A Systematic Translation of Guarded Recursive Data Types to Existential Types

Martin Sulzmann
School of Computing, National University of Singapore
S16 Level 5, 3 Science Drive 2, Singapore
117543
sulzmann@comp.nus.edu.sg

Meng Wang
School of Computing, National University of Singapore
S16 Level 5, 3 Science Drive 2, Singapore
117543
wangmeng@comp.nus.edu.sg

ABSTRACT
Guarded recursive data types (GRDT) are a new language feature which allows to type check the different branches of case expressions under different type assumptions. We show that GRDT can be translated to type classes with existential types (TCET). The translation to TCET might be problematic in the sense that common implementations such as the Glasgow Haskell Compiler (GHC) fail to accept the translated program. We establish some sufficient conditions under which we can provide for a refined translation from TCET to existential types (ET) based on a novel proof term construction method. The resulting ET program is accepted by GHC. The sufficient conditions are met by all GRDT examples we have found in the literature. Our work can be seen as the first formal investigation to relate the concepts of guarded recursive data types and (type classes with) existential types.

Categories and Subject Descriptors
D.3.2 [Programming Languages]: Language Classifications—Applicative (functional) languages; D.3.3 [Programming Languages]: Language Constructs and Features—Polymorphism, Constraints; F.3.3 [Logics and Meanings of Programs]: Studies of Program Constructs—Type structure

General Terms
Languages, Theory

Keywords
Type systems, type-directed translation, proof-term construction, constraint solving.

1. INTRODUCTION
Guarded recursive data types (GRDT) [28] introduced by Xi, Chen and Chen are a new language feature which allows to type check more programs. The basic idea is to use different type assumptions for each branch of a case expression. E.g., consider the following (toy) GRDT program. We will use Haskell-style syntax [8] throughout the paper.

Example 1 We introduce a GRDT Erk a where a may be refined depending on the constructor. Function f takes advantage of the temporary equality assumptions enabled by pattern matching.

\[
\text{data Erk } a \Rightarrow (a = \text{Int}) \Rightarrow I \ a \\
| \forall b . (a = [b]) \Rightarrow L \ a \\
f :: \text{Erk } a \rightarrow a \\
f (I \ x) = x + 1 \\
f (L \ x) = \text{tail } x
\]

In detail, the data type definition introduces two constructors belonging to data type Erk a. The novelty of GRDT is that in case of constructor I we refine the type to Erk Int. We present type refinement in terms of equations. In case of L we refine the type to Erk Int. Note that GRDT imply existential types [14]. Constructor L has type \( \forall b . (a = [b]) \Rightarrow a \rightarrow \text{Erk } a \). Therefore, all variables not appearing in the result type are bound by the forall keyword. Note that some presentations [4] write \( \text{I } a \Rightarrow (a = \text{Int}) \) instead of \( (a = \text{Int}) \Rightarrow \text{I } a \). The important point is that when pattern matching over values we can make use of these additional type assumptions. Consider the function definition where in the first clause we temporarily add a \( a = \text{Int} \) to our assumptions (assuming that x has type a). Thus, we can verify that the \( x + 1 \) has type a. A similar observation applies to the second clause. Hence, function f is type correct.

GRDT have been recognized as a very useful language feature, e.g., consider [20, 17, 18]. Hence, it is desirable to extend existing languages with GRDT. In fact, a number of authors [1, 2, 3, 27] have recognized that GRDT-style behavior can be expressed in terms of some existing language features already available in Haskell. All of these encodings share the same idea and represent type equalities by Haskell terms.

Example 2 Here is an encoding of Example 1 in terms of existential types [14]. We introduce a special data type \( E a b \) to represent equality assumption among types. E.g., we represent \( a = \text{Int} \) by \( E a \text{ Int} \) where the associated value E
2. PRELIMINARIES

We write \( f(o) \) to denote the set of free variables in some object \( o \).

We assume that the reader is familiar with the concepts of substitution, unifiers, most general unifiers (m.g.u.) etc [12]. E.g., \( f[a] \) denotes the substitution which has the effect of replacing each occurrence of \( a \) by \( t \). Often, we abbreviate \( [t_1/a_1, \ldots, t_n/a_n] \) by \( [t]_a \).

We make use of constraints \( C \) consisting of conjunction of primitive constraints such as \( t_1 = t_2 \) describing equality among \( t_1 \) and \( t_2 \). We often treat constraints as sets, therefore, we use \( \cup \) as a short-hand for Boolean conjunction.

We also assume basic familiarity with first-order logic. We write \( \models \) to denote the model-theoretic entailment relation, \( \vDash \) to denote Boolean implication and \( \iff \) to denote Boolean equivalence. We let \( \exists_F \) denote the formula \( \exists x_1 \ldots \exists x_n F \) where \( \{x_1, \ldots, x_n\} = \text{fv}(F) = W \). We refer to [21] for details.

3. GUARDED RECURSIVE DATA TYPES

In this section, we define the set of well-typed GRDT programs. Note that there exist several variations of GRDT such as Cheney's and Hinze's first-class phantom types [4], Peyton-Jones's, Washburn's and Weirich's generalized algebraic data types [10] and equality-qualified types by Sheard and Pasalic [9]. Our formulation is closest to the system described by Simonet and Pottier [22].

First, we define the set of expressions and types.

Expressions: \( e ::= K \{ x \mid \lambda x.e \} \mid e \mid \text{case } e \text{ of } [p_1 \rightarrow e_1]_{i \in I} \)

Patterns: \( p ::= x \mid (p,p) \mid K p \)

Types: \( t ::= a \mid t \rightarrow t \mid T \)

Type Schemes: \( \sigma ::= t \downarrow \forall x.C \rightarrow t \)

For simplicity, we leave out let-definitions and type annotations but may make use of them in examples. Note that pattern matching syntax used in examples can be straightforwardly expressed in terms of case expressions.

GRDT definitions in example programs such as

\[
data \text{Erk } a = (a = \text{Int}) \rightarrow I \text{a } \mid \text{forall } b.(a = [b]) \rightarrow L \text{a}
\]

imply constructors \( I : \forall a.a = \text{Int} \rightarrow a \rightarrow \text{Erk } a \) and \( L : \forall a,b.a = [b] \Rightarrow a \rightarrow \text{Erk } a \). We prohibit "invalid" definitions such as \( \text{data \ unsat } a = (a = (a, \text{Int})) \Rightarrow \text{U } a \) which yields a constructor with an unsatisfiable set of equations. We assume that booleans, integers, pairs and lists are pre-defined.

The typing rules describing well-typing of GRDT expressions are in Figure 1. We introduce judgments \( C, \Gamma \vdash^L e : t \) to denote that expression \( e \) has type \( t \) under constraint \( C \) and environment \( \Gamma \). We assume that \( C \) consists of conjunction of equations. A judgment is valid if we find a derivation w.r.t. the typing rules. Note that in \( \Gamma \) we record the types of lambda-bound variables and primitive functions such as \text{head} : \forall a.a \rightarrow a, \text{tail} : \forall a.a \rightarrow [a] \text{ etc. Rules (Abs), (App) and (Var-x) are standard. Rule (K) seems somewhat redundant and could be modeled by rules (App) and (Var-x) assuming that constructors are recorded in \( I_{\text{init}} \). Our intention is that constructors are always fully applied. Rule (Case) deals with case expression. Nothing unusual so far. Next, we consider the GRDT specific rules. In rule (Eq) we are able to change the type of an expression. Note that the side condition \( C \supseteq t_1 = t_2 \) holds if (1) \( C \) does not have a unifier, or (2) for any unifier \( \phi \) of \( C \) we have
that $\phi(t_1) = \phi(t_2)$ holds. In rule (Pat) we make use of an auxiliary judgment $p : t \vdash \forall \bar{y} . (D, \Gamma_p)$ which establishes a relation among pattern $p$ of type $t$ and the binding $\Gamma_p$ of variables in $p$. Variables $\bar{y}$ refer to all “existential” variables.

Logically, these variables must be considered as universally quantified. Hence, we write $\forall \bar{y}$. The side condition $\bar{y} \cap \text{fv}(C, \Gamma, t_2) = \emptyset$ prevents existential variables from escaping. In rule (P-Pair), we assume that there are no name clashes between variables $\bar{y}_1$ and $\bar{y}_2$. Constraint $D$ arises from constructor uses in $p$. The other rules are standard.

Let’s consider the first clause of $f$ in Example 1 again. According to rule (Pat), the pattern $\varpi x$ provides the additional type assumption $a = \text{Int}$ which is used in typing of the body $x+1$. Note that because of this additional assumption, rule (Eq) is able to turn the type of $x$ from $a$ to $\text{Int}$. Thus, the expression $x+1$ is well typed. Similarly, rule (Eq) also turns the type of $x+1$ to $a$. Hence, the annotation given to $f$ is correct. Rule (Eq) has some other surprising consequences.

**Example 3** Consider the following variation of Example 1

```latex
\text{data} \text{Erk a = (a=\text{Int}) \rightarrow I a}
\text{g :: Erk Bool \rightarrow b}
\text{g (1 x) = x + 'a'}
```

We make use of $\text{Bool = Int}$ which is equivalent to $\text{False}$ to type the body of the clause. Hence, we can derive anything. Hence, $g$ has type $\text{Erk Bool \rightarrow b}$ for any $b$. Note that we only temporarily make use of $\text{False}$. The constraint in the final judgment is satisfiable.

As already observed by Cheney and Hinze [4] such meaningless programs can always be replaced by “undefined”. Note that we never ever construct a value of type $\text{Erk Bool}$. Hence, w.r.t. $g$ we slightly restrict the set of typable programs and replace logical by constructive entailment. Effectively, we rule out GRDT programs where $\text{False}$ occurs in (intermediate) typing judgments. The definition of constructive entailment among type equality is as follows:

<table>
<thead>
<tr>
<th>$t = t'$ in $C$</th>
<th>$C \vdash t = t'$</th>
<th>$C \vdash t = t_1 \wedge C \vdash t = t_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C \vdash t = t_2$</td>
<td>$C \vdash t_1 = t_3$</td>
<td></td>
</tr>
<tr>
<td>$C \vdash t_1 = t_2$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$C \vdash t_3 = t_4$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$C \vdash t_1 \rightarrow t_3 = t_2 \rightarrow t_4$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$C \vdash t_1 \rightarrow t_3 = t_2 \rightarrow t_4$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$C \vdash t_3 \rightarrow t_4$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

We obtain the constructive GRDT system $\vdash^{c} e$ by replacing (Eq) with the following rule.

