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Stability of stationary solutions for nonintegrable peakon equations

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Abstract

The Camassa-Holm equation with linear dispersion was originally derived as an asymptotic equation in shallow water wave theory. Among its many interesting mathematical properties, which include complete integrability, perhaps the most striking is the fact that in the case where linear dispersion is absent it admits weak multi-soliton solutions - "peakons" - with a peaked shape corresponding to a discontinuous first derivative. There is a one-parameter family of generalized Camassa-Holm equations, most of which are not integrable, but which all admit peakon solutions. Numerical studies reported by Holm and Staley indicate changes in the stability of these and other solutions as the parameter varies through the family.

In this article, we describe analytical results on one of these bifurcation phenomena, showing that in a suitable parameter range there are stationary solutions - "leftons" - which are orbitally stable.

1 Introduction

The family of partial differential equations

$$u_t - u_{xxt} + (b+1)uu_x = bu_xu_{xx} + uu_{xxx}, \tag{1}$$

labelled by the parameter b, is distinguished by the fact that it includes two completely integrable equations, namely the Camassa-Holm equation (the case $b = 2$ [2, 3]), and the Degasperis-Procesi equation (the case $b = 3$ [13, 15]). Each of the two integrable equations arises as the compatibility condition for an associated pair of linear equations (a Lax pair), and the latter leads to other hallmarks of integrability, namely the inverse scattering transform, multi-soliton solutions [3, 27, 35], an infinite number of conservation laws, and a bi-Hamiltonian structure. (The latter structure for the case $b = 2$ was found in [19].) According to various tests for integrability, the cases $b = 2, 3$ are the only integrable equations within this family [15, 28, 29, 38].

The Camassa-Holm equation was originally proposed as a model for shallow water waves [2, 3], and it is explained in [16, 17] that the members of the family of equations (1), apart from the case $b = -1$, are asymptotically equivalent by means of an appropriate Kodama transformation. The results of [8] (see Proposition 2 therein, and also equation (3.8) in [31]) show that, in a model of shallow water flowing over a flat bed, the solution u of (1) corresponds to the horizontal component of velocity evaluated at the level line $\theta \in [0, 1]$, where $\theta = \sqrt{\frac{11b-10}{12b}}$, which requires either $b \geq 10/11$ or $b \leq -10$. However, there continues to be debate in the literature about the precise range of validity of such models [1].

Another aspect of the equations (1) that makes them the focus of much interest is the special solutions that they admit. Although (as already mentioned) there are multi-soliton solutions for $b = 2, 3$, these smooth solutions only exist on a zero background in the case where the equation has additional linear dispersion terms (terms proportional to u_x and/or u_{xxx}, that is); such terms can be removed by a combination of a

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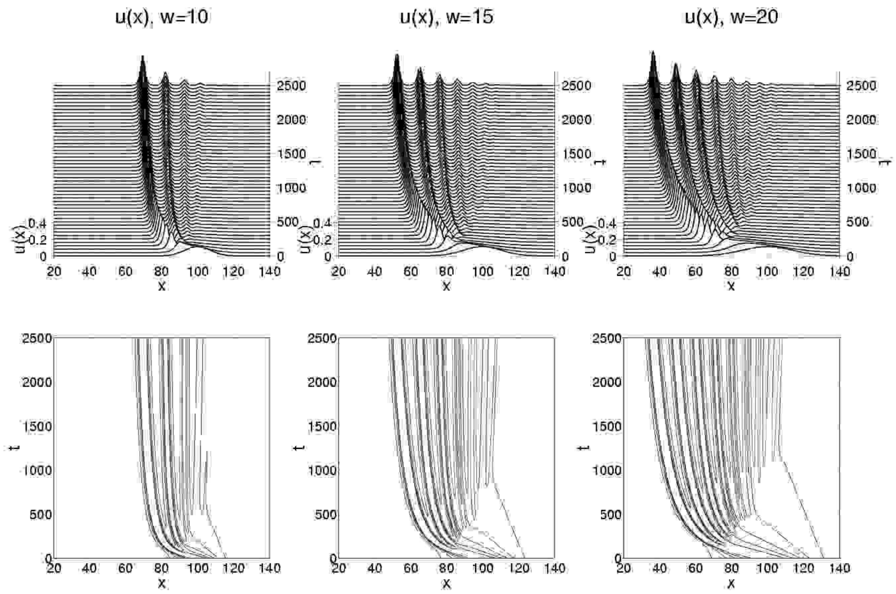

Figure 1: Leftons evolve from Gaussian initial profiles of different widths in the case $b = -3$. (Reproduced with kind permission from [26].)

Galilean transformation together with a shift $u \to u+u_0$, which (for $u_0 \neq 0$) changes the boundary conditions at spatial infinity. In the case of vanishing boundary conditions at infinity, there are no smooth multi-soliton solutions, but Camassa and Holm noticed that for $b = 2$ and any positive integer N there are instead weak solutions given by

$$u(x,t) = \sum_{j=1}^{N} p_j(t) e^{-|x-q_j(t)|}, \tag{2}$$

which have the form of a linear superposition of N peaked waves whose positions q_j and amplitudes p_j are respectively the canonically conjugate coordinates and momenta in a finite-dimensional Hamiltonian system that is completely integrable in the Liouville-Arnold sense. When $b = 2$, Hamilton's equations correspond to the geodesic equations for an N-dimensional manifold with coordinates $q_1, \ldots, q_N$ and co-metric $g^{ij} = e^{-|q_i-q_j|}$. The form of the multi-peakon solutions (2) persists for all values of b, although in general the Hamiltonian system governing the time evolution of the positions and amplitudes is non-canonical [30], and for $N > 2$ this finite-dimensional dynamics is expected to be integrable only when $b = 2, 3$.

In the case $b = 2$, it is known that the Camassa-Holm equation is of Euler-Poincaré type, corresponding to a geodesic flow with respect to the H^1 metric on a suitable diffeomorphism group [37]; the geodesic equations for the N-peakon solutions (2) are a finite-dimensional reduction of this flow [24]. Although the standard geodesic interpretation, in terms of a metric, is lost for other values of b, it was recently shown that the periodic case of the Degasperis-Procesi and the other equations in the b family can be regarded as geodesic equations for a non-metric connection on the diffeomorphism group of the circle [18].

Holm and Staley made an extensive numerical study of solutions of (1) for different values of b, by starting with different initial profiles and observing how they evolved with time and with changing b [25, 26]. They observed that, broadly speaking, there are three distinct parameter regimes with quite different behaviour, separated by bifurcations at $b = 1$ and $b = -1$, as follows:

Peakon regime: For $b > 1$, arbitrary initial data asymptotically separates out into a number of peakons as $t \to \infty$.

Ramp-cliff regime: For $-1 < b < 1$, solutions behave asymptotically like a combination of a "ramp"-like solution of Burgers equation (proportional to x/t), together with an exponentially-decaying tail ("cliff").

Lefton regime: For $b < -1$, arbitrary initial data moves to the left and asymptotically separates out into a number of "leftons" as $t \to \infty$, which are smooth stationary solitary waves. (See Figure 1.)

