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# Bäcklund transformations for the $sl(2)$ Gaudin magnet

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## Abstract

Elementary, one- and two-point, Bäcklund transformations are constructed for the generic case of the  $sl(2)$  Gaudin magnet. The spectrality property is used to construct these explicitly given, Poisson integrable maps which are time-discretizations of the continuous flows with any Hamiltonian from the spectral curve of the  $2 \times 2$  Lax matrix.

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# 1. Introduction

Bäcklund transformations (BT's) are an essential tool used to generate new solutions out of given solutions to integrable equations. This is by now a well-developed area, with elegant BT's having been found and studied for almost all integrable hierarchies, see [1, 2].

The theory of BT's for evolution equations had entered the subject of finite-dimensional integrability through the discretization of time variable(s). One of the most important and earliest accounts on this subject is the papers by Veselov [3] where integrable Lagrange correspondences were introduced as discrete-time analogs of integrable continuous flows. Veselov clarified geometric meaning of these correspondences as finite shifts on Jacobians and gave several important examples. Reader is referred to an extensive literature which has appeared since then, see for instance [4, 5, 6, 7] and references therein.

In the present paper, following the approach of [8], we look at BT's for finite-dimensional (Liouville) integrable systems as special canonical transformations, thereby taking a Hamiltonian point of view. We introduce and study several new properties of BT's which appear to be very natural in such approach.

Bäcklund transformations for finite-dimensional integrable systems are defined in this paper as symplectic, or more generally Poisson, integrable maps which are explicit maps (rather than implicit multi-valued correspondences of [3]) and which can be viewed as time discretizations of particular continuous flows. The most characteristic features of such maps are: i) BT's preserve the same set of integrals of motion as does the continuous flow which they discretize, ii) they depend on a (Bäcklund) parameter  $\lambda$  that specifies the corresponding shift on a Jacobian or on a generalized Jacobian [9], iii) a spectrality property holds with respect to  $\lambda$  and to the 'conjugate' variable  $\mu$ , which means that the point  $(\lambda, \mu)$  belongs to the spectral curve [8, 10].

Because of the above properties, the constructed BT's are suitable as explicit (symplectic) geometric integrators. Namely, explicitness makes these maps to be pure iterative, while the importance of the parameter  $\lambda$  is that it allows an adjustable discrete time step. The spectrality property is strongly related to the symplecticness of the map. Finally, numerical integrators which exactly preserve the level set of integrals and at the same time are symplectic proved to be impossible to find for generic Hamiltonian dynamics [11], but for integrable flows they do exist and so are in demand.

In this paper we consider a generic (diagonal) case of the  $\mathfrak{sl}(2)$  XXX Gaudin magnet which is an algebraic completely integrable system. We study the problem of constructing elementary (one- and two-point) Bäcklund transformations for this system. In Section 3 we construct an elementary (one-point) BT which gives an exact discretization of a specific continuous flow. By making a two-point composite map in Section 7 we are then able to discretize any of the independent commuting flows with the Hamiltonians from the spectral curve of the  $2 \times 2$  Lax matrix.

## 2. Gaudin magnet

The  $\mathfrak{sl}(2)$  Gaudin magnet is derived from the Lax matrix

$$L(u) = \sum_{j=1}^n \frac{1}{u - a_j} \begin{pmatrix} s_j^3 & s_j^- \\ s_j^+ & -s_j^3 \end{pmatrix} + \alpha \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} A(u) & B(u) \\ C(u) & -A(u) \end{pmatrix}, \quad (2.1)$$

$$A(u) = \alpha + \sum_{j=1}^n \frac{s_j^3}{u - a_j}, \quad B(u) = \sum_{j=1}^n \frac{s_j^-}{u - a_j}, \quad C(u) = \sum_{j=1}^n \frac{s_j^+}{u - a_j}. \quad (2.2)$$

Local variables in this model are generators of the direct sum of  $n$   $\mathfrak{sl}(2)$  spins,  $s_j^3, s_j^\pm$ ,  $j = 1, \dots, n$ , with the following Poisson brackets:

$$\{s_j^3, s_k^\pm\} = \mp i \delta_{jk} s_k^\pm, \quad \{s_j^+, s_k^-\} = -2i \delta_{jk} s_j^3. \quad (2.3)$$

We denote Casimir operators (spins) as  $s_j$ :

$$s_j^2 = (s_j^3)^2 + s_j^+ s_j^-. \quad (2.4)$$

Fixing Casimirs  $s_j$  we go to a symplectic leaf where the Poisson bracket is non-degenerate, so that the symplectic manifold is a collection of  $n$  spheres.

Let us also introduce the total spin  $\vec{J}$  which will be used later, as follows:

$$J_3 = \sum_{j=1}^n s_j^3, \quad J_+ = \sum_{j=1}^n s_j^+, \quad J_- = \sum_{j=1}^n s_j^-. \quad (2.5)$$

The Lax matrix (2.1) satisfies the linear  $r$ -matrix Poisson algebra,

$$\{L_1(u), L_2(v)\} = [r(u - v), L_1(u) + L_2(v)], \quad (2.6)$$

with the permutation matrix as the  $r$ -matrix

$$r(u - v) = \frac{i}{u - v} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad (2.7)$$

Here we use standard notations  $L_1$  and  $L_2$  for tensor products:

$$L_1(u) = L(u) \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad L_2(v) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes L(v). \quad (2.8)$$

Equation (2.6) is equivalent to the following Poisson brackets for the rational functions  $A(u)$ ,  $B(u)$ , and  $C(u)$ :

$$\{A(u), A(v)\} = \{B(u), B(v)\} = \{C(u), C(v)\} = 0, \quad (2.9)$$

$$\{A(u), B(v)\} = \frac{i}{u - v} (B(v) - B(u)), \quad (2.10)$$

$$\{A(u), C(v)\} = \frac{-i}{u - v} (C(v) - C(u)), \quad (2.11)$$

$$\{C(u), B(v)\} = \frac{-2i}{u - v} (A(v) - A(u)). \quad (2.12)$$

The spectral curve  $\Gamma$ ,

$$\Gamma : \quad \det(L(u) - v) = 0, \quad (2.13)$$

is a hyperelliptic, genus  $n - 1$  curve,

$$v^2 = A^2(u) + B(u)C(u) = \alpha^2 + \sum_{j=1}^n \left( \frac{H_j}{u - a_j} + \frac{s_j^2}{(u - a_j)^2} \right), \quad (2.14)$$

with the Hamiltonians (integrals of motion)  $H_j$  of the form

$$H_j = \sum_{k \neq j} \frac{2s_j^3 s_k^3 + s_j^+ s_k^- + s_j^- s_k^+}{a_j - a_k} + 2\alpha s_j^3. \quad (2.15)$$