<table>
<thead>
<tr>
<th>$C, \Gamma \vdash^{c} e : t$</th>
<th>$C, \Gamma \vdash^{c} e : t'$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C, \Gamma \vdash^{c} e : t'$</td>
<td></td>
</tr>
</tbody>
</table>

Note that Example 3 is not typable anymore in the constructive system.

**4. TRANSLATING GRDT TO TCET**

The main result of this section is that GRDT can be encoded by type classes with existential types (TCET). This will form an important intermediate step in our translation to ET. For this purpose, we introduce a type class $\text{Ct a b}$ to convert a term of type $\text{a}$ into a term of type $\text{b}$. In essence, we model directed equality. The following instance declarations implement this idea.

```latex
\text{class Ct a b where cast :: a \rightarrow b}
\text{instance Ct a a where cast x = x -- (Id) instance (Ct b1 a1, Ct b2 a2) \Rightarrow Ct (a1 \rightarrow a2) (b1 \rightarrow b2) where cast f x = cast (f (cast x)) -- (Arrow) instance (Ct a1 a2, Ct a2 a3) \Rightarrow Ct a1 a3 where cast a1 = cast (cast a1) -- (Trans)}
```

Operationally, the conversion functions performs the identity operation for all monomorphic instances derivable w.r.t. the above rules.
We translate GRDT programs to TCET by replacing each equation \( t_1 = t_2 \) in a data type definition by \( C t_1 t_2 \) and \( CT t_1 t_2 \). Additionally, we apply cast to all sub-expressions.

**Example 4** Here is the translation of Example 1.

\[
\text{data } \text{Erk}_H \text{ a } = (C t \text{ a } \text{ Int}, C t \text{ Int a}) \Rightarrow I_H \text{ a } \\
| \text{forall } b. (Ct (a \cdot b), C (b \cdot a)) \Rightarrow I_H \text{ a }
\]

\[
f_H :: \text{Erk}_H \text{ a } \Rightarrow a \cdot f_H (I_H \text{ x }) = \\
(\text{cast } ((\text{cast } (+)) (\text{cast x}))) (\text{cast } 1)
\]

\[
f_H (I_H \text{ x }) = (\text{cast } ((\text{cast } (\cdot)) (\text{cast x})))
\]

When typing the second clause we temporarily make use of \( C t \cdot a \cdot b \) and \( C t [b] a \). Thus, \( \text{cast x} \) can be given type \([b] \).

We make use of instance \((\text{Id})\) to show that \( \text{cast tail} \) has type \([b] \to [b] \). Hence, \((\text{cast tail}) (\text{cast x}) \) can be given type \([a] \). Similar reasoning applies to the first clause where we make use of instance \((\text{Arrow})\). Hence, function \( f_H \) is type correct. □

The connection between GRDT and TCET becomes obvious when considering their underlying formal systems. A formal description of TCET covering the single-parameter case is given by Läuffer [13]. In our own work [23], we recently formalized the general case including multi-parameter type classes which will make use of in the following.

Briefly, in the TCET system we find new type (multi-paramater) class constraints \( TC t_1 \ldots t_n \) instead of equality constraints \( t_1 = t_2 \). For simplicity, we assume that instance declarations are preprocessed and the relations they describe are translated to logic formulae. We commonly denote these logic formulae by \( P \) and refer to \( P \) as the program theory. E.g., the instance declarations from above can be described by the following first-order formulae.

\[
\forall a. (C t a \leftrightarrow \text{True})
\]

\[
\forall a, t_i, t_j. (C t (a \to a) (t_i \to t_j) \leftrightarrow C t a \cdot t_i \land C t a \cdot t_j)
\]

where \( \leftrightarrow \) denotes Boolean equivalence. We refer the interested reader to [24] for more details on the translation of instances to logic formulae.

For each class declaration \( \text{class } C \text{ a } \ldots \text{ a } \text{ m }\text{ : } \text{ t } \) we assume a new primitive \( m : \forall a. TC a \Rightarrow t \). For simplicity, we restrict ourselves to monomorphic methods. That is, we require that \( f(t) \subseteq a \). Note that the restriction to monomorphic methods is sufficient for the purpose of translating GRDT to TCET.

The typing rules for TCET are almost the same as those for GRDT in Figure 1. We adopt rules \((\text{App})\), \((\text{Abs})\), \((\text{Var-x})\), \((\text{Case})\), \((\text{Pat})\), \((\text{P-Var})\), \((\text{P-Pair})\) and \((\text{P-K})\) from Figure 1. However, we drop rule \((\text{Eq})\). Furthermore, we adjust rule \((\text{K})\) and introduce a new rule \((\text{M})\) to take care of class methods.

\[
(K) \quad \text{K} : \forall a, \text{ TC } a \Rightarrow t \Rightarrow T \text{ a} \\
C, \Gamma, \text{ t } \Rightarrow e : [\text{[a]}\text{\tilde{a}}] \Rightarrow P_e \Rightarrow C \supset [\text{[\tilde{a}}[\text{\tilde{a}}][\text{\tilde{a}}]D] \\
C, \Gamma, \text{ t } \Rightarrow K e : T \text{ t} \\
\]

\[
(M) \quad m : \forall a. TC a \Rightarrow t \Rightarrow f(t) \subseteq a \Rightarrow P_e \Rightarrow C \supset TC \text{ t} \\
C, \Gamma, \text{ t } \Rightarrow m : [\text{[a]}\text{\tilde{a}}]\text{[\tilde{a}}[\text{\tilde{a}}]D] \\
\]

Note that entailment is now defined w.r.t. the program theory. The side condition \( P_e \supset C \supset [\text{[\tilde{a}}[\text{\tilde{a}}][\text{\tilde{a}}]D] \) denotes that any model satisfying \( P_e \) and \( C \) also satisfies \( [\text{[\tilde{a}}[\text{\tilde{a}}][\text{\tilde{a}}]D] \).

To distinguish the two systems we write \( C, \Gamma, \text{ t } \Rightarrow e : t \) to denote that expression \( e \) has type \( t \) under constraint \( C \) and environment \( \Gamma \) in the TCET system. In case of \( \text{True}, \Gamma, \text{ t } \Rightarrow e : t \) we sometimes write \( \Gamma, \text{ t } \Rightarrow e : t \) for short.

We are in the position to define the formal translation from GRDT to TCET. In order to model the constructive entailment relation \( \text{ t } \Rightarrow e \) among equalities we need to impose some conditions on the program theory.

**Definition 1 (Full and Faithful)** We say that the program theory \( P_e \) is full and faithful w.r.t. constructive equality iff (1) for each \( n \)-ary type constructor \( C \) there is some appropriate instance such that

\[
P_e \Rightarrow (C t a_1 \ldots a_n) (T t b_1 \ldots b_n) \land C t (T t b_1 \ldots b_n) (T a_1 \ldots a_n) \supset (C t a_1 b_1 \land C t a_1 b_1 \land \ldots C t a_n b_n \land C t b_n a_n)
\]

and (2) all monomorphic cast instances are equivalent to the identity. Equality among expressions is defined in terms of a standard denotational semantics, e.g., consider [15].

To turn GRDT typeable expressions into TCET typeable expressions, we perform a syntactic transformation by applying the cast function to each (sub-)expression. We write \( e[C] \) to denote an occurrence of \( e' \) in \( e \).

**Definition 2 (Fully Casted)** Let \( e \) be an GRDT expression. We construct a fully casted expression \( e' \) out of \( e \) by applying cast on every subexpression of \( e \). A single transformation step is defined as

\[
\text{e}[C] \mapsto \text{e'}[\text{cast } \text{e}_1] \quad \text{where } \text{e}_1 \text{ is syntactically different from } \text{cast } \text{e}_2 \text{ for some expression } \text{e}_2.
\]

The transformation of GRDT constructors is simple. Each GRDT constructor

\[
K : \forall a, b. \{ t_1 = t_1', \ldots, t_n = t_n' \} \Rightarrow t \Rightarrow T \tilde{a} 
\]

implies a TCET constructor

\[
K' : \forall a, b. \{ C t_1 t_1', C t_1 t_1', \ldots, C t_n t_n', C t_n t_n' \} \Rightarrow t \Rightarrow T \tilde{a}
\]

We can state the following formal connection between GRDT and TCET.

**Theorem 1 (GRDT to TCET)** Let \( e \) be a GRDT expression and \( e' \) be its fully casted version. For each GRDT constructor \( K \) we introduce its TCET equivalent \( K' \). Let \( P_e \) a full and faithful program theory representing all GRDT type constructors mentioned in \( e \). Then, we have that \( \text{True}, \Gamma, \text{ t } \Rightarrow e : t \iff \text{True}, \Gamma, \text{ t } \Rightarrow e' : t \).

A proof can be found in Appendix B.1.

As already pointed out the restriction to the \( Inc \) system is not onerous. Note that in order to directly translate Example 3 the program theory would need to be strengthened by including additional “improvement” rules such as \( P_e \Rightarrow C t \text{ Bool } \supset \text{False}, P_e \Rightarrow C t \text{ Int } \supset \text{False} \) etc.

The above result is constructive in the sense that we can type-check the resulting TCET program if the entire GRDT
typing derivation (including $C \vdash t_1 = t_2$ derivations) is available. We can also give a meaning to translated TCET program based on the scheme presented in [24]. However, GHC fails to accept the TCET program because instance declarations are potentially “non-terminating”.

E.g., consider instance (Trans) from above. When performing context reduction we need to guess the intermediate type when applying instance (Trans). Hence, context-reduction may or may not terminate. Hence, the check whether $C \supset Ct$, holds where $C$ is a set of $Ct$ assumptions may not terminate. On the other hand, $C' \supset t_1 = t_2$ is decidable assuming that $C'$ is derived from $C$ by turning each $Ct t i$ into an equation $t = t'$. We conclude that we further need to redefine our transformation method for GRDT. The translation to TCET represents an important intermediate step to achieve a translation to ET which is finally accepted by GHC.

5. TRANSLATING GRDT TO ET

The result from the previous section allows us to assume that GRDT programs have been translated to TCET by fully casting expressions and transforming GRDT constructors into TCET constructors. Hence, it is sufficient to consider the translation from TCET to ET. We establish some sufficient conditions under which we achieve a type-directed translation translation scheme from TCET to ET based on a proof system to construct terms connected to type class constraints $Ct t i$.