The behaviour observed separately in each of the parameter ranges $b > 1$ and $b < -1$ can be understood as particular instances of the *soliton resolution conjecture* [40], a vaguely defined conjecture which states that for suitable dispersive wave equations, solutions with "generic" initial data will decompose into a finite number of solitary waves plus a radiation part which decays to zero. In this article, our aim is to provide a first step towards explaining this phenomenon analytically for the equation (1) in the "lefton" regime $b < -1$. We show that in this parameter range a single lefton solution is orbitally stable, by applying the approach of Grillakis, Shatah and Strauss in [21]. The main ingredients required for our stability analysis are the Hamiltonian structure and conservation laws for (1). The lefton solutions are a critical point for a functional which is combination of the Hamiltonian and a Casimir, but the second variation has some negative spectrum, so it is not possible to apply the energy-Casimir method as in [23].

In the next section we describe the Hamiltonian structure and conservation laws of (3) that exist for all b. After that we consider orbital stability of stationary waves when $b < -1$: see Theorem 2 in section 3 for the main result of the paper. We make some remarks about other values of b in our conclusions.

2 Conserved quantities and Hamiltonian structure

In order to better understand the properties of each equation in the family (1), it is convenient to rewrite it in the following way:

$$m_t + um_x + bu_x m = 0, \qquad m = u - u_{xx}. \tag{3}$$

This can be regarded as a nonlocal evolution equation for m, where (at each time t) the field u is obtained from m by the convolution

$$u(x) = g * m(x) = \int_{-\infty}^{\infty} g(x-y)m(y)dy, \qquad g(x) = \frac{1}{2}\exp(-|x|). \tag{4}$$

Henceforth we use the symbol $\int$ without limits to denote an integral $\int_{\mathbb{R}}$ over the whole real line.

From the equation (1) written in the nonlocal form (3) it is straightforward to verify that, for any value of $b \neq 0, 1$, there are at least three different functionals that are formally conserved by the time evolution of m [14], namely

$$E = \int m\, dx, \qquad C_1 = \int m^{1/b}\, dx, \tag{5}$$

and

$$C_2 = \int m^{-1/b} \left(\frac{m_x^2}{b^2 m^2} + 1 \right) dx. \tag{6}$$

In saying that these quantities, each of which has the form $\int \mathcal{T}\, dx$ for some density $\mathcal{T}$, are formally conserved, we mean that there is a flux $\mathcal{F}$ such that the conservation law $\frac{\partial \mathcal{T}}{\partial t} = \frac{\partial \mathcal{F}}{\partial x}$ holds for any smooth solution of the equation (1). If the integral $\int \mathcal{T}\, dx = \int_{\mathbb{R}} \mathcal{T}\, dx$ exists, and the flux $\mathcal{F}$ vanishes at infinity, then clearly $d/dt \int \mathcal{T}\, dx = 0$ for strong solutions that decay sufficiently fast at infinity.

The smooth solutions of (1) can be derived from a variational principle $\delta S = 0$, by starting from the conservation law $(m^{1/b})_t + (um^{1/b})_x = 0$ associated with C_1 and introducing a potential φ such that $\varphi_x = m^{1/b}$, $\varphi_t = -um^{1/b}$. The action is

$$S = \int \int_{\mathbb{R}^2} \mathcal{L}\, dx\, dt, \qquad \mathcal{L} = \frac{\varphi_t}{2\varphi_x}\Big((\log \varphi_x)_{xx} + 1 \Big) - \frac{\varphi_x^b}{b-1}$$

(where we have changed a sign compared with [14]). After rearranging, the Euler-Lagrange equation gives

$$\frac{\partial}{\partial x}\left(\frac{\varphi_t}{\varphi_x}-\left(\frac{\varphi_t}{\varphi_x}\right)_{xx}+\varphi_x^b\right)=0,$$

which (up to an integration with respect to x) is equivalent to (3). Noether's theorem applied to the time translation symmetry $t\to t+s$ leads to the conserved density

$$\varphi_t\,\frac{\partial\mathcal{L}}{\partial\varphi_t}-\mathcal{L}=\frac{\varphi_x^b}{b-1},$$

which (up to scaling) corresponds to E above; the space translation $x\to x+s$ leads to an equivalent density. Applying Noether's theorem to the symmetry of shifting the potential $\varphi\to\varphi+s$ gives the density

$$\frac{\partial\mathcal{L}}{\partial\varphi_t}=\frac{1}{2\varphi_x}\Big((\log\varphi_x)_{xx}+1\Big),$$

which corresponds to C_2. The same action S is also valid for $b=0$, but needs to be modified slightly for $b=1$.

There is another type of conservation law which holds for solutions of (3), which is the fact that

$$m(q,t)q_x^b=m(x,0) \tag{7}$$

for all t in the domain of existence, where $x\mapsto q(x,t)$ is a diffeomorphism of the line defined from the solution of the initial value problem

$$q_t=u(q,t),\qquad q(x,0)=x. \tag{8}$$

(See [45], and also Proposition 9 in [18] for the case of the circle.) By adapting McKean's argument for the case $b=2$ [36], this implies that if a solution is initially positive, then $m(x,t)>0$ everywhere as long as the solution exists. In the next section we shall restate a stonger result along these lines in the case $m\in H^1$, which is proved by Zhou in [45].

The choice of nomenclature for the above functionals comes from the fact that, for any b, the skew-symmetric operator

$$B=-(bmD_x+m_x)(D_x-D_x^3)^{-1}(bD_xm-m_x) \tag{9}$$

is a Hamiltonian operator [28, 30], in the sense that it defines a Poisson bracket

$$\{\,F,G\,\}=\left\langle\frac{\delta F}{\delta m},B\,\frac{\delta G}{\delta m}\right\rangle,$$

between any pair of smooth functionals F,G, where $<f,g>=\int fg\,dx$ denotes the usual pairing between real functions on the line. Note that in (9) and elsewhere we use D_x to mean differentiation with respect to x. For suitable functions f the inverse operator in (9) is defined by $(D_x-D_x^3)^{-1}f=G*f$, taking the convolution with $G(x)=\frac{1}{2}\mathrm{sgn}(x)(1-\exp(-|x|))$.

The quantities C_1 and C_2 are the Casimirs for this bracket, satisfying $\{\,F,C_j\,\}=0$ for any F, for $j=1,2$. For any $b\neq 1$, the equation (3) can be written in Hamiltonian form as

$$m_t=\frac{1}{b-1}\,B\,\frac{\delta E}{\delta m}, \tag{10}$$

with E as in (5) being the Hamiltonian (up to scale); for $b=1$ (when $E=C_1$) one should take $\int m\log m\,dx$ as the Hamiltonian.

In fact, depending on the solutions considered, one or more of these functionals may not be defined. For example, in the case of the leftons, which are the solutions of interest here, we have that $b<-1$ and m is smooth, rapidly decaying and everywhere positive, so that E and C_2 both exist while C_1 does not. In the case where b is positive, on the other hand, if m is sufficiently smooth, rapidly decaying and positive, then we would have that only E and C_1 exist, and C_2 does not. In the case of a single peakon given by $u = c\exp(-|x-ct|)$, the field m is given by a delta function, $m = 2c\delta(x-ct)$, and similarly for the multi-peakon solution (2) it is $m = 2\sum_j p_j(t)\delta(x-q_j(t))$, so that the functional E makes sense, but C_1 and C_2 do not.