These are integrals of motion, or Hamiltonians, of the  $\mathfrak{sl}(2)$  Gaudin magnet, which are Poisson commuting:

$$\{H_j, H_k\} = 0, \quad j, k = 1, \dots, n. \quad (2.16)$$

Notice that there is one linear integral:

$$\sum_{j=1}^n H_j = 2\alpha J_3. \quad (2.17)$$

We can bring the curve  $\Gamma$  into the canonical form by scaling the variable  $v \mapsto \hat{v}$ :

$$\hat{v} = v \prod_{j=1}^n (u - a_j). \quad (2.18)$$

The equation of the curve becomes

$$\begin{aligned} \hat{v}^2 &= \left[ \alpha^2 + \sum_{j=1}^n \left( \frac{H_j}{u - a_j} + \frac{s_j^2}{(u - a_j)^2} \right) \right] \prod_{j=1}^n (u - a_j)^2 \\ &= \alpha^2 u^{2n} + f_1 u^{2n-1} + f_2 u^{2n-2} + \dots + f_{2n}. \end{aligned} \quad (2.19)$$

When  $\alpha = 0$  the genus of the curve drops to  $n - 2$ , because  $f_1 = 0$  in such case. The Gaudin magnet then becomes  $\mathfrak{sl}(2)$ -invariant: apart from integrals (2.15) all three components of the total spin  $\vec{J}$  are integrals too,

$$\alpha = 0 : \quad \{H_j, J_k\} = 0, \quad j = 1, \dots, n, \quad k = 1, 2, 3. \quad (2.20)$$

We will not consider this case, but concentrate on the generic case of  $\alpha \neq 0$  when there is only one linear integral  $f_1 = 2\alpha(J_3 - \alpha \sum_{j=1}^n a_j)$ . The latter case is called generic (diagonal) case of the  $\mathfrak{sl}(2)$  XXX Gaudin magnet. It is known that all its flows are linearized on generalized Jacobian of the hyperelliptic curve (2.19), see references in [9].

### 3. One-point basic map

The  $\mathfrak{sl}(2)$  Gaudin magnet with the  $2 \times 2$  Lax matrix (2.1) is within the class of systems that was considered recently in [9], namely it belongs to the (even) case of the generalized Jacobian. Hence, its Bäcklund transformations can be extracted from that paper. However, we

want to present here an independent derivation of those BT's as well as to give more detailed exposition of their various properties. The reader is referred to [9] also for explanation of geometric meaning of Bäcklund transformations.

A Bäcklund transformation should act on the Lax matrix as a similarity transform:

$$L(u) \mapsto M(u)L(u)M(u)^{-1} \quad \forall u, \quad (3.1)$$

with some non-degenerate  $2 \times 2$  matrix  $M(u)$ , simply because a BT should preserve the spectrum of  $L(u)$ .

Let us introduce new ( $\tilde{\cdot}$ ) notations for the up-dated variables

$$\tilde{L}(u) = \sum_{j=1}^n \frac{1}{u - a_j} \begin{pmatrix} \tilde{s}_j^3 & \tilde{s}_j^- \\ \tilde{s}_j^+ & -\tilde{s}_j^3 \end{pmatrix} + \alpha \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} \tilde{A}(u) & \tilde{B}(u) \\ \tilde{C}(u) & -\tilde{A}(u) \end{pmatrix}, \quad (3.2)$$

$$\tilde{A}(u) = \alpha + \sum_{j=1}^n \frac{\tilde{s}_j^3}{u - a_j}, \quad \tilde{B}(u) = \sum_{j=1}^n \frac{\tilde{s}_j^-}{u - a_j}, \quad \tilde{C}(u) = \sum_{j=1}^n \frac{\tilde{s}_j^+}{u - a_j}, \quad (3.3)$$

$$\{\tilde{s}_j^3, \tilde{s}_k^\pm\} = \mp i \delta_{jk} \tilde{s}_k^\pm, \quad \{\tilde{s}_j^+, \tilde{s}_k^-\} = -2i \delta_{jk} \tilde{s}_k^3. \quad (3.4)$$

We are looking for a Poisson map that intertwines two Lax matrices  $L(u)$  and  $\tilde{L}(u)$ :

$$M(u)L(u) = \tilde{L}(u)M(u) \quad \forall u. \quad (3.5)$$

Because spins  $s_j$ ,  $j = 1, \dots, n$ , appear as coefficients of the curve, they are not changed by the map, i.e.  $\tilde{s}_j = s_j$ . Hence, we can talk about a symplectomorphism ( $s_j = \text{const}$ ) instead of a Poisson map.

Now we should choose an ansatz for the dependence of the matrix  $M(u)$  on the spectral parameter  $u$ . Let us fix the simplest case of a *linear* function:

$$M(u) = M_1 u + M_0. \quad (3.6)$$

Taking limit  $u \rightarrow \infty$  in (3.5) we conclude that  $M_1$  must be diagonal. Moreover, the most elementary (one-point) BT should correspond to the case when  $\det M(u)$  has only one zero  $u = \lambda$ , which will lead to having only one Bäcklund parameter (cf. the spectrality property in [8, 10]). So, we should choose either

$$M_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{or} \quad M_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \quad (3.7)$$

We will consider the first case, as the second one will produce a similar BT (cf. moving in discrete time in positive and in negative direction). Finally, we arrive at the following parameterisation of the unknown matrix  $M(u)$ :

$$M(u) = \begin{pmatrix} u - \lambda + pq/\gamma & p \\ q & \gamma \end{pmatrix}, \quad (3.8)$$

with the  $c$ -number determinant

$$\det M(u) = \gamma(u - \lambda). \quad (3.9)$$

Here the variables  $p$  and  $q$  are indeterminate dynamical variables, but  $\lambda$  and  $\gamma$  are  $c$ -number Bäcklund parameters<sup>4</sup>.