We start off by describing our proof system. We assume that constraints such as $f : Ct a b$ carry now a proof term $f$ representing “evidence” for $Ct a b$. We silently drop $f$ in case proof terms do not matter. We introduce judgments of the form $f : Ct a b \leftrightarrow F$ to denote that $f$ is the proof term corresponding to $Ct a b$ under the assumption $F$ where $F$ refers to a (possibly existentially quantified) conjunction of type class constraints. The rules describing the valid judgments are in Figure 2. Note that we write the actual definition of $f$ as part of the premise. Rules (Id), (Var) and (Trans) are straightforward. Rules (Arrow) and (Pair) deal with function and pair types. We assume that the proof rules will be extended accordingly for user-defined types. Rule (s) allows for the structural composition of proof terms. Rules (VE) and (GE) deal with universal and existential quantifiers. In essence, we make the construction rules represented by $Ct$ instance declarations explicit.

Example 5 We give the derivation tree for $f : Ct a (Int, Bool)$ $\leftrightarrow g_1 : Ct a [b, c], g_2 : Ct b Int, g_3 : Ct c Bool$ in Figure 2. For convenience, we combine rule (VE) with rules (Id), (Var), (Arrow). We conclude that

$$\begin{align*}
    f \ x \ = \ & \ \text{let } g_1 \ x \ y - (g_2 \ x, g_3 \ y) \\
    \text{in } g_1 \ (g_1 \ x) \\
\end{align*}$$

A simple observation of our proof rules shows that the proof system is sound w.r.t. the logical reading of instance declarations.

Lemma 1 (Soundness) Let $P_f$ be the program theory. Let $C = \{ f_1 : Ct a_1 b_1, \ldots, f_n : Ct a_n b_n \}$ such $f : Ct a b \leftrightarrow C$ is valid. Then, $P_f \models C \supset Ct a b$.

We can also state that proof terms are well-typed.

Definition 3 Let $C = \{ f_1 : Ct a_1 b_1, \ldots, f_n : Ct a_n b_n \}$. We construct an environment $\Gamma$ out of $C$, written as $C \sim \Gamma$, by mapping each $g : Ct a b \in C$ to $g : a \mapsto b \in \Gamma$.

Lemma 2 (Well-Typed) Let $C = \{ f_1 : Ct a_1 b_1, \ldots, f_n : Ct a_n b_n \} \Gamma$ such that $C \sim \Gamma$ and $f : Ct a b \leftrightarrow C$ is valid. Then $\Gamma \vdash f : a \mapsto b$.

Proofs can be found in Appendix B.2. Note that the proof term $f$ is equivalent to the identity assuming $f_i, \ldots, f_n$ are equivalent to the identity as well.

As presented, our proof term construction rules in Figure 2 are still non-terminating (see rule (Trans)). In the upcoming Section 6, we give a decidable procedure to compute $f : Ct a b \leftrightarrow C$ given $Ct a b$ and $C$.

We are in the position to systematically translate TCET to ET. Each TCET constructor $K ^ { \sigma }$ is turned into an ET constructor $K ^ { \sigma ' }$, written $(K : \sigma) \sim (K ^ { \sigma ' } : \sigma ')$.

We have that $[(K : v a, b, D) \Rightarrow t \mapsto t \bar{a}] \sim (K : v a, b, t \mapsto E \ t_1 \ t_1' \ \ldots \ \mapsto E \ t_n \ t_n' \ t_n = T \ a) \ where \ D = \{Ct t_1 \ t_1', Ct t_1 \ t_2, \ldots, Ct t_n \ t_n', Ct t_n', t_n \}$. Silently, we assume a fixed order among $Ct$ constraints. Note that the type constructor $E$ is defined in Example 2.

For the translation of expressions we introduce judgments of the form $C, \Gamma \vdash t : t \mapsto \bar{c} \ where \ C$ holds $C t$ assumptions, $e$ is a TCET expression, and $\bar{c}$ is an ET expression. The translation rules can be found in Figure 3. Our main tasks are to resolve cast functions (see rule (Reduce)) based on our proof system and to explicitly insert proof terms in constructors (see rule (P-K)). In rule (K), we define $P_f \models C \supset (g, h) : [\bar{f} / \bar{a}] D$ iff $g : Ct t_i \ t_i' \mapsto C$ and $h_i : Ct t_i', t_i' \mapsto C$ for $i = 1, \ldots, n$ where $[\bar{f} / \bar{a}] D = \{Ct t_i \ t_i', Ct t_i \ t_1, \ldots, Ct t_n \ t_n', Ct t_n' \ t_n \}$. Note that $P_f \models C \supset (g, h) : [\bar{f} / \bar{a}] D$ implies that $P_f \models C \supset [\bar{f} / \bar{a}] D$ (see Lemma 1). As will see the other direction (which is crucial for completeness) does not hold necessarily.

We can state soundness of our translation scheme given that the TCET program is typable. Note that the ET system is a special instance of TCET. We write $\Gamma \vdash ^T e : t$ to denote a judgment in the ET system.

Theorem 2 (TCET to ET Soundness) Let $\text{True}, \Gamma \vdash ^T e : t \ and \ True, \Gamma \vdash ^T e \mapsto e' \ then \ \exists \ \Gamma' \vdash ^E e' : t$.

We also find that $e$ and $e'$ are equivalent assuming the program theory and proof system is full and faithful.

In combination with Theorem 1 we obtain a systematic translation from GRDT to ET. We do rely on full type information for the GRDT program such that our proof term construction method is able to insert the appropriate evidence values.
Proof Term Construction Rules:

(Id) \[ \forall a. \lambda x. x : C t a a \iff \text{True} \]

(Var) \[ \forall a, b, f : C t a b \iff f : C t a b \]

(Trans) \[ f = \lambda x. f_2 (f_1 x) \]

\[ \forall a, b, f : C t a b \iff g (f_1 x) \]

(Arrow) \[ f = \lambda g. \lambda x. f_2 (g (f_1 x)) \]

\[ \forall a, b, f : C t (a_1 \rightarrow a_2) (b_1 \rightarrow b_2) \iff f_1 : C t b_1 a_1, f_2 : C t a_2 b_2 \]

(Pair) \[ f = \lambda (x, y). (f_1 x, f_2 y) \]

\[ C = \{ f_1 : C t a_1 b_1, f_2 : C t a_2 b_2 \} \]

\[ \forall a, b, f : C t (a_1, a_2) ((b_1, b_2) \iff C \}

\[ f : C t a b \iff f \]

(\forall) \[ \forall a, b, f : C t \phi (t_1), \phi (t_2) \iff \phi (f) \]

(\exists) \[ f : C t \iff \exists a, f \phi (f) \]

Example:

(data Foo a) -> Ct (Foo a) (Foo b)

We have that \( P_e \models g : C t (Foo a) (Foo b) \supset h : C t a b \)

But \( h : C t a b \iff g : C t (Foo a) (Foo b) \) does not exist.

Hence, our translation scheme gets possibly stuck in rules (K) and (Reduce).

Note that the instance declaration implies that \( C t (Foo a) (Foo b) \iff C t a b \).

The instance context seems somewhat redundant but necessary to ensure that the program theory models fully and faithfully the entailment relation \( \vdash \).

Clearly, we can build \( g \) on type \( \text{Foo a} \rightarrow \text{Foo b} \) given \( h \) on type \( a \rightarrow b \) whereas for the other direction we would need to decompose proof terms which is not possible here.

The above is not surprising. Similar situations arise for simple type class programs. E.g., we cannot decompose \( E q \ [a] \) into \( E q \ a \) for any \( a \). All what we can do is to identify some sufficient conditions which allow us to extend the rules in Figure 2 faithfully.

Definition 4 (Decomposable Types) Let \( T \) be a \( n \)-ary type constructor. We say that \( T \) is decomposable at position \( i \) where \( i \in \{ 1, \ldots, n \} \) iff a proof term construction rule \( f_i : C t a_i b_i \iff g_i : C t (T a_1 \ldots a_n) (T b_1 \ldots b_n) \)

\( h : C t (T b_1 \ldots b_n) (T a_1 \ldots a_n) \) exists such that (1) \( f_i \) is well-typed under \( \{ g_i : C t a_i \ldots a_n \rightarrow T b_1 \ldots b_n \} \)

and (2) \( f_i \) is equivalent to the identity if \( g_i \) and \( h \) are equivalent to the identity.

We say that \( T \) is decomposable iff \( T \) is decomposable at all positions.

We find that pairs are decomposable.

Example 7 We make use of \( \_ : \forall a. a. \) Consider

\[ g_1 = \lambda x. \text{fst} (f (x, \_)) \]

\[ g_2 = \lambda x. \text{snd} (f (\_, x)) \]

\( g_1 : C t a_1 b_1 \iff f : C t (a_1, a_2) (b_1, b_2) \)

\( g_2 : C t a_2 b_2 \iff f : C t (a_1, a_2) (b_1, b_2) \)

However, function types seem only to be decomposable in their co-variant position under a non-strict semantics.
Example 8

\[ (\text{Arrow}) \quad g = \lambda x. (f \, (\lambda y. x)) \quad \perp \]
\[ g : \text{Ct} \, a_2 \, b_2 \leftrightarrow f : \text{Ct} \, (a_1 \rightarrow a_2) \, (b_1 \rightarrow b_2) \]

Note that \( g \) is the identity under a non-strict semantics. However, it seems that \( h : \text{Ct} \, b_1 \, a_1 \leftrightarrow f : \text{Ct} \, (a_1 \rightarrow a_2) \, (b_1 \rightarrow b_2) \) does not exist.

Example 9  The Either data type is decomposable:

data Either a b = Left a | Right b

The construction rules are as follow:

\[ (\text{EitherL}) \quad g = \lambda x. \text{project}_L \, (f \, (\text{inject}_L \, x)) \]
\[ g : \text{Ct} \, a_1 \, b_1 \leftrightarrow f : \text{Ct} \, (\text{Either} \, a_1 \, a_2) \, (\text{Either} \, b_1 \, b_2) \]

(EitherR↓)

\[ g = \lambda x. \text{project}_R \, (f \, (\text{inject}_R \, x)) \]
\[ g : \text{Ct} \, a_2 \, b_2 \leftrightarrow f : \text{Ct} \, (\text{Either} \, a_1 \, a_2) \, (\text{Either} \, b_1 \, b_2) \]

where

\[ \text{inject}_L \, x = \text{Left} \, x \]
\[ \text{project}_L \, (\text{left} \, x) = x \]
\[ \text{inject}_R \, x = \text{Right} \, x \]
\[ \text{project}_R \, (\text{right} \, x) = x \]

Note that the decomposition conditions (Definition 4) are satisfied. Consider the (EitherL↓) case. Expressions are well-typed. Assume \( f \) is the identity. Then, \( f \, (\text{inject}_L \, x) \) must yield \( L \, x \). Hence, application of \( \text{project}_L \) is safe. Hence, \( g \) is the identity. A similar reasoning applies (EitherR↓).