It is worth mentioning that for the integrable cases of (1), the nonlocal Hamiltonian operator (9) defines just one of a set of compatible Hamiltonian structures. When $b = 2$ the first Hamiltonian operator is given by $B_1 = D_x(1-D_x^2)$, and the second is $B_2 = mD_x + D_x m$, where the latter defines the Lie-Poisson bracket (the dual of the Euler-Poincaré structure); in that case, the nonlocal operator (9) defines the third Hamiltonian structure, and is given by $B_2 B_1^{-1} B_2$ up to scaling. For $b = 3$, B in (9) defines the second Hamiltonian structure, while $B_1 = D_x(1-D_x^2)(4-D_x^2)$ is the first Hamiltonian operator [13].

3 Stability of the stationary solution

In what follows we shall primarily be interested in the "lefton" solutions. A single lefton is a stationary solution of (1) given by the explicit formula [14]

$$u = A\Big(\cosh\gamma(x-x_0)\Big)^{-\frac{1}{\gamma}}, \qquad \gamma = -\frac{b+1}{2} \tag{11}$$

(independent of t), where the position x_0 and amplitude A are arbitrary constants. For $b < -1$, corresponding to $\gamma > 0$, when $A > 0$ this is a positive, smooth solution decaying like $e^{-|x|}$ as $|x| \to \infty$; so asymptotically it has the same shape as a peakon solution. From (3), stationary solutions satisfy $u^b m$ =constant.

3.1 Overview of the theory

Following [21], we consider orbital stability, which means nonlinear stability for solutions of Hamiltonian systems up to drifts along the action of Hamiltonian symmetries. Suppose that a system in Hamiltonian form is defined on a real Hilbert space $\mathcal{X}$, with energy functional (Hamiltonian) E, and admits a one-parameter Lie group of Hamiltonian symmetries $T_s : \mathcal{X} \to \mathcal{X}$ (where s is the parameter of the group), with infinitesimal generator T_0', where T_s is a unitary operator on $\mathcal{X}$. The Hamiltonian system is

$$w_t = J\,\frac{\delta E}{\delta w}, \tag{12}$$

with the (skew-symmetric) Hamiltonian operator $J : \mathcal{X}^* \to \mathcal{X}$, but one considers weak solutions in $\mathcal{X}$, namely w which satisfy

$$\frac{d}{dt}\langle\psi, w\rangle = -\left\langle \frac{\delta E}{\delta w}, J\,\psi \right\rangle, \tag{13}$$

for all $\psi \in D(J) \subset \mathcal{X}^*$, where $<,>$ denotes the pairing between $\mathcal{X}$ and $\mathcal{X}^*$. The natural isomorphism $I : \mathcal{X} \to \mathcal{X}^*$ is defined by $< Iu, v >= (u, v)$, where $(,)$ is the inner product on $\mathcal{X}$.

Then one considers the stability of particular solutions, which physically correspond to *bound states* or *solitary waves*, for which $w(t)$ takes the form

$$T_{\omega t}\,\phi, \tag{14}$$

for some fixed $\phi \in \mathcal{X}$, depending on the parameter $\omega \in \mathbb{R}$, which is a critical point of the functional

$$F = E - \omega Q. \tag{15}$$

The conserved functional Q (often identified as the *charge*) arises from the symmetry via an infinite-dimensional version of Noether's theorem [39]; this means that the Hamiltonian vector field associated with Q generates the symmetry T_s, in the sense that $w_s = J\,\frac{\delta Q}{\delta w} \equiv T_0'\,w$. Both E and Q are invariant under the symmetry group.

The three main assumptions in [21] can be paraphrased thus:

(i) Local existence: For each $w_0 \in \mathcal{X}$ the solution of (13) with initial data $w(0) = w_0$ exists in some time interval $t \in [0, t_0)$, for some $t_0 > 0$, and both E and Q are conserved: $E(w(t)) = E(w_0)$, $Q(w(t)) = Q(w_0)$ for all t in this interval.

(ii) Existence of bound states: There is a smooth map $\omega \mapsto \phi = \phi_\omega$ from some parameter interval (ω_1, ω_2) into $\mathcal{X}$, such that ϕ is a critical point for the functional F in (15), i.e. $\frac{\delta E}{\delta w}(\phi) - \omega \frac{\delta Q}{\delta w}(\phi) = 0$; also $\phi \in D((T_0')^2)$ and $T_0'\phi \neq 0$.

(iii) For each $\omega \in (\omega_1, \omega_2)$, the second variation $\frac{\delta^2 F}{\partial w^2}(\phi)$ has exactly one negative simple eigenvalue, has its kernel spanned by $T_0'\phi$, and the rest of its spectrum is positive and bounded away from zero.

In the above, of central importance is the second variation of F,

$$\mathtt{H} := \frac{\delta^2 F}{\partial w^2}(\phi) = \frac{\delta^2 E}{\partial w^2}(\phi) - \omega \frac{\delta^2 Q}{\partial w^2}(\phi),$$

which is a self-adjoint operator from $\mathcal{X}$ to $\mathcal{X}^*$. Its spectrum is defined to be the set of $\lambda \in \mathbb{R}$ such that $\mathtt{H} - \lambda I$ is not invertible. Evaluation of the functional F at ϕ defines a function $d(\omega)$. The solution (14) is the ϕ-orbit $\{T_{\omega t}\phi \,|\, t \in \mathbb{R}\}$, and its stability is defined in terms of the norm $\|\cdot\|$ on $\mathcal{X}$, as follows.

Definition 1. *The ϕ-orbit is stable if for all $\epsilon > 0$ there exists $\delta > 0$ with the following property. If $w(t)$ is a solution to (12) in some time interval $[0, t_0)$, such that $\|w(0) - \phi\| < \delta$, then $w(t)$ is defined for $0 \leq t < \infty$ and*

$$\sup_{0<t<\infty} \inf_{s \in \mathbb{R}} \|w(t) - T_s\phi\| < \epsilon.$$

The above definition can be modified in the case where the solution $w(t)$ may exhibit blow up in finite time, but this will not be needed for our purposes. One of the main results of [21] is the following.

Theorem 1. *Given assumptions (i)-(iii) above, for $\omega \in (\omega_1, \omega_2)$ the ϕ-orbit is stable if the function d satisfies $d''(\omega) > 0$.*

As stated above, the aforementioned three sets of assumptions are enough to obtain a sufficient condition for stability. In order to make the stronger statement that the convexity of the function $d(\omega)$ is both necessary and sufficient for stability, as in Theorem 2 of [21], a slightly more stringent version of the assumptions in (ii) and a fourth condition on the Hamiltonian operator are required. (See the remark made on p.167 of [21].) In this paper, we will only require the sufficient condition for the stability of the leftons.

Grillakis et al. also explain how (with minor alterations) their approach is valid for solutions defined in a Banach space. In the rest of this section, we consider appropriate modifications of their approach for the lefton solutions (11). In particular, it will be necessary to consider stability in a certain Banach subspace of a particular Hilbert space, and we will discuss below how the approach of [21] applies in this case.

3.2 Choice of a suitable Banach space

For the lefton solution (11), the corresponding field m has the form $m = m_0(x)$, where

$$m_0 = A\frac{1-b}{2}\Big(\cosh\gamma(x - x_0)\Big)^{\frac{b}{\gamma}}, \qquad \text{with} \quad \gamma = -\frac{b+1}{2} > 0 \tag{16}$$

for $b < -1$, which is positive and smooth, and decays like $e^{-|bx|}$ at infinity.