Comparing asymptotics in  $u$  in both sides of (3.5) we readily get

$$p = \frac{J_-}{2\alpha}, \quad q = \frac{\tilde{J}_+}{2\alpha}. \quad (3.10)$$

If we want an explicit single-valued map from  $L(u)$  to  $\tilde{L}(u)$

$$L(u) \mapsto \tilde{L}(u) = M(u)L(u)M^{-1}(u), \quad (3.11)$$

then we must express  $M(u)$ , and therefore  $p$  and  $q$ , in terms of the old variables, i.e. the entries of  $L(u)$  only. There is however a problem, since from (3.10) we have only the expression for the  $p$ , but the variable  $q$  is given in terms of new, and therefore *unknown* variable  $\tilde{J}_+$ . To overcome this difficulty we will use an extra piece of data, namely the spectrality. Apart from equations (3.5) that our map satisfies, it will be parameterised by the point  $P$  on the curve  $\Gamma$ ,

$$P = (\lambda, \mu) \in \Gamma. \quad (3.12)$$

Notice that there are two points on the curve  $\Gamma$ ,  $P = (\lambda, +\mu)$  and  $Q = (\lambda, -\mu)$ , corresponding to a fixed  $\lambda$  and sitting one above the other because of the hyperelliptic involution:

$$(\lambda, \mu) \in \Gamma : \quad \det(L(\lambda) - \mu) = 0 \quad \Leftrightarrow \quad \mu^2 + \det(L(\lambda)) = 0. \quad (3.13)$$

Because  $\det(M(\lambda)) = 0$  the matrix  $M(\lambda)$  has a one-dimensional kernel

$$M(\lambda)\Omega = \begin{pmatrix} pq/\gamma & p \\ q & \gamma \end{pmatrix} \Omega = 0, \quad \Omega = \begin{pmatrix} \gamma \\ -q \end{pmatrix}. \quad (3.14)$$

The equality

$$M(\lambda)L(\lambda)\Omega = \tilde{L}(\lambda)M(\lambda)\Omega = 0 \quad (3.15)$$

implies that  $L(\lambda)\Omega \sim \Omega$ , so that  $\Omega$  is an eigenvector of  $L(\lambda)$ . Let us fix the corresponding point of the spectrum as  $P = (\lambda, \mu)$ :

$$\begin{pmatrix} A(\lambda) - \mu & B(\lambda) \\ C(\lambda) & -A(\lambda) - \mu \end{pmatrix} \begin{pmatrix} \gamma \\ -q \end{pmatrix} = 0. \quad (3.16)$$

This gives us the formula for the variable  $q$ :

$$q = \gamma \frac{A(\lambda) - \mu}{B(\lambda)} = -\gamma \frac{C(\lambda)}{A(\lambda) + \mu}. \quad (3.17)$$

The two last expressions are equivalent since  $(\lambda, \mu) \in \Gamma$ .

Now, the formulas (3.11), (3.8), (3.10), and (3.17), give a one-point Poisson integrable map ( $\equiv$  one-point BT) from  $L(u)$  to  $\tilde{L}(u)$ . The map is parameterised by one point  $(\lambda, \mu)$  on the spectral curve  $\Gamma$  (and by an extra parameter  $\gamma$ ).

Explicitly, it reads

$$\tilde{A}(u) = \frac{\gamma(u - \lambda + 2pq/\gamma)A(u) - q(u - \lambda + pq/\gamma)B(u) + p\gamma C(u)}{\gamma(u - \lambda)}, \quad (3.18)$$

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<sup>4</sup> $M(u)$  comes from a  $L$ -operator of the quadratic  $r$ -matrix algebra whose det is a Casimir, cf. [12].

$$\tilde{B}(u) = \frac{(u - \lambda + pq/\gamma)^2 B(u) - 2p(u - \lambda + pq/\gamma)A(u) - p^2 C(u)}{\gamma(u - \lambda)}, \quad (3.19)$$

$$\tilde{C}(u) = \frac{\gamma^2 C(u) + 2q\gamma A(u) - q^2 B(u)}{\gamma(u - \lambda)}. \quad (3.20)$$

Equating residues at  $u = a_j$  in both sides of the above equations, we obtain the map in terms of the local spin variables:

$$\tilde{s}_j^3 = \frac{\gamma(a_j - \lambda + 2pq/\gamma)s_j^3 - q(a_j - \lambda + pq/\gamma)s_j^- + p\gamma s_j^+}{\gamma(a_j - \lambda)}, \quad (3.21)$$

$$\tilde{s}_j^- = \frac{(a_j - \lambda + pq/\gamma)^2 s_j^- - 2p(a_j - \lambda + pq/\gamma)s_j^3 - p^2 s_j^+}{\gamma(a_j - \lambda)}, \quad (3.22)$$

$$\tilde{s}_j^+ = \frac{\gamma^2 s_j^+ + 2q\gamma s_j^3 - q^2 s_j^-}{\gamma(a_j - \lambda)}. \quad (3.23)$$

Recall that  $\alpha$ ,  $a_j$  and  $s_j$ ,  $j = 1, \dots, n$ , are parameters of the model;  $\gamma$  and  $\lambda$  are parameters of the map; and variables  $p$  and  $q$  are as follows:

$$p = \frac{J_-}{2\alpha}, \quad q = \gamma \frac{A(\lambda) - \mu}{B(\lambda)} = -\gamma \frac{C(\lambda)}{A(\lambda) + \mu}, \quad (3.24)$$

$$\mu^2 = \alpha^2 + \sum_{j=1}^n \left( \frac{H_j}{\lambda - a_j} + \frac{s_j^2}{(\lambda - a_j)^2} \right), \quad (3.25)$$

$$H_j = \sum_{k \neq j} \frac{2s_j^3 s_k^3 + s_j^+ s_k^- + s_j^- s_k^+}{a_j - a_k} + 2\alpha s_j^3. \quad (3.26)$$

## 4. BT as a discrete-time map

In this Section we will show that the BT constructed above can be seen as a time-discretization of a specific Hamiltonian flow where the parameter  $\lambda$  plays a role of inversion of the time step.

