On a Schwarzian PDE associated with the KdV Hierarchy

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Abstract

We present a novel integrable non-autonomous partial differential equation of the Schwarzian type, i.e. invariant under Möbius transformations, that is related to the Korteweg-de Vries hierarchy. In fact, this PDE can be considered as the generating equation for the entire hierarchy of Schwarzian KdV equations. We present its Lax pair, establish its connection with the SKdV hierarchy, its Miura relations to similar generating PDE's for the modified and regular KdV hierarchies and its Lagrangian structure. Finally we demonstrate that its similarity reductions lead to the *full* Painlevé VI equation, i.e. with four arbitary parameters.

1 Introduction

In this note we introduce a non-autonomous 1+1-dimensional nonlinear evolution equation for the variable $z(s,t)$, namely:

$$\begin{aligned} z_{sstt} &= z_{sst}\Big(\frac{z_{st}}{z_s}+\frac{z_{tt}}{z_t}\Big)+z_{stt}\Big(\frac{z_{st}}{z_t}+\frac{z_{ss}}{z_s}\Big)-z_{st}\Big(\frac{z_{st}z_{ss}}{z_s^2}+\frac{z_{st}z_{tt}}{z_t^2}+\frac{z_{ss}z_{tt}}{z_sz_t}\Big)\\ &\quad+\frac{1}{s-t}\Big[\frac{s}{t}\Big(z_{sst}-\frac{z_{st}z_{ss}}{z_s}-\frac{1}{2}\frac{z_{st}^2}{z_t}\Big)-\frac{t}{s}\Big(z_{stt}-\frac{z_{st}z_{tt}}{z_t}-\frac{1}{2}\frac{z_{st}^2}{z_s}\Big)\Big]\\ &\quad-\frac{1}{(s-t)^2}\Big[n^2\frac{s^2}{t^2}\frac{z_s}{z_t}\Big(z_{st}-\frac{z_sz_{tt}}{z_t}\Big)+m^2\frac{t^2}{s^2}\frac{z_t}{z_s}\Big(z_{st}-\frac{z_tz_{ss}}{z_s}\Big)\Big]\\ &\quad-\frac{1}{2}\frac{1}{(s-t)^3}\Big[n^2\frac{s}{t}z_s\Big(1+\frac{(4t-3s)s}{t^2}\frac{z_s}{z_t}\Big)-m^2\frac{t}{s}z_t\Big(1+\frac{(4s-3t)t}{s^2}\frac{z_t}{z_s}\Big)\Big]\ . \end{aligned} \tag{1.1}$$

We will refer to this equation simply as *the* Schwarzian PDE (SPDE). It is straightforward to verify that it is invariant under Möbius transformations

$$z(s,t)\ \mapsto\ \frac{\alpha z(s,t)+\beta}{\gamma z(s,t)+\delta}\ ,$$

with constant α,β,γ,δ. It is straightforward to verify also that the equation (1.1) arises as the Euler-Lagrange equations from the following Lagrangian density:

$$\mathcal{L}[z(s,t)]=\frac{1}{2}\frac{(s-t)}{st}\frac{z_{st}^2}{z_sz_t}+\frac{1}{2(s-t)}\Big(n^2\frac{s}{t^3}\frac{z_s}{z_t}+m^2\frac{t}{s^3}\frac{z_t}{z_s}\Big)\ . \tag{1.2}$$

Note that the lagrange density $\mathcal{L}$ depends explicitly on the independent variables s and t. We further assert the following properties of the SPDE:

1. The SPDE (1.1) is integrable in the conventional soliton sense: there exists a Lax pair and an infinity of conservation laws.
2. The SPDE (1.1) is part of a *compatible* parameter-family of PDEs, meaning that the same dependent variable z obeys eq. (1.1) simultaneously for all possible pairs of parameters n,m (with each parameter being associated with a different independent variable). We will make this statement more precise in what follows.
3. This infinite family of PDEs is equivalent to the Schwarzian KdV hierarchy via expansions on the *independent* variables.
4. As indicated already the system carries a Lagrangian structure. We expect that from this Lagrangian one can infer that the system is multi-Hamiltonian, although we will not discuss this point further here.
5. There exists a Miura Chain, i.e. a short sequence of differential relations connecting the SPDE with other generating PDEs sharing the above properties with the SPDE.
6. The similarity reduction of the SPDE under scaling symmetry reduces the SPDE to the full sixth Painlevé equation (PVI) with four arbitrary parameters.
7. The SPDE has a fully discrete counterpart which is integrable in its own right.