To begin with, we need to show that m_0 is a critical point of a functional defined on an appropriate space. As noted above, on the real line the functional C_1 diverges for b negative, so we want to realize the solution above as the critical point for a specific linear combination of the functionals E and C_2. The main technical difficulty in this case concerns the functional C_2, which is only defined for certain positive (or non-negative) m. We can redefine C_2 for negative m by introducing modulus signs, but the integrand will not be smooth wherever m has a zero; for this reason we would like to consider solutions of (3) with $m > 0$ everywhere.

The crucial observation to make is the fact that the solution (16) satisfies the first order differential equation

$$m_x^2 = b^2\left(m^2 - \frac{m^{3+1/b}}{k}\right). \tag{17}$$

From this it follows that $m = m_0$ is a critical point for the functional

$$F = -E + k\,C_2, \tag{18}$$

corresponding to the value

$$k = \left(A\frac{1-b}{2}\right)^{1+1/b}. \tag{19}$$

Indeed, for suitable smooth m, v such that $m + \epsilon v$ is positive and F is defined there whenever $|\epsilon|$ is small enough, the first variation in the direction v is $\delta F(m)\,v = \lim_{\epsilon\to 0}\frac{d}{d\epsilon}F(m+\epsilon v) =< \frac{\delta F}{\delta m}, v >$, hence

$$\begin{aligned}\frac{\delta F}{\delta m} &= -\frac{\delta E}{\delta m} + k\,\frac{\delta C_2}{\delta m}\\ &= -1 + km^{-1/b-1}\left(\frac{(1+2b)}{b^3}\frac{m_x^2}{m^2} - \frac{2}{b^2}\frac{m_{xx}}{m} - \frac{1}{b}\right).\end{aligned} \tag{20}$$

The latter expression vanishes for the lefton solution: $\frac{\delta F}{\delta m}(m_0) = 0$ for m_0 given by (16) and k given by (19).

In order to apply the results of [21] we must restrict to a suitable space in which the functionals E and C_2 are twice differentiable, at least near to m_0. To do so, we first introduce the weight

$$\alpha := m_0^{-2-1/b} = \left(A\frac{1-b}{2}\right)^{-2-1/b}\Big(\cosh\gamma(x-x_0)\Big)^{-\frac{2b+1}{\gamma}}, \tag{21}$$

and consider the equation (3) defined in the space $L^2_\alpha := L^2(\mathbb{R}, \alpha\,dx)$. With the standard pairing $<,>$, this gives the isomorphism $u \mapsto \alpha u$ from L^2_α to its dual. The reason for this choice of the weight α will become clear shortly when we consider the second variation of F.

The second variation of E is zero, so the entire contribution to the second variation of F comes from C_2. Assuming that $C_2(m+\epsilon v)$ is defined for suitably smooth m, v, we have

$$\delta^2 C_2(m, v) := \lim_{\epsilon\to 0}\frac{d^2}{d\epsilon^2}F(m+\epsilon v) = \int (\mathrm{P}\,v_x^2 + \mathrm{Q}\,v^2)\,dx, \tag{22}$$

after performing an integration by parts, where $\mathrm{P} = 2m^{-2-1/b}/b^2$, and

$$\mathrm{Q} = \frac{m^{-4-1/b}}{b^4}\Big(2b(1+2b)m_{xx}m - (1+2b)(1+3b)m_x^2 + b^2(1+b)m^2\Big).$$

Evaluating this at $m = m_0$ and using (17) gives

$$\delta^2 C_2(m_0, v) = \frac{2}{b^2}\int\Big(\alpha\,v_x^2 - b(b+1)\alpha\,v^2\Big)\,dx,$$

with α as in (21). To ensure that the second variation of C_2 in the direction v is defined at m_0, we require that v belongs to the Hilbert space $H^1_\alpha := H^1(\mathbb{R}, \alpha\,dx)$ which has the inner product

$$(v, w)_\alpha = \int (vw + v_x w_x)\,\alpha\,dx,$$

and corresponding norm $||\,v\,||_\alpha = \sqrt{(v,v)_\alpha}$, so that $|\delta^2 C_2(m_0, v)| \le K\,||v||^2_\alpha$ for a universal constant K. Hence we should consider solutions of (3) with $m \in H^1_\alpha$. Since α is uniformly bounded away from zero, H^1_α is a subspace of H^1.

However, as we shall explain further below, there is no neighbourhood of m_0 in H^1_α where C_2 exists, which leads us to consider a subspace $\mathcal{Z} \subset H^1_\alpha$, defined by

$$\mathcal{Z} := \{\, f \in H^1_\alpha \,|\, f = O(m_0) \quad \text{as} \quad |x| \to \infty \,\}. \tag{23}$$

Since functions in H^1 are continuous, by the Sobolev embedding theorem, it follows that $f \in \mathcal{Z}$ is a continuous function, so there exists some $\mathrm{K} \geq 0$ such that $|f(x)| \leq \mathrm{K}\, m_0(x)$ for all $x \in \mathbb{R}$. Then for any such f one can define

$$\mathrm{K}_f := \sup_{x\in\mathbb{R}} \frac{|f(x)|}{m_0(x)}. \tag{24}$$

With this definition, it is easy to check that $\mathcal{Z}$ is a Banach space with respect to the norm

$$||f||_{\mathcal{Z}} := ||f||_\alpha + \mathrm{K}_f. \tag{25}$$

3.3 Definition of stability and verification of assumptions

Henceforth we are going to consider the orbital stability of the solution m_0 in the Banach space $\mathcal{Z}$, allowing for translations in the independent variable x, so we begin with a precise definition of this, analogous to Definition 1. One of the requirements in [21] is that T_s should be a unitary operator; but the norm $\|\cdot\|_{\mathcal{Z}}$ is not invariant under translations, while the H^1 norm is, so this is involved in the definition of stability adopted here.

Definition 2. *The solution m_0 is stable if for all $\epsilon > 0$ there exists $\delta > 0$ with the following property. If there is a solution $m(\cdot, t)$ to (3) in some time interval $[0, t_0)$, such that $\|m(\cdot, 0) - m_0\|_{\mathcal{Z}} < \delta$, then m is defined for $0 \leq t < \infty$ and*

$$\sup_{0<t<\infty} \inf_{\xi\in\mathbb{R}} \|m(\cdot, t) - m_0(\cdot - \xi)\|_{H^1} < \epsilon.$$

Our main result is the following.

Theorem 2. *The lefton solution (16) is stable in the sense of Definition 2.*

In order to prove the above result, we now explain how the assumptions of [21] hold, up to appropriate modifications. While our proof uses the tools developed in [21], our presentation follows that of [11] quite closely. Here the relevant one-parameter symmetry group is spatial translation,

$$T_s\, m(x, t) = m(x + s, t), \tag{26}$$

which has the infinitesimal generator $T'_0 = D_x$. Both E and C_2 are invariant under this symmetry, which commutes with the time evolution, i.e. $[D_t, D_x] = 0$. We wish to show the stability of the m_0-orbit for this group. However, unlike the charge Q, the functional C_2 cannot generate the flow of m_0 along the orbit, because it is a Casimir. Nevertheless, the fact that m_0 is a critical point of the functional (18), with nontrivial dependence on the parameter k, is sufficient for the methods in [21] to work. Note that, since b is negative, compared with (15) we have taken $E \to -E$ in (18), and for the Hamiltonian operator we have $J \to (1 - b)^{-1}B$.

