Discrete Painlevé equations from Y-systems

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Abstract. We consider T-systems and Y-systems arising from cluster mutations applied to quivers that have the property of being periodic under a sequence of mutations. The corresponding nonlinear recurrences for cluster variables (coefficient-free T-systems) were described in the work of Fordy and Marsh, who completely classified all such quivers in the case of period 1, and characterized them in terms of the skew-symmetric exchange matrix B that defines the quiver. A broader notion of periodicity in general cluster algebras was introduced by Nakanishi, who also described the corresponding Y-systems, and T-systems with coefficients.

A result of Fomin and Zelevinsky says that the coefficient-free T-system provides a solution of the Y-system. In this paper, we show that in general there is a discrepancy between these two systems, in the sense that the solution of the former does not correspond to the general solution of the latter. This discrepancy is removed by introducing additional non-autonomous coefficients into the T-system. In particular, we focus on the period 1 case and show that, when the exchange matrix B is degenerate, discrete Painlevé equations can arise from this construction.

1. Introduction

The theory of cluster algebras, introduced by by Fomin and Zelevinsky more than ten years ago [12], has found a wide range of connections with different parts of mathematics and theoretical physics, especially in Lie theory, Teichmüller theory, Poisson geometry, discrete integrable systems and string theory. Some of the inspiration for the development of cluster algebras resulted from observations of Somos and others [19], concerning the *Laurent phenomenon* for nonlinear recurrences of the form

$$x_{n+N}\,x_n = F(x_{n+1},\ldots,x_{n+N-1}). \tag{1.1}$$

For certain special choices of F, a polynomial in $N-1$ variables, all iterates are Laurent polynomials in the initial data with integer coefficients. In particular, this means that if all N initial values are chosen to be 1, then an integer sequence is produced. A very well-known example is the Somos-4 recurrence

$$x_{n+4}\,x_n = x_{n+3}\,x_{n+1} + x_{n+2}^2; \tag{1.2}$$

the integer sequence beginning $1, 1, 1, 1, 2, 3, 7, 23, 59, 314, 1529, 8209, 83313, \ldots$ (for details, see `http://oeis.org/A006720`) is generated by (1.2) starting from four initial 1s, while if the initial data x_1, x_2, x_3, x_4 are viewed as variables then the iterates x_n belong to the Laurent polynomial ring $\mathbb{Z}[x_1^{\pm1}, x_2^{\pm1}, x_3^{\pm1}, x_4^{\pm1}]$.

In general, the clusters in a cluster algebra of rank N are generated by sequences of cluster mutations, defined by exchange relations of the form $\tilde{x}x = M_1 + M_2$, where (as in (1.2) above) the right hand side is a sum of two monomials M_1, M_2, and it is known that all clusters consist of Laurent polynomials in the cluster variables $\mathbf{x} = (x_1, x_2, \ldots, x_N)$ from any initial seed [12]. However, the Laurent phenomenon can arise in a broader context, including recurrences (1.1) where F is a sum of more than two monomials [13], which has led to the development of so-called LP algebras [44].

Cluster algebras and the Laurent phenomenon appear in connection with integrable systems in various different ways, both at the classical and quantum level, and for both continuous and discrete systems. On the one hand, the Hirota-Miwa equation (also known as the discrete Hirota equation, the discrete KP equation, or the octahedron recurrence) is an example of a partial difference equation with the Laurent property [13]; it appears as an identity for transfer matrix elements in quantum integrable models [42], and with particular boundary conditions it is obtained from sequences of cluster mutations in a cluster algebra of infinite rank [5]. On the other hand, for the continuous KP equation there is a cluster algebra structure in the graphs describing the combinatorics of the interaction of solitons [41]; the latter connection can be understood from the fact that the tau-functions of KP soliton solutions are encoded by points in a finite-dimensional Grassmannian, whose cluster algebra structure was explained by Scott [52]. Examples such as these suggest that tau-functions of integrable systems should admit an interpretation as cluster variables in a suitable cluster algebra. This interpretation can be very useful: there are natural Poisson/symplectic structures associated with cluster algebras [11, 20, 21], which may be used to obtain Poisson brackets for discrete integrable systems, both partial difference equations [37] and integrable maps [34].

There are many other connections between cluster algebras and integrability. The original Y-systems of Zamolodchikov [55], which are functional relations arising from the thermodynamic Bethe ansatz for certain integrable quantum field theories, were the prototype for the dynamics of coefficients in cluster algebras, and periodicities in Y-systems continue to be the subject of active research [36]. In the context of cluster algebras, there are also various examples of discrete integrable systems that are linearizable, in the sense that the variables satisfy linear recurrences. In particular, Fordy and Marsh considered cluster algebras obtained from quivers which are periodic under sequences of mutations [15], and showed that in certain cases, corresponding to affine A-type Dynkin quivers, linear relations hold between the cluster variables; for types A and D this was found independently in the context of frieze patterns [2], and linear relations for all affine Dynkin types were proved in [40]. Starting with results in [30], Poisson-commuting first integrals were constructed for affine A-type systems, as well as for other linearizable systems obtained from cluster algebras with periodicity [16, 18]. For certain linearizable systems, first integrals are also known in the noncommutative case [6]. Outside the context of cluster algebras, there are other linearizable systems with the Laurent property (see e.g. [24, 31]), within the LP algebra framework [1, 35].

In this paper we point out a new link between cluster algebras and integrable systems, by showing how certain discrete Painlevé equations, as well as their higher-order analogues, can be constructed from Y-systems. Our starting point is Nakanishi's construction of generalized T-systems and Y-systems associated with cluster algebras with periodicity under mutations [47], which is based on a broader notion of periodicity than that considered by Fordy and Marsh [15]. Nevertheless, for the most part we concentrate on the simplest case of Y-systems arising from quivers with period 1 under mutation; such quivers were completely classified by Fordy and Marsh, who showed that they generate recurrences (coefficient-free T-systems) of the general form

$$x_{n+N}\, x_n = \prod_{a_j\geq 0} x_{n+j}^{a_j} + \prod_{a_j\leq 0} x_{n+j}^{-a_j}, \tag{1.3}$$

where the indices in each product lie in the range $1 \leq j \leq N-1$, with the exponents $(a_1, ..., a_{N-1})$ forming an integer $(N-1)$-tuple which is palindromic, so that $a_j = a_{N-j}$. We discuss the Y-systems associated with (1.3), which have the form

$$y_{n+N}\, y_n = \frac{\prod_{a_j\leq 0} (1+y_{n+j})^{-a_j}}{\prod_{a_j\geq 0} (1+y_{n+j}^{-1})^{a_j}}, \tag{1.4}$$

and describe the relation between the solutions of (1.3) and (1.4) as explicitly as possible: in general, the solution space of (1.3) corresponds to a space of lower dimension within the solutions of (1.4). In particular, when the recurrence (1.3) is of Somos-N type, i.e.

$$x_{n+N}\, x_n = x_{n+N-p}\, x_{n+p} + x_{n+N-q}\, x_{n+q} \tag{1.5}$$

with $N \geq 2$ and $1 \leq p < q \leq \lfloor N/2 \rfloor$, then the general solution of the associated Y-system corresponds to a non-autonomous difference equation of q-Painlevé type.

1.1. Outline of the paper

In section 2 we consider the Somos-4 recurrence and its connections with various topics in mathematical physics (QRT maps and dimer models in particular), before explaining how the corresponding Y-system is related to a q-Painlevé I equation. Section 3 is concerned with coefficient-free cluster algebras and cluster mutation-periodicity, and especially the period 1 case as classified in [15]. Following [18], we describe how symplectic maps arise in the latter context, and introduce a special family of recurrences (each member of which is equivalent to iteration of a symplectic map) called U-systems (Definition 3.11). The fourth section gives a very brief description of cluster algebras with coefficients, setting the scene for Nakanishi's more general notion of periodicity, and the Y-systems and T-systems that arise in that context. A modified version of T-systems is introduced, called T_z-systems, together with so-called Z-systems, given by equations (4.5) and (4.7) respectively. The period 1 case is considered in more detail, examining the precise relation between T_z-systems and Y-systems in that case. Section 5 is devoted to some specific examples of T_z-systems in the period 1 case, in particular showing how discrete Painlevé equations, and their higher-order analogues, appear from this construction. We make a few conclusions in section 6, and the proof of a technical result (Proposition 3.9) is reserved for an appendix.

A few of our results appeared in unpublished sections of `arXiv:1207.6072v1`, the original preprint version of [18].

2. Motivating example: Somos-4 and q-Painlevé I

It was noted in [37] that for the case of cluster algebras associated with Lotka-Volterra systems the solution of the T-system corresponds to a system of lower order than the Y-system. Nakanishi pointed out to one of us that the same phenomenon occurs for the Somos-4 recurrence (1.2), which serves as motivation for the rest of the paper.