2 Properties of the SPDE and related generating PDEs

We now demonstrate or give arguments for the properties that we listed in the Introduction.

Lax pair

The following linear overdetermined set of equations constitute a Lax pair for the SPDE:

$$
\begin{aligned}
2t\varphi_t &= n\begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}\varphi + \frac{1}{\kappa - t}\begin{pmatrix} -\kappa n(1+a) & 2t^2 z_t \\ \kappa n^2(1-a^2)/(2tz_t) & -nt(1-a) \end{pmatrix}\varphi, \qquad (2.1a)\\
2s\varphi_s &= m\begin{pmatrix} 1 & 0 \\ 0 & b \end{pmatrix}\varphi + \frac{1}{\kappa - s}\begin{pmatrix} -\kappa m(1+b) & 2s^2 z_s \\ \kappa m^2(1-b^2)/(2sz_s) & -ms(1-b) \end{pmatrix}\varphi. \qquad (2.1b)
\end{aligned}
$$

In the above $a = a(s,t)$ and $b = b(s,t)$ are auxiliary variables. The compatibility conditions arising from the consistency of cross-differentiating (2.1a) and (2.1b) are

$$
nsa_s = mtb_t = \frac{1}{2(s-t)}\left[n^2\frac{s^2}{t}\frac{z_s}{z_t}(1-a^2) - m^2\frac{t^2}{s}\frac{z_t}{z_s}(1-b^2)\right] , \qquad (2.2a)
$$

$$
stz_{st} = \frac{mt^2 z_t b - ns^2 z_s a}{s-t} . \qquad (2.2b)
$$

The SPDE (1.1) follows immediately from the relations (2.2a) and (2.2b). In fact, differentiating (2.2b) with respect to s and with respect to t yields relations for a_t and b_s, whilst (2.2a) gives a_s and simultaneously b_t. Thus, with these expressions we can cross-differentiate, e.g. $\partial_t a_s = \partial_s a_t$, to obtain an equation where we can use the previous obtained expressions to to eliminate all a's and b's (miraculously, it turns out that the single remaining a or b drops out entirely at the end).

We note that the Lax pair (2.1) falls in the general class of Lax pairs of *non-isospectral type*[1] that were treated in [1]. However, although general families of Lax equations were postulated in that paper, the precise reduction which leads to (2.1) was not investigated there.

Consistency and Integrability

The SPDE (1.1) is compatible with itself in the following sense: if one considers for one and the same function $z = z(t_1, t_2, t_3, \dots)$ of many independent variables $t_1, t_2, t_3, \dots$ copies of the equation (1.1) in terms of any two distinct variables t_i, t_j replacing s and t in(1.1), each associated with its own parameters n_i, n_j replacing n and m in (1.1), then all these copies of the SPDE on one and the same function z are consistent. Thus, we have effectively an infinite family of commuting flows all given by the same equation, but with different parameters: in other words the SPDE represents a *parameter-family of consistent PDE's*.

[1] This does not mean that the spectral parameter κ depends on s,t: κ is constant w.r.t. the independent variables. However, the poles in the Lax matrices depend explicitly on s and t.

This statement can be verified by direct calculation, but since (1.1) is not in evolution form the verification is most easily done using the Lax pair. In fact, suppose we augment the system (2.1) with a third part, i.e. an equation similar to (2.1a) and (2.1b) but now in terms of a third independent variable (u say) associated with a new parameter ℓ (instead of n or m) and a third auxiliary variable c. Obviously the compatibility relations with the original members of the Lax pair, namely with (2.1a) or (2.1b) lead to extra relations of the same type as (2.2) involving now relations for a_u, b_u, c_s, c_t as wel as z_{su} and z_{tu}. We have to check the consistency of these relations, i.e. the validity of expressions such as $\partial_u a_s = \partial_s a_u$ and $\partial_u z_{st} = \partial_t z_{su} = \partial_s z_{tu}$. This follows by tedious but straightforward direct calculation.