Henceforth we assume that k is a free parameter while A is specified by the relation (19); this is to simplify the notation for the stability analysis. Furthermore, we assume that k is positive. To consider positive solutions m, we require $A > 0$, which follows from the inverse formula $A(k) = 2k^{b/(b+1)}/(1-b)$ with $k > 0$.

We now consider the three main assumptions from subsection 3.1 in more detail. The first of these assumptions concerns local existence and conservation laws.

To begin with we briefly discuss local existence of solutions, which is the first part of assumption (i) above. There are several papers which prove results on local existence and blow up of the solutions of (1), either for particular values of b, e.g. in [7] for $b = 2$, and in [42] for $b = 3$, or for the whole family of equations on the line [45]; another family of equations that includes the case $b = 2$ is treated in [44]. Analytic solutions of (1) are considered in [6], while the equation with additional linear dispersion (terms proportional to u_x and u_{xxx}) is treated in [32] and [34]; for the periodic case, see also [5, 20]. Following the approach for the Camassa-Holm equation in [7], one can rewrite (3) as a quasi-linear evolution equation in L^2, that is

$$m_t + \mathtt{A}(m)\, m = 0,$$

with the operator $\mathtt{A}(m) = (g * m)\, D_x + b(g * m)_x\, \mathrm{id}$, where $\mathtt{A}(m) \in L(H^1, L^2)$ for $m \in H^1$. This way of presenting the equation allows the application of Kato's theorem, which gives local well-posedness for $m \in H^1$. Alternatively, one can use the local existence result for (1) with solutions $u \in H^s$ for $s > 3/2$ as stated by Zhou (see [45], Theorem 1.1), which is equivalent to $m \in H^{s-2}$; so for $u \in H^3$ this also gives local existence of solutions $m \in H^1$.

Although the preceding results could be modified to give a local existence result for $m \in \mathcal{Z}$, this is not necessary for our purposes. Instead, we will show that taking suitable initial data $m(\cdot, 0) \in \mathcal{Z} \subset H^1$ gives global existence of the solution in $\mathcal{Z}$, as in Theorem 4 below, and the arguments leading up to this only require local existence in H^1.

The other part of assumption (i) concerns the conservation laws.

Lemma 1. *Suppose the initial data $m(\cdot, 0) \in H^1 \cap L^1$ is everywhere non-negative. Then the energy E is constant, with*

$$E = \|m\|_{L^1} < \infty$$

as long as the solution of (3) exists.

Proof. The fact that $E = \int m\, dx$ is conserved follows immediately upon noting that (3) takes the Hamiltonian form (10), and for non-negative solutions E is the same as the L^1 norm of m. (For a direct proof that does not use the Hamiltonian property, the proof of Lemma 3.4 in [7], for the case $b = 2$, can be adapted to all values of b.) $\square$

Now $m \in H^1_\alpha$ implies that $m = o(\alpha^{-1/2})$ for large $|x|$, hence $m \in H^1 \cap L^1$, so in particular the preceding lemma applies to initial data in $\mathcal{Z} \subset H^1$, as does the next result on global existence in H^1.

Theorem 3. *If $m(\cdot, 0) \in H^1 \cap L^1$ is positive then the solution to (3) exists globally in H^1 and remains positive for all $t > 0$.*

Proof. This is essentially just a restatement of Theorem 2.5 in the paper by Zhou [45], but here we sketch a different proof. Note that from (4) we have $u = g * m$, so that $u_x = g_x * m$, and hence

$$\|u_x\|_{L^\infty} \leq \|g_x\|_{L^\infty}\, \|m\|_{L^1} = E/2, \tag{27}$$

where we have used Lemma 1. Then an application of integration by parts together with (3) yields

$$\begin{array}{rcl} \frac{d}{dt}\int m^2\, dx & = & 2\int m m_t\, dx \\ & = & -2\int u m m_x - 2b\int u_x m^2\, dx \\ & = & (1-2b)\int u_x m^2\, dx. \end{array}$$

A similar calculation shows that

$$\frac{d}{dt}\int m_x^2\, dx = -(2b+1)\int u_x m_x^2\, dx + b\int u_x m^2\, dx,$$

which combines with the previous one to give

$$\begin{array}{rcl} \frac{d}{dt}\|m\|^2_{H^1} & = & (1-b)\int u_x m^2\, dx - (2b+1)\int u_x m_x^2\, dx \\ & \leq & \frac{E}{2}\max(1-b, -2b-1)\, \|m\|^2_{H^1}, \end{array}$$

where we have used (27) (and note that both $1-b$ and $-2b-1$ are positive for $b<-1$, which is the case of interest here). Thus from Gronwall's inequality we see that $\|m\|_{H^1}$ remains bounded for all $t>0$.

Now for the diffeomorphism $q(x,t)$ defined by (8), it follows that $q_x(x,t)$ is positive for all t. To be precise,

$$0<q_x(x,t)=\exp\Big(\int_0^t u_x(q(x,s),s)\,ds\Big)\leq \exp(Et/2) \tag{28}$$

holds for all t, so from (7) it is clear that for every $t>0$, $m(x,t)>0$ holds for all x, since $m(x,0)>0$ everywhere. □

To see that the functional C_2 is conserved, note that as long as $\frac{\delta C_2}{\delta m}$ is defined we have

$$\frac{dC_2}{dt}=\left\langle\frac{\delta C_2}{\delta m},m_t\right\rangle=-\frac{1}{(b-1)}\left\langle\frac{\delta E}{\delta m},B\,\frac{\delta C_2}{\delta m}\right\rangle=0$$

for a dense class of $m\in H^1$. To be precise, from the coefficient of k on the right hand side of (20) one sees that $B\,\frac{\delta C_2}{\delta m}=0$ for sufficiently smooth m - this is just the statement that C_2 is a Casimir - and for general m one should approximate by smooth functions. This means that for initial data in the subspace $\mathcal{Z}\subset H^1$, the value of C_2 remains constant, and it turns out that if C_2 is finite then positive solutions remain in this subspace.

Theorem 4. *Suppose that the initial data $m(\cdot,0)\in\mathcal{Z}$ is everywhere positive with $C_2<\infty$. Then the solution of (3) exists globally in $\mathcal{Z}$.*

Proof. Writing $\mathrm{K}_m(t)$ to denote the supremum part (24) of the $\mathcal{Z}$ norm of $m(\cdot,t)$, we have

$$\mathrm{K}_m(t)=\sup_{x\in\mathbb{R}}\frac{|m(x,t)|}{m_0(x)}=\sup_{x\in\mathbb{R}}\frac{|m(q,t)|}{m_0(q)}=\sup_{x\in\mathbb{R}}\frac{|q_x^{-b}m(x,0)|}{m_0(q)},$$

using the fact that the solution $q=q(x,t)$ of (8) is a diffeomorphism for all $t\geq 0$, followed by (7). From this it is clear that

$$\mathrm{K}_m(t)\leq \sup_{x\in\mathbb{R}}\frac{|q_x^{-b}|m_0(x)}{m_0(q)}\,\mathrm{K}_m(0). \tag{29}$$

