

Kent Academic Repository

Full text document (pdf)

Citation for published version

Villa, Cristiano (2014) A cautionary note on using the scale prior for the parameter N of a binomial distribution. *Ecology*, 95 (9). pp. 2674-2677. ISSN 0012-9658.

DOI

Link to record in KAR

<http://kar.kent.ac.uk/47159/>

Document Version

Author's Accepted Manuscript

Copyright & reuse

Content in the Kent Academic Repository is made available for research purposes. Unless otherwise stated all content is protected by copyright and in the absence of an open licence (eg Creative Commons), permissions for further reuse of content should be sought from the publisher, author or other copyright holder.

Versions of research

The version in the Kent Academic Repository may differ from the final published version.

Users are advised to check <http://kar.kent.ac.uk> for the status of the paper. **Users should always cite the published version of record.**

Enquiries

For any further enquiries regarding the licence status of this document, please contact:

researchsupport@kent.ac.uk

If you believe this document infringes copyright then please contact the KAR admin team with the take-down information provided at <http://kar.kent.ac.uk/contact.html>

1 A cautionary note on using the scale prior for the
2 parameter N of a binomial distribution

3 Cristiano Villa and Stephen G. Walker

4 January 28, 2014

5 **Abstract**

6 Statistical analysis of ecological data may require the estimation of the size of
7 a population, or of the number of species with a certain population. This task fre-
8 quently reduces to estimating the discrete parameter N representing the number of
9 trials in a binomial distribution. In Bayesian methods, there has been a substantial
10 amount of discussion on how to select the prior for N . We propose a prior for
11 N based on an objective measure of the *worth* that each value of N has in being
12 included in the model space. This prior is compared (through the analysis of the
13 popular snowshoe hare dataset) with the scale prior which, in our opinion, cannot
14 be understood from solid objective considerations.

15 **Keywords** abundance, binomial, Kullback–Leibler divergence, loss function,
16 objective prior

17 **1 Introduction**

18 In this paper we discuss objective prior distributions for the discrete parameter N of a
19 binomial distribution, with specific applications to estimation of population or species

20 sizes. In particular, we argue that in statistical applications the scale prior $\pi(N) \propto 1/N$
21 should not be employed for it is lacking a probabilistic interpretation.

22 In the statistical analysis of ecological data it is frequent to deal with data that comes
23 from binomial outcomes, such as the size of a population or the number of species. The
24 capture-recapture models for closed population introduced by Otis et al. (1978) represent
25 an example on how the estimation of N proceeds in wildlife data analysis.

26 A common choice of objective prior for N is the scale prior, that is $\pi(N) \propto 1/N$.
27 Recently, (Link, 2013) has shown its support to the scale prior for N on the basis of
28 its better performance in comparison to the uniform prior and, in addition, that it has
29 been proposed by Berger et al. (2012). We argue that there is no real motivation in the
30 use of the scale prior; on the contrary, it appears to be an *ad-hoc* solution rather than
31 the result of specific probabilistic considerations. In other words, $\pi(N) \propto 1/N$ has no
32 “meaning”. We believe that a way of defining an objective prior for N has to take into
33 considerations the reason why a particular value of the parameter has been included in
34 the parameter space $\mathcal{N} = \{1, 2, \dots\}$. In particular, the objective approach defines losses
35 instead of probabilities. This idea is discussed in Villa & Walker (2013a) and Villa &
36 Walker (2013b).

37 It is noteworthy to point out that the scale prior has been used by Wang et al. (2007),
38 King & Brooks (2008) (and the references therein), for applications in ecology, and by
39 Basu & Ebrahimi (2001), for an example of an application in capture-recapture models
40 in software reliability.

41 The organisation of the paper is as follows. In Section 2 we discuss some background
42 on objective priors for N , and define the prior we propose. Section 3 shows a comparison
43 of the scale prior with our by analysing the popular snowshoe hare data. Finally, Section
44 4 includes some discussion points and general considerations.

2 Objective priors for N

Consider $x \sim \text{Bin}(N, p)$, where $N \in \mathcal{N} = \{1, 2, \dots\}$ represents the number of independent Bernoulli trials, and $p \in (0, 1)$ the probability of success at each trial. The aim is to make inference on the discrete parameter N , assuming p is unknown.

The task of assigning an objective prior to a discrete parameter is not a trivial and, in the past, has represented an interesting challenge. The main reason comes from the fact that common objective approaches such as Jeffreys' rule (Jeffreys, 1961) and reference analysis (Berger et al., 2009) are not suitable for discrete parameters and, when they are, they do not provide sensible results. Note that the uniform prior $\pi(N) \propto 1$, which may appear to be a natural choice to represent ignorance about N , is not suitable for inference as, for when p is unknown, leads to an improper posterior (Berger et al., 1999, 2012).