As is clear from the variational equations from the Lagrangian (1.2), namely

$$\frac{\delta \mathcal{L}}{\delta z} = \frac{\partial^2}{\partial s \partial t}\left(\frac{\partial \mathcal{L}}{\partial z_{st}}\right) - \frac{\partial}{\partial s}\left(\frac{\partial \mathcal{L}}{\partial z_s}\right) - \frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial z_t}\right) = 0 \ , \tag{2.3}$$

the SPDE can be written as a conservation law. Since we know from the above argument that the SPDE can be imposed on $z = z(s, t, u, \dots)$ in as many independent variables as we want (each associated with its own parameter n,m,ℓ, $\dots$), we derive in this way an infinity of conservation laws all for the same object z. What is potentially nontrivial is to rewrite these conservation laws in terms of only one pair of preferred variables s and t say. It is only at this point that possible nonlocalities in the explicit expressions for the conservation laws may occur. We will not discuss this point here in detail any further, but postpone this discussion to future investigations of the SPDE.

Connection with SKdV Hierarchy

Using the expansion of the independent variables:

$$\frac{\partial}{\partial t} = -\frac{n}{t^{1/2}} \sum_{j=1}^{\infty} \frac{1}{t^j} \frac{\partial}{\partial x_j} \quad , \quad \frac{\partial}{\partial s} = -\frac{m}{s^{1/2}} \sum_{j=1}^{\infty} \frac{1}{s^j} \frac{\partial}{\partial x_j} \ , \tag{2.4}$$

where $x_1, x_2, \dots$ is the infinite sequence of higher times associated with the KdV hierarchy, we can expand the Lagrangian (1.2) in powers of s and t as follows:

$$\begin{aligned} \mathcal{L} \quad = \quad & \frac{1}{2}\frac{mn}{(st)^{3/2}}\left\{\frac{s+t}{s-t} + \left(\frac{1}{s} - \frac{1}{t}\right)\left[\frac{z_{x_2}}{z_{x_1}} - \frac{z_{x_1x_1}^2}{z_{x_1}^2}\right]\right. \\ & \left. + \left(\frac{1}{s^2} - \frac{1}{t^2}\right)\left[\frac{z_{x_3}}{z_{x_1}} - 2\frac{z_{x_1x_1}z_{x_1x_2}}{z_{x_1}^2} + \frac{z_{x_1x_1}^2 z_{x_2}}{z_{x_1}^3}\right] + \dots\right\} \ . \end{aligned} \tag{2.5}$$

Thus, order by order we obtain a sequence of Lagrangians which turns out to be exactly the Lagrangians for the SKdV hierarchy, the first equations in the sequence reading

$$\frac{z_{x_2}}{z_{x_1}} = \{z, x_1\} \equiv \frac{z_{x_1x_1x_1}}{z_{x_1}} - \frac{3}{2}\frac{z_{x_1x_1}^2}{z_{x_1}^2} \ , \tag{2.6a}$$

which is the Schwarzian KdV equation, cf. [2, 3], and

$$\frac{z_{x_3}}{z_{x_1}} = 2\frac{\partial^2}{\partial x_1^2}\left(\frac{z_{x_2}}{z_{x_1}}\right) + \frac{3}{2}\frac{z_{x_2}^2}{z_{x_1}^2} \ , \tag{2.6b}$$

which is the first higher-order SKdV equation. (Strictly speaking the sequence of Lagrangians appearing in (2.6) yield the x_1 derivatives of this hierarchy of equations.) The expansions (2.4) amount exactly to the transition from the higher time variables in the hierarchy to so-called Miwa coordinates, [4]. We mention that the notion of a generating PDE for a hierarchy of integrable nonlinear evolution equations, and their corresponding Lagrangians, is in spirit the same as the idea of "compounding hierarchies" that was put forward some years ago by one of the authors, cf. [5]. We refer also to [6] for related issues concerning the Baker-Akhiezer function for the KP hierarchy in terms of Miwa coordinates.

Miura Chain

The different PDEs in the Miura chain are related via the sequence:

$$\textsf{Schwarzian PDE} \quad \longrightarrow \quad \textsf{Modified PDE} \quad \longrightarrow \quad \textsf{Regular PDE}$$

The MPDE (Modified PDE) is the equation for a variable $v(s,t)$ that is connected to the auxiliary variables $a(s,t)$ and $b(s,t)$ via:

$$na = -2t\partial_t \log v \quad , \quad mb = -2s\partial_s \log v \ , \tag{2.7}$$

and the relevant equation can be obtained from (2.2a) in the form

$$\partial_s\partial_t \log v = \frac{nm}{4st(s-t)}\left[t(1-a)(1+b)Y - s(1+a)(1-b)\frac{1}{Y}\right] \ , \tag{2.8a}$$

together with

$$\begin{aligned} 2st\partial_s\partial_t \log Y \quad &= \quad ns\partial_s\left[(1-Y)\frac{2tY+(s-tY)(1+a)}{(t-s)Y}\right] \\ &\quad - mt\partial_t\left[(1-Y)\frac{2s-(s-tY)(1+b)}{(t-s)Y}\right] . \end{aligned} \tag{2.8b}$$