2.1. Symplectic coordinates and Liouville integrability

The Somos-4 recurrence (1.2) is the simplest three-term bilinear recurrence of the form (1.5). In can be obtained as a particular reduction of the discrete Hirota equation, which leads to a Lax pair and associated spectral curve for it [18]. In subsequent sections we shall explain how it arises in the context of cluster algebras as a (generalized) T-system, but here we just describe particular properties of this equation.

Due to its bilinear nature, it is clear that the equation (1.2) is invariant under the two-dimensional group of scaling transformations

$$x_n \longrightarrow \lambda\, \mu^n\, x_n, \qquad (\lambda, \mu) \in (\mathbb{C}^*)^2. \tag{2.1}$$

This is a simple example of the gauge transformations that appear in Hirota's theory of tau-functions for soliton equations [26]. The "physical variables" are the gauge-invariant quantities

$$Y_n = \frac{x_n x_{n+2}}{x_{n+1}^2}, \tag{2.2}$$

which satisfy the second-order recurrence

$$Y_{n+2}\, Y_n = \frac{Y_{n+1}+1}{Y_{n+1}^2}. \tag{2.3}$$

The above recurrence for Y_n is solved in elliptic functions, which leads to the explicit analytic solution of (1.2), as in [27].

There is another way to obtain the variables (2.2). Iterations of the Somos-4 recurrence preserve a degenerate quadratic Poisson bracket in the variables x_n [30], but of more importance for the dynamics is the presymplectic form

$$\begin{aligned} \omega \;\; = \;\; & -\left(\tfrac{\mathrm{d}x_1\wedge\mathrm{d}x_2}{x_1x_2}+\tfrac{\mathrm{d}x_1\wedge\mathrm{d}x_4}{x_1x_4}+\tfrac{\mathrm{d}x_3\wedge\mathrm{d}x_4}{x_3x_4}\right) \\ & +2\left(\tfrac{\mathrm{d}x_1\wedge\mathrm{d}x_3}{x_1x_3}+\tfrac{\mathrm{d}x_2\wedge\mathrm{d}x_4}{x_2x_4}\right)-3\,\tfrac{\mathrm{d}x_2\wedge\mathrm{d}x_3}{x_2x_3}, \end{aligned} \tag{2.4}$$

which is also preserved by the map $\varphi\,:\,(x_1,x_2,x_3,x_4)\mapsto(x_2,x_3,x_4,x_5)$. This closed two-form in four dimensions is degenerate, but (Y_1,Y_2) are coordinates on the space of leaves of the null foliation for ω, and on this space ω reduces to the symplectic form

$$\hat{\omega} = (Y_1Y_2)^{-1}\mathrm{d}Y_2\wedge\mathrm{d}Y_1,$$

which gives a log-canonical Poisson bracket between these coordinates:

$$\{Y_1,Y_2\} = Y_1Y_2. \tag{2.5}$$

Iterates of (1.2) produce iterates of (2.3), and in the (Y_1,Y_2)-plane this corresponds to iteration of the symplectic map

$$\hat{\varphi}: \qquad \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} \longmapsto \begin{pmatrix} \tilde{Y}_1 \\ \tilde{Y}_2 \end{pmatrix} = \begin{pmatrix} Y_2 \\ (Y_2+1)/(Y_1Y_2^2) \end{pmatrix}, \tag{2.6}$$

which is of QRT type [48]. A geometrical construction of this map, which goes back to Chasles (or even Euler) [54], is achieved by starting from the following family of biquadratic curves:

$$(Y_1Y_2)^2 - H\,Y_1Y_2 + Y_1 + Y_2 + 1 = 0. \tag{2.7}$$

For fixed H, the above curve admits the pair of involutions

$$\iota_1\,:\,Y_1\leftrightarrow Y_2, \qquad \iota_2\,:\,(Y_1,Y_2)\mapsto(Y_1^\dagger,Y_2),$$

where the "horizontal switch" ι_2 gives the conjugate point $(Y_1^\dagger,Y_2)$ obtained by intersecting the curve with a horizontal line. The formula for the product of the roots of (2.7), viewed as a quadratic in Y_1, gives the expression $Y_1^\dagger=(Y_2+1)/(Y_1Y_2^2)$, from which it is clear that the map (2.6) is the composition $\hat{\varphi}=\iota_1\cdot\iota_2$. Moreover, for generic H the curve (2.7) has genus one, and all formulae are independent of this parameter, so solving (2.7) for H gives a first integral; hence the map $\hat{\varphi}$ is an integrable system with one degree of freedom, in the Liouville sense [46, 54], and the level sets of the Hamiltonian H are elliptic curves. By performing a sequence of blowups at singularities, this map lifts to a morphism of a smooth elliptic surface [7].

Remark 2.1. In terms of the Rogers dilogarithm function

$$L(\zeta) = -\frac{1}{2}\int_0^{\zeta} \left(\frac{\log(1-y)}{y} + \frac{\log y}{(1-y)} \right) \mathrm{d}y = Li_2(\zeta) + \frac{1}{2}\log\zeta\,\log(1-\zeta),$$

the symplectic map (2.6) has the generating function

$$\hat{G}(Y_2,\tilde{Y}_2) = L\left(\frac{1}{1+Y_2}\right) + \log Y_2 \log \tilde{Y}_2 + (\log Y_2)^2 - \frac{1}{2}\log Y_2 \log(1+Y_2),$$

such that $\log \tilde{Y}_1 \,\mathrm{d}\log \tilde{Y}_2 - \log Y_1 \,\mathrm{d}\log Y_2 = \mathrm{d}\hat{G}$. We present dilogarithmic generating functions for T-systems coming from periodic quivers in Lemma 3.2 below; the particular function $\hat{G}$ above follows from a particular case of this result. It is interesting to note that dilogarithms also appear in the Lagrangians for integrable lattice equations [45], while the main subject of [47] is the identities for the Rogers dilogarithm which are associated with the Y-systems of general cluster algebras with periodicity.

2.2. Dimer models and relativistic Toda lattices

Recently, Goncharov and Kenyon have shown that dimer models on a torus give rise to certain quantum integrable systems, referred to as cluster integrable systems [22]. Each of these systems has a classical limit whose Lax matrix is the Kasteleyn matrix of the dimer model, with a combinatorial construction giving the Poisson-commuting Hamiltonians as sums of weighted dimer covers. The partition function of the dimer model is the spectral curve of the classical system. Furthermore, these systems have discrete symmetries given by cluster exchange relations, which are bilinear (Somos-type) equations. Thus the Somos recurrences of the form (1.5) can be understood as discrete integrable systems arising as discrete symmetries (or Bäcklund transformations [43]) of these continuous systems. In a further development, Eager et al. [8] have observed that in certain cases, associated with $Y^{\mathrm{p},0}$ toric surfaces, the classical cluster integrable systems arising in this way are equivalent to the relativistic Toda lattices in [50], while the dimer models associated with other $Y^{\mathrm{p},\mathrm{q}}$ geometries apparently produce new relativistic Toda systems. The Somos-4 recurrence (1.2) being discussed here corresponds to $Y^{2,0}$ (that is, $\mathrm{p}=2$); in [9] it is explained that it also arises from $Y^{2,1}$ (del Pezzo 1).

The classical cluster integrable systems corresponding to $Y^{\mathrm{p},0}$ geometries are equivalent to the relativistic Toda lattices, which live on a 2p-dimensional phase space with coordinates $(c_j,d_j)_{j=1}^{\mathrm{p}}$ whose non-vanishing Poisson brackets are given by

$$\{c_j,c_{j+1}\} = -c_jc_{j+1}, \quad \{c_j,d_j\} = c_jd_j, \quad \{c_j,d_{j+1}\} = -c_jd_{j+1}, \tag{2.8}$$

where indices are read mod p. The first Hamiltonian is

$$H_1 = \sum_j c_j + d_j, \tag{2.9}$$

and there are p − 2 commuting higher Hamiltonians as well as two Casimirs, given by $\prod_{j=1}^{\mathrm{p}} c_j$ and $\prod_{j=1}^{\mathrm{p}} d_j$ for $\mathrm{p} \neq 2$, so that the system is integrable with symplectic leaves of dimension 2p − 2. From [10] there is a Bäcklund transformation (in the sense of [43]) given by

$$\tilde{c}_j = c_j \left(\frac{d_j + c_{j-1}}{d_{j+1} + c_j} \right), \qquad \tilde{d}_j = d_{j+1} \left(\frac{d_j + c_{j-1}}{d_{j+1} + c_j} \right) \tag{2.10}$$

for $\mathrm{p} \neq 2$; this is a Poisson map that preserves the Hamiltonians and Casimirs.