Consider the limit  $\lambda \rightarrow \infty$  then

$$\mu = \alpha + O\left(\frac{1}{\lambda}\right). \quad (4.1)$$

Assume that

$$\gamma = -\lambda + \gamma_0 + O\left(\frac{1}{\lambda}\right), \quad (4.2)$$

then we have the following expansion for the matrix  $M(u)$ :

$$M(u) = -\lambda \left( 1 - \frac{1}{2\lambda} M_0(u) \right) + O\left(\frac{1}{\lambda}\right). \quad (4.3)$$



The equation of the map,  $M(u)L(u) = \tilde{L}(u)M(u)$ , turns in this limit into the Lax pair of a continuous flow:

$$\dot{L}(u) = [L(u), M_0(u)], \quad M_0(u) = \begin{pmatrix} u - \gamma_0 & J_-/\alpha \\ J_+/\alpha & -u + \gamma_0 \end{pmatrix}, \quad (4.4)$$

where  $1/(2\lambda)$  is a time-step and  $\dot{L}(u) \equiv \lim_{\lambda \rightarrow \infty} 2\lambda(\tilde{L}(u) - L(u))$  is the time-derivative.

The flow (4.4) is a Hamiltonian flow

$$\dot{L}(u) = \{H, L(u)\}, \quad (4.5)$$

with the Hamiltonian function  $H$  as

$$H = \frac{i}{\alpha} \left( J_+ J_- + 2\alpha \sum_{j=1}^n (a_j - \gamma_0) s_j^3 \right). \quad (4.6)$$

Therefore, the constructed Bäcklund transformation is a two-parameter  $(\lambda, \gamma)$  time-discretization of this continuous flow.

## 5. Symplecticity

In this Section we give a simple proof of symplecticity of the constructed map by finding an explicit generating function of the corresponding canonical transformation from the old to new variables.

First, because the spin variables (Casimirs) do not change,

$$s_j^2 = (s_j^3)^2 + s_j^+ s_j^- = (\tilde{s}_j^3)^2 + \tilde{s}_j^+ \tilde{s}_j^-, \quad (5.1)$$

we can exclude the variables  $s_j^+$  and  $\tilde{s}_j^-$ ,  $j = 1, \dots, n$ ,

$$s_j^+ = \frac{s_j^2 - (s_j^3)^2}{s_j^-}, \quad \tilde{s}_j^- = \frac{s_j^2 - (\tilde{s}_j^3)^2}{\tilde{s}_j^+}, \quad (5.2)$$

expressing everything in terms of  $2n$  ‘canonical’ variables  $(s_j^3, s_j^-)_{j=1}^n$  and  $(\tilde{s}_j^3, \tilde{s}_j^+)_{j=1}^n$  with the following Poisson brackets:

$$\{s_j^3, s_k^-\} = i\delta_{jk} s_k^-, \quad \{\tilde{s}_j^3, \tilde{s}_k^+\} = -i\delta_{jk} \tilde{s}_k^+. \quad (5.3)$$

We want to represent our Bäcklund transformation as a canonical transformation defined by the generating function  $F(\tilde{s}^+ | s^-) \equiv F_{\lambda, \gamma}(\tilde{s}^+ | s^-)$  such that

$$s_j^3 = i s_j^- \frac{\partial F(\tilde{s}^+ | s^-)}{\partial s_j^-}, \quad \tilde{s}_j^3 = i \tilde{s}_j^+ \frac{\partial F(\tilde{s}^+ | s^-)}{\partial \tilde{s}_j^+}. \quad (5.4)$$

Notice that we have chosen the arguments of the generating function as  $(\tilde{s}_j^+ | s_j^-)_{j=1}^n$ . Because the symplecticity property does not depend on the choice of the arguments of its generating

function, these arguments are fixed in order to get a simpler expression for the function  $F_{\lambda,\gamma}$  (recall that the variables  $p$  and  $q$  (3.10) depend exactly on these variables).

Rewrite now the equations of the map (3.21)–(3.23) in the form

$$\left(\gamma s_j^3 - \frac{\tilde{J}_+}{2\alpha} s_j^-\right)^2 + \gamma(a_j - \lambda)\tilde{s}_j^+ s_j^- - \gamma^2 s_j^2 = 0, \quad (5.5)$$

$$\left(\gamma \tilde{s}_j^3 - \frac{J_-}{2\alpha} \tilde{s}_j^+\right)^2 + \gamma(a_j - \lambda)\tilde{s}_j^+ s_j^- - \gamma^2 s_j^2 = 0. \quad (5.6)$$

Resolving them with respect to  $s_j^3$  and  $\tilde{s}_j^3$ , we obtain

$$s_j^3 = \frac{\tilde{J}_+}{2\alpha\gamma} s_j^- + z_j, \quad \tilde{s}_j^3 = \frac{J_-}{2\alpha\gamma} \tilde{s}_j^+ + z_j, \quad (5.7)$$

$$z_j^2 = s_j^2 - \frac{a_j - \lambda}{\gamma} \tilde{s}_j^+ s_j^-, \quad j = 1, \dots, n. \quad (5.8)$$

It is now easy to check that the function

$$F_{\lambda,\gamma}(\tilde{s}^+ | s^-) = -i \frac{\tilde{J}_+ J_-}{2\alpha\gamma} - i \sum_{j=1}^n \left( 2z_j + s_j \log \frac{z_j - s_j}{z_j + s_j} \right) \quad (5.9)$$

satisfies equations (5.4). Thereby symplecticity of the map is proven.

## 6. Spectrality

The map depends on two parameters,  $\lambda$  and  $\gamma$ . Let us first concentrate on its  $\lambda$ -dependence.

Spectrality, which was introduced in [8], is an interesting property of Bäcklund transformations. It usually holds for any Bäcklund transformation which has a parameter. Technically, it means that the components of the point  $P = (\lambda, \mu) \in \Gamma$  which parameterises the map are conjugated variables, namely:

$$\mu = \frac{\partial F_{\lambda,\gamma}(\tilde{s}^+ | s^-)}{\partial \lambda}. \quad (6.1)$$

To prove this formula, use (3.10) and (3.17) to find the formula for the  $\mu$ ,

$$\mu = A(\lambda) - \frac{\tilde{J}_+}{2\alpha\gamma} B(\lambda). \quad (6.2)$$

Now, with the help of (5.7) and (5.9) we easily check the needed formula for the spectrality property (6.1).