Decomposable types ensure that our proof term construction system is not only sound but also complete.
Lemma 3 (Decomposition) Let $P_t$ be a full and faithful proof theory. If $C = \{f_1 : \text{ct} t_1 t_2 \text{ a constraint and } C = \{f_1, f_2, \ldots, f_n : \text{ct} a_n b_n\}$ such that $P_t \vdash C \supset C t_1 t_2$ and all types appearing in constraints are decomposable. Then, $f : \text{ct} t_1 t_2 \Rightarrow C$ for some proof term $f$.

The proof is straightforward and proceeds by induction over $P_t \vdash C \supset C t_1 t_2$.

We are able to state completeness of our translation from TCET to ET given that the types appearing in assumption constraints are decomposable. By assumption constraints we refer to constraints $D$ in rule (Pat).

Theorem 3 (TCET to ET Completeness) Let $\text{true}, \Gamma \vdash \vdash 1 : t$ and all types appearing in assumption constraints in intermediate derivations are decomposable. Then, $\text{true}, \Gamma \vdash \vdash 1 \sim e'$ for some $e'$.

6. DECIDABLE PROOF TERM CONSTRUCTION METHOD

We introduce a method to decide $f : \text{ct} t_1 t_2 \Rightarrow C$ (see Figure 2). That is, given $C$ and $\text{ct} t_1 t_2$ construct a derivation for some $f$. The main challenge is to find a decidable representation for rule (Trans). In the above statement, $C$ contains the set of CT assumptions whereas $\text{ct} t_1 t_2$ refers to a use site (see rule (Reduce) in Figure 3). In order to distinguish between CT uses and assumptions we write $\text{ct} M t_1 t_2$ to refer to a use of CT. Our task is to construct $\text{ct} M$ out of a given set of CT assumptions. Note that CT constraints can be viewed as directed edges. Hence, the successful construction of a CT use is equivalent to finding a path in the graph of CT edges. However, we do not rely on our method on graph algorithms. We would like our method to work even under some additional side conditions such as $\text{ct} M t_1 t_2$, $\text{ct} M t_3 t_4 t_5$ or $t_4 \rightarrow a$. That is, construct $\text{ct} M t_1 t_2$ and $\text{ct} M t_3 t_4$ out of some assumption set $C$ under the side condition that $t_2 = t_3 \rightarrow a$ for some $a$. Therefore, we view proof term construction as constraint solving where we rewrite constraint stores until all CTMs have been resolved.

The formal development is as follows. We assume that CTMs uses are attached to “locations”. The idea is that $i : \text{ct} M a b$ refers to some program text cast, where cast is used at type $a \rightarrow b$ and $i$ refers to the location (e.g., position in the abstract syntax tree). As before, we write $f : \text{ct} a b$ to refer to the proof term $f$ associated to a $\text{ct} a b$ assumption.

We employ Constraint Handling Rules (CHR) [5] to construct CTMs uses of CT assumptions. CHRs are a rule-based language for specifying transformations among constraints. A CHR simplification rule (R) $\phi(\tilde{c}) \sim \tilde{d}$ states that if we find a constraint matching the lhs of a rule we replace this constraint by the rhs. We assume that $\phi$ refers to type class constraints and $d$ is ref to either type class constraints or equations. Formally, we write $C \Rightarrow R C \sim \phi(\tilde{c}) \sim \tilde{d}$ where $\phi(\tilde{c}) = \tilde{d}$ for some substitution $\phi$. Silently, we assume the variables in CHRs are renamed before rule application.

A CHR propagation rule (R) $\phi(\tilde{c}) \sim \tilde{d}$ states that if we find a constraint matching the lhs of a rule we add the rhs to the store. Formally, we write $C \Rightarrow_R C, \phi(\tilde{d})$ where $\phi(\tilde{c}) = \tilde{d}$ such that $\phi(\tilde{c}) = \tilde{d}$. CHRs also have a logical reading which is not relevant here.

The CHR-based representation of the proof term construction rules can be found in Figure 4. Note that each CR simplification rule also introduces a transformation rule among expressions written $e \sim e'$. We write $C \Rightarrow R$ to denote an $n$ number of application of CHRs starting with the initial store $C$ yielding store $D'$. We write $e \sim R e'$ to denote a reduction sequence among expressions.

Proof rules (Arrow) and (Pair) from Figure 2 can be straightforwardly encoded in terms of CHRs. Note that rule (Trans) from Figure 2 has been split into rules (Trans1) and (Id). Our idea is to incrementally build CTM uses out of CT assumptions. A naive CHR-translation of transitivity such as

$$(\text{Trans}) \quad i : \text{ct} M a' b' \iff j : \text{ct} M a' b', k : \text{ct} M b b' \quad \text{cast}_i \sim \text{cast}_j \circ \text{cast}_k$$

leads to problems because we need to guess $b$. In CHR terminology, the above CHR is not range-restricted. We say a CHR is range-restricted if grounding the lhs grounds the rhs. Note that there is no rule (Var). The same effect can be achieved by rule (Trans1) in combination with rule (Id).

Example 10 Here is a sample derivation. We underline constraints involved in rule applications and silently perform equivalence transformations, replacing equals by equals. For brevity, we leave out castm transformations.

\[
\begin{align*}
g_1 : \text{ct} a [b, c], g_2 : \text{ct} b \text{ Int}, g_3 : \text{ct} c \text{ Bool}, \quad i : \text{ct} M a [\text{Int}, \text{Bool}] \\
\Rightarrow \text{Trans1} \quad g_1 : \text{ct} a [b, c], g_2 : \text{ct} b \text{ Int}, g_3 : \text{ct} c \text{ Bool}, \quad j : \text{ct} M b \text{ Int}. \\
\Rightarrow \text{Pair} \quad g_1 : \text{ct} a [b, c], g_2 : \text{ct} b \text{ Int}, g_3 : \text{ct} c \text{ Bool}, \quad k : \text{ct} M b \text{ Int}. l : \text{ct} M \text{ Int} \text{ Bool} \\
\Rightarrow \text{Trans1} \quad g_1 : \text{ct} a [b, c], g_2 : \text{ct} b \text{ Int}, g_3 : \text{ct} c \text{ Bool}, \quad m : \text{ct} M \text{ Int} \text{ Int}. n : \text{ct} M \text{ Int} \text{ Bool} \\
\Rightarrow \text{Trans1} \quad g_1 : \text{ct} a [b, c], g_2 : \text{ct} b \text{ Int}, g_3 : \text{ct} c \text{ Bool}, \quad m : \text{ct} M \text{ Int} \text{ Int}. n : \text{ct} M \text{ Int} \text{ Bool} \\
\Rightarrow * \quad g_1 : \text{ct} a [b, c], g_2 : \text{ct} b \text{ Int}, g_3 : \text{ct} c \text{ Bool} \\
\end{align*}
\]

In the above derivation, $\Rightarrow *$ represents a step derivation. □

There is also another set of rules which exclusively manipulates CT assumptions. In rule (Trans1) we make use of a CHR propagation rule to build the closure of all available CT assumptions. Note that we silently avoid to apply propagation rules twice on the same constraints (to avoid infinite propagation). Note that for each “decomposition” rule we introduce a propagation rule. The CHR representation of the rules from Example 7 and 8 can be found in Figure 4.

It should be clear now that simplification rules incrementally resolve CTM uses whereas propagation rules build the closure of all available CT assumptions. The following example stresses the importance of propagation rules.
CtM Simplification Rules:

\[
\begin{align*}
(Id) & \quad i : CtM a b \iff a = b \\
(Trans) & \quad g : Ct a b, i : CtM a' b' \quad \leadsto \quad g : Ct a b, a = a', j : CtM b' b \\
(Arrow) & \quad i : CtM (a_1 \rightarrow a_2) (b_1 \rightarrow b_2) \quad \leadsto \quad i_1 : CtM b_1, i_2 : CtM a_2 b_2 \\
(Pair) & \quad i : CtM (a_1, a_2) (b_1, b_2) \quad \leadsto \quad \lambda (x,y).((castm_i x), (castm_i y))
\end{align*}
\]

Figure 4: CHR-based Proof Term Construction

Example 11 Consider

\[
\begin{align*}
\text{Trans:} & \quad g : Ct a b, h : Ct a (b \rightarrow d), i : CtM c d \\
\implies & \quad g : Ct a b, a h : Ct a (b \rightarrow d), i : CtM c d \\
\text{Arrow:} & \quad g : Ct a b, h : Ct a (b \rightarrow d), i : CtM c d \\
\implies & \quad \lambda x. ((h \circ g) (\lambda y x)) \downharpoonright : Ct c d, i : CtM c d
\end{align*}
\]

Note that we can only apply (Arrow) after we have applied (Trans). □

Another important observation is that CHRs are “indeterministic”.

Example 12 Recall Example 10. We find the following alternative derivation.

\[
\begin{align*}
g_1 & : Ct a (b, c), g_2 : Ct b \text{Int}, g_3 : Ct c \text{Bool}, i : CtM a (\text{Int, Bool}) \\
\implies & \quad g_1 : Ct a (b, c), g_2 : Ct b \text{Int}, g_3 : Ct c \text{Bool}, i : CtM a (\text{Int, Bool}) \\
& \quad \text{Int, } c = \text{Bool}
\end{align*}
\]

Note that the final stores differ. Indeed, CHRs are nonequivalent. □

We say a set of CHRs is confluent if any sequence of derivation steps on the same initial store leads to the same (logically equivalent) final store. In Figure 4 rules (Id) and (Trans) overlap and therefore we might discover derivations with same initial store but different final stores.

However, we rule out derivations which yield “bad” final stores. Let \( C = \{ f_1 : Ct a_1 b_1, \ldots, f_n : Ct a_n b_n \} \) and \( i : CtM a b, C \implies D' \). We say that the CHR derivation is good if \( C \) and \( D' \) are logically equivalent, i.e., \( \models C \iff \exists f(D') = f(D) \). That is, we rule out derivations yielding stores with unresolved \( CtM \) uses, \text{False} and further instantiated \( Ct \) assumptions. Note that the derivation in Example 12 is bad because the \( Ct \) assumptions have been further instantiated in the final store.

We can state that our CHR-based method in Figure 4 is sound w.r.t. the system described in Figure 2. That is, each good derivation implies a valid proof. We can also guarantee to find a good derivation if a proof exists. Furthermore, any good derivation yields equivalent expressions.

Lemma 4 (Sound CHR Construction) Let \( C = \{ f_1 : Ct a_1 b_1, \ldots, f_n : Ct a_n b_n \} \) and \( i : CtM a b, C \implies D' \) and \( castm_i \sim_e e \) such that the \( Ct \) derivation is good. Then, \( f : Ct a b \iff C \) such that \( f \) and \( e \) are equivalent.