In the case p = 2, the Poisson tensor has rank 2, with $c_1c_2d_1$ and d_1d_2 being the two independent Casimirs. If we set

$$Y_1 = \sqrt{\frac{c_1d_1}{c_2}}, \qquad Y_2 = \sqrt{\frac{c_2d_1}{c_1}},$$

then (2.8) gives the log-canonical bracket (2.5) for the Y_j, and on the symplectic leaves

$$\alpha = \sqrt{c_1c_2d_1} = \text{constant}, \qquad \beta = d_1d_2 = \text{constant} \tag{2.11}$$

the Hamiltonian H_1 in (2.9) becomes precisely the first integral for Somos-4, obtained by solving (2.7) for H. For p = 2, the Poisson map (2.10) must be modified as

$$\tilde{c}_1 = \frac{c_2d_1}{c_2 + d_2}, \quad \tilde{c}_2 = c_1, \quad \tilde{d}_1 = c_2 + d_2, \quad \tilde{d}_2 = \frac{d_1d_2}{c_2 + d_2}.$$

It is easy to verify that, on the level of the symplectic leaves (2.11), the iteration of this map is identical to (2.6).

2.3. Somos-4 Y-system and a discrete Painlevé equation

The Somos-4 recurrence (1.2) is a generalized T-system, in the sense of [47]; it has the form (1.3), with $N = 4$ and the palindromic triple $(a_1, a_2, a_3) = (-1, 2, -1)$. The generalized Y-system corresponding to Somos-4, of the form (1.4), is given by

$$y_{n+4}\, y_n = \frac{(1 + y_{n+3})(1 + y_{n+1})}{(1 + y_{n+2}^{-1})^2}. \tag{2.12}$$

It turns out that the general solution of this Y-system is related to non-autonomous versions of (1.2) and (2.3), which is how a discrete Painlevé equation appears.

There is an algebraic connection between the solutions of T- and Y-systems, which corresponds to a general link between cluster variables (denoted by $\mathbf{x}$) and coefficient variables (denoted by $\mathbf{y}$) that was presented in Proposition 3.9 of [14] (cf. Proposition 5.11 in [47]). In the particular case at hand, it means that substituting $y_n = Y_n$ with Y_n as in (2.2) gives a solution of the Y-system (2.12) whenever x_n satisfies (1.2). Yet there is a discrepancy between the Y-system (2.12), which is a fourth-order recurrence, and (2.3), which is only second-order; although (1.2) is fourth-order, the gauge symmetry

(2.1) reduces the dimension by two. Every solution of (2.3) is a solution of (2.12), but the converse is not true: the recurrence (2.12) requires four initial values, so the general solution should depend on four arbitrary constants, whereas the general solution of (2.3), which is given explicitly in terms of elliptic functions in [27], depends on only two parameters.

In order to better understand the general solution of (2.12), consider the quantity

$$\mathcal{Z}_n := \frac{y_{n+2}\, y_{n+1}^2\, y_n}{1 + y_{n+1}},$$

which is defined so that $\mathcal{Z}_n = 1$ for all n whenever $y_n = Y_n$ satisfies (2.3). In general, the Y-system (2.12) holds if and only if $\mathcal{Z}_n$ satisfies

$$\frac{\mathcal{Z}_n\, \mathcal{Z}_{n+2}}{\mathcal{Z}_{n+1}^2} = 1, \tag{2.13}$$

and by taking logarithms this gives a linear difference equation which implies that $\log \mathcal{Z}_n$ is a linear function of n, hence $\mathcal{Z}_n = \beta\, \mathfrak{q}^n$ for constants $\beta\,, \mathfrak{q} \in \mathbb{C}^*$. Upon rewriting the definition of $\mathcal{Z}_n$, the Y-system can be rewitten as a second-order non-autonomous recurrence for y_n.

Proposition 2.2. *The general solution of the Somos-4 Y-system is given by a solution of the second-order recurrence*

$$y_{n+2}\, y_n = \beta\, \mathfrak{q}^n\, \frac{(1 + y_{n+1})}{y_{n+1}^2}, \tag{2.14}$$

which is a $\mathfrak{q}$*-difference analogue of the first Painlevé equation.*

The above statement is obvious from the foregoing discussion, apart from the identification of (2.14) as a known example of a discrete Painlevé equation. To see this, one can make a gauge transformation of the dependent variable in the equation (2.14), setting $y_n = \beta^{-1}\, \alpha_n\, \upsilon_n$ for a suitable function α_n, which should be chosen so that one of the coefficients on the right hand side of the recurrence becomes constant. We obtain

$$\upsilon_{n+2}\, \upsilon_n = \frac{(\alpha_n\, \upsilon_{n+1} + \beta)}{\upsilon_{n+1}^2}, \qquad \text{where} \quad \alpha_{n+2}\, \alpha_{n+1}^2\, \alpha_n = \beta^4\, \mathfrak{q}^n. \tag{2.15}$$

The above equation for u_n is one of the discrete Painlevé I equations derived using the singularity confinement method in [49], where it is shown that the non-autonomous coefficient α_n must satisfy $\alpha_{n+4}\, \alpha_n = \alpha_{n+2}^2$ in order for the singularitities to be confined. The latter condition is a consequence of the second-order relation for α_n in (2.15); it implies that this coefficient takes the alternating form $\alpha_n = \alpha_\pm\, \mathfrak{q}_\pm^n$ for even/odd n. To see the link with Painlevé differential equations, one should take a continuum limit $\upsilon_n = h^{-2} - U(nh)$ with $s = nh$ held fixed as $h \to 0$, and scale α_n and β suitably, to obtain the differential equation

$$\frac{d^2U}{ds^2} = 6U^2 + s,$$

which is the Painlevé I equation. (See the last section of [32] for more details of the continuum limit.) The special case $\alpha_n = \alpha =$ constant for all n corresponds to the general Somos-4 recurrence with constant coefficients α, β, which was solved analytically in [27], and was obtained from a quiver with frozen nodes in [15].

Remark 2.3. The substitution (4.7) gives a particular solution of the Y-system in terms of a solution of the T-system, but one can use the same substitution in the general case, without the assumption that x_n satisfies the coefficient-free T-system (1.2). Indeed, setting $y_n = x_n x_{n+2}/x_{n+1}^2$ in (2.14) implies that x_n is a solution of the non-autonomous T-system

$$x_{n+4}\, x_n = \beta\, \mathfrak{q}^n\, (x_{n+3}\, x_{n+1} + x_{n+2}^2), \tag{2.16}$$

which is equivalent to one of the bilinear forms of discrete Painlevé I equations obtained in [49]. The Laurent property is preserved in the presence of the non-autonomous coefficients, in the sense that the iterates of (2.16) belong to the ring $\mathbb{Z}[x_0^{\pm 1}, x_1^{\pm 1}, x_2^{\pm 1}, x_3^{\pm 1}, \mathfrak{q}^{\pm 1}, \beta]$ for all n; some related observations appear in the unpublished preprint `arXiv:0807.2538`.

Remark 2.4. Each iteration of the non-autonomous recurrence (2.14) preserves the symplectic form $\hat{\omega} = (y_n\, y_{n+1})^{-1}\, \mathrm{d}y_{n+1} \wedge \mathrm{d}y_n$, which is the same as for the QRT map (2.3) in the autonomous case.

Note that the exponents of $\mathscr{Z}_n$ on the left hand side of (2.13) are the same as those in the substitution (2.2) that gives a solution of the Somos-4 Y-system. In section 4 we derive analogous results for other Y-systems obtained from cluster algebras with periodicity. Our main examples are associated with cluster algebras from quivers, which we introduce in the next section.

3. Cluster recurrences and symplectic maps from quivers

In this section we introduce the basic notions of coefficient-free cluster algebras obtained from quivers, before describing cluster mutation-periodic quivers and associated (autonomous) recurrence relations and symplectic maps.