A new, comparing to [8], observation is that there is also an analogous property with respect to the parameter  $\gamma$ , only now it is the integral  $J_3$  that plays the role of the conjugated variable:

$$J_3 = -\gamma \frac{\partial F_{\lambda,\gamma}(\tilde{s}^+ | s^-)}{\partial \gamma}. \quad (6.3)$$

The proof is very simple, once we notice that (5.7) entails

$$J_3 = \frac{\tilde{J}_+ J_-}{2\alpha\gamma} + \sum_{j=1}^n z_j. \quad (6.4)$$

Concluding this Section we want to remark that because of the second ‘spectrality’ property (6.3), which was somehow built into the Bäcklund transformation from the very beginning, one could recover the generating function of the map just taking one integral,

$$F_{\lambda,\gamma}(\tilde{s}^+|s^-) = \int^\gamma \left( -\frac{J_3}{\gamma} \right) d\gamma + \text{const}, \quad (6.5)$$

without needing to solve the system of  $2n$  differential equations (5.7).

## 7. Inverse map and a two-point map

In this Section we will first construct the inverse Bäcklund transformation and then use it to derive the two-point Bäcklund transformation which will be a composition of the direct map parameterised by the point  $P_1 = (\lambda_1, \mu_1) \in \Gamma$  and the inverse map parameterised by the point  $Q_2 = (\lambda_2, -\mu_2) \in \Gamma$ .

### 7.1 The inversion of the Bäcklund transformation

Let us call the direct map by  $B_P$ . The inverse map acts from  $\tilde{L}(u)$  to  $L(u)$ . We can rewrite the equations for the  $B_P$  in the inverse form

$$M^\wedge(u)\tilde{L}(u) = L(u)M^\wedge(u), \quad M^\wedge(u) = \begin{pmatrix} \gamma & -p \\ -q & u - \lambda + pq/\gamma \end{pmatrix}. \quad (7.1)$$

To define the inverse map we must find expressions for the co-factor matrix  $M^\wedge(u)$ , or for the variables  $p$  and  $q$ , in terms of  $\tilde{\phantom{u}}$ -variables, i.e. in terms of the entries of  $\tilde{L}(u)$ . We have already the expressions (3.10),

$$p = \frac{J_-}{2\alpha}, \quad q = \frac{\tilde{J}_+}{2\alpha}, \quad (7.2)$$

which define  $q$ . To obtain the formula for the variable  $p$  we will use again the spectrality property. The matrix  $M^\wedge(\lambda)$  has a one-dimensional kernel  $\tilde{\Omega}$ ,

$$M^\wedge(\lambda)\tilde{\Omega} = \begin{pmatrix} \gamma & -p \\ -q & pq/\gamma \end{pmatrix} \tilde{\Omega} = 0, \quad \tilde{\Omega} = \begin{pmatrix} p \\ \gamma \end{pmatrix}. \quad (7.3)$$

The main difference comparing to the formulas of the direct map is that the inverse map will be parametrized by the point  $Q = (\lambda, -\mu) \in \Gamma$ , not the  $P = (\lambda, \mu) \in \Gamma$ . Therefore,  $\tilde{\Omega}$  is an eigenvector of the matrix  $\tilde{L}(u)$  with the eigenvalue  $Q = (\lambda, -\mu)$ :

$$\begin{pmatrix} \tilde{A}(\lambda) + \mu & \tilde{B}(\lambda) \\ \tilde{C}(\lambda) & -\tilde{A}(\lambda) + \mu \end{pmatrix} \begin{pmatrix} p \\ \gamma \end{pmatrix} = 0. \quad (7.4)$$

This gives us the needed formula for the variable  $p$ ,

$$p = \gamma \frac{\tilde{A}(\lambda) - \mu}{\tilde{C}(\lambda)} = -\gamma \frac{\tilde{B}(\lambda)}{\tilde{A}(\lambda) + \mu}. \quad (7.5)$$

To prove that this does indeed give the inverse map, we have to show that the two formulas, (3.17) and (7.5), define in fact the same variable  $\mu$ :

$$\mu = \tilde{A}(\lambda) - \frac{p}{\gamma} \tilde{C}(\lambda) \stackrel{?}{=} A(\lambda) - \frac{q}{\gamma} B(\lambda). \quad (7.6)$$

It is easy to see that this equation is the (11)-element of the matrix identity:

$$M^\wedge(\lambda) \tilde{L}(\lambda) = L(\lambda) M^\wedge(\lambda). \quad (7.7)$$

We will denote as  $\mathcal{B}_Q$  the map which is inverse to the map  $B_P$ . Generally speaking, we have constructed 4 different maps,  $B_P$ ,  $B_Q$ ,  $\mathcal{B}_Q$ , and  $\mathcal{B}_P$ , with two pairs of maps which are inverse to each other:

$$\mathcal{B}_Q \circ B_P = B_P \circ \mathcal{B}_Q = \mathcal{B}_P \circ B_Q = B_Q \circ \mathcal{B}_P = \text{Id}. \quad (7.8)$$

## 7.2 The two-point map $B_{P_1, Q_2}$

We now construct a composite map which is a product of the map  $B_{P_1} \equiv B_{(\lambda_1, \mu_1)}$  and  $\mathcal{B}_{Q_2} \equiv \mathcal{B}_{(\lambda_2, -\mu_2)}$ :

$$B_{P_1, Q_2} = \mathcal{B}_{Q_2} \circ B_{P_1}. \quad (7.9)$$

The second parameter of the basic map, namely the  $\gamma$ , is taken the same in both maps, so  $\gamma_1 = \gamma_2$ . Obviously, when  $\lambda_1 = \lambda_2$  (and  $\mu_1 = \mu_2$ ) this composite map will turn into an identity map.

The first map  $B_{P_1}$  reads as follows:

$$M_1(u)L(u) = \tilde{L}(u)M_1(u), \quad M_1(u) = \begin{pmatrix} u - \lambda_1 + p_1 q_1 / \gamma & p_1 \\ q_1 & \gamma \end{pmatrix}, \quad (7.10)$$

where the formulas for the variables  $p_1$  and  $q_1$  are

$$p_1 = \frac{J_-}{2\alpha} = \gamma \frac{\tilde{A}(\lambda_1) - \mu_1}{\tilde{C}(\lambda_1)} = -\gamma \frac{\tilde{B}(\lambda_1)}{\tilde{A}(\lambda_1) + \mu_1}, \quad (7.11)$$

$$q_1 = \frac{\tilde{J}_+}{2\alpha} = \gamma \frac{A(\lambda_1) - \mu_1}{B(\lambda_1)} = -\gamma \frac{C(\lambda_1)}{A(\lambda_1) + \mu_1}. \quad (7.12)$$