Lemma 5 (Complete CHR Construction) Let \( C = \{ f_1 : Ct a_1 b_1, \ldots, f_n : Ct a_n b_n \} \) such that \( f : Ct a b \iff C \). Then, \( i : CtM a b, C \implies C \) such that \( castm_i \sim_e e \) and \( f \) and \( e \) are equivalent.

Lemma 6 (Sound Term Construction) Let \( C = \{ f_1 : Ct a_1 b_1, \ldots, f_n : Ct a_n b_n \} \) and \( i : CtM a b, C \implies D_1 \) and \( castm_i \sim_e e_1 \) and \( i : CtM a b, C \implies D_2 \) and \( castm_i \sim_e e_2 \) such that both \( CHR \) derivations are good. Then, \( e_1 \) and \( e_2 \) are equivalent.

Proofs can be found in Appendix B.5

Note that in order to find a good derivation we might need to backtrack. See Examples 12 and 10. To obtain a decidable proof method we yet need to rule out certain \( CHR \) derivations. E.g., consider

\[
\begin{align*}
g & : Ct a b, h : Ct b a, i : CtM a b \\
\implies & \quad g : Ct a b, h : Ct b a, i : CtM a b \\
\implies & \quad g : Ct a b, h : Ct b a, i : CtM a b
\end{align*}
\]

Fortunately, we are able to rule out such non-terminating derivations by imposing stronger restrictions on good derivations. The crucial point is that we disallow “cyclic” \( Ct \) assumptions of the form \( g : Ct a (a, b) \). Such assumptions must result from invalid \( GRDT \) definitions which we generally rule out.

Lemma 7 We can impose a complete termination condition on good derivations.
We conclude that we obtain a decidable CHR-based proof term construction method. Our method is exponential in the worst-case. However, we believe that such cases will rarely appear in practice. An advantage of our method is that we can perform proof term construction under side conditions. This feature allows us to integrate our method with a general solving method for constructing typing derivations. Details are discussed in the next section.

7. COMBINING PROOF TERM CONSTRUCTION AND BUILDING TYPING DERIVATIONS

Our current translation method assumes full type annotations for the GRDT program. Type inference for GRDT is a challenging problem. However, it is mostly sufficient to provide annotations for function definitions only and omit type annotations for sub-expressions. In [23], we introduced a general type inference method for type classes with existential types. The idea is to generate "implication" constraints out of the program text. Solving of these constraints allows us to construct a typing derivation. The solving procedure for implication constraints is phrased as an extension to CHR solving. Hence, we can easily combine the inference method introduced in [23] with our CHR-based proof term construction method. Due to space limitations, we explain the approach by example only.

Consider the following TCET program from Example 4. For simplicity, we only consider one clause.

\[
\text{data Erk} \text{H} \ a \rightarrow \forall b. \ (\text{Ca} \ [b], \ C \ [b] \ a) \rightarrow \text{L} \text{J} \ a \\
\text{fJL} :: \text{Erk} \text{H} \ a \rightarrow \ a \\
\text{fJL} \ (\text{L} \text{J} \ x) = \text{cast} \ ((\text{cast tail}) \ (\text{cast x}))
\]

In a first step, we translate data types and patterns according to Figure 3 and replace all occurrences of \text{cast} in the program text by \text{castm} where each \text{castm} occurrence is attached to distinct locations.

\[
\text{data Erk} \text{H} \ ^i \ a = \forall b. \text{L} \text{J} \ a \ (\text{E} \ a \ [b]) \\
\text{fJL} :: \text{Erk} \text{H} \ ^i \ a \rightarrow \ a \\
\text{fJL} \ (\text{L} \text{J} \ x) = \text{castm} \ ((\text{castm} \ _{\text{tail}}) \ (\text{castm} \ _{\text{x}}))
\]

According to [23], we generate the following "implication" constraint out of the above program text.

\[
t = \text{Erk} \ a \rightarrow a, a = \text{Sk} \ _{1}, b = \text{Sk} \ _{2}, a, \ (g : \text{Ca} \ [b], \ h : \text{Ca} \ [b] \ a) \supset \ (1 : \text{CtM} \ a \ _{b}, \ b_{1} = a, \ 2 : \text{CtM} \ a \ _{b}, \ b_{2} = a \ [a \ [g] \rightarrow \ [a \ [h]], \ 3 : \text{CtM} \ a \ _{b}, \ b_{3} = a, \ b_{2} = b_{3} \rightarrow \ a_{1})
\]

Annotate \text{fJL} :: \text{Erk} \text{H} \ a \rightarrow \ a implies \text{fJL} :: \forall a. \text{Erk} \text{H} \ a \rightarrow \ a. Hence, we substitute \( b \) by the skolem constructor \text{Sk} \ _{1}. Similarly, we substitute \( b \) by \text{Sk} \ _{2}. Each \text{castm} expression gives rise to \( i : \text{CtM} \ a \ _{b} \) where \text{castm} :: \( a \rightarrow \ b \). To each \text{Ct} assumption we attach proof terms (see rule (P-K)). We make use of the TCET representation of GRDT but connect the constraints to ET proof terms. The interesting bit is the use of Boolean implication \( \supset \) to state that under the \text{Ct} assumptions we can derive the \text{CtM} uses.

The constraint in (1) represents all possible typing derivations. We simply solve this constraint by applying CHR in Figure 4 until all \text{CtM} uses have been resolved. Thus, all locations in the function body referring to proof terms are defined in terms of proof terms attached to \text{Ct} assumptions. In general, we solve \( \text{C} \_0, D \rightarrow C \_1 \) by running \( \text{C} \_0, D \rightarrow \ \supset \text{C} \_1 \) and check that \( \text{C} \_1 \) and \( \text{C} \_2 \) are logically equivalent (modulo variables in the initial store). We refer the interested reader to [23] for more details.

For the above constraint (1) we proceed as follows. We find that \( t = \text{Erk} \ a \rightarrow a, a = \text{Sk} \ _{1}, b = \text{Sk} \ _{2}, a, g : \text{Ca} \ [b], h : \text{Ca} \ [b] \ a \) (2) is immediately final. Consider,

\[
t = \text{Erk} \ a \rightarrow a, a = \text{Sk} \ _{1}, b = \text{Sk} \ _{2}, a, g : \text{Ca} \ [b], h : \text{Ca} \ [b] \ a, \ 1 : \text{CtM} \ a \ _{b}, b_{1} = a, \ 2 : \text{CtM} \ a \ _{b}, a \ [a \ [g] \rightarrow \ [a \ [h]], \ 3 : \text{CtM} \ a \ _{b}, b_{3} = a, \ b_{2} = b_{3} \rightarrow \ a_{1}, \ fJL \ (\text{L} \text{J} \ x) = \text{castm} \ _{\text{tail}} \ (\text{castm} \ _{\text{x}})
\]

We refer the interested reader to [23] for more details.

Note that we simultaneously transform constraints and program text. Constraints involved in rule applications are underlined. Silently, we extend \( e' \rightarrow e'' \) to \( e'[\cdot] \rightarrow e''[\cdot] \) where \( e[\cdot] \) denotes an expression with a hole. For clarity, we use let definitions instead of textually replacing expressions. Note that final constraints (2) and (3) are logically equivalent.
Hence, the translation is successful. Note that the final program text for the second derivation can be simplified to the second clause in Example 2. We note that several other derivations are possible. E.g., consider the following where we apply rule (I1) instead of (Trans).

\[
t = \text{Erk} a \rightarrow a, a = S k_1, b = S k_2 a, g : \text{Cl} a [b], h : \text{Cl} [b] a, 1 : \text{Cl} M a_1 b_1 b_1 = a,
\]

1. \(\vdash t = \text{Erk} a \rightarrow a, a = S k_1, b = S k_2 a, g : \text{Cl} a [b], h : \text{Cl} [b] a, b_1 = a, a_3 = [a_2] \rightarrow [a_2'], a_3 = a,
\]

2. \(\vdash b_1 = b_1 \rightarrow a_1, 1 : \text{Cl} M a_1 a_2.
\]

\(\vdash t = \text{Erk} a \rightarrow a, a = S k_1, b = S k_2 a, g : \text{Cl} a [b], h : \text{Cl} [b] a, a = [a_2] \rightarrow [a_2'], a_3 = a,
\]

2. \(\vdash b_1 = b_1 \rightarrow a_1, a = b_1, 1 : \text{Cl} M a_1 a_2.
\]

\[\vdash t = \text{Erk} a \rightarrow a, a = S k_1, b = S k_2 a, g : \text{Cl} a [b], h : \text{Cl} [b] a, b_1 = a, a_3 = [a_2] \rightarrow [a_2'], a_3 = a,
\]

2. \(\vdash b_1 = b_1 \rightarrow a_1, a = b_1, 1 : \text{Cl} M a_1 a_2.
\]

\[\vdash t = \text{Erk} a \rightarrow a, a = S k_1, b = S k_2 a, g : \text{Cl} a [b], h : \text{Cl} [b] a, b_1 = a, a_3 = [a_2] \rightarrow [a_2'], a_3 = a,
\]

2. \(\vdash b_1 = b_1 \rightarrow a_1, a = b_1, 1 : \text{Cl} M a_1 a_2.
\]

Note that skolem variable \(S k_1\) is unified with \([a_2]'\) which immediately yields failure. That is, we obtain a “bad” final store (see Appendix B.6 for details). However, there might be other derivations which yield “good” final stores. Each of them corresponds to a valid solution and all of them are equivalent (see Lemma 6). The following is another possible translation of Example 2.

\[
\forall \beta' (L \beta' = (E (g, b))) =
\]

let cast\_m \_g \_x = cast\_m (g (cast\_m \_x))

cast\_m = \_

let cast\_m = h

cast\_m \_x = \_

in cast\_m ((cast\_m tail) (cast\_m \_x))

8. RELATED METHOD

Our systematic translation method is inspired by the work by Baars and Swierstra [1], Chen, Zhu and Xi [2], Hinze and Guch [3]. These works showed by example how to express GRDT-style behavior by representing type equalities by Haskell terms and insert appropriate conversion functions into the program text. We note that none of these works considers a systematic translation scheme.