A motivation behind the choice of $1/N$ is that, although Jeffreys himself never discussed the prior for N when p is unknown, the choice of $\pi(N) \propto 1/N$ is assumed as natural (Berger et al., 2012), as it is the prior Jeffreys recommends for (continuous) scale parameters. Link (2013), in addition to the above motivation, recommends the scale prior as it solves estimation problems related to the use of the uniform prior (when \mathcal{N} is finite).

The choice of $1/N$ as an objective prior for N is questionable for the following reasons. The motivation for Jeffreys prior in a discrete setting is obsolete. Jeffreys rule is based on invariance property under one-to-one transformations of the parameter of interest, and this notion has no meaning for a discrete parameter space. Furthermore, Kahn (1987), shows that if we assign a Beta prior to p , $\pi(p) \sim \text{Be}(a, b)$, and assume the parameters of the binomial independent a priori, then $\pi(N) \propto 1/N^c$ yields a proper posterior for N if $a + c > 1$. It is therefore legitimate to wonder why c has to be chosen as equal to one. Why not, for example, $\pi(N) \propto 1/N^2$ or $\pi(N) \propto 1/N^3$? This fact adds a level of

70 subjectivity and arbitrariness to the whole procedure, making the process not as objective
71 as intended.

One may argue that the scale prior is the result of a different objective procedure as well. Berger et al. (2012) use an approach which consists in embedding the discrete problem into a continuous one and then apply reference analysis. However, as there exist more than one embedding procedure, they obtain two different priors: the scale prior and

$$\pi(N) \propto \sqrt{N\{N + 4/(n + 3)\}}$$

72 where n is the size of independent and identically distributed random variables: $X_i \sim$
73 $Bin(N, p)$, $i = 1, \dots, n$. As both priors have similar properties, the recommendation of
74 $1/N$ lays in the simplicity of its functional form. Again, the choice of the scale prior does
75 not appear to be truly objective.

76 It is fundamental to highlight that in an applied (statistical) setting, such as in ecology,
77 an objective prior needs an idea which is well supported. Unlike academic statisticians,
78 who can discourse on objective priors on theoretical grounds, applied statisticians have to
79 put the motive first: an objective prior needs to have a meaning. In fact, the derivation
80 of $\pi(N)$ should be the result of a process where there is a clear explanation on why a
81 particular prior is chosen and what it represents; we find, for example, that in Link (2013)
82 this explanation is missing, and that the justification in adopting the scale prior is just a
83 reminder to someone else's work.

84

85 The prior we propose is based on the idea of assigning a *worth* to each element $N \in \mathcal{N}$.
86 The *worth* is objectively measured by assessing what is lost if that parameter value is
87 removed from \mathcal{N} , and it is the true one. Once the *worth* has been determined, this will
88 be linked to the prior probability by means of the self-information loss function (Merhav

89 & Feder, 1998) $-\log \pi(N)$. A detailed illustration of the idea can be found in Villa &
 90 Walker (2013a) and Villa & Walker (2013b), but here is an overview.

91 Let us indicate by f_N the binomial distribution with parameters N , give p (for the
 92 moment assumed to be known). The utility (i.e. *worth*) to be assigned to f_N is a
 93 function of the Kullback–Leibler divergence (Kullback & Leibler, 1951) measured from
 94 the model to the nearest one; where the nearest model is the one defined by $N' \neq N$
 95 such that $D_{KL}(f_N \| f_{N'})$ is minimised. In fact (see Berk (1966)) N' is where the posterior
 96 asymptotically accumulates if N is excluded from \mathcal{N} . The objectivity of how the utility
 97 of f_N is measured is obvious, as it depends on the choice of the model only.

98 Let us now write $u_1(N) = \log \pi(N)$ and let the minimum divergence from f_N be
 99 represented by $u_2(N)$. Note that $u_1(N)$ is the utility associated with the prior probability
 100 for model f_N , and $u_2(N)$ is the utility in keeping N in \mathcal{N} . We want $u_1(N)$ and $u_2(N)$ to
 101 be matching utility functions, as they are two different ways to measure the same utility
 102 in N . As it stands, $-\infty < u_1 \leq 0$ and $0 \leq u_2 < \infty$, while we actually want $u_1 = -\infty$
 103 when $u_2 = 0$. The scales are matched by taking exponential transformations; so $\exp(u_1)$
 104 and $\exp(u_2) - 1$ are on the same scale. Hence, we have

$$e^{u_1(N)} = \pi(N) \propto e^{g\{u_2(N)\}}, \quad (1)$$

105 where

$$g(u) = \log(e^u - 1). \quad (2)$$

106 By setting the functional form of g in (1), as it is defined in (2), we derive the proposed
 107 objective prior for the discrete parameter N

$$\pi(N) \propto \exp \left\{ \min_{N \neq N' \in \mathcal{N}} D_{KL}(f_N \| f_{N'}) \right\} - 1. \quad (3)$$