The second map  $\mathcal{B}_{Q_2}$  reads as follows:

$$M_2(u)\tilde{L}(u) = \tilde{\tilde{L}}(u)M_2(u), \quad M_2(u) = \begin{pmatrix} \gamma & -p_2 \\ -q_2 & u - \lambda_2 + p_2 q_2 / \gamma \end{pmatrix}, \quad (7.13)$$

where the formulas for the variables  $p_2$  and  $q_2$  are

$$p_2 = \frac{\tilde{\tilde{J}}_-}{2\alpha} = \gamma \frac{\tilde{\tilde{A}}(\lambda_2) - \mu_2}{\tilde{\tilde{C}}(\lambda_2)} = -\gamma \frac{\tilde{\tilde{B}}(\lambda_2)}{\tilde{\tilde{A}}(\lambda_2) + \mu_2}, \quad (7.14)$$

$$q_2 = \frac{\tilde{J}_+}{2\alpha} = \gamma \frac{\tilde{A}(\lambda_2) - \mu_2}{\tilde{B}(\lambda_2)} = -\gamma \frac{\tilde{C}(\lambda_2)}{\tilde{A}(\lambda_2) + \mu_2}. \quad (7.15)$$

Notice that  $q_1$  is equal to  $q_2$ , hence we omit the sub-index,  $q_1 = q_2 = q$ .

The composite map  $B_{P_1, Q_2}$  acts from  $L(u)$  to  $\tilde{L}(u)$ ,

$$M(u)L(u) = \tilde{L}(u)M(u), \quad (7.16)$$

$$M(u) = \frac{1}{\gamma} M_2(u)M_1(u) = \begin{pmatrix} u - \lambda_1 + \frac{q}{\gamma}(p_1 - p_2) & p_1 - p_2 \\ \frac{q}{\gamma}(\lambda_1 - \lambda_2 - \frac{q}{\gamma}(p_1 - p_2)) & u - \lambda_2 - \frac{q}{\gamma}(p_1 - p_2) \end{pmatrix}. \quad (7.17)$$

In order to get rid of the intermediate  $\tilde{\cdot}$ -variables, we will use the spectrality property with respect to two points,  $P_1 = (\lambda_1, \mu_1)$  and  $Q_2 = (\lambda_2, -\mu_2)$ . Obviously, both spectralities are still valid after composing the maps. For the point  $P_1$  we get the following equations:

$$M(\lambda_1)\Omega_1 = 0, \quad \Omega_1 = \begin{pmatrix} \gamma \\ -q \end{pmatrix}, \quad L(\lambda_1)\Omega_1 = \mu_1\Omega_1 \quad \Rightarrow$$

$$q = \gamma \frac{A(\lambda_1) - \mu_1}{B(\lambda_1)} = -\gamma \frac{C(\lambda_1)}{A(\lambda_1) + \mu_1}; \quad (7.18)$$

$$M^\wedge(\lambda_1)\tilde{\tilde{\Omega}}_1 = 0, \quad \tilde{\tilde{\Omega}}_1 = \begin{pmatrix} p_1 - p_2 \\ \lambda_1 - \lambda_2 - \frac{q}{\gamma}(p_1 - p_2) \end{pmatrix}, \quad \tilde{L}(\lambda_1)\tilde{\tilde{\Omega}}_1 = -\mu_1\tilde{\tilde{\Omega}}_1 \quad \Rightarrow$$

$$p_1 - p_2 = \frac{\gamma(\lambda_2 - \lambda_1)\tilde{B}(\lambda_1)}{\gamma(\tilde{A}(\lambda_1) + \mu_1) - q\tilde{B}(\lambda_1)} = \frac{\gamma(\lambda_1 - \lambda_2)(\tilde{A}(\lambda_1) - \mu_1)}{q(\tilde{A}(\lambda_1) - \mu_1) + \gamma\tilde{C}(\lambda_1)}. \quad (7.19)$$

For the point  $Q_2$  we get the second set of equations:

$$M(\lambda_2)\Omega_2 = 0, \quad \Omega_2 = \begin{pmatrix} p_1 - p_2 \\ \lambda_1 - \lambda_2 - \frac{q}{\gamma}(p_1 - p_2) \end{pmatrix}, \quad L(\lambda_2)\Omega_2 = -\mu_2\Omega_2 \quad \Rightarrow$$

$$p_1 - p_2 = \frac{\gamma(\lambda_2 - \lambda_1)B(\lambda_2)}{\gamma(A(\lambda_2) + \mu_2) - qB(\lambda_2)} = \frac{\gamma(\lambda_1 - \lambda_2)(A(\lambda_2) - \mu_2)}{q(A(\lambda_2) - \mu_2) + \gamma C(\lambda_2)}; \quad (7.20)$$

$$M^\wedge(\lambda_2)\tilde{\tilde{\Omega}}_2 = 0, \quad \tilde{\tilde{\Omega}}_2 = \begin{pmatrix} \gamma \\ -q \end{pmatrix}, \quad \tilde{L}(\lambda_2)\tilde{\tilde{\Omega}}_2 = \mu_2\tilde{\tilde{\Omega}}_2 \quad \Rightarrow$$

$$q = \gamma \frac{\tilde{A}(\lambda_2) - \mu_2}{\tilde{B}(\lambda_2)} = -\gamma \frac{\tilde{C}(\lambda_2)}{\tilde{A}(\lambda_2) + \mu_2}. \quad (7.21)$$

Equations (7.18) and (7.21) are already known to us (cf. (7.12) and (7.15)). The formulas (7.19) and (7.20) for the variable  $p_1 - p_2$  are new. They are equivalent to the formulas (7.11) and (7.14) expressed in terms of entries of  $L(u)$  and  $\tilde{L}(u)$ .

