay $d(t)$ is chosen as $d(t) = 5 + 2\sin t$. The delay related initial condition is chosen as

$$
\phi(t) = col(\cos(t), 1 - \sin(t))
$$

The simulation results shown in figure 1 confirm that the proposed approach is effective.

Fig. 1 The time responses of the system states, observer states, estimation errors and control signal

6 Conclusion

A sliding mode observer-based control design approach has been proposed for a class of nonlinear time delay systems. The sliding mode observer can estimate the system state uniformly asymptotically and is insensitive to the uncertainty. Sufficient conditions have been derived using the Lyapunov-Razumikin approach under which the observer-based control law can stabilize the corresponding closed-loop system uniformly asymptotically. There is no limitation to the rate of change of the time delay. The accessible parts have been employed in the control design to reduce conservatism.

References

1. Angulo, M., Fridman, L., Levant, A.: Output-feedback finite-time stabilization of disturbed LTI systems, Automatica, **48** (4), 606–611 (2012)
2. Basin, M., Rodriguez-Gonzalez, J., Fridman, L., Acosta, P.: Integral sliding mode design for robust filtering and control of linear stochastic time-delay systems, Internat. J. Robust Nonlinear Control, **15** (9), 407–421 (2005)
3. Bhat, K. P. M., Koivo, H. N.: Observer theory for time-delay systems, IEEE Trans. on Automat. Control, **21** (2), 266–269 (1976)
4. Cacace, F., Germani, A., Manes, C.: An observer for a class of nonlinear systems with time varying observation delay, Systems & Control Letters, **59**(5), 2305–2312 (2010)
5. Dinuzzo, F., Ferrara, A.: Higher order sliding mode controllers with optimal reaching, IEEE Trans. on Automat. Control, **54** (9), 2126–2136 (2009)
6. Edwards, C., Yan, X. G., Spurgeon, S. K.: On the solvability of the constrained Lyapunov problem, IEEE Trans. on Automat. Control, **52** (10), 1982–1987 (2007)
7. Efimov, D., Fridman, L.: Global sliding-mode observer with adjusted gains for locally Lipschitz systems, Automatica, **47**(3), 565–570 (2011)
8. Galimidi, A. R., Barmish, B. R.: The constrained Lyapunov problem and its application to robust output feedback stabilization, IEEE Trans. on Automat. Control, **31** (5), 410–419 (1986)
9. Gu, K., Kharitonov, V. L., Chen, J.: Stability of time-delay systems, Boston: Birkhäuser (2003)
10. Hale, J. K., Lunel, S. M. V.: Introduction to functional differential equations, New York: Springer-Verlag (1993)
11. Hua, C., Guan, X. & Shi, P.: Robust backstepping control for a class of time-delayed systems, IEEE Trans. on Automat. Control, **50**(6), 894–899 (2005)
12. Ibrir, S.: Observer-based control of a class of time-delay nonlinear systems having triangular structure, *Automatica*, **47** (2), 388–394 (2011)
13. Jafarov, E. M.: Design modification of sliding mode observers for uncertain MIMO systems without and with time-delay, Asian J. Control, **7**(4), 380–392 (2005)
14. Khalil, H. K.: Nonlinear Systems *(Third Edition)*, New Jersey: Prentice Hall, Inc. (2002)
15. Luo, N., De La Sen, M. & Rodellar, J.: Robust stabilization of a class of uncertain time delay systems in sliding mode, Internat. J. Robust Nonlinear Control, **7** (1), 59–74 (1997)
16. Marino, R., Tomei, P.: Adaptive observers with arbitrary exponential rate of convergence for nonlinear systems, IEEE Trans. on Automat. Control, **40**, 1300–1304 (1995)
17. Niu, Y., Lam, J., Wang, X., Ho, D.: Observer-based sliding mode control for nonlinear state-delayed systems, Internat. J. Systems Sci., **35**(2), 139–150 (2004)
18. Nounou, M. N., Nounou, H. N., Mahmoud, M. S.: (2007) Robust adaptive sliding-mode control for continuous time-delay systems, IMA J. Math. Control Inform. **24** (3), 299–313.
19. Orlov, Y.: Discontinuous Systems: Lyapunov analysis and robust synthesis under uncertainty conditions, London: Springer-Verlag (2009)
20. Orlov, Y., Aoustin, Y., Chevallereau, C.: Finite time stabilization of a perturbed double integrator – Part I: Continuous sliding mode-based output feedback synthesis, IEEE Trans. on Automat. Control, **56**(3), 614–618 (2011)
21. Richard, J. P.: Time-delay systems: An overview of some recent advances and open problems, Automatica, **39**(10), 1667–1694 (2003)
22. Spurgeon, S. K.: Sliding mode observers: a survey, Internat. J. Systems Sci., **39**(8), 751–764 (2008)
23. Veluvolu, K., Soh, Y.: Fault reconstruction and state estimation with sliding mode observers for Lipschitz non-linear systems, IET Control Theory & Appl., **11**, 1255–1263 (2011)

24. Wang, Z., Huang, B., Unbehauen, H.: Robust H_∞ observer design of linear state delayed systems with parametric uncertainty: the discrete-time case, Automatica, **35**(6), 1161–1167 (1999)
25. Xia, X. H., Gao, W. B.: Nonlinear observer design by observer error linearization, SIAM J. Control Optim., **27**, 199–216 (1989)
26. Yan, X. G., Lam, J., Li, H. S., Chen, I. M.: Decentralized control of nonlinear large-scale systems using dynamic output feedback, J. Optim. Theory Appl., **104**(2), 459–475 (2000)
27. Yan, X. G., Lam, J., Xie, L.: Robust observer design for non-linear interconnected systems using structural characteristics, Int. J. of Control, **76**(7), 741746 (2003)
28. Yan, X. G., Spurgeon, S. K., Edwards, C. State and parameter estimation for nonlinear delay systems using sliding mode techniques, IEEE Trans. on Automat. Control, **58**(4), 1023-1029 (2013)
29. Yan, X. G., Spurgeon, S. K., Edwards, C. Sliding mode control for time-varying delayed systems based on a reduced-order observer, Automatica, **46**(8), 1354–1362 (2010)