3.1. Recurrences from periodic quivers

In a coefficient-free cluster algebra of rank N, a seed $(B, \mathbf{x})$ consists of an *exchange matrix* $B = (b_{ij}) \in \mathrm{Mat}_N(\mathbb{Z})$, which is skew-symmetrizable (i.e. there is a positive diagonal integer matrix D such that $B^T D = -DB$), and a *cluster* $\mathbf{x}$, which is an N-tuple of cluster variables $\mathbf{x} = (x_1, x_2, \ldots, x_N)$. For $j, k \in \mathbb{Z}$ such that $j \leq k$, we write $[j, k]$ for $\{j, j+1, \ldots, k-1, k\}$. For $k \in [1, N]$, the mutation μ_k gives a new seed $(\tilde{B}, \tilde{\mathbf{x}}) = (\mu_k(B), \mu_k(\mathbf{x}))$, with $\tilde{B} = (\tilde{b}_{ij})$ and $\tilde{\mathbf{x}} = (\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_N)$, where matrix mutation is defined by

$$\tilde{b}_{ij} = \begin{cases} -b_{ij} & \text{if} \quad i = k \quad \text{or} \quad j = k, \\ b_{ij} + \frac{1}{2}(|b_{ik}|b_{kj} + b_{ik}|b_{kj}|) & \text{otherwise,} \end{cases} \tag{3.1}$$

while cluster mutation is given by

$$\tilde{x}_i = \begin{cases} \dfrac{\prod_{i\in[1,N]} x_i^{[b_{ik}]_+} + \prod_{i\in[1,N]} x_i^{[-b_{ik}]_+}}{x_k}, & i = k, \\ x_i, & i \neq k, \end{cases} \tag{3.2}$$

where $[b]_+ = \max(b, 0)$. The *cluster algebra* $\mathcal{A} = \mathcal{A}(B)$ is the algebra over $\mathbb{Z}$ generated by all of the cluster variables obtained by all possible sequences of mutations, and the Laurent property means that all cluster variables belong to $\mathbb{Z}[\mathbf{x}^{\pm 1}] := \mathbb{Z}[x_1^{\pm 1}, \ldots, x_N^{\pm 1}]$, the ring of Laurent polynomials in the initial cluster [12].

A quiver is a graph consisting of a number of nodes together with arrows between the nodes. To each quiver Q with N nodes, without 1- or 2-cycles, there corresponds a skew-symmetric integer matrix $B \in \mathrm{Mat}_N(\mathbb{Z})$, and vice-versa. For any such quiver, one can apply quiver mutation μ_k at node k, which acts as follows: (i) reverse all arrows in/out of node k; (ii) if there are p arrows from node j to node k, and q arrows from node k to node ℓ, then add pq arrows from node j to node ℓ; (iii) remove any 2-cycles created in step (ii). The latter operation sends Q to $\tilde{Q} = \mu_k(Q)$, and is equivalent to the formula (3.1) for matrix mutation in the skew-symmetric case $b_{ij} = -b_{ji}$.

In general, iteration of (3.2) cannot be interpreted as a discrete dynamical system, because there are N possible directions for mutation at each step, and the matrix mutation (3.1) alters the exponents that appear in the two monomials on the right hand side of the exchange relation. However, in the skew-symmetric case, where the exchange matrix is associated with a quiver, Fordy and Marsh [15] defined B to be *cluster mutation-periodic* with period m if (for a suitable labelling of indices) $\mu_m \cdot \mu_{m-1} \cdot \ldots \cdot \mu_1(B) = \rho^m(B)$, where ρ is a cyclic permutation. In this setting, the *cluster map* $\varphi = \rho^{-m} \cdot \mu_m \cdot \mu_{m-1} \cdot \ldots \cdot \mu_1$ acts as the identity on B, but generically $\mathbf{x} \mapsto \varphi(\mathbf{x})$ has a non-trivial action on the cluster, and generates an infinite sequence of cluster variables; thus one has discrete dynamics corresponding to mutations in a special sequence of directions.

For the case of period $m = 1$, cluster mutation-periodicity for a quiver Q means that (with appropriately labelled nodes) the action of mutation μ_1 at node 1 on Q has the same effect as the action of $\rho: (1, 2, 3, \ldots, N) \mapsto (N, 1, 2, \ldots, N-1)$, such that the number of arrows from j to k in Q is the same as the number of arrows from $\rho^{-1}(j)$ to $\rho^{-1}(k)$ in $\rho(Q)$. Then the action of $\varphi = \rho^{-1} \cdot \mu_1$ on the cluster $\mathbf{x}$ takes the form of the birational map

$$\varphi: \quad (x_1, x_2, \ldots, x_{N-1}, x_N) \longmapsto (x_2, x_3, \ldots, x_N, x_{N+1}), \tag{3.3}$$

where

$$x_{N+1} = \frac{\prod_{j=1}^{N-1} x_{j+1}^{[b_{1,j+1}]_+} + \prod_{j=1}^{N-1} x_{j+1}^{[-b_{1,j+1}]_+}}{x_1}.$$

Iterating this map is equivalent to the iteration of a single scalar recurrence relation, that is

$$x_{n+N} x_n = \prod_{j=1}^{N-1} x_{n+j}^{[b_{1,j+1}]_+} + \prod_{j=1}^{N-1} x_{n+j}^{[-b_{1,j+1}]_+}, \qquad n = 1, 2, 3, \ldots. \tag{3.4}$$

Example 3.1. The exchange matrix

$$B = \begin{pmatrix} 0 & -1 & 2 & -1 \\ 1 & 0 & -3 & 2 \\ -2 & 3 & 0 & -1 \\ 1 & -2 & 1 & 0 \end{pmatrix}, \tag{3.5}$$

with $N = 4$, satisfies $\mu_1(B) = \rho(B)$, and the map $\varphi = \rho^{-1} \cdot \mu_1$ acting on the cluster $\mathbf{x} = (x_1, x_2, x_3, x_4)$ is equivalent to an iteration of the Somos-4 recurrence (1.2), which has the form (3.4).

A complete classification of period 1 quivers is given in [15]. Cluster mutation-periodicity with period 1 holds if and only if the matrix elements of B satisfy the relations

$$b_{j,N} = b_{1,j+1}, \quad j \in [1, N-1], \tag{3.6}$$

and

$$b_{j+1,k+1} = b_{j,k} + b_{1,j+1}[-b_{1,k+1}]_+ - b_{1,k+1}[-b_{1,j+1}]_+, \quad j,k \in [1, N-1]. \tag{3.7}$$

The above formulae entail that a matrix B associated with a period 1 cluster mutation-periodic quiver is completely determined by the elements in its first row, where the integers $a_j = b_{1,j+1}$ for $j = 1, \ldots, N-1$ form a palindromic integer $(N-1)$-tuple $\mathbf{a} = (a_1, \ldots, a_{N-1})$, i.e. $a_j = a_{N-j}$ (cf. Theorem 6.1 in [15]). Moreover, apart from being skew-symmetric, such a matrix B is also symmetric about the skew diagonal, i.e. $b_{jk} = b_{N-k+1,N-j+1}$. Hence each recurrence of the form (1.3) with palindromic exponents a_j corresponds to a matrix B of this kind, and conversely.

3.2. Symplectic forms for cluster maps

For a skew-symmetric integer matrix B, it was shown in [21] (see also [11]) that the two-form

$$\omega = \sum_{j<k} \frac{b_{jk}}{x_j x_k} \mathrm{d}x_j \wedge \mathrm{d}x_k \tag{3.8}$$

transforms covariantly with respect to cluster mutations μ_i. In the special case of cluster mutation-periodicity with period 1, it turns out that (3.8) is invariant under the action of the cluster map (3.3). Upon introducing the one-form

$$\vartheta = \sum_{j<k} b_{jk}\, z_j\, \mathrm{d}z_k, \qquad \text{with} \qquad z_j = \log x_j,$$

so that $\omega = \mathrm{d}\vartheta$, the fact that $\varphi^*\omega = \omega$ can be seen from the existence of a generating function for φ, given in terms of the Rogers dilogarithm function (as in Remark 2.1). For the rest of this section we assume that B is a skew-symmetric integer matrix corresponding to a cluster mutation-periodic quiver with period 1.

Proposition 3.2. *The map (3.3) has the generating function $G = G_0 + G_L$, such that $\varphi^*\vartheta - \vartheta = \mathrm{d}G$, where*

$$G_0 = \sum_{1\leq j<k\leq N-1} [-b_{1,j+1}]_+ \, b_{1,k+1} \, z_{j+1} z_{k+1} + \sum_{j=1}^{N-1} b_{1,j+1} \, z_{j+1} \left(-z_1 + \frac{1}{2}[-b_{1,j+1}]_+ z_{j+1} \right),$$

and

$$G_L = -L(\zeta) + \frac{1}{2}\log(1-\zeta)\,\log\left(\frac{1-\zeta}{\zeta}\right)$$

for $\zeta = \Big(1 + \exp(-\sum_{k=1}^{N-1} b_{1,k+1} z_{k+1})\Big)^{-1}$.

The proof of the preceding result follows from a direct calculation, using the conditions (3.6) and (3.7). (For another proof that $\varphi^*\omega = \omega$, see Lemma 2.3 in [18].)