Concluding, we have constructed a two-point Bäcklund transformation which is factorised to two one-point Bäcklund transformations and which is explicitly given, together with its inverse, by the formulas:

$$M(u)L(u) = \tilde{L}(u)M(u), \quad M(u) = \begin{pmatrix} u - \lambda_1 + xX & X \\ -x^2X + (\lambda_1 - \lambda_2)x & u - \lambda_2 - xX \end{pmatrix}, \quad (7.22)$$

where

$$x := \frac{A(\lambda_1) - \mu_1}{B(\lambda_1)} = -\frac{C(\lambda_1)}{A(\lambda_1) + \mu_1} = \frac{\tilde{A}(\lambda_2) - \mu_2}{\tilde{B}(\lambda_2)} = -\frac{\tilde{C}(\lambda_2)}{\tilde{A}(\lambda_2) + \mu_2}, \quad (7.23)$$

$$X := \frac{(\lambda_2 - \lambda_1)B(\lambda_1)B(\lambda_2)}{B(\lambda_1)(A(\lambda_2) + \mu_2) - B(\lambda_2)(A(\lambda_1) - \mu_1)}; \quad (7.24)$$

$$= \frac{(\lambda_1 - \lambda_2)B(\lambda_1)(A(\lambda_2) - \mu_2)}{(A(\lambda_1) - \mu_1)(A(\lambda_2) - \mu_2) + B(\lambda_1)C(\lambda_2)} \quad (7.25)$$

$$= \frac{(\lambda_2 - \lambda_1)B(\lambda_2)(A(\lambda_1) + \mu_1)}{(A(\lambda_1) + \mu_1)(A(\lambda_2) + \mu_2) + B(\lambda_2)C(\lambda_1)} \quad (7.26)$$

$$= \frac{(\lambda_1 - \lambda_2)(A(\lambda_1) + \mu_1)(A(\lambda_2) - \mu_2)}{(A(\lambda_1) + \mu_1)C(\lambda_2) - (A(\lambda_2) - \mu_2)C(\lambda_1)} \quad (7.27)$$

$$= \frac{(\lambda_2 - \lambda_1)\tilde{B}(\lambda_2)\tilde{B}(\lambda_1)}{\tilde{B}(\lambda_2)(\tilde{A}(\lambda_1) + \mu_1) - \tilde{B}(\lambda_1)(\tilde{A}(\lambda_2) - \mu_2)} \quad (7.28)$$

$$= \frac{(\lambda_1 - \lambda_2)\tilde{B}(\lambda_2)(\tilde{A}(\lambda_1) - \mu_1)}{(\tilde{A}(\lambda_2) - \mu_2)(\tilde{A}(\lambda_1) - \mu_1) + \tilde{B}(\lambda_2)\tilde{C}(\lambda_1)} \quad (7.29)$$

$$= \frac{(\lambda_2 - \lambda_1)\tilde{B}(\lambda_1)(\tilde{A}(\lambda_2) + \mu_2)}{(\tilde{A}(\lambda_2) + \mu_2)(\tilde{A}(\lambda_1) + \mu_1) + \tilde{B}(\lambda_1)\tilde{C}(\lambda_2)} \quad (7.30)$$

$$= \frac{(\lambda_1 - \lambda_2)(\tilde{A}(\lambda_2) + \mu_2)(\tilde{A}(\lambda_1) - \mu_1)}{(\tilde{A}(\lambda_2) + \mu_2)\tilde{C}(\lambda_1) - (\tilde{A}(\lambda_1) - \mu_1)\tilde{C}(\lambda_2)}. \quad (7.31)$$

The above formulas give several equivalent expressions for the variables  $x$  and  $X$  since the points  $(\lambda_1, \mu_1)$  and  $(\lambda_2, -\mu_2)$  belong to the spectral curve  $\Gamma$ , i.e. are bound by the following relations:

$$\mu_1^2 = A^2(\lambda_1) + B(\lambda_1)C(\lambda_1), \quad \mu_2^2 = A^2(\lambda_2) + B(\lambda_2)C(\lambda_2), \quad (7.32)$$

$$\mu_1^2 = \tilde{A}^2(\lambda_1) + \tilde{B}(\lambda_1)\tilde{C}(\lambda_1), \quad \mu_2^2 = \tilde{A}^2(\lambda_2) + \tilde{B}(\lambda_2)\tilde{C}(\lambda_2). \quad (7.33)$$

### 7.3 Two-point map as a discrete-time map

We will see in this Section that the two-point map constructed above is a one-parameter,  $\lambda_1$ , time-discretization of a family of flows parameterised by the point  $Q_2 = (\lambda_2, -\mu_2)$ , with the difference  $\lambda_1 - \lambda_2$  playing the role of the time-step.

Indeed, consider the limit  $\lambda_1 \rightarrow \lambda_2$ ,

$$\lambda_1 = \lambda_2 + \varepsilon, \quad \varepsilon \rightarrow 0. \quad (7.34)$$

It is easy to see from the formulas of the previous subsection that

$$x = x_0 + O(\varepsilon), \quad x_0 = \frac{A(\lambda_2) - \mu_2}{B(\lambda_2)} = -\frac{C(\lambda_2)}{A(\lambda_2) + \mu_2} \quad (7.35)$$

and

$$X = \varepsilon X_0 + O(\varepsilon^2), \quad X_0 = -\frac{B(\lambda_2)}{2\mu_2}. \quad (7.36)$$

Then we derive that the matrix  $M(u)$  has the following asymptotics:

$$M(u) = (u - \lambda_2) \left( 1 - \frac{\varepsilon}{2\mu_2(u - \lambda_2)} \begin{pmatrix} A(\lambda_2) + \mu_2 & B(\lambda_2) \\ C(\lambda_2) & -A(\lambda_2) + \mu_2 \end{pmatrix} \right) + O(\varepsilon^2). \quad (7.37)$$

If we now define the time-derivative  $\dot{L}(u)$  as

$$\dot{L}(u) = \lim_{\varepsilon \rightarrow 0} \frac{\tilde{\tilde{L}}(u) - L(u)}{\varepsilon}, \quad (7.38)$$

then in the limit we obtain from the equation of the map,  $M(u)L(u) = \tilde{\tilde{L}}(u)M(u)$ , the Lax equation for a corresponding continuous flow that our Bäcklund transformation discretizes, namely:

$$\dot{L}(u) = \left[ L(u), \frac{L(\lambda_2)}{2\mu_2(u - \lambda_2)} \right]. \quad (7.39)$$