Solving from the quadratic equation (2.8a) for

$$Y \equiv \frac{m}{n}\frac{tz_t}{sz_s}\frac{(1-b)}{(1-a)} \tag{2.9}$$

in terms of v and its derivatives (which enter in a and b via (2.7)) and substituting the result into eq. (2.8b) we obtain a complicated-looking PDE in terms of v, of second degree in the highest derivative v_{sstt}: that equation is the MPDE, but because of its length we omit the explicit formula.

Finally, the Regular PDE, by which we mean the PDE that upon expansion yields actually the entire hierarchy of KdV equations, is given by

$$
\begin{aligned}
U_{sstt} &= U_{sst}\Big(\frac{1}{s-t}+\frac{U_{st}}{U_s}+\frac{U_{tt}}{U_t}\Big)+U_{stt}\Big(\frac{1}{t-s}+\frac{U_{st}}{U_t}+\frac{U_{ss}}{U_s}\Big)-U_{st}\frac{U_{ss}U_{tt}}{U_sU_t} \\
&+U_{ss}\Big(\frac{m^2}{(s-t)^2}\frac{U_t^2}{U_s^2}-\frac{U_{st}^2}{U_s^2}-\frac{1}{s-t}\frac{U_{st}}{U_s}\Big)+U_{tt}\Big(\frac{n^2}{(s-t)^2}\frac{U_s^2}{U_t^2}-\frac{U_{st}^2}{U_t^2}+\frac{1}{s-t}\frac{U_{st}}{U_t}\Big) \\
&+\frac{m^2}{2(s-t)^3}\frac{U_t}{U_s}\Big(U_s+U_t+2(t-s)U_{st}\Big)-\frac{n^2}{2(s-t)^3}\frac{U_s}{U_t}\Big(U_s+U_t+2(s-t)U_{st}\Big) \\
&+\frac{1}{2(s-t)}U_{st}^2\Big(\frac{1}{U_s}-\frac{1}{U_t}\Big)\ .
\end{aligned}
\tag{2.10}
$$

This equation derives from the Lagrangian:

$$
\mathcal{L}[U(s,t)]=\frac{1}{2}(s-t)\frac{U_{st}^2}{U_sU_t}+\frac{1}{2(s-t)}\Big(n^2\frac{U_s}{U_t}+m^2\frac{U_t}{U_s}\Big)\ . \tag{2.11}
$$

It is remarkable that the only apparent difference between the Lagrangians (1.2) and (2.11) resides in the way the variables s and t enter.

The PDE (2.10) arises from the following Lax pair:

$$
\phi_t = \frac{n}{2t^{1/2}}\begin{pmatrix}0&0\\1&0\end{pmatrix}\phi-\frac{1}{\kappa-t}\begin{pmatrix}n-AU_t & U_t\\ A(n-AU_t) & AU_t\end{pmatrix}\phi \tag{2.12a}
$$

$$
\phi_s = \frac{m}{2s^{1/2}}\begin{pmatrix}0&0\\1&0\end{pmatrix}\phi-\frac{1}{\kappa-s}\begin{pmatrix}m-BU_s & U_s\\ B(m-BU_s) & BU_s\end{pmatrix}\phi \tag{2.12b}
$$

where $A(s,t)$ and $B(s,t)$ are some auxiliary variables. The non-isospectral Lax pair (2.12) is gauge-equivalent to the Lax representation (2.1) via the gauge transformation

$$
\varphi=\begin{pmatrix}S & v\\ \kappa/v & 0\end{pmatrix}\phi\ , \tag{2.13}
$$

in which S is yet another auxiliary variable which is determined by the consistency relation of (2.13) with both Lax relations (2.1) and (2.12).

The compatibility conditions between (2.12a) and (2.12b) yield the coupled relations

$$
U_{st} = \frac{1}{s-t}\Big[mU_t-nU_s+2(A-B)U_sU_t\Big] \tag{2.14a}
$$

$$
A_s = \frac{m}{2s^{1/2}}+\frac{1}{t-s}\Big[(A-B)^2U_s+m(A-B)\Big] \tag{2.14b}
$$

$$
B_t = \frac{n}{2t^{1/2}}+\frac{1}{s-t}\Big[(A-B)^2U_t-n(A-B)\Big]\ . \tag{2.14c}
$$

Combining (2.14b) and (2.14c) we obtain a coupled equation for $A-B$ and U, which yields the PDE (2.10) after eliminating $A-B$ by solving for the latter in (2.14a).