The two-form (3.8) is log-canonical: it is constant in the logarithmic coordinates z_j, so it is evidently closed, but may be degenerate. If $\det B \neq 0$ then the map φ is symplectic, but in general to obtain a symplectic map it is necessary to consider the null distribution of ω, which is generated by commuting vector fields of the form $\sum_j u_j x_j \frac{\partial}{\partial x_j}$, where the integer vector $\mathbf{u} = (u_1, u_2, \ldots, u_N) \in \ker B$. Each such vector field integrates to yield a scaling transformation

$$\mathbf{x} \longrightarrow \lambda^{\mathbf{u}} \cdot \mathbf{x} = (\lambda^{u_1} x_1, \lambda^{u_2} x_2, \ldots, \lambda^{u_N} x_N), \qquad \lambda \in \mathbb{C}^*, \tag{3.9}$$

so that overall there is an action of the algebraic torus $(\mathbb{C}^*)^{N-r}$, where $r = \mathrm{rank}\, B$, and a complete set of independent invariants under these scaling transformations provides coordinates for the space of leaves of the null foliation for ω. Due to the skew-symmetry of the integer matrix B, r is even, and the vector space $\mathbb{Q}^N$ has an orthogonal direct sum decomposition $\mathbb{Q}^N = \mathrm{im}\, B \oplus \ker B$ with respect to the standard scalar product, denoted $(\ ,\ )$. Given an integer vector $\mathbf{v} = (v_1, \ldots, v_N)$, the scaling action on the Laurent monomial $\mathbf{x}^{\mathbf{v}} = \prod_j x_j^{v_j}$ gives $\lambda^{\mathbf{u}} \cdot \mathbf{x}^{\mathbf{v}} = \lambda^{(\mathbf{u},\mathbf{v})}\mathbf{x}^{\mathbf{v}}$, hence $\mathbf{x}^{\mathbf{v}}$ is invariant under the overall action of $(\mathbb{C}^*)^{N-r}$ if and only if $(\mathbf{u}, \mathbf{v}) = 0$ for all $\mathbf{u} \in \ker B$, so $\mathbf{v} \in \mathrm{im}\, B$. Thus a choice of basis $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ for $\mathrm{im}\, B$ defines a set of symplectic coordinates via the map

$$\begin{array}{cccc} \pi: & \mathbb{C}^N & \longrightarrow & \mathbb{C}^r \\ & \mathbf{x} & \longmapsto & \mathbf{U} := (\mathbf{x}^{\mathbf{v}_1}, \mathbf{x}^{\mathbf{v}_2}, \ldots, \mathbf{x}^{\mathbf{v}_r}). \end{array} \tag{3.10}$$

For what follows, it will be convenient to choose a $\mathbb{Z}$-basis for the $\mathbb{Z}$-module $\mathrm{im}\, B_{\mathbb{Z}} = \mathrm{im}\, B \cap \mathbb{Z}^N$, which guarantees that φ induces a birational map in the coordinates $\mathbf{U} = (U_1, U_2, \ldots, U_r)$. (Different choices of basis are possible, such as taking any set of r independent rows of B; cf. the τ-coordinates in [20, 21].)

Theorem 3.3. *Let $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ be a $\mathbb{Z}$-basis for $\mathrm{im}\, B_{\mathbb{Z}}$. Then given π as in (3.10), there is an associated symplectic birational map $\hat{\varphi} : \mathbb{C}^r \to \mathbb{C}^r$ such that $\pi \cdot \varphi = \hat{\varphi} \cdot \pi$, with $\pi^*\hat{\omega} = \omega$, where the symplectic form $\hat{\omega}$ is log-canonical in the coordinates $(U_1, U_2, \ldots, U_r)$.*

Proof: This is equivalent to Theorem 2.6 in [18], corresponding to a choice of basis as in case (a) of Lemma 2.9 therein. □

The rest of this section is taken up with presenting a technical result, Proposition 3.9 below, which shows that there is a special choice of $\mathbb{Z}$-basis in Theorem 3.3 such that the map $\hat{\varphi}$ is equivalent to a recurrence relation, which we will refer to as the *U-system.*

Definition 3.4. *For a non-zero vector* $\mathbf{v} \in \mathbb{Q}^N$*, we say that* $\mathbf{v}$ *has* support $\mathrm{supp}(\mathbf{v}) = [j,k] \subset [1,N]$ *whenever* $v_j v_k \neq 0$ *and* $v_i = 0$ *for all* $i < j$ *and* $i > k$*; and in that case the* length *of the support is* $|\mathrm{supp}(\mathbf{v})| = |[j,k]| = k - j + 1$*. Moreover, we say that a vector* $\mathbf{v}$ *with* $\mathrm{supp}(\mathbf{v}) = [j,k]$ *has* palindromic support *if* $v_i = v_{k+j-i}$ *for all* $i \in [j,k]$*.*

Example 3.5. The exchange matrix (3.5) has rows $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3, \mathbf{b}_4$ with $\mathrm{supp}(\mathbf{b}_1) = [2,4]$, $\mathrm{supp}(\mathbf{b}_2) = \mathrm{supp}(\mathbf{b}_3) = [1,4]$ and $\mathrm{supp}(\mathbf{b}_4) = [1,3]$. The vectors $\mathbf{b}_1$ and $\mathbf{b}_4$ have palindromic support, but $\mathbf{b}_2$ and $\mathbf{b}_3$ do not.

Lemma 3.6. *For* $\mathbf{v} \in \mathbb{Q}^N$*, define the reversal map* $\mathrm{r} : \mathbf{v} = (v_j) \mapsto \mathrm{r}(\mathbf{v}) = (v_{N-j+1})$*; and for* $\mathbf{v}$ *with* $\mathrm{supp}(\mathbf{v}) \subset [1, N-1]$ *define the shift map* $\mathrm{s} : \mathbf{v} = (v_j) \mapsto \mathrm{s}(\mathbf{v}) = (v'_j)$*, where* $v'_1 = 0$ *and* $v'_j = v_{j-1}$ *for* $j = 2, \ldots, N$*. If* $\mathbf{v} \in \mathrm{im}\, B$ *then* $\mathrm{r}(\mathbf{v}) \in \mathrm{im}\, B$*, and if also* $\mathrm{supp}(\mathbf{v}) \subset [1, N-1]$ *then* $\mathrm{s}(\mathbf{v}) \in \mathrm{im}\, B$ *as well.*

Proof: Let $\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_N$ denote the rows of B. From the conditions (3.6) and (3.7) on the matrix elements of B, it follows that $\mathrm{r}(\mathbf{b}_j) = -\mathbf{b}_{N-j+1}$ for $j \in [1,N]$. Any $\mathbf{v} \in \mathrm{im}\, B$ can be written as a linear combination $\mathbf{v} = \sum_j c_j \mathbf{b}_j$, which means that the linear map r acts as $\mathrm{r}(\mathbf{v}) = -\sum_j c_j \mathbf{b}_{N-j+1} \in \mathrm{im}\, B$, proving the first claim.

For the second claim, note that whenever $\mathbf{v} \in \mathrm{im}\, B$ with $\mathrm{supp}(\mathbf{v}) \subset [1, N-1]$, the map φ acts on the monomial $\mathbf{x}^{\mathbf{v}}$ to yield $\varphi^*(\mathbf{x}^{\mathbf{v}}) = \mathbf{x}^{\mathrm{s}(\mathbf{v})}$. Picking a $\mathbb{Z}$-basis as in Theorem 3.3 means that $\mathbf{v} = \sum_i w_i \mathbf{v}_i$ for a vector $\mathbf{w} = (w_i) \in \mathbb{Z}^r$, and then $\varphi^*(\mathbf{x}^{\mathbf{v}}) = \varphi^* \pi^*(\mathbf{U}^{\mathbf{w}}) = \pi^* \hat{\varphi}^*(\mathbf{U}^{\mathbf{w}})$. So $\mathbf{x}^{\mathrm{s}(\mathbf{v})}$ is the pullback of a function of the coordinates U_i, implying that this monomial is invariant under the action of the scaling transformations (3.9), and $(\mathbf{u}, \mathrm{s}(\mathbf{v})) = 0$, for all $\mathbf{u} \in \ker B$. Hence $\mathrm{s}(\mathbf{v}) \in \mathrm{im}\, B$, as required. □

Definition 3.7. *A* palindromic basis *for an* r*-dimensional subspace of* $\mathbb{Q}^N$ *is a basis* $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ *such that* $\mathbf{v}_j = \mathrm{s}^{j-1}(\mathbf{v})$*,* $j \in [1,r]$*, for a vector* $\mathbf{v}$ *with palindromic support, where* $\mathrm{supp}(\mathbf{v}) \subset [1, N-r+1]$*.*

Example 3.8. For the matrix B in Example 3.1, the subspace $\mathrm{im}\, B \subset \mathbb{Q}^4$ has the palindromic basis $\{\mathbf{v}_1, \mathbf{v}_2\}$, where $\mathbf{v}_1 = \mathbf{b}_4 = (1,-2,1,0)$ and $\mathbf{v}_2 = -\mathbf{b}_1 = (0,1,-2,1)$.