To bound this further, consider $\rho(x,t)=m_0(x)/m_0(q)$, which is seen to satisfy

$$\frac{\partial\rho}{\partial t}=-\rho\,q_t\,\frac{m_0'(q)}{m_0(q)}=-b\rho\,u(q,t)\tanh\gamma(q-x_0),$$

upon using (8) once again, as well as the explicit formula (16). The hyperbolic tangent is bounded above by 1, and $u=g*m$ implies $\|u\|_{L^\infty}\leq\|g\|_{L^\infty}\|m\|_{L^1}=E/2$, so overall this gives

$$\frac{\partial\rho}{\partial t}\leq-\frac{bE}{2}\rho,$$

and hence Gronwall's inequality (together with $\rho(x,0)=1$) yields $\rho(x,t)\leq\exp(-bEt/2)$. Similarly, the term q_x^{-b} in (29) has an upper bound obtained from (28), so that overall $\mathrm{K}_m(t)\leq\exp(-bEt)\,\mathrm{K}_m(0)$ for all $t\geq 0$. Now rewriting the integrand in (6) as $(m/m_0)^{-2-1/b}(b^{-2}m_x^2+m^2)\,\alpha$ gives the inequality $\|m\|_\alpha^2\leq b^2\,C_2\mathrm{K}_m^{2+1/b}<\infty$, so $\|m\|_{\mathcal{Z}}$ remains bounded for all $t>0$. □

For the assumption (ii) above, we note that the bound state $\phi=m_0$ is a smooth function of k, it satisfies $T_0'\,m_0\neq 0$, and we have also shown that m_0 is a critical point of the functional (18). The main point to discuss is whether the functionals E and C_2 (and hence F) are twice differentiable in a neighbourhood of m_0. This turns out to require further restrictions on the allowed variations. There is no problem with E, as for $m=m_0+v$ we have

$$|E(m)-E(m_0)|=\left|\int v\,dx\right|\leq||v||_{L_1}\leq(\alpha^{-1},|v|)_\alpha\leq||\alpha^{-1}||_\alpha\,||v||_\alpha,$$

using the H^1_α inner product $(,)$ followed by Cauchy-Schwarz, which gives continuity of E in H^1_α; and E is also clearly a smooth functional. For C_2, it is worthwhile to consider piecewise smooth functions $v \in H^1_\alpha$ which are asymptotic to a multiple of $e^{-c|x|}$ as $|x| \to \infty$, where $c > -b - \frac{1}{2}$ to ensure $||v||_\alpha < \infty$. For all such functions, $C_2(v)$ exists provided $v > 0$. However, if $-b > c$ then there is at least one choice of sign of ϵ such that $m_0(x) + \epsilon v(x) < 0$ for either positive or negative x of large enough magnitude. For example, for the family of positive functions $v_c = \exp(-c|x|)$ we have

$$C_2(v_c) = -2\left(\frac{b}{c} + \frac{c}{b}\right) \qquad \text{and} \qquad ||v_c||_\alpha < \infty \qquad \text{for} \qquad c > -b - \frac{1}{2}.$$

However, for all $\epsilon < 0$ and $c < -b$ it is clear that $m_0(x) + \epsilon v_c(x) < 0$ whenever $|x|$ is sufficiently large. This means that there is no neighbourhood of m_0 in H^1_α where C_2 is a smooth functional.

To rectify this problem, we consider a neighbourhood of m_0 in $\mathcal{Z}$.

Lemma 2. *For all $v \in \mathcal{Z}$ there exists an R such that $C_2(m_0 + \epsilon v)$ is a smooth function of ϵ for $|\epsilon| < R$.*

Proof. By replacing v by ϵv for suitably small ϵ if necessary, one can assume that $K_v < 1$, which implies that $m = m_0 + v$ is a positive function, and

$$\frac{||m||^2_\alpha}{b^2(1 + \mathrm{K}_v)^{2+\frac{1}{b}}} \leq C_2(m) \leq \frac{||m||^2_\alpha}{(1 - \mathrm{K}_v)^{2+\frac{1}{b}}}.$$

Then for all v with $\mathrm{K}_v < 1$, $C_2(m_0 + \epsilon v)$ is defined, and the integrand in (6) is a bounded differentiable function of ϵ, provided that $|\epsilon| < 1$. □

In fact the above proof shows that, with respect to the norm $||\cdot||_{\mathcal{Z}}$, the functional C_2 is smooth in a ball of radius 1 around m_0. The preceding considerations make it clear that in order to apply the results in [21] we should consider initial data in $\mathcal{Z}$, as in Definition 2, and all variations must be taken in this subspace of H^1_α. Having found a suitably restricted class of variations, we proceed to verify assumption (iii) above, which concerns the operator

$$\mathtt{H} \equiv \frac{\delta^2 F}{\delta m^2}(m_0) = k\Big(-D_x \mathrm{P}_0 D_x + \mathrm{Q}_0\Big), \tag{30}$$

where P_0 and Q_0 are, respectively, the coefficients functions P and Q defined after (22) evaluated at $m = m_0$.

Lemma 3. *On H^1_α the operator $\mathtt{H}$ defined in (30) has only one negative eigenvalue, its kernel is one-dimensional, and the positive part of the spectrum is bounded below away from zero.*

Proof. Upon evaluating the quantities depending on m at $m = m_0$, the eigenvalue problem $(\mathtt{H} - \lambda\alpha)y = 0$ associated to $\mathtt{H}$ can be written as

$$k\left(-D_x \mathrm{P}_0 D_x + \mathrm{Q}_0\right) y = \lambda \alpha y, \tag{31}$$

where $\mathrm{P}_0 = 2\alpha/b^2$, $\mathrm{Q}_0 = -2(b+1)\alpha/b$. We now make the change of variables $\widetilde{y} = \sqrt{\alpha} y$, so that the eigenvalue problem becomes

$$\mathtt{L}\widetilde{y} = \lambda \widetilde{y}, \qquad \mathtt{L} = k\left(-\frac{2}{b^2}D_x^2 + \widetilde{\mathrm{Q}}\right), \tag{32}$$

with $\widetilde{\mathrm{Q}} = \alpha^{-1/2}(\alpha^{-1/2}\alpha_x)_x/b^2 - 2(b+1)/b$. To find the continuous spectrum of $\mathtt{L}$, we define

$$\mathtt{L}^\infty = \lim_{x \to \infty} \mathtt{L} = k\left(-\frac{2}{b^2}D_x^2 + \frac{1}{2b^2}\right),$$

where we have used the fact that α grows like $\exp(-(2b+1)|x|)$ as $|x| \to \infty$. The continuous spectrum of $\mathtt{L}$ is then given by the set

$$\{\lambda \in \mathbb{C} \mid \mathtt{L}^\infty(\sigma) = \lambda \ \text{ for some } \ \sigma \in \mathbb{R}\} \tag{33}$$

(see [22], Theorem A.2, p. 140), where $\mathtt{L}^\infty(\sigma)$ is obtained from $\mathtt{L}^\infty$ by replacing D_x with $i\,\sigma$. The continuous spectrum of $\mathtt{L}$ in H^1 thus consists of the interval $\left[\frac{1}{2}k/b^2, \infty\right)$ (recalling that k is positive). The result then carries over to $\mathtt{H}$ through the change of variables.

The eigenvalue problem associated with $\mathtt{L}$ is an irregular Sturm-Liouville problem, with endpoints $\pm\infty$ both being limit-points. Due to this and the fact that the continuous spectrum is bounded below, the discrete spectrum below the continuous spectrum consists of simple eigenvalues which are ordered according to the number of zeros of the corresponding eigenvector, with no two eigenvectors having the same number of zeros and with the lowest eigenvalue corresponding to an eigenvector with no zero (case 8.iii of Theorem 10.12.1 in [43]).