108 We note that in this way the Bayesian approach is conceptually consistent, as we
 109 update a prior utility assigned to N , through the application of Bayes theorem, to ob-
 110 tain the resulting posterior utility expressed by $\log \pi(N|x)$. Indeed, there is an elegant
 111 procedure akin to Bayes which works from a utility point of view, namely that

$$\log \pi(N|x) = K + \log f_N(x|N) + \log \pi(N),$$

112 which has the interpretation of

$$\text{Utility}(N|x, \pi) = K + \text{Utility}(N|x) + \text{Utility}(N|\pi),$$

113 where K does not depend on N . There is then a retention of meaning between the prior
 114 and the posterior information (here represented as utilities). This property is not shared
 115 by the usual interpretation of Bayes theorem when priors are objectively obtained; in
 116 fact, the prior would usually be improper, hence not representing probabilities, whilst
 117 the posterior is (and has to be) a proper probability distribution.

118 In Villa & Walker (2013a) we show that the nearest model to f_N is at $N' = N + 1$.

119 Thus, the prior for N is given by

$$\pi(N) \propto \frac{1}{(N+1)(1-p)} \exp \left\{ \sum_{x=0}^N \log(N+1-x) \binom{N}{x} p^x (1-p)^{N-2} \right\} - 1. \quad (4)$$

120 The prior in (4) is improper but, with just one observation, yields a proper posterior.

121 If p is unknown, the joint prior distribution for the parameters of the binomial is given



Figure 1: Snowshoes hare in its natural habitat.

122 by

$$\pi(N, p) = \pi(N|p)\pi(p), \quad (5)$$

123 where $\pi(N|p)$ is the prior in (4) above, and $\pi(p)$ a suitable prior for the probability of
124 success at each trial.

125 **3 Snowshoes hares analysis**

126 To illustrate the objective prior we propose, and to compare it with the scale prior, we
127 analyse a popular capture-recapture data set. The problem has been originally discussed
128 in Otis et al. (1978) and, from a Bayesian perspective, for example in Royle et al. (2007)
129 and Link (2013). In particular, Link (2013) has analysed the data using a scale prior for
130 N (although using a data augmentation approach).

131

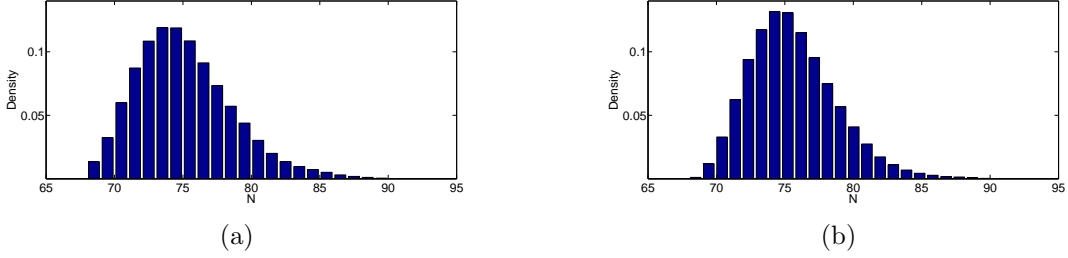


Figure 2: Histogram of the posterior distribution for the parameter N for the hare data using scale prior (a) and our prior (b).

132 The data consists of a sample of $n = 68$ hares captured-recaptured, over $T = 6$ days.
 133 The encounter frequencies, over the 6 days, gives the set $\{25, 22, 13, 5, 1, 2\}$, that is for a
 134 total of 145 capture-recapture occurrences. For this illustration, we consider model for
 135 closed populations M_0 , as defined in Otis et al. (1978), which assumes that the capture-
 136 recapture probabilities are constant for all the animals and across the 6 days. Thus,
 137 indicating by y_i the detection frequency of animal i , with $i = 1, \dots, N$, the likelihood
 138 function is given by

$$L(N, p|y) \propto \frac{N!}{(N-n)!} p^{\sum_i y_i} (1-p)^{T \cdot N - \sum_i y_i}. \quad (6)$$

139 We analyse the data by considering both the scale prior and our prior for N . For the
 140 scale prior, we have $\pi(N, p) = \pi(N)\pi(p)$, assuming prior independence of the parameters.
 141 When we use the prior (4), the joint prior has the form of (5). In both circumstances we
 142 set $\pi(p) \sim Be(1/2, 1/2)$, that is Jeffreys' prior. As the posterior distributions are analyt-
 143 ically intractable, we obtain the marginal distribution for N through MCMC methods.