Note that in \([1, 3, 16]\) equality is represented in terms of the following definition.

\[
\text{newtype EEq a b = EEq (forall f. f a -> f b)}
\]

The above encodes Leibnitz’ law which states that if \(a\) and \(b\) are equivalent then we may substitute one for the other in any context. By construction this ensures that the only inhabitant of \(\text{Eq} a b\) is the identity (excluding non-terminating functions which might break this property). Our representation of equality makes it necessary to postulate that all values attached to monomorphic instances of \(E \times \tau\) represent the identity to ensure preservation of the semantics of programs (see Definition 1). On the other hand, the \(\text{Eq}\) representation faces problems when trying to manipulate proof terms. E.g., there are situations where we need to "decompose" a value of type \(\text{Eq}(a, b)\) into a value of type \(\text{Eq}(a, c)\) which is impossible based on the above definition. Example 6 shows that our representation of type equality shares the same problem. However, we believe that our representation is more likely to be decomposable.

Weirich [27] also considered a type class encoding based on single-parameter type classes. Our use of multi-parameter type classes in combination with existential types appears to be novel and more natural to mimic GRDT-style behavior.

Kislev [11] suggests an alternative type class encoding of GRDT. The gist of his idea is to turn each (value) pattern clause into an (type class) instance declaration. We believe that in addition to the already “problematic” instance declaration for transitivity such an encoding scheme may create further potentially non-terminating instances. We are not aware of any formal results which match the results stated in this paper.

Pottier and Gauthier [17] give a type-preserving definition-alization of polymorphic programs to System F extended with GRDT. Their formal results (proofs of Lemmas 4.1 and 4.2 in [17]) let us conjecture that resulting GRDT programs can be translated to ET based on our translation method.

Our proof term construction method can be seen as a refined version of the type-directed evidence-translation scheme [7] for Haskell. We could achieve a decidable construction for a seemingly non-terminating set of instances. There are some connections to methods for finding paths in graphs and “ask” constraints which appear in the context of constraintlogic programming [9]. We yet need to work out the exact details.

9. CONCLUSION

The primary goal of our work was to concisely study and restate the concepts of guarded recursive data types (GRDT), existential types (ET) and type classes (TSET). We could achieve this goal by giving for the first time a systematic translation method from GRDT to ET (Section 5) based on an intermediate translation to TSET (Section 4). For the translation method to be complete we require that types appearing in assumption constraints must be decomposable (Definition 4). We also assume full GRDT type information but are able to construct ET expressions automatically based on a novel CHR-based proof term construction method (Section 6). We can even combine our method with an independently developed type inference scheme for GRDT (Section 7). Hence, we obtain a fully automatic tool to translate GRDT to ET where the final program is accepted by GHC. In our experience, the decomposition condition which is crucial for translation is met by all GRDT examples found in the literature. A comprehensive list of examples can be found under \(^3\).

\(^3\)Examples are also part of the technical report version [25].
repeatedly apply such functions to elements of lists etc. A “smart” compiler may be able to avoid such redundant computations (either statically or dynamically). In this context, we would like to mention that GRDT have been recently added to Haskell. Implementations are available in the latest release of GHC [6] and Chameleon [23] (experimental version of Haskell). In case of GHC, the Core back-end has been extended with GRDT as a primitive feature. Clearly, we expect “native” GRDT code to run faster than “source-to-source translated” GRDT code. However, the advantage of our work is that we could identify a large class of GRDT programs which can be implemented by a source-to-source translation. Thus, our work offers a light-weight approach to write GRDT-style programs based on some existing language features.

Our proof term construction method is of independent interest and may prove to be useful to advance the state of art in type-directed translations for languages such as Haskell. This is another interesting avenue which we plan to explore in the future.

Acknowledgements

We thank Gregory Duck, Simon Peyton-Jones and Jeremy Wazny for their comments. In particular, we would like to thank James Cheney and Oleg Kiselyov for fruitful discussions.

10. REFERENCES


APPENDIX

A. SEMANTICS OF EXPRESSIONS

We follow the ideal semantics of MacQueen, Plotkin and Sethi [15]. The meaning of a term is a value in the CPO $\mathcal{V}$, where $\mathcal{V}$ contains all continuous functions from $\mathcal{V}$ to $\mathcal{V}$ and an error element $\mathbf{W}$, usually pronounced “wrong”. Depending on the concrete type system used, $\mathcal{V}$ might contain other elements as well. We assume that the values of additional type constructors are representable in the CPO $\mathcal{V}$. Then $\mathcal{V}$ is the least solution of the equation

$$\mathcal{V} = \mathbf{W} + \mathcal{V} \to \mathcal{V}.$$  

The meaning function on terms is as follows:

$$[x] = \eta(x)$$

$$[\lambda u.e] = \lambda v.[e]_v[v := v]$$

$$[e]_v = \begin{cases} [e]_v & \text{if } [e]_v \in \mathcal{V} \to \mathcal{V} \land [e']_v \notin \mathbf{W} \\ \text{then } [e]_v & \text{else } \mathbf{W} \end{cases}$$

$$[\text{let } x = e \text{ in } e'] = \begin{cases} [e]_v & \text{if } [e]_v \notin \mathbf{W} \\ \text{then } [e']_v[v := [e]_v] & \text{else } \mathbf{W} \end{cases}$$

Note that the above semantics is call-by-value.

B. PROOFS

B.1 Proof of Theorem 1 (GRDT to TCET)

First, we introduce an auxiliary definition and lemma to establish a connection between constructive type equality entailment and entailment among type classes.

**Definition 5** Let $C$ be a set of term equality constraints and $C'$ be a set of type class constraints. We say that $C$ is equivalent to $C'$, written as $C \sim C'$, iff $\forall t. t. t' \in C$ iff $(C t t' \in C \land C t t' \in C')$. We call $C'$ the "Ct" equivalent of $C$; and $C$ the "Eq" equivalent of $C'$.

**Lemma 8** Let $P_\varepsilon$ be a full and faithful type class theory. Let $C$ be a set of equality constraints and $C'$ its "Ct" equivalent. We have $C \vdash^e t_1 = t_2$ iff $P_\varepsilon \models C' \supset (C t t_1 t_2)$.

**Proof.** The proof is done in two directions. (Direction $\Rightarrow$): We proof by induction on derivations.

- **Case:**

  $$t = t' \in C$$

  $$C \vdash^e t = t'$$

  Because we have $t = t' \in C$, we know $C t t' \in C'$ and $C t t' \in C'$. Thus $P_\varepsilon \models C' \supset (C t t', C t t')$.

- **Case:**

  $$C \vdash^e t_1 = t_2 \quad C \vdash^e t_2 = t_3$$

  $$C \vdash^e t_1 = t_3$$

  By induction, we have

  $$P_\varepsilon \models C' \supset (C t t_1, C t t_2, C t t_3, C t t_3)$$

  We conclude

  $$P_\varepsilon \models C' \supset (C t t_1 t_2, C t t_2 t_3, C t t_3 t_4)$$

  Other cases are similar. □

The next lemma follows immediately from the rule (M).

**Lemma 9** $C, \Gamma \vdash^T \text{cast } : t \rightarrow t'$ iff $P_\varepsilon \models C \supset C t t'$

We obtain Theorem 1 as a special instance from the following lemma.

**Lemma 10** Let $e$ be a GRDT expression and $e'$ be its fully casted version. Let $P_\varepsilon$ a full and faithful program theory representing all GRDT type constructors mentioned in $e$. Silently, we transform the GRDT constructors mentioned in $e$ to TCET constructors. We have that $C, \Gamma \vdash^{G_e} e : t$ iff $C', \Gamma \vdash^T e' : t$ where $C'$ is the "Ct" equivalent of $C$.

**Proof.** The proof is done in two directions. (Direction $\Rightarrow$): We proof by induction on derivation.

- **Case (Eq):**

  $$C, \Gamma \vdash^{G_e} e : t$$

  $$C \vdash^e t = t'$$

  By the induction hypothesis, we have

  $$C', \Gamma \vdash_t e' : t$$ (1)
Also by Lemma 8 and $C \vdash_{t} t' = t''$ we have
\[ P_{p} \models C' \supset (C t t', C t' t'). \tag{2} \]
From (1) and (2), we conclude that
\[ C', \Gamma \vdash_{t} (\text{cast } e') : t'. \]

W.l.o.g. We can assume $e' \equiv (\text{cast } e'')$. Thus we obtain
\[ C', \Gamma \vdash_{t} ((\text{cast } \circ \text{cast }) e') : t'. \]
We assume $C', \Gamma \vdash_{t} e'' : t''$. In the above case, the first cast is of type $t \to t'$ and the second $t' \to t$. Thus by Lemma 9, we know that $C t t'$ and $C t' t'$ can be derived from the context. By the (Trans) type class instance, we can derive $C t t' t'$. Then by Lemma 9, we know there exists a cast of type $t'' \to t'$. After replacing the cast composition $\text{cast} \circ \text{cast}$ in the above judgement by the new cast, we obtain
\[ C', \Gamma \vdash_{t} (\text{cast } e') : t'. \]
This is equivalent to
\[ C', \Gamma \vdash_{t} e' : t'. \]

Case (App):
\[
C, \Gamma \vdash_{t}^{G_{c}} e_1 : t_2 \to t \quad C, \Gamma \vdash_{t}^{G_{c}} e_2 : t_2
\]
\[
\frac{C', \Gamma \vdash_{t} e_1 : t \quad C', \Gamma \vdash_{t} e_2 : t_2}{C', \Gamma \vdash_{t}^{G_{c}} e_1 e_2 : t}
\]

By the induction hypothesis, we have
\[ C', \Gamma \vdash_{t} e_1 : t \quad C', \Gamma \vdash_{t} e_1' : t_2 \]
By application of rule (App), we obtain
\[ C', \Gamma \vdash_{t} (\text{cast } (e_1', e_2)) : t \tag{1} \]
Note that we always have $C \vdash_{t} t = t$. Thus we conclude
\[ C', \Gamma \vdash_{t} (\text{cast } (e_1', e_2)) : t \]

Other cases are similar.