The kernel of $\mathtt{L}$ is found by making the observation that $\widetilde{y} = \sqrt{\alpha} m_{0,x}$ solves (32) for $\lambda = 0$. Equivalently, it is easy to verify that $T_0' m_0 = m_{0,x}$ is in the kernel of $\mathtt{H}$. Since the eigenvalue zero is simple, there is nothing else in the kernel of $\mathtt{L}$. Furthermore, since $m_{0,x}$ has one zero, there is one and only one negative eigenvalue. By multiplying (31) by y and integrating over $\mathbb{R}$, it follows that the negative eigenvalue is bounded below by $-2k(b+1)/b < 0$. As before, these results carry over to $\mathtt{H}$ through the change of variables. $\square$

3.4 Proof of stability

In order to carry out the proof of Theorem 2, we introduce another notion from [21].

Definition 3. *The tubular neighbourhoods of m_0 in H^1_α and $\mathcal{Z}$ are given by*

$$U_\epsilon = \{f \in H^1_\alpha | \inf_{s\in\mathbb{R}} \|f(\cdot + s) - m_0\|_\alpha < \epsilon\}$$

and

$$U^{\mathcal{Z}}_\epsilon = \{f \in \mathcal{Z} | \inf_{s\in\mathbb{R}} \|f(\cdot + s) - m_0\|_{\mathcal{Z}} < \epsilon\},$$

respectively.

Lemma 4. *There exist $\epsilon > 0$ and a C^1 map $s : U_\epsilon \to \mathbb{R}$ such that for every $v \in U_\epsilon$,*

$$\left(v(\cdot + s(v)),\, m_{0,x}\right)_\alpha = 0. \tag{34}$$

Proof. Consider the function $\rho(s) = \left(v(\cdot + s),\, m_{0,x}\right)_\alpha$. We have that $\rho'(s) = \left(v_x(\cdot + s),\, m_{0,x}\right)_\alpha$. Thus, when evaluated at $v = m_0$ and $s = 0$, we have $\rho(0) = 0$ and $\rho'(0) = \|m_{0,x}\|^2_\alpha > 0$. By the implicit function theorem, there is a ball $B_\epsilon \subset H^1_\alpha$ of radius ϵ around m_0, an open interval $\mathcal{I}$ around the origin in $\mathbb{R}$, and a C^1 map $s : B_\epsilon \to \mathcal{I}$ such that that the equation $\rho(s) = 0$ has a unique solution $s = s(v) \in \mathcal{I}$ for all $v \in B_\epsilon$. The result follows by noting that the tubular neighborhood in H^1_α is equivalently defined by $U_\epsilon = \{v(\cdot + r) | v \in B_\epsilon,\ r \in \mathbb{R}\}$, and the map s extends to the whole of U_ϵ by setting $s(v(\cdot+r)) = s(v) - r$. $\square$

Now we define the scalar function

$$d(k) = F(m_0) \equiv -E(m_0) + k\,C_2(m_0), \tag{35}$$

where the lefton m_0, as in (16), depends on k via $A(k) = 2k^{b/(b+1)}/(1-b)$.

Lemma 5. *Suppose that $d''(k) > 0$. Then there exists a constant $\zeta > 0$ such that if $y \in H^1_\alpha$ and $\langle C_2'(m_0),\, y\rangle = 0 = \left(m_{0,x}, y\right)_\alpha$, then*

$$\langle \mathtt{H}y,\, y\rangle \geq \zeta \|y\|^2_{H^1}. \tag{36}$$

Proof. Using primes to denote variational derivatives $\delta/\delta m$, we differentiate the relation $F'(m_0) = -E'(m_0) + kC_2'(m_0) = 0$ with respect to k, to find $C_2'(m_0) = -\mathtt{H}m_{0,k}$. Furthermore, $d'(k) = C_2(m_0)$ and thus $d''(k) = \langle C_2'(m_0),\, m_{0,k}\rangle = -\langle \mathtt{H}m_{0,k},\, m_{0,k}\rangle > 0$.

We now consider the spectral decomposition with respect to the eigenvalue problem $(\mathtt{H} - \lambda I)y = 0$, where $I = \alpha - D_x \alpha D_x$ is the isomorphism from H^1_α to its dual; the properties of the associated spectrum are as described in Lemma 3 (even if the non-zero eigenvalues and corresponding eigenvectors are different from those of the problem $(\mathtt{H}-\lambda\alpha)y=0$ treated in its proof). Letting χ denote the negative eigenvector, such that $\mathtt{H}\chi=-\mu^2 I\chi$, with $\|\chi\|_\alpha=1$, we expand $m_{0,k}=a_0\chi+b_0 m_{0,x}+p_0$ for some $p_0\in P$, where P is the positive subspace for $\mathtt{H}$, and a_0 and b_0 are constants. Then $\langle \mathtt{H}m_{0k},\, m_{0k}\rangle<0$, as above, implies $\langle \mathtt{H}p_0,p_0\rangle<a_0^2\mu^2$.

Next take y belonging to the subspace $\mathcal{S}\subset H^1_\alpha$ defined by the pair of conditions $\langle C_2'(m_0),\, y\rangle=0=(m_{0,x},y)_\alpha=0$. On the one hand, by the second condition, every such y has the unique representation $y=a\chi+p$ for some $p\in P$ and constant a. The first condition then yields

$$0=-\langle C_2'(m_0),\, y\rangle=\langle \mathtt{H}m_{0,k},\, y\rangle=-a_0 a\mu^2+\langle \mathtt{H}p_0,\, p\rangle\,,$$

and a direct calculation (as in the proof of Theorem 3.3 in [21]) shows that $\langle \mathtt{H}y,\, y\rangle>0$ for all non-zero y. On the other hand, observe that there is the direct sum decomposition $\mathcal{S}=\tilde{P}\oplus\mathrm{Span}\{\psi\}$, where $\tilde{P}$ is the subspace consisting of all $\tilde{p}\in P$ such that $\langle \mathtt{H}p_0,\tilde{p}\rangle=0$, and $\psi=\tilde{a}\chi+p_0$ with $\tilde{a}=\frac{\langle \mathtt{H}p_0,p_0\rangle}{a_0\mu^2}$. Moreover, this is an orthogonal direct sum with respect to the bilinear form defined by $\mathtt{H}$. This form is positive definite on $\mathcal{S}$ and coercive on $\tilde{P}$, since by Lemma 3 there exists some $\tilde{\zeta}>0$ such that $\langle \mathtt{H}p,p\rangle>\tilde{\zeta}\|p\|^2_\alpha$ for all $p\in P$. Upon writing any $y\in\mathcal{S}$ as $y=\tilde{p}+\tau\psi$ for some constant τ, $\langle \mathtt{H}y,y\rangle=\langle \mathtt{H}\tilde{p},\tilde{p}\rangle+\tau^2\langle \mathtt{H}\psi,\psi\rangle$ and $\|y\|^2_\alpha\le 2\|\tilde{p}\|^2_\alpha+2\tau^2\|\psi\|^2_\alpha$ together imply $\langle \mathtt{H}y,y\rangle\ge\hat{\zeta}\|y\|^2_\alpha$, where $\hat{\zeta}=\frac{1}{2}\min(\tilde{\zeta},\|\psi\|^2_\alpha/\langle \mathtt{H}\psi,\psi\rangle)$. The inequality (36) follows by noting that

$$\|y\|^2_{H^1}\le k^{(2b+1)/(b+1)}\|y\|^2_\alpha \tag{37}$$

for all $y\in H^1_\alpha$. □

We are now ready to prove Theorem 2. First of all, to verify that $d''(k)>0$, note that using (17) in (35) yields

$$d(k)=2\int(km_0^{-1/b}-m_0)\,dx,$$

and this is just proportional to $k^{b/(b+1)}$. Then we compute

$$d''(k)=\frac{-2\hat{K}bk^{-1-1/(b+1)}}{(b+1)^2}, \tag{38}$$

where

$$\hat{K}=\int\Big(\mathrm{sech}\gamma x\Big)^{1/\gamma}\tanh^2\gamma x\,dx>0,$$

with $\gamma=-(b+1)/2>0$ (recalling that $b<-1$ for the leftons). We conclude that the quantity in (38) is positive, and thus Lemma 5 is applicable.