144 The histogram of the posterior distributions are plotted in Figure 2. The posterior for
 145 N obtained by applying the scale prior $\pi(N) \propto 1/N$ is shown in (a), while the posterior
 146 obtained by applying the prior we propose in (4) is shown in (b). Both distributions are
 147 positively skewed and accumulate on the same values of N . When the scale prior is used,
 148 the median is $N = 81.5$, with 95% credible interval (68.7, 94.3). When our prior is used,

149 we have a median of $N = 81.0$ and 95% credible interval (68.7, 93.4); note that prior (4)
150 gives a smaller credible interval than the one obtained by adopting the scale prior.

151 For completeness, we note that for p we have medians $p = 0.33$ in both cases, with
152 95% credible intervals (0.24, 0.43) and (0.24, 0.42), for the scale and our prior respectively.

153 4 Discussion

154 The choice of an objective prior for N must be based not only on performance, but also on
155 solid motivation. If this assumption is not met, it may appear that an objective approach
156 is justifiable as long as the adopted prior leads to a posterior distribution that is suitable
157 for inference (i.e. proper) and that has appealing performances. In the example of the
158 hare data, we have shown that the prior based on losses results in a credible interval that
159 is narrower than the one obtained by applying the scale prior for N . Additionally, while
160 the latter prior has no probabilistic justification, the former one is the result of a clear
161 objective motivation.

162 The prior for N can be applied to any of the remaining capture-recapture models
163 (Otis et al., 1978), that is when either one or more effects (time effects, behavioral effects,
164 heterogeneity effects) are considered. We have not included any example, either simulated
165 or based on real data, for models including time, behavioral or heterogeneity effects.
166 However, the implementation is similar to the one outlined.

167 References

168 BASU, S. & EBRAHIMI, N. (2001). Bayesian capture-recapture methods for error de-
169 tection and estimation of population size: Heterogeneity and dependence. *Biometrika*
170 **88**, 269–79.

- 171 BERGER, J. O., BERNARDO, J. M., & SUN, D. (2009). The formal definition of
172 reference priors. *Annals of Statistics* **37**, 905–38.
- 173 BERGER, J. O., BERNARDO, J. M., & SUN, D. (2012). Objective priors for discrete
174 parameter spaces. *Journal of the American Statistical Association* **107**, 636–48.
- 175 BERGER, J. O., LISEO, B., & WOLPERT, R. L. (1999). Integrated likelihood methods
176 for eliminating nuisance parameters. *Statistical Science* **18**, 1–28.
- 177 BERK, R. H. (1966). Limiting behaviour of posterior distributions when the model is
178 incorrect. *Annals of Mathematical Statistics* **37**, 51–8.
- 179 JEFFREYS, H. (1961). *Theory of Probability*. University Press, Oxford.
- 180 KAHN, W. D. (1987). A cautionary note for Bayesian estimation of the binomial pa-
181 rameter n . *American Statistician* **41**, 38–39.
- 182 KING, R. & BROOKS, S. P. (2008). On the Bayesian estimation of a closed population
183 size in the presence of heterogeneity and model uncertainty. *Biometrics* **64**, 816–24.
- 184 KULLBACK, S & LEIBLER, R. A. (1951). On information and sufficiency. *Annals of*
185 *Mathematical Statistics* **22**, 79–86.
- 186 LINK, W. A. (2013). A cautionary note on the discrete uniform prior for the binomial
187 N . *Ecology* **94**, 2173–9.
- 188 MERHAV, N. & FEDER, M. (1998). Universal prediction. *IEEE Transactions on Infor-*
189 *mation Theory* **44**, 2124–47.
- 190 OTIS, D. L., BURNHAM, K. P., WHITE, G. C., & ANDERSON, D. R. (1978). *Sta-*
191 *tistical inference from capture data on closed animal populations*. *Wildlife Monographs*
192 **64**, 1–135.

- 193 ROYLE, J. A., DORAIO, R. M. & LINK, W. A. (2007). *Analysis of multinomial models*
194 *with unknown index using data augmentation. Journal of Computational and Graphical*
195 *Statistics* **16**, 67–85.
- 196 VILLA, C. & WALKER, S. G. (2013a). An objective approach to prior mass functions
197 for discrete parameter spaces. *Journal of the American Statistical Association* Revision
198 submitted.
- 199 VILLA, C. & WALKER, S. G. (2013b). Objective prior for the number of degrees of
200 freedom of a t distribution. *Bayesian Analysis* To appear.
- 201 WANG, X, HE, C. Z. & SUN, D. (2007). Bayesian population estimation for small sam-
202 ple capture-recapture data using noninformative priors. *Journal of Statistical Planning*
203 *and Inference* **137**, 1099–118.