\textit{(Direction $\Leftarrow$):} We proceed by structural induction. We denote by $[e']$ the "erasure" of expression $e'$, i.e. we erase all cast occurrences from $e'$. W.l.o.g. We can assume $e' \equiv (\text{cast } e'')$.

\[ e'' = x \]
\[ \frac{C', \Gamma \vdash_{t} \text{cast } : t \to t' \quad C', \Gamma \vdash_{t} e'' : t}{C', \Gamma \vdash_{t} \text{cast } e'' : t'} \]

Because $e'' = x$, then $[e''] = e''$. Therefore, we have
\[ C, \Gamma \vdash_{t}^{G_{c}} [e''] : t \tag{1} \]
By $C', \Gamma \vdash_{t} \text{cast } : t \to t'$ and Lemma 9, we obtain
\[ P_{p} \models C' \supset (C t t', C t' t') \]
Together with Lemma 8, we have
\[ C \vdash_{t} t = t' \tag{2} \]
By (1), (2) and rule (Eq), we conclude
\[ C, \Gamma \vdash_{t}^{G_{c}} [e''] : t' \]
Because $[\text{cast } e''] = [e'']$, then we have
\[ C, \Gamma \vdash_{t}^{G_{c}} [\text{cast } e''] : t' \]
This is equivalent to
\[ C, \Gamma \vdash_{t}^{G_{c}} [e''] : t' \]

\[ o e'' = \lambda x.e'' \]
\[ \frac{C', \Gamma \vdash_{t} \text{cast } : t \to t' \quad C', \Gamma, x : t_1 \vdash_{t} e'' : t_2}{C', \Gamma, x : t_1 \vdash_{t}^{G_{c}} e'' : t_2} \]
\[ C', \Gamma \vdash_{t}^{G_{c}} (\text{cast } e'') : t' \]

In the above derivation $t = t_1 \to t_2$. By the induction hypothesis, we have
\[ C, \Gamma, x : t_1 \vdash_{t}^{G_{c}} [e'''] : t_2 \]
By applying the (Abs) rule, we obtain
\[ C, \Gamma \vdash_{t}^{G_{c}} [e'''] : t \tag{1} \]
By $C', \Gamma \vdash_{t} \text{cast } : t \to t'$ and Lemma 9, we obtain
\[ P_{p} \models C' \supset (C t t', C t' t') \]
Together with Lemma 8, we have
\[ C \vdash_{t} t = t' \tag{2} \]
By (1), (2) and rule (Eq), we conclude
\[ C, \Gamma \vdash_{t}^{G_{c}} [e'''] : t' \]
Because $[\text{cast } e''] = [e''']$, then we have
\[ C, \Gamma \vdash_{t}^{G_{c}} [\text{cast } e''] : t' \]
This is equivalent to
\[ C, \Gamma \vdash_{t}^{G_{c}} [e''] : t' \]

Other cases are similar. \qed
B.2 Proof of Lemma 2 (Well-Typed)

Our assumptions are: Let $\mathcal{C} = \{f_1 : \mathsf{Cl} \ a_1 \ b_1, ..., f_n : \mathsf{Cl} \ a_n \ b_n\}$ and $\Gamma$ such that $\mathcal{C} \succsim \Gamma$ and $f : \mathsf{Cl} \ a \ b \rightarrow \mathcal{C}$ is valid. Then $\Gamma \vdash f : a \rightarrow b$.

**Proof:** The proof proceeds by induction over the proof term construction derivation. W.l.o.g we combine rule ($\forall \mathsf{E}$) with rules ($\exists \mathsf{Id}$),($\exists \mathsf{Var}$),($\exists \mathsf{Arrow}$) etc. We also combine ($\exists \mathsf{E}$) with ($\mathsf{Trans}$).

- **Case ($\exists \mathsf{Id}$):**

  $\lambda x. x : \mathsf{Cl} \ a a \leftrightarrow \mathsf{True}$

  We know that $\Gamma = \emptyset$. Thus we conclude $\Gamma \vdash \lambda x. x : a \rightarrow a$.

- **Case ($\exists \mathsf{Var}$):**

  $f : \mathsf{Cl} \ a b \leftrightarrow f : \mathsf{Cl} \ a b$

  We know that $\Gamma = \{f : a \rightarrow b\}$. Thus we conclude $\Gamma \vdash f : a \rightarrow b$.

- **Case ($\mathsf{Trans}$):**

  $f = \lambda g, \lambda x. f g (f_1 x)$

  $f : \mathsf{Cl} \ a_1 a_3 \leftrightarrow f_1 : \mathsf{Cl} \ a_1 a_2, f_2 : \mathsf{Cl} \ a_2 a_3$

  We know that $\Gamma = \{f_1 : a_1 \rightarrow a_2, f_2 : a_2 \rightarrow a_3\}$. Thus by typing derivation we can easily conclude $\Gamma \vdash f : a_1 \rightarrow a_3$.

- **Case ($\mathsf{Arrow}$):** Similar to ($\mathsf{Trans}$).

- **Case ($\eta$):**

  $f : \mathsf{Cl} \ a b \leftrightarrow f_1 : c_1, ..., f_n : c_n, f_1 : c_i \leftrightarrow F_i$

  $F = F_i$ for $i = 1, ..., n$

  $f : \mathsf{Cl} \ a b \leftrightarrow F$

  By induction, we have $\bigcup_i^\Gamma \Gamma_i \vdash f : a \rightarrow b$. Because $\bigcup_i^\Gamma \Gamma_i \subseteq \Gamma$ derived from $F = F_i$, then we conclude $\Gamma \vdash f : a \rightarrow b$.

\[\square\]

B.3 Proof of Theorem 2 (TCET to ET Soundness)

Theorem 2 follows directly from the following more general lemma.

**Lemma 11** Let $\mathcal{C}, \Gamma \vdash^E e : t$, $\mathcal{C}, \Gamma \vdash^E e : t \sim e'$ and $\Gamma'$ such that $\mathcal{C} \succsim \Gamma'$. Then $\Gamma \cup \Gamma' \vdash^E e' : t$.

**Proof:** The proof proceeds by induction on derivations.

- **Case ($\mathcal{K}$):**

  $(K' : \forall a \cdot \tilde{b} \cdot t \rightarrow E \ t_1 \ t_2 \rightarrow ... \rightarrow E \ t_n \ t'_n \rightarrow T \ \tilde{a})$

  $C, \Gamma \vdash^E e : \tilde{[a/\tilde{a}]}t \sim e'$

  $P_p \vdash C \supset (g, h) : \mathcal{C} \vdash^E (E \ t_1 \ t_2 \rightarrow ... \rightarrow E \ t_n \ t'_n \rightarrow T \ \tilde{a})$

  $C, \Gamma \vdash^E Ke : T \ t \sim K' e' \ E \ (g, h_1) \ ... \ (g_n, h_n)$

  By the induction hypothesis, we have

  $\Gamma \cup \Gamma' \vdash^E e' : \tilde{[a/\tilde{a}]}t$ (1)

  Also we have

  $K' : \forall a \cdot \tilde{b} \cdot t \rightarrow E \ t_1 \ t_2 \rightarrow ... \rightarrow E \ t_n \ t'_n \rightarrow T \ \tilde{a}$ (2)

  Note that here we assume an ordering among the constraints. $P_p \vdash C \supset (g, h) : \mathcal{C} \vdash^E (E \ t_1 \ t_2 \rightarrow ... \rightarrow E \ t_n \ t'_n)\ \implies$

  $g_k : \mathsf{Cl} \ t_i \ t'_i \rightarrow C$ and $h_i : \mathsf{Cl} \ t_i \ t'_i \rightarrow C$

  W.l.o.g we can assume $g_k, h_i \notin \Gamma$. Hence by Lemma 2, we have

  $\Gamma \cup \Gamma' \vdash^E e \ (g_k, h_i) : E \ t_i \ t'_i \ \text{where} \ i = 1, ..., n$ (3)

  From (1),(2),(3) and rule ($\mathcal{K}$), we conclude

  $\Gamma \cup \Gamma' \vdash^E K' e' \ E \ (g_1, h_1) \ ... \ (g_n, h_n) : T \ \tilde{t}$

  **Case ($\mathsf{Reduce}$):**

  $D \subseteq C \ f : \mathsf{Cl} \ t_1 \ t_2 \rightarrow D$

  $C, \Gamma \vdash^E \mathsf{cast} : t_1 \rightarrow t_2 \sim f$

  Given $D \subseteq C \ f : \mathsf{Cl} \ t_1 \ t_2 \rightarrow D$, W.l.o.g we assume $f \notin \Gamma$. Thus we conclude by Lemma 2

  $\Gamma \cup \Gamma' \vdash^E f : t_1 \rightarrow t_2$

  **Case (Pat):**

  $p : t_1 \vdash \forall b \cdot (D \mathcal{C} p \ p') \ \tilde{b} \cap f v \mathcal{C} \Gamma, t_2 \ = \emptyset$

  $C \land D, \Gamma \cup \Gamma' \vdash^E e : t_2 \sim e'$

  $C, \Gamma \vdash^E p \rightarrow e : t_1 \rightarrow t_2 \sim p' \rightarrow e'$

  By the induction hypothesis, we have

  $\Gamma \cup \Gamma' \vdash^E \ p' \ ightarrow e' : t_1 \rightarrow t_2$

  Where $C \sim \Gamma \ C$ and $D \sim \Gamma \ D$.

  Also by Lemma 12 (see below), we have $p' \vdash \forall b \cdot (\Gamma' \cup \Gamma \ D)$. Thus we conclude

  $\Gamma \cup \Gamma' \sim^E \ p' \ ightarrow e' : t_1 \rightarrow t_2$

  Other cases are standard. \[\square\]

**Lemma 12** Given $p : t_1 \vdash \forall b \cdot (D \mathcal{C} p \ p')$ then $p' \vdash \forall b \mathcal{C} \Gamma' \cup \Gamma \ D$.

**Proof:** Standard by induction on derivation. \[\square\]
B.4 Proof of Theorem 3 (TCET to ET Completeness)

Theorem 3 follows directly from the following lemma.

Lemma 13 Let $C, \Gamma \vdash^T e : t$ and all types appearing in assumption constraints in intermediate derivations are decomposable. The $C, \Gamma \vdash^T e : t \rightsquigarrow e'$ for some $e'$.

Proof. The proof is done by construction of $e'$.

- Case (Reduce):
  Note that cast is a class method of type $\forall t, t'. Ct t' \Rightarrow t \rightarrow t'$. Since we have $C, \Gamma \vdash^T \text{cast} : t_1 \rightarrow t_2$, by rule (M), we can derive $P_{b} \vdash C \supset C t_1 t_2$.

  Given all the types are decomposable, by Lemma 3, we know $f : Ct t_1 t_2 \leftrightarrow C$ for some $f$ if $P_{b} \vdash C \supset C t_1 t_2$. Thus the rule (Reduce) always produces a $f$.

- Case:
  Other rules are standard.

B.5 Proofs of Lemmas 4, 5 and 6

B.5.1 Proof of Lemma 4 (Sound CHR Construction)

Our assumptions are: Let $C = \{f_1 : Ct a_1 b_1, ..., f_n : Ct a_n b_n\}$ and $i : CtM a, b, C \Rightarrow^* D'$ and $\text{cast}_{m_i} \rightsquigarrow e$ such that the CHR derivation is good. Then, $f : Ct a b \leftrightarrow C$ such that $f$ and $e$ are equivalent.