Proposition 3.9. *The subspace* $\mathrm{im}\, B \subset \mathbb{Q}^N$ *admits a palindromic basis, which yields a* $\mathbb{Z}$*-basis for* $\mathrm{im}\, B_{\mathbb{Z}}$ *(unique up to an overall sign).*

Proof: See Appendix. □

Example 3.10. The 6×6 exchange matrix

$$B = \begin{pmatrix} 0 & -1 & 1 & 0 & 1 & -1 \\ 1 & 0 & -2 & 1 & -1 & 1 \\ -1 & 2 & 0 & -2 & 1 & 0 \\ 0 & -1 & 2 & 0 & -2 & 1 \\ -1 & 1 & -1 & 2 & 0 & -1 \\ 1 & -1 & 0 & -1 & 1 & 0 \end{pmatrix} \qquad (3.11)$$

has $\mathbf{a} = (-1, 1, 0, 1, -1)$, and the vectors $\mathbf{b}_6 = (-\mathbf{a}, 0)$ and $\mathrm{s}(\mathbf{b}_6) = (0, -\mathbf{a}) = -\mathbf{b}_1$ span a 2-dimensional subspace of $\operatorname{im} B$, but $\operatorname{rank} B = 4$. The vector $\mathbf{v} = (1, -2, 1, 0, 0, 0)$ generates the palindromic $\mathbb{Z}$-basis $\{\, \mathbf{v}, \mathrm{s}(\mathbf{v}), \mathrm{s}^2(\mathbf{v}), \mathrm{s}^3(\mathbf{v}) \,\}$ for $\operatorname{im} B_{\mathbb{Z}}$.

Definition 3.11. *Let B be a skew-symmetric integer matrix satisfying the conditions (3.6) and (3.7), and let $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ be the unique palindromic $\mathbb{Z}$-basis for $\operatorname{im} B_{\mathbb{Z}}$ such that the first component of $\mathbf{v}_1$ is positive. Then the iteration of the corresponding map $\hat{\varphi}$, as in Theorem 3.3, is equivalent to iterating a recurrence of the form*

$$U_{n+r}\, U_n = \mathcal{F}(U_{n+1}, U_{n+2}, \ldots, U_{n+r-1}), \tag{3.12}$$

for a certain rational function $\mathcal{F}$. We refer to (3.12) as the U-system *associated with the exchange matrix B.*

Example 3.12. The recurrence (2.3), with $Y_n \to U_n$, is the U-system for the Somos-4 exchange matrix (3.5).

4. Y-systems and Z-systems

In this section we give a rapid introduction to cluster algebras with coefficients, in order to describe Nakanishi's notion of periodicity in cluster algebras [47], which includes the cluster mutation-periodicity of [15] as a special case. For all cluster algebras admitting such periodicity, it is possible to define associated T- and Y-systems. After a brief summary of the general situation, the focus returns to the case of cluster mutation-periodicity with period 1, as considered in the last section.

4.1. Cluster algebras with coefficients

Let $\mathbb{P}$ be a semifield, an abelian multiplicative group endowed with a binary operation $\oplus$ which is commutative, associative, and distributive with respect to the group multiplication. In general, a cluster algebra with coefficients, of rank N, is the algebra $\mathcal{A} = \mathcal{A}(B, \mathbf{x}, \mathbf{y})$ generated by the clusters of seeds $(B', \mathbf{x}', \mathbf{y}')$ produced from all possible sequences of mutations starting from an initial seed $(B, \mathbf{x}, \mathbf{y})$, where B is an $N \times N$ skew-symmetrizable matrix, $\mathbf{x}$ is a cluster, and $\mathbf{y}$ is a coefficient tuple $\mathbf{y} = (y_1, \ldots, y_N) \in \mathbb{P}^N$. Under the mutation μ_k, the elements of B are mutated according to (3.1), while the coefficients have the exchange relation

$$\tilde{y}_j = \begin{cases} y_k^{-1}, & j = k, \\ y_j\Big(1 \oplus y_k^{-\mathrm{sgn}(b_{kj})}\Big)^{-b_{kj}}, & j \neq k. \end{cases} \tag{4.1}$$

The exchange relation for cluster variables is

$$\tilde{x}_j = \begin{cases} \dfrac{y_k \prod_{j=1}^N x_j^{[b_{k,j}]_+} + \prod_{j=1}^N x_j^{[-b_{k,j}]_+}}{(1 \oplus y_k) x_k}, & j = k, \\ x_i, & j \neq k, \end{cases} \tag{4.2}$$

and this reduces to the relation (3.2) in the coefficient-free case by taking the projection from $\mathbb{P}$ to the trivial semifield consisting of a single element, $\{1\}$.

Y-systems, to be introduced shortly, correspond to relations between elements of the universal semifield $\mathbb{P}_{univ}(\mathbf{y})$, consisting of subtraction-free rational functions in the variables y_j. Working in this semifield, the addition $\oplus$ can be replaced by the usual addition $+$ in the field $\mathbb{Q}(\mathbf{y})$.

4.2. Periodicity and Y-systems

Nakanishi's general notion of periodicity can be stated as follows.

Definition 4.1. *Let the matrix B belong to a seed in a cluster algebra $\mathcal{A}$ of rank N, and let $\mathbf{i} = (i_1, \ldots, i_h)$ be a sequence of indices $i_j \in I := [1, N]$. For the composition $\mu_{\mathbf{i}} = \mu_{i_h} \cdot \ldots \cdot \mu_{i_2} \cdot \mu_{i_1}$, set $\tilde{B} = \mu_{\mathbf{i}} B$, and let $\nu \in S_N$ be a permutation. Then the sequence $\mathbf{i}$ is said to be a ν-*period *of B if $\tilde{B} = \nu(B)$, i.e. $\tilde{b}_{\nu(j)\nu(k)} = b_{jk}$ for $j, k = 1, \ldots, N$; moreover, if $\nu =$*id *then it is just called a* period *of B.*

Example 4.2. A cluster mutation-periodic quiver Q with period m, as defined in [15], is equivalent to a skew-symmetric matrix B for which the sequence $\mathbf{i} = (1, 2, \ldots, m)$ is ρ^m-periodic, for a cyclic permutation ρ, in the sense of the above definition. In particular, a period 1 quiver is equivalent to a matrix B for which $\mathbf{i} = (1)$ is a ρ-period; any such matrix satisfies the conditions (3.6) and (3.7).

Remark 4.3. Note that [15] and [47] adopt opposite conventions for labelling permutations, corresponding to $\rho \leftrightarrow \rho^{-1}$ above. In this section we follow [47].

For a symmetrizable matrix B with a ν*-period* $\mathbf{i}$, the Y-system is, roughly speaking, a set of algebraic relations between the coefficients $\tilde{y}_i$ obtained by applying mutations $\mu_{\mathbf{i}}, \ \mu_{\nu(\mathbf{i})}, \ \ldots, \ \mu_{\nu^{g-1}(\mathbf{i})}$ in this order, where g is the order of ν. To be more precise, we recall some definitions in [47].

Definition 4.4. *Let B be a symmetrizable matrix with a ν-*period $\mathbf{i}$. *We say* $\mathbf{i}$ *is* regular *if all the components of* $\mathbf{i}|\nu(\mathbf{i})|\cdots|\nu^{g-1}(\mathbf{i})$ *exhaust I, and if all the components of* $\mathbf{i}$ *belong to distinct ν-orbits in I. We decompose* $\mathbf{i}$ *into t parts,* $\mathbf{i}(0)|\mathbf{i}(1)|\cdots|\mathbf{i}(t-1)$ *and write* $\mathbf{i}(p) = (i(p)_1, \ldots, i(p)_{r_p})$ *for $p = 0, \ldots, t-1$. This decomposition is called a* slice *of* $\mathbf{i}$*, of length t, if the mutations $\mu_{i(p)_1}, \ldots, \mu_{i(p)_{r_p}}$ form a commuting set for each p.*