This is a Hamiltonian flow with  $\mu_2$ ,

$$\mu_2 = \sqrt{A^2(\lambda_2) + B(\lambda_2)C(\lambda_2)} = \sqrt{\alpha^2 + \sum_{j=1}^n \left( \frac{H_j}{\lambda_2 - a_j} + \frac{s_j^2}{(\lambda_2 - a_j)^2} \right)},$$

as the Hamiltonian function,

$$\dot{L}(u) = -i\{\mu_2, L(u)\}. \quad (7.40)$$

This means that the two-point map discretizes a one-parameter family of flows. Having chosen the parameter  $\lambda_2$  to be equal to any of the poles of the Lax matrix (parameters of the model)  $a_j$ ,  $j = 1, \dots, n$ , the map leads to  $n$  different maps, each discretizing the flow with the corresponding Hamiltonian  $H_j$ ,  $j = 1, \dots, n$ . Indeed, take the limit  $\lambda_2 \rightarrow a_j$ ,

$$\lambda_2 = a_j + \varepsilon, \quad \varepsilon \rightarrow 0. \quad (7.41)$$

Then we have

$$\mu_2 = \frac{s_j}{\varepsilon} + \frac{H_j}{2s_j} + O(\varepsilon), \quad (7.42)$$

and in this limit the Lax equation (7.39)–(7.40) turns into

$$\dot{L}(u) = -\frac{i}{2s_j} \{H_j, L(u)\} = \left[ L(u), \frac{1}{2s_j(u - a_j)} \begin{pmatrix} s_j^3 & s_j^- \\ s_j^+ & -s_j^3 \end{pmatrix} \right]. \quad (7.43)$$

Let us denote a collection of these maps by  $\{B_{P_1}^{H_j}\}_{j=1}^n$ . The map  $B_{P_1}^{H_k}$  discretizes the flow governed by the Hamiltonian  $H_k$  with  $\lambda_1 - a_k$  playing the role of the discrete time-step

parameter. The map (and its inverse) is defined by the two-point matrix  $M(u)$  (7.22) with the following expressions for the variables  $x$  and  $X$ :

$$x = \frac{A(\lambda_1) - \mu_1}{B(\lambda_1)} = \frac{\tilde{s}_k^3 - s_k}{\tilde{s}_k^-}, \quad (7.44)$$

$$X = \frac{(a_k - \lambda_1)B(\lambda_1)s_k^-}{B(\lambda_1)(s_k^3 + s_k) - s_k^-(A(\lambda_1) - \mu_1)} = \frac{(a_k - \lambda_1)\tilde{B}(\lambda_1)\tilde{s}_k^-}{\tilde{s}_k^- \left( \tilde{A}(\lambda_1) + \mu_1 \right) - \tilde{B}(\lambda_1) \left( \tilde{s}_k^3 - s_k \right)}. \quad (7.45)$$

All these maps are explicit Poisson maps, preserving Hamiltonians and having the spectrality property with respect to the pair of variables  $(\lambda_1, \mu_1)$ .

## 8. Concluding remarks

One of the very important branches of the theory of finite-dimensional integrable systems is the area of discrete-time integrable systems. The interest to this area was revived in the beginning of 90's by Veselov in the series of works (see [3]). He defined integrable Lagrange correspondences as discrete-time analogs of integrable continuous flows, clarified their geometric meaning as finite shifts on Jacobians and gave several important examples. Since then the subject has got a boost and has been developed further by many authors. It has not been our intention to give a review of many important recent contributions made to the area, because it would require much more space. Instead, here we only mention in brief the main features of a new recent approach to constructing integrable maps which was introduced in [8, 10, 12], developed in [9], which has also been used in this paper and which will be referred to as Bäcklund transformations for finite-dimensional integrable systems.

One of the new features of this approach to discrete-time integrability is *the spectrality property* which is a projection on the classical case of the famous (quantum) Baxter equation. It was discovered on the examples of Toda lattice and elliptic Ruijsenaars system in [8] and was generalized to the integrable case of the DST model in [10]. Later on, it was observed that the property is universal and that, in effect, it gives a canonical way of parameterising the corresponding shift on the Jacobian which is characterized by adding a point  $(\lambda, \mu)$  to a divisor of points on the spectral curve  $\Gamma$  (cf. [9]).

A direct consequence of the spectrality property is the *explicitness* of the constructed maps. This new point, which is an obvious advantage because explicit iterative maps are much more useful than implicit maps (given as a system of non-linear equations), was clearly demonstrated in [9]. This new aspect of constructing *explicitly given maps* has been also adopted and illustrated in detail in the present paper.

There were several examples of explicit maps known before, like McMillan's map, but all those cases were exceptional, for in the generic situation, according to Veselov's approach, integrable Lagrange/Poisson correspondences are multi-valued maps, i.e. *correspondences* rather than maps. Using the spectrality property as extra data allows to overcome this drawback and to construct discrete-time integrable flows as genuine maps.

Another new feature of the proposed construction of integrable time-discretizations is an identification of the most elementary, one-point, basic map and construction of composite



maps, like the two-point map, as compositions of the one-point map and its inverse. The choice of the matrix  $M(u)$  (3.8) generating the one-point map is dictated by the algebraic considerations explained in [12]. In brief, the matrix  $M(u)$  should be a simple  $L$ -operator of the quadratic algebra,

$$\{L_1(u), L_2(v)\} = [r(u-v), L_1(u)L_2(v)], \quad (8.1)$$

with the same rational  $r$ -matrix (2.7) as in the linear algebra (2.6). The number of zeros of the  $\det M(u)$  is the number of essential Bäcklund parameters, so that the matrix  $M(u)$  in (3.8) is one-point and the matrix  $M(u)$  in (7.22) is two-point. The fact that the right ansatz for the matrix  $M(u)$  obeys the algebra (8.1) usually guarantees that the resulted map will be Poisson, see [12] for details.

In the present paper we have observed a new ‘spectrality’ property of the basic one-point map with respect to the parameter  $\gamma$  in

$$\det M(u) = \gamma(u - \lambda). \quad (8.2)$$

We have also shown that the two-point map factorises to two one-point maps.

The two-point map constructed above is probably most general map for the considered  $\mathfrak{sl}(2)$  Gaudin model, meaning that it gives a discretization of continuous flows given by any Hamiltonian  $H_j$ ,  $j = 1, \dots, n$ , from the spectral curve,

$$v^2 = A^2(u) + B(u)C(u) = \alpha^2 + \sum_{j=1}^n \left( \frac{H_j}{u - a_j} + \frac{s_j^2}{(u - a_j)^2} \right). \quad (8.3)$$

So, at least in principle, any other integrable map for this model should be a function of the  $n$  maps constructed in this paper.

There is no established name for integrable maps with all the qualities mentioned above, namely: i) spectrality, ii) explicitness, iii) Poissonicity, iv) limits to continuous flows, v) preservation of the same integrals as for the continuous flows which these maps discretize. We are using for them the same name, *Bäcklund transformations*, as was used in the references [8, 10, 9, 12].

The application of the constructed maps as exact numerical integrators of the continuous flows is considered in [13].

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