The Miura transformation between the Regular PDE (2.10) and the Modified PDE given by the system (2.8) can be constructed on the basis of the various intermediate relations between the objects U,Y,a,b,A and B and is given by

$$\frac{(s-t)U_{st}-mU_t-nU_s}{(s-t)U_{st}+mU_t+nU_s}=\frac{(n+2tw_t)U_s+(m-2sw_s)U_t}{(n-2tw_t)U_s+(m+2sw_s)U_t}\ , \tag{2.15}$$

where $w \equiv \log v$.

Finally, we remark that the analogue of the famous Cole-Hopf transformation that interpolates between the SKdV equation and the MKdV equation is given by an implicit transformation that interpolates between the SPDE (1.1) and the Modified PDE (2.8) and that can be straightforwardly inferred from (2.9) in combination with (2.8a).

Connection with PVI

The similarity reduction of the SPDE is obtained via the constraint

$$\mu z + t z_t + s z_s = 0\ . \tag{2.16}$$

Using the constraint to elminate the derivatives with respect to one of the independent variables (s, say) and introducing the new variable [2]

$$W(t) = 1 + \frac{t}{\mu}\frac{z_t}{z}$$

we obtain from the SPDE a third-order ODE for W which reads:

$$\begin{aligned} W''' = {} & 2\left(\frac{1}{W}+\frac{1}{W-1}\right)W'W'' - 2\left(\frac{1}{t}+\frac{1}{t-s}\right)W'' \\ & -\left(\frac{1}{W^2}+\frac{1}{(W-1)^2}+\frac{1}{W(W-1)}\right)W'^3 - \frac{(11t-5s)W+2s-5t}{2(s-t)tW(W-1)}W'^2 \\ & + \frac{n^2s^2W^3-m^2t^2(W-1)^3+\mu^2(s-t)^2W^3(W-1)^3+2(s-t)tW^2(W-1)^2}{(s-t)^2t^2W^2(W-1)^2}W' \\ & + \frac{n^2s^2W^2[(s-t)W-t]-m^2t^2(W-1)^2[(s-t)W+t-2s]-\mu^2(s-t)^2W^2(W-1)^2[(s+t)W-t]}{2(s-t)^3t^3W(W-1)}\ , \end{aligned} \tag{2.17}$$

where the prime denotes differentiation with respect to t. Eq. (2.17) can be integrated once using the integrating factor

$$\frac{t(t-s)[(t-s)W-t]}{W(W-1)}$$

leading to the following second-order equation for W:

$$\begin{aligned} W'' = {} & \frac{1}{2}\left(\frac{1}{W}+\frac{1}{W-1}+\frac{t-s}{(t-s)W-t}\right)W'^2 - \left(\frac{1}{t}+\frac{1}{t-s}-\frac{W-1}{(t-s)W-t}\right)W' \\ & + \frac{W(W-1)[(t-s)W-t]}{2t(t-s)}\left(\frac{\mu^2}{t}-\frac{m^2}{(t-s)W^2}+\frac{n^2s}{t(t-s)(W-1)^2}-\frac{\nu^2 s}{((t-s)W-t)^2}\right) \\ & , \end{aligned} \tag{2.18}$$

[2] Alternatively, the similarity reduction can be obtained by explicitly solving the constraint (2.16) as $z(s,t) = (st)^{-\mu/2}Z(s/t)$. This leads to a third order equation for $Z(s/t)$, namely the Schwarzian PVI which we found in [11] via a different approach.