We assume that $\mathbf{i}$ is regular and has a slice $\mathbf{i} = \mathbf{i}(0)|\mathbf{i}(1)|\ldots|\mathbf{i}(t-1)$. Then we define a sequence of seeds $\{(B(u), \mathbf{x}(u), \mathbf{y}(u))\}_{u\in\mathbb{Z}}$ by

$$\begin{aligned}
\cdots\cdots \overset{\mu_{\nu^{-1}(\mathbf{i}(t-2))}}{\longleftrightarrow} (B(-1), \mathbf{x}(-1), \mathbf{y}(-1)) \overset{\mu_{\nu^{-1}(\mathbf{i}(t-1))}}{\longleftrightarrow} \\
(B(0), \mathbf{x}(0), \mathbf{y}(0)) \overset{\mu_{\mathbf{i}(0)}}{\longleftrightarrow} (B(1), \mathbf{x}(1), \mathbf{y}(1)) \overset{\mu_{\mathbf{i}(1)}}{\longleftrightarrow} \cdots \overset{\mu_{\mathbf{i}(t-1)}}{\longleftrightarrow} \\
(B(t), \mathbf{x}(t), \mathbf{y}(t)) \overset{\mu_{\nu(\mathbf{i}(0))}}{\longleftrightarrow} (B(2), \mathbf{x}(t+1), \mathbf{y}(t+1)) \overset{\mu_{\nu(\mathbf{i}(1))}}{\longleftrightarrow} \cdots\cdots
\end{aligned}$$

where $(B(0), \mathbf{x}(0), \mathbf{y}(0)) := (B, \mathbf{x}, \mathbf{y})$. The set of *forward mutation points* $P_+ \subset I \times \mathbb{Z}$ are pairs $(i, u) \in I \times \mathbb{Z}$ such that i is a component of $\nu^m(\mathbf{i}(k))$ for $u = mt + k$ ($m \in \mathbb{Z}$, $k \in [0, t-1]$). Take g_i to be the smallest positive integer such that $\nu^{g_i}(i) = i$. Then the Y-system takes the form

$$y_i(u)\, y_i(u + tg_i) = \frac{\prod_{(j,v)\in P_+} (1 + y_j(v))^{G'_+(j,v;i,u)}}{\prod_{(j,v)\in P_+} (1 + y_j(v)^{-1})^{G'_-(j,v;i,u)}}, \tag{4.3}$$

where

$$G'_{\pm}(j,v;i,u) = \begin{cases} \mp b_{ji}(v) & \text{if} \quad v \in (u, u+tg_i),\ b_{ji}(u) \lessgtr 0, \\ 0 & \text{otherwise.} \end{cases}$$

The corresponding (coefficient-free) T-system is

$$x_i(u)\, x_i(u+tg_i) = \prod_{(j,v)\in P_+} x_j(v)^{H'_+(j,v;i,u)} + \prod_{(j,v)\in P_+} x_j(v)^{H'_-(j,v;i,u)}, \tag{4.4}$$

where

$$H'_{\pm}(j,v;i,u) = \begin{cases} \pm b_{ji}(u) & \text{if} \quad u \in (v-tg_j, v),\ b_{ji}(u) \gtrless 0, \\ 0 & \text{otherwise.} \end{cases}$$

Here we modify the T-system by introducing a coefficient that multiplies both terms on the right hand side. (However, this is different from the way that coefficients appear in the general exchange relation (4.2) in a cluster algebra.)

Proposition 4.5. *Let $x_i(u)$ satisfy the modified T-system*

$$x_i(u)\, x_i(u+tg_i) = \mathcal{Z}_i(u)\Big(\prod_{(j,v)\in P_+} x_j(v)^{H'_+(j,v;i,u)} + \prod_{(j,v)\in P_+} x_j(v)^{H'_-(j,v;i,u)}\Big). \tag{4.5}$$

Then

$$\overline{y}_i(u) = \prod_{(j,v)\in P_+} \frac{x_j(v)^{H'_+(j,v;i,u)}}{x_j(v)^{H'_-(j,v;i,u)}} \tag{4.6}$$

satisfies the Y-system (4.3) if and only if

$$\prod_{(j,v)\in P_+} \frac{\mathcal{Z}_j(v)^{H'_+(j,v;i,u)}}{\mathcal{Z}_j(v)^{H'_-(j,v;i,u)}} = 1. \tag{4.7}$$

Proof: Upon substituting $y_i(u) = \overline{y}_i(u)$ into each side of (4.3), this follows from a direct calculation that is almost identical to the proof of Proposition 5.11 in [47]. □

We refer to (4.7) as the *Z-system*, while we say that the Z-system together with the modified T-system (4.5) in Proposition 4.5 constitute the *T_z-system*.

Henceforth we focus on the special case of cluster mutation-periodicity with period 1, which we describe in more detail. In order to define the Y-system in the period 1 case, one starts from $B(0) = B$ and defines a sequence $B(u)$ for $u \in \mathbb{Z}$ by applying an infinite sequence of mutations, starting with $B(1) = \mu_1(B(0))$, then $B(2) = \mu_{\rho(1)}(B(1))$, $B(3) = \mu_{\rho^2(1)}(B(2))$, and so on, and similarly extending backwards with $B(-1) = \mu_{\rho^{-1}(1)}(B(0))$, etc. In this way, one has $B(n) = \rho^n(B)$ for $n \in \mathbb{Z}$, hence $B(N) = B$ since $\rho^N = $id; so this sequence of matrices is periodic. Similarly, this infinite sequence of mutations produces a sequence of clusters $\mathbf{x}(u)$ and a sequence of coefficient tuples $\mathbf{y}(u)$ for $u \in \mathbb{Z}$, but in general the latter two sequences are not periodic.

In the period 1 case, all the vertices of the quiver corresponding to B cycle with period N under the action of ρ, and the Y-system relates $y_j(u)$ to $y_j(u+N)$. In fact, in this case, by replacing $y_j(n-1) \to y_n$ whenever $n = j \mod N$, one can write a single recurrence relation for y_n, $n \in \mathbb{Z}$, as follows.

Proposition 4.6. *The Y-system associated with a cluster mutation-periodic quiver Q with period 1 corresponding to a skew-symmetric matrix B can be written as the recurrence*

$$y_{n+N}\, y_n = \frac{\prod_{j=1}^{N-1}(1+y_{n+j})^{[-b_{1,j+1}]_+}}{\prod_{j=1}^{N-1}(1+y_{n+j}^{-1})^{[b_{1,j+1}]_+}}. \tag{4.8}$$

Proof: This follows from (4.3), which is the same as equation (5.8) in [47]. □

Note that (4.8) is the same as (1.4), with the exponents written in terms of the coefficients in the first row of B.

The T-system with coefficients corresponding to the ρ-period (1) consists of the analogous set of formulae for the combinations $x_j(u+N)x_j(u)$ of cluster variables, generated by the exchange relation (4.2), and the coefficient-free case is induced by the projection $\pi_1 : \mathbb{P}_{univ}(\mathbf{y}) \to \{1\}$ from the coefficient semifield to the trivial semifield. As for $y_j(u)$, the period 1 property of B means that we can replace $x_j(n-1) \to x_n$ whenever $n = j \mod N$, and write a single recurrence for x_n.

Proposition 4.7. *In the case of cluster mutation-periodicity with period 1, the T-system without coefficients is equivalent to the recurrence (3.4). If x_n is a solution of (3.4), then taking $y_n = \bar{y}_n$, where*

$$\bar{y}_n = \prod_{j=1}^{N-1} x_{n+j}^{-b_{1,j+1}}, \tag{4.9}$$

yields a solution of the Y-system (4.8).

Proof: This is a special case of Proposition 5.11 in [47], using the formula (5.29) therein. □

As we saw in section 2, for the example of Somos-4, the substitution (4.9) does not provide the general solution of the Y-system (4.8). The key to understanding the discrepancy between the T-system and the Y-system is the following.

Proposition 4.8. *Let x_n be a solution of the modified T-system*

$$x_{n+N}x_n = \mathcal{Z}_n\Big(\prod_{j=1}^{N-1} x_{n+j}^{[b_{1,j+1}]_+} + \prod_{j=1}^{N-1} x_{n+j}^{[-b_{1,j+1}]_+}\Big). \tag{4.10}$$

Then taking $y_n = \bar{y}_n$, where

$$\bar{y}_n = \prod_{j=1}^{N-1} x_{n+j}^{-b_{1,j+1}}, \tag{4.11}$$

yields a solution of the Y-system if and only if $\mathcal{Z}_n$ satisfies

$$\prod_{j=1}^{N-1} \mathcal{Z}_{n+j}^{-b_{1,j+1}} = 1. \tag{4.12}$$

The space of solutions of the T_z-system consisting of (4.10) together with (4.12) has dimension $N+\tilde{r}$, where $\tilde{r} = |\mathrm{supp}(\mathbf{b}_1)| - 1$.

Proof: The first claim above is just a special case of Proposition 4.5. To prove the second claim, observe that the exponents appearing in (4.12) are (up to sign) just the components of the vector $\mathbf{b}_1$ in the first row of B, so that in this case the Z-system is a difference equation of order $\tilde{r}$. Then $\tilde{r}$ initial values are required for (4.12), say $\mathcal{Z}_0, \mathcal{Z}_1, \ldots, \mathcal{Z}_{\tilde{r}-1}$, together with a further N values $x_0, x_1, \ldots, x_{N-1}$ in (4.10), giving a total of $N + \tilde{r}$ initial data. □

Remark 4.9. In general, unless the first non-zero component of $\mathbf{b}_1$ (the leading exponent) is ± 1, the Z-system (4.12) is an algebraic recurrence relation rather than a rational one. When the leading exponent is ± 1, the Z-system just generates a sequence of monomials in the initial data, and in all such examples it appears that the Laurent property holds for (4.10), in the sense that $x_n \in \mathbb{Z}[x_0^{\pm 1}, \ldots, x_{N-1}^{\pm 1}, \mathcal{Z}_0^{\pm 1}, \ldots, \mathcal{Z}_{\tilde{r}-1}^{\pm 1}]$.