We now show that there exists $\epsilon>0$ such that

$$E(m_0)-E(m)\ge\frac{\zeta}{4}\|m(\cdot+s(m))-m_0\|^2_{H^1} \tag{39}$$

for all $m\in U^{\mathcal{Z}}_\epsilon\subset U_\epsilon$ satisfying $C_2(m)=C_2(m_0)$.

To see this, set $m(\cdot+s(m))=(1+a)m_0+y$, for some $a\in\mathbb{R}$, where $y\in H^1_\alpha$ is such that $\langle E'(m_0),\, y\rangle=k\,\langle C_2'(m_0),\, y\rangle=\int y\,dx=0$. Then Taylor's theorem with $v=m(\cdot+s(m))-m_0=am_0+y$ gives

$$\begin{aligned}C_2(m(\cdot+s(m))&=C_2(m_0)+\langle C_2'(m_0),\, v\rangle+O(\|v\|^2_{\mathcal{Z}})\\&=C_2(m_0)+\frac{a}{k}\int m_0\,dx+O(\|v\|^2_{\mathcal{Z}}),\end{aligned}$$

(where we used the fact that $C_2'(m_0)=1/k$), and also $C_2(m_0)=C_2(m)=C_2(m(\cdot+s(m))$ by translation invariance of C_2, from which it follows that $a=O\left(\|v\|^2_{\mathcal{Z}}\right)$. A Taylor expansion of $F=k\,C_2-E$, with $F'(m_0)=0$ and $F''(m_0)=\mathtt{H}$, gives $F(m)=F(m(\cdot+s(m))=F(m_0)+\frac{1}{2}\langle \mathtt{H}v,\, v\rangle+o\left(\|v\|^2_{\mathcal{Z}}\right)$. Using the fact that $C_2(m)=C_2(m_0)$ once more, together with the estimate of the magnitude of a, the previous relation yields

$E(m_0) - E(m) = \frac{1}{2}\langle \mathtt{H}v, v\rangle + o\left(\|v\|^2_{\mathcal{Z}}\right) = \frac{1}{2}\langle \mathtt{H}y, y\rangle + o\left(\|v\|^2_{\mathcal{Z}}\right)$. But $(y, m_{0,x})_\alpha = (m(\cdot + s(m)), m_{0,x})_\alpha = 0$ using Lemma 4 and $(m_0, m_{0,x})_\alpha = 0$. Therefore Lemma 5 applies to y, giving $E(m_0) - E(m) \geq \frac{\zeta}{2}\|y\|^2_{H^1} + o\left(\|v\|^2_{\mathcal{Z}}\right)$, and $\|y\|_{H^1} = \|v - am_0\|_{H^1} \geq \|v\|_{H^1} - |a|\|m_0\|_{H^1} \geq \|v\|_{H^1} - O\left(\|v\|^2_{H^1}\right)$, so for $\|v\|_{\mathcal{Z}}$ small enough we have $E(m_0) - E(m) \geq \frac{\zeta}{4}\|v\|^2_{H^1}$, which proves (39).

To complete the proof, suppose that m_0 is unstable. Then there exists a sequence of initial data $m_n(\cdot, 0) \in \mathcal{Z}$ for $n = 1, 2, \ldots$ and $\eta > 0$ such that

$$\|m_n(\cdot, 0) - m_0\|_{\mathcal{Z}} \to 0 \quad \text{but} \quad \sup_{t>0} \inf_{\xi\in\mathbb{R}} \|m_n(\cdot, t) - m_0(\cdot - \xi)\|_{H^1} \geq \eta,$$

were $m_n(\cdot, t)$ is the solution with initial datum $m_n(\cdot, 0)$. Let t_n be the first time so that

$$\inf_{\xi\in\mathbb{R}} \|m_n(\cdot, t_n) - m_0(\cdot - \xi)\|_{H^1} = \eta. \tag{40}$$

Then as $n \to \infty$, $E(m_n(\cdot, t_n)) = E(m_n(\cdot, 0)) \to E(m_0)$, and $C_2(m_n(\cdot, t_n)) = C_2(m_n(\cdot, 0)) \to C_2(m_0)$. Picking a sequence $v_n \in \mathcal{Z}$ such that $C_2(v_n) = C_2(m_0)$ and $\|v_n - m_n(\cdot, t_n)\|_{\mathcal{Z}} \to 0$, it follows that $\|v_n - m_n(\cdot, t_n)\|_{H^1} \to 0$. Then for η sufficiently small, we deduce from (39) that

$$\frac{\zeta}{4}\|v_n(\cdot + s(v_n)) - m_0\|^2_{H^1} \leq E(m_0) - E(v_n) \to 0,$$

by continuity of E. By the translation invariance of the H^1 norm, this means that $\|v_n - m_0(\cdot - s(v_n))\|_{H^1} \to 0$, which further implies

$$\|m_n(\cdot, t_n) - m_0(\cdot - s(v_n))\|_{H^1} \to 0.$$

This contradicts (40) and completes the proof.

4 Conclusions

We have established the stability of the lefton solution when $b < -1$. These results are a first step towards understanding how the soliton resolution conjecture, as described in [40], should hold for (1); this would be consistent with the numerical results of Holm and Staley (see Figure 1). However, our notion of stability is rather limited, in that it requires solutions that are initially close to the lefton with respect to the Banach space norm $\|\cdot\|_{\mathcal{Z}}$, in order to be close in H^1 at subsequent times. We expect that stability should hold more generally, for all initial data that is close to the lefton in H^1, at least up to the blow up time [45]; but in that context, the methods of [21] cannot be applied, because C_2 is not defined everywhere. It would be interesting to carry out further numerical studies to test these ideas (by considering perturbations proportional to $v_c = e^{-c|x|}$, for instance); the numerical integration of (1) is a challenging problem in itself [4].

It would also be interesting to see whether similar methods could be used to derive stability results for other ranges of b values, and for other explicit solutions (see e.g. [41]). However, for $-1 < b < 1$ there is the problem that explicit analytic formulae for the "ramp-cliff" profiles are unknown. In the peakon regime $b > 1$, there is an explicit formula: the peakon solution is given by $u = c\exp(|x - ct|)$, with m being given by a delta function. For the integrable cases $b = 2, 3$ the orbital stability of the peakons has been proved, but the arguments used in [10, 33] make essential use of some of the higher conserved quantities for the Camassa-Holm and Degasperis-Procesi equations, respectively. As far as we know, for other values of b the only conserved quantities are E, C_1 and C_2, and only E makes sense for the peakons.

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