where the integration constant is conveniently chosen as $\nu^2 s$ with ν independent of s, t (noting that W depends on s, t through the combination s/t). It is not hard to see that eq. (2.18) is equivalent to the Painlevé VI equation ($\mathrm{P_{VI}}$) after the trivial change of variables $t/(t-s) \mapsto \tau$. Thus, we obtain PVI in the standard form

$$\begin{aligned}\frac{d^2w}{d\tau^2} &= \frac{1}{2}\left(\frac{1}{w}+\frac{1}{w-1}+\frac{1}{w-\tau}\right)\left(\frac{dw}{d\tau}\right)^2 - \left(\frac{1}{\tau}+\frac{1}{\tau-1}+\frac{1}{w-\tau}\right)\frac{dw}{d\tau} \\ &\quad + \frac{w(w-1)(w-\tau)}{2\tau^2(\tau-1)^2}\left(\mu^2 - m^2\frac{\tau}{w^2} + n^2\frac{\tau-1}{(w-1)^2} - \nu^2\frac{\tau(\tau-1)}{(w-\tau)^2}\right) .\end{aligned} \qquad \mathrm{P_{VI}}$$

for $w(\tau) = W(t)$. with the identification of the parameters as follows:

$$\alpha = \mu^2 \quad , \quad \beta = m^2 \quad \gamma = n^2 \quad , \quad \delta = \nu^2 \tag{2.19}$$

(we have rescaled the parameters compared with [11]). PVI, which incidentally was first found by R. Fuchs in [7] in 1905, (and not by either Painlevé or Gambier as is ocasionally claimed in the recent literature), was found to be connected to the lattice KdV systems in [8]. It is known that PVI is related to the Toda lattice, cf. e.g. [9, 10]. In a recent paper [11] we presented the full Miura chain associated with PVI, consisting of ordinary differential as well as ordinary difference equations, showing that in fact the parameter-family of ODEs that constitute PVI has a natural description in terms of both discrete as well as continuous equations. The Miura chain for PVI is just the reduction of the Miura chain for the SPDE under the similarity constraint (2.16), although we implimented this constraint differently in [11].

Thus the SPDE (1.1) yields PVI as similarity reduction with four arbitrary parameters. As far as we are aware this is the first case of an integrable scalar PDE that reduces to the full PVI equation. In the literature, cf. e.g. [12]-[15] PVI was obtained in several circumstances, either from the reduction of the three-wave resonant interaction system or in the context of the Einstein equations. However, in these instances only special parameter-cases of PVI were obtained. In [16] a connection between PVI and so-called generalised Ernst equations was studied, which seems to give a connection with full PVI, but the starting point is a complicated system of equations. What we have demonstrated here is that the full PVI equation is directly connected to the KdV hierarchy.

Connection with Equations on the Lattice

We finish by pointing out that the SPDE (1.1) as well as the other generating PDEs in the Miura chain are not only compatible as parameter-families of PDEs in the sense pointed out above, but also are compatible with a system of *discrete* equations. In fact, the actual derivation of the SPDE was based on this underlying structure which involves continuous as well discrete components. The following statement holds:

Proposition: The SPDE (1.1) is consistent with the following set of differential-difference equations for $z(s,t) = z_{n,m}(s,t)$:

$$-\,t\frac{\partial z_{n,m}}{\partial t} \;=\; n\frac{(z_{n+1,m}-z_{n,m})(z_{n,m}-z_{n-1,m})}{z_{n+1,m}-z_{n-1,m}} \tag{2.20a}$$

$$-s\frac{\partial z_{n,m}}{\partial s} \;=\; m\frac{(z_{n,m+1}-z_{n,m})(z_{n,m}-z_{n,m-1})}{z_{n,m+1}-z_{n,m-1}}\ , \tag{2.20b}$$

in which the parameters of the SPDE play the role of independent discrete variables.

Actually one can show by direct calculation that the operations of shifting in the variables n and m governed by the equations (2.20) commute with the continuous evolution according to the flows in terms of the variables s and t. A similar statement holds for the other generating PDEs, where the differential-difference equations for $v(s,t) = v_{n,m}(s,t)$ read

$$-2t\frac{\partial}{\partial t}\log v_{n,m} = n a_{n,m}\quad ,\quad -2s\frac{\partial}{\partial s}\log v_{n,m} = n b_{n,m}\ , \tag{2.21}$$

with the auxiliary variables $a(s,t) = a_{n,m}(s,t)$ and $b(s,t) = b_{n,m}(s,t)$ which appeared in the Lax representation (2.1) given explicitly in terms of $v_{n,m}$ by the relations

$$a_{n,m} \equiv \frac{v_{n+1,m}-v_{n-1,m}}{v_{n+1,m}+v_{n-1,m}}\quad ,\quad b_{n,m} \equiv \frac{v_{n,m+1}-v_{n,m-1}}{v_{n,m+1}+v_{n,m-1}}\ . \tag{2.22}$$