Proof. The proof is done through induction on the CHR derivation. W.l.o.g we combine rule ($\forall E$) with rules (Id), (Var), (Arrow) and (Pair). We also combine ($\exists E$) with (Trans).

- Suppose the rule applied is (Id):
  
  $i : CtM a, b, C \Rightarrow a = b, C \Rightarrow^* D'$
  
  $\text{cast}_{m_i} \rightsquigarrow \lambda x. x$

  Note that the above derivation unifies $a$ and $b$. Thus we have

  $\lambda x. x : Ct a a \leftrightarrow \text{True}$.

- Suppose the rule applied is (Trans1):

  $i : CtM a, b, C \Rightarrow a_2 = a, j : CtM b_2, b, C \Rightarrow^* D'$
  
  $\text{cast}_{m_i} \rightsquigarrow \text{cast}_{m_j} \circ g$

  Note that the above derivation unifies $a$ and $a_2$. Thus we have

  $f = \text{cast}_{m_j} \circ g$

  (Trans)

  $f : Ct a b \leftrightarrow g : Ct a_2 b_2, \text{cast}_{m_j} : Ct b_2 b$

  $f : Ct a b \leftrightarrow D$.

  where $g : Ct a_2 b_2 \subseteq C$. Also by induction, we know $j: Ct b_2 b \leftrightarrow D'$ for some $D' \subseteq C$. Take $D$ as $D'$, we have $D \subseteq C$.

- Suppose the rule applied is (Arrow):

  $i : CtM (a_1 \rightarrow a_2) (b_1 \rightarrow b_2), C \Rightarrow i_1 : CtM b_1 a_1, i_2 : CtM a_2 b_2, C \Rightarrow^* D'$
  
  $\text{cast}_{m_i} \rightsquigarrow \lambda g. \lambda x. \text{cast}_{m_{i_1}} (g (\text{cast}_{m_{i_2}} x))$

  Also we have

  $f = \lambda g. \lambda x. \text{cast}_{m_{i_1}} (g (\text{cast}_{m_{i_2}} x))$

  (Trans)

  $f : Ct (a_1 \rightarrow a_2) (b_1 \rightarrow b_2) \leftrightarrow \text{cast}_{m_{i_1}} : Ct b_1 a_1, \text{cast}_{m_{i_2}} : Ct a_2 b_2$

  Also by induction, we know $j : Ct b_1 a_1 \leftrightarrow D'$ and $j : Ct a_2 b_2 \leftrightarrow D''$ for some $D' \subseteq C$ and $D'' \subseteq C$. Take $D$ as $D' \cup D''$, we have $D \subseteq C$.

- Case (Arrow) is similar to (Arrow).

B.5.2 Proof of Lemma 5 (Complete CHR Construction)

Our assumptions are: Let $C = \{f_1 : Ct a_1 b_1, ..., f_n : Ct a_n b_n\}$ such that $f : Ct a b \leftrightarrow C$. Then, $i : CtM a, b, C \Rightarrow^* C$ such that $\text{cast}_{m_i} \rightsquigarrow e$ and $f$ and $e$ are equivalent.

Proof. W.l.o.g we combine rule ($\forall E$) with rules (Id), (Var), (Arrow) and (Pair). We also combine ($\exists E$) with (Trans).

- Case (Id):

  $\lambda x. x : Ct a a \leftrightarrow \text{True}$

  Then we have

  $i : CtM a, a, C \Rightarrow i_{id} a = a, C$

  $\text{cast}_{m_i} \rightsquigarrow \lambda x. x$

  - Case (Var).

    $f : Ct a b \leftrightarrow f : Ct a b$

    Then we have, given $f : Ct a b \in C$

    $i : CtM a, b, C \Rightarrow_{\text{Trans1}} j : CtM b, b, C \Rightarrow_{\text{id}} C$

    $\text{cast}_{m_i} \rightsquigarrow \text{cast}_{m_j} \circ f \rightsquigarrow \lambda x. x \circ f$

  - Case (Trans).

    $f = \lambda g. f_1 (f_1 x)$

    (Trans)

    $f : Ct a_1 a_2 \leftrightarrow f_1 : Ct a_1 b_2, f_2 : Ct a_2 a_3$

    We have

    $i : CtM a_1 a_3, f_1 : Ct a_1 a_2, f_2 : Ct a_2 a_3$

    $\Rightarrow_{\text{Trans1}} j : Ct a_2 a_3, f_1 : Ct a_1 a_2, f_2 : Ct a_2 a_3$

    $\Rightarrow_{\text{Trans1}} k : CtM a_3 a_4, f_1 : Ct a_1 a_2, f_2 : Ct a_2 a_3$

    $\Rightarrow_{\text{id}} f_2 \circ f_1 : Ct a_1 a_2, f_2 : Ct a_2 a_3$

    $\Rightarrow_{\text{id}} \lambda x. x \circ f_2 \circ f_1$

- Case (Arrow).
By induction,
\[ C, i_1 : CtM b_1 a_1 \Rightarrow^* D_1 \]
\[ castm_{i_1} \Rightarrow^* f_1 \]
\[ C, i_2 : CtM a_2 b_2 \Rightarrow^* D_2 \]
\[ castm_{i_2} \Rightarrow^* f_2 \]

Therefore
\[ i : CtM (a_1 \rightarrow a_2) (b_1 \rightarrow b_2), f_1 : Ct b_1 a_1, f_2 : Ct a_2 b_2 \]
\[ \Rightarrow^*_{\text{Arrow}} \]
\[ i_1 : CtM b_1 a_1, i_2 : CtM a_2 b_2, f_1 : Ct b_1 a_1, f_2 : Ct a_2 b_2 \]
\[ \Rightarrow^*_{\text{pair}} \]
\[ \lambda g. \lambda x. castm_{i_2} (g (castm_{i_1} x)) \]
\[ \Rightarrow^*_{\text{pair}} \]
\[ f_1 : Ct b_1 a_1, f_2 : Ct a_2 b_2 \]
\[ \Rightarrow^*_{\text{pair}} \]
\[ \lambda g. \lambda x. f_2 (g (f_1 x)) \]
\[ \text{Pair is similar to } (\text{Arrow}). \]

\[ \Box \]

B.5.3 Proof of Lemma 6 (Sound Term Construction)

Our assumptions are:
Let \( \{ f_1 : Ct a_1 b_1, \ldots, f_n : Ct a_n b_n \} \), \( i : CtM a b, C \Rightarrow^* D_1 \) and \( castm_i \Rightarrow^* e_1 \) and \( i : CtM a_1 b_1, C \Rightarrow^* D_2 \) and \( castm_i \Rightarrow^* e_2 \) such that both CHR derivations are good. Then, \( e_1 \) and \( e_2 \) are equivalent.

Proof: Let \( f : Ct a b \leftrightarrow C \), from Lemma 4, we know that \( e_1 \) is equivalent to \( f \) and \( e_2 \) is equivalent to \( f \). Thus we conclude that \( e_1 \) is equivalent to \( e_2 \).

B.6 Termination of CHRs

We impose an order among derivations. Let \( C = \{ f_1 : Ct a_1 b_1, \ldots, f_n : Ct a_n b_n \} \), \( i : CtM a b, C \Rightarrow^* D_1 \) and \( castm_i \Rightarrow^* e_1 \) and \( i : CtM a b, C \Rightarrow^* D_2 \) and \( castm_i \Rightarrow^* e_2 \) such that both CHR derivations are good. We say that \( i : CtM a b, C \Rightarrow^* D_1 \) is shorter than \( i : CtM a b, C \Rightarrow^* D_2 \) if the size of \( e_1 \) is shorter than the size of \( e_2 \) where the size function returns the number of nodes in the syntax tree of an expression. In case of initial stores with multiple CtMs we compare the sum of the individual sizes of resulting expressions.

Lemma 14 Let \( i : CtM t t, C \Rightarrow^* D \) be a good derivation. Then, \( castm_i \Rightarrow^* e \) where \( e \) is equivalent to the identity.

Lemma 15 Any good derivation which violates the termination condition can be shortened.

Proof. We assume a good derivation which violates the termination condition where we consider the ‘earliest’ violation in the derivation.

\[ C \]
\[ \Rightarrow \]
\[ C_1, (g : Ct t_1 t_2)_{t_1, i_1 : CtM t_1 t_2} _{t_1, l_1 \notin L_1} \]
\[ \Rightarrow_{\text{Trans1}} C_1, (g : Ct t_1 t_2)_{t_1, l_1 \notin L_1} \]
\[ \Rightarrow \]
\[ C_2, (g : Ct t_1 t_2)_{t_1, l_1 \notin L_2} \]
\[ \Rightarrow_{\text{Trans2}} C_2, (g : Ct t_1 t_2)_{t_1, l_1 \notin L_2} \]
\[ \Rightarrow \]
\[ D \]

W.l.o.g., in the derivation steps between (1) and (2) we only apply CHRrs on \( j : CtM t_1 t_2 \) or its successors, i.e. those resulting from (Trans1) and (Arrow) rules.

First, we show that only (Trans1) or (Id) rules could have been applied on \( j : CtM t_1 t_2 \) or its successors. Assume the contrary, that is some (Pair) or (a similar type constructor) rule has been applied on \( j : CtM t_1 t_2 \). Then,
\[ \Rightarrow_{\text{pair}} \]
\[ \ldots, (g : Ct t_1 t_2)_{t_1 = (t_5, t_6), t_2 = (t_6, t_7), (j : CtM t_5 t_6 \cup t_7, j_1 : CtM t_5 t_6 \cup t_7) \]
However, then we obtain a cycle among types. E.g., assume that \( (j_1 : CtM t_5 t_6 \cup t_7) \) equals \( (k : CtM t_5 t_6 \cup t_7) \). We find that \( t_1 = t_5, t_2 = (t_5, t_6), t_3 = (t_6, t_7) \), which implies \( (g : Ct t_1 t_2)_{t_1, l_1} \). Thus, we obtain a contradiction. Note that by assumption the type equations resulting from Ct constraints \( Ct a b \) must be satisfied. Otherwise, the GRDT definition is invalid.

Hence, we only find (Trans1) or (Id) applications in between (1) and (2). Effectively, we generate a cast function to convert \( t_5 \) into some \( b \) which we then convert back into \( t_1 \). However, any such transformation yields a cast function which is equivalent to the identity. See Lemma 14. Hence, the steps between (1) and (2) are redundant. Hence, we obtain a shorter derivation.

Lemma 16 CHRs are terminating under the termination condition.
PROOF. Follows immediately. Note that we disallow $Ct$ assumptions of the form $g : Ct a \ (a, b)$. Hence, any non-terminating derivation must violate the termination condition. □