Also for the object $U(s,t) = U_{n,m}(s,t)$ we have a set of differential-difference equations compatible with the generating PDE (2.10), namely

$$\frac{\partial U_{n,m}}{\partial t} = \frac{n}{U_{n+1,m}-U_{n-1,m}}\quad ,\quad \frac{\partial U_{n,m}}{\partial s} = \frac{m}{U_{n,m+1}-U_{n,m-1}}\ . \tag{2.23}$$

Finally, to complete the picture we mention the associated fully discrete equation for the variable $z_{n,m}(s,t)$, i.e the partial *difference* equation (PΔE)

$$\frac{(z_{n,m}-z_{n+1,m})(z_{n,m+1}-z_{n+1,m+1})}{(z_{n,m}-z_{n,m+1})(z_{n+1,m}-z_{n+1,m+1})} = \frac{s}{t}\ . \tag{2.24}$$

The equation (2.24), which was first given in [17], is possibly the most fundamental equation in the entire structure, and it has been studied at length in the context of discrete conformal maps, see e.g. the recent monograph [18]. Note that in (2.24) the independent variables s,t of the SPDE now enter as parameters of the discrete equation. In [19] (see also [20]) the similarity reduction on the lattice was formulated, namely by imposing the following constraint which is compatible on the lattice with the equation (2.24):

$$n\frac{(z_{n+1,m}-z_{n,m})(z_{n,m}-z_{n-1,m})}{z_{n+1,m}-z_{n-1,m}} + m\frac{(z_{n,m+1}-z_{n,m})(z_{n,m}-z_{n,m-1})}{z_{n,m+1}-z_{n,m-1}} = \bar{\mu} z_{n,m}\ , \tag{2.25}$$

It was, however only in the recent paper [11] that a closed-form third-order ordinary difference equation (OΔE) was found that represents the similarity reduced equation, namely

$$
\begin{aligned}
&(r^2-1)(z_{n+1}-z_n)^2 = \\
&= \left[2r^2\frac{\mu z_{n+1}(z_{n+2}-z_n)-(n+1)(z_{n+2}-z_{n+1})(z_{n+1}-z_n)}{(m-\mu-\nu)(z_{n+2}-z_n)+(n+1)(z_{n+2}-2z_{n+1}+z_n)}+z_{n+1}-z_n\right]\times \\
&\quad\times\left[2r^2\frac{\mu z_n(z_{n+1}-z_{n-1})-n(z_{n+1}-z_n)(z_n-z_{n-1})}{(m-\mu+\nu)(z_{n+1}-z_{n-1})+n(z_{n+1}-2z_n+z_{n-1})}+z_n-z_{n+1}\right]\,.
\end{aligned}
\tag{2.26}
$$

Eq. (2.26) is related to the *discrete Painlevé equation* that was studied at length in [8]. Discrete Painlevé form an exciting class of ordinary difference equations which is becoming increasingly an area of intense activity (see [21] for a recent review). In [11] we presented the discrete and continuous Miura chains of both discrete and continuous equations of Painlevé type that are associated with the third-order OΔE (2.26). As is obvious from what we said above on the similarity reduction of the SPDE to the full PVI equation, these discrete and continuous equations are directly associated with PVI.

3 Conclusions and Outlook

It has been a long-standing conjecture whether all differential equations of Painlevé type could be derived as reductions from a larger integrable nonlinear equation or system of PDE's by similarity reduction. This is, in a sense, the converse of the famous ARS hypothesis [22]. The programme developed in the papers [19, 20, 8, 11] has demonstrated that there is an intimate interplay between discrete and continuous structures behind the Painlevé equations and that notably PVI and its discrete counterparts are obtainable from either a system of PΔEs on the full lattice or of a very rich parameter-family of PDEs in the continuum, which we have exhibited in the present paper. An obvious conjecture is that these connections also extend to discrete Painlevé equations which are of a possibly richer type, in particular the so-called q-Painlevé equations, which seemingly are "beyond" the PVI equation. In that case, we may expect that there exist partial difference equations of q-type, generalizing the SPDE (1.1), whose similarity reduction leads to those discrete Painlevé equations. The ultimate aim would be to find the "big equation" that sits above the recently found most general discrete Painlevé equation by Sakai, [23], which would be a partial difference equation of elliptic type.

References

[1] S.P Burtsev, A. V. Mikhailov and V. E. Zakharov, *Theor. Math. Phys.* **70**:227–240 (1987).

[2] J. Weiss. The Painlevé property for partial differential equations II: Bäcklund transformation, Lax pairs, and the Schwarzian derivative. *J. Math. Phys.*, 24:1405–1413, 1983. Bäcklund transformation and the Henon-Heiles system. *Phys. Lett.*, 105A:387–389, 1984.

[3] O. Mokhov, Symplectic and Poisson Geometry on Loop Spaces of Manifolds and Nonlinear Equations, Preprint `hep-th/9504076`.

[4] T. Miwa, *Proc. Japan Acad.* Ser. A **58**:9 (1982).

[5] F.W. Nijhoff, Integrable Hierarchies, Lagrangian Structures and Non-commuting Flows, in: *Topics in Soliton Theory and Exactly Solvable Nonlinear Equations*, Eds. M. Ablowitz, B. Fuchssteiner and M. Kruskal, (World Scientific, 1987), 150–181.

[6] B. Konopelchenko and L. Martinez Alonso, The KP hierarchy in Miwa coordinates, *Phys. Lett.* **A258**: 272–278 (1999).

[7] R. Fuchs, Sur quelques équations différentielles linéaires du second ordre, *C.R. Acad. Sci. (Paris)* 141:555-558 (1905); Über lineare homogene Differentialgleichungen zweiter Ordnung mit drei im Endlichen gelegene wesentlich singuläre Stellen, *Math. Ann.* 63:301–321 (1907).

[8] F.W. Nijhoff, A. Ramani, B. Grammaticos and Y. Ohta, On discrete Painlevé equations associated with the lattice KdV systems and the Painlevé VI equation, `solv-int/9812011`, (submitted to Stud. Appl. Math.).

[9] K. Okamoto, Studies on the Painlevé Equations. I. Sixth Painlevé Equation P_{VI}, *Ann. Mat. Pura Appl.* 146:337–381 (1986).

[10] K. Okamoto, Sur les échelles associées aux fonctions spéciales et l'équation de Toda, *J. Fac. Sci. Univ. Tokyo* **34** IA: 709–740 (1987).

[11] F. W. Nijhoff, N. Joshi and A. Hone, On the discrete and continuous Miura chain associated with the sixth Painlevé equation, preprint `solv-int/9906006`.

[12] A.S. Fokas, R.A. Leo, L. Martina and G. Soliani, The scaling reduction of the three-wave resonant system and the Painlevé VI equation, *Phys. Lett.* **115A**: 329–332 (1986).

[13] V.I. Gromak and V.V. Tsegel'nik, System of Threewave Resonant Interaction and P-Type Equations, *Theor. Math. Phys.* **78**: 14–23 (1989).

[14] L. Martina and P. Winternitz, Analysis and applications of the symmetry group of the multidimensional three-wave interaction problem,

[15] K. P. Tod, Self-dual Einstein Metrics from the Painlevé VI equation, *Phys. Lett.* A190:221–224 (1994).

[16] R. Halburd, PhD thesis, University of New SiSo

[17] F.W. Nijhoff and H.W. Capel, The Discrete Korteweg-de Vries Equation, *Acta Applicandae Mathematicae* 39:133–158 (1995).

[18] A.I. Bobenko and R. Seiler, *Discrete Integrable Geometry and Physics*, Oxford University Press (1999).

[19] F.W. Nijhoff, On some "Schwarzian" Equations and their Discrete Analogues, in: *Algebraic Aspects of Integrable Systems; In Memory of Irene Dorfman*, Eds. A.S. Fokas and I.M. Gel'fand, Birkhäuser Verlag, 1997, pp. 237–260.

[20] F.W. Nijhoff, Discrete Painlevé Equations and Symmetry Reduction on the Lattice, in: *Discrete Integrable Geometry and Physics*, Eds. A.I. Bobenko and R. Seiler, (Oxford Univ. Press, 1999) 209–234.

[21] B. Grammaticos, F.W. Nijhoff and A. Ramani, Discrete Painlevé Equations, Lectures presented at the School: *The Painlevé Property, One Century Later*, Ed. R. Conte, CRM Series in Mathematical Physics (Springer Verlag, 1999), pp. 417–522.

[22] M.J. Ablowitz, A. Ramani and H. Segur, A connection between nonlinear evolution equations and ordinary differential equations of P-type, I,II, *J. Math. Phys.* 21:715–721,1006–1015 (1980).

[23] H. Sakai, Rational surfaces associated with affine root systems and geometry of the Painlevé equations, Preprint Kyoto University, May 1999.