The discrepancy between the solutions of the Y-system and the coefficient-free T-system can be seen by looking at the fibres of the map from $\mathbf{x}$ variables to $\mathbf{y}$ variables defined by (4.11): generically, these have dimension $\tilde{r}$, as there is an action of a torus $(\mathbb{C}^*)^{\tilde{r}}$ by scalings $\mathbf{x} \to \lambda^{\mathbf{u}} \cdot \mathbf{x}$ for integer vectors $\mathbf{u}$ that are orthogonal to $\mathbf{b}_1$ and the shifts $\mathrm{s}^k(\mathbf{b}_1)$ (for all k where this is defined), and the value of $\mathbf{y}$ is preserved by any such scaling. The solutions of the Y-system can also be understood by introducing symplectic coordinates in the same way as for the coefficient-free T-system, which leads to the following.

Definition 4.10. *The* U_z-system *associated with (4.10) consists of the recurrence*

$$U_{n+r}\, U_n = \mathcal{Z}_n\, \mathcal{F}(U_{n+1}, U_{n+2}, \ldots, U_{n+r-1}), \tag{4.13}$$

with the same rational function $\mathcal{F}$ as in (3.12), together with (4.12).

5. Examples of T_z-systems and equations of q-Painlevé type

In this section we give some examples of T_z-systems and U_z-systems for some particular Y-systems associated with quivers that are mutation periodic with period 1.

5.1. Affine A-type quivers and linear relations

The 4×4 exchange matrix

$$B = \begin{pmatrix} 0 & -1 & 0 & -1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$$

corresponds to the $A_3^{(1)}$ affine Dynkin diagram, oriented such that one of the arrows is clockwise and the other three are anticlockwise; this quiver is sometimes denoted $\tilde{A}_{1,3}$, and it is also one of the primitive quivers involved in the classification of period 1 quivers by Fordy and Marsh [15], who denote it $P_4^{(1)}$. The T_z-system in this case has the form

$$x_{n+4}\, x_n = \mathcal{Z}_n\, (x_{n+3}\, x_{n+1} + 1), \qquad \text{with} \qquad \mathcal{Z}_{n+2}\, \mathcal{Z}_n = 1. \tag{5.1}$$

The simple form of the Z-system above means that the coefficients of the non-autonomous T-system are periodic, being given by the sequence $\mathcal{Z}_0, \mathcal{Z}_1, \mathcal{Z}_0^{-1}, \mathcal{Z}_1^{-1}$, repeated with period 4. Since the matrix B is of full rank, the U$_z$-system is the same as the T$_z$-system in this case, while the Y-system is given by

$$y_{n+4}\, y_n = (1 + y_{n+3})(1 + y_{n+1}). \tag{5.2}$$

In [30] it was shown that for the autonomous version of the T-system, obtained by taking $\mathcal{Z}_n = 1$ for all n in (5.1), the iterates satisfy the fourth-order linear recurrence $x_{n+4} - K x_{n+2} + x_n = 0$, where the coefficient K is a first integral (conserved quantity), constant along orbits. Using computer algebra (with Maple) we have verified the following result for the T$_z$-system.

Proposition 5.1. *The iterates x_n of the T_z-system (5.1) satisfy the linear relation*

$$x_{n+24} - \mathrm{C}\, x_{n+12} + x_n = 0, \tag{5.3}$$

where the first integral C *is a Laurent polynomial in* $\mathbb{Z}[x_0^{\pm 1}, x_1^{\pm 1}, x_2^{\pm 1}, x_3^{\pm 1}, \mathcal{Z}_0^{\pm 1}, \mathcal{Z}_1^{\pm 1}]$ *consisting of 67 terms with positive coefficients.*

The existence of such a linear relation means that (5.1), and hence (5.2), can be solved explicitly, and from the form of (5.3) the solution can be written in terms of Chebyshev polynomials with argument $\mathcal{K}/2$.

For cluster algebras obtained from affine Dynkin quivers, linear relations between cluster variables have been derived in various ways [2, 15, 16, 40]. In particular one can take the $A_{N-1}^{(1)}$ Dynkin diagram with one clockwise arrow and $N-1$ anticlockwise arrows to get the quiver $\tilde{A}_{1,N-1}$, which is the same as the primitive $P_N^{(1)}$ in [15]. For this family of quivers, the T$_z$-system has the form

$$x_{n+N}\, x_n = \mathcal{Z}_n\,(x_{n+N-1}\, x_{n+1} + 1), \qquad \text{with} \qquad \mathcal{Z}_{n+N-2}\, \mathcal{Z}_n = 1, \tag{5.4}$$

while the Y-system is

$$y_{n+N}\, y_n = (1 + y_{n+N-1})(1 + y_{n+1}). \tag{5.5}$$

Extensive numerical experiments suggest that there are linear relations with constant coefficients for all members of this family, given as follows.

Conjecture 5.2. *For each $N \geq 3$, the iterates of the $\tilde{A}_{1,N-1}$ T_z-system (5.4), associated with the Y-system (5.5), satisfy a constant-coefficient linear recurrence relation of order $4(N-1)(N-2)$, having the form*

$$x_{n+4s} - \mathrm{A}\, x_{n+3s} + \mathrm{B}\, x_{n+2s} - \mathrm{A}\, x_{n+s} + x_n = 0$$

for N odd, and

$$x_{n+4s} - \mathrm{C}\, x_{n+2s} + x_n = 0$$

for N even, where A, B, C *denote first integrals, and $s = (N-1)(N-2)$.*

5.2. Somos-type T_z-systems

Rather than considering the most general Somos-type recurrence, of the form (1.5), here we focus on some particular examples which indicate how the results in section 2 generalize to other q-Painlevé type equations and their higher-order analogues.

To begin with, we look at the family of bilinear T_z-systems given by

$$x_{n+N}\,x_n = \mathcal{Z}_n\,(x_{n+N-1}\,x_{n+1} + x_{n+N-2}\,x_{n+2}), \qquad \frac{\mathcal{Z}_{n+N-2}\,\mathcal{Z}_n}{\mathcal{Z}_{n+N-3}\,\mathcal{Z}_{n+1}} = 1, \tag{5.6}$$

corresponding to $p = 1$, $q = 2$ in (1.5). After solving the Z-system (2.13), the case $N = 4$ of (5.6) is equivalent to (2.16). In general, just as for Somos-4, it is always possible to make a gauge transformation to move the non-autonomous coefficient entirely onto either the first or the second term on the right hand side of the recurrence for x_n; we illustrate this in some specific examples below. A special case of the family (5.6), with purely periodic coefficients, arose by a reduction of the discrete KdV equation in [34].

Example 5.3 (Somos-5)**.** For $N = 5$, the T_z-system (5.6) becomes

$$x_{n+5}\,x_n = \mathcal{Z}_n\,(x_{n+4}\,x_{n+1} + x_{n+3}\,x_{n+2}), \qquad \frac{\mathcal{Z}_{n+3}\,\mathcal{Z}_n}{\mathcal{Z}_{n+2}\,\mathcal{Z}_{n+1}} = 1,$$

which corresponds to the Y-system

$$y_{n+5}\,y_n = \frac{(1+y_{n+4})(1+y_{n+1})}{(1+y_{n+3}^{-1})(1+y_{n+2}^{-1})}.$$

The exchange matrix B in this case has rank 2 (see [17] for details), and taking $U_n = x_{n+3}x_n/(x_{n+2}x_{n+1})$ gives the associated U_z-system

$$U_{n+2}\,U_n = \mathcal{Z}_n\,(1 + U_{n+1}^{-1}), \qquad \text{with} \qquad \mathcal{Z}_n = \beta_n\,\mathfrak{q}^n, \quad \beta_{n+2} = \beta_n. \tag{5.7}$$

Each iteration of the latter is symplectic, preserving the log-canonical 2-form $\hat{\omega} = (U_n\,U_{n+1})^{-1}\,\mathrm{d}U_{n+1} \wedge \mathrm{d}U_n$, and (5.7) is an alternative discrete Painlevé I equation (although not one of the ones derived in [49]). Note that it is possible to use a gauge transformation G_n in the T_z-system, setting $x_n = G_n x'_n$, to move all the non-autonomous part onto the first term on the right hand side; in that case, by taking $U'_n = x'_{n+3}x'_n/(x'_{n+2}x'_{n+1})$, the equation (5.7) can be transformed to

$$U'_{n+2}\,U'_n = \alpha_n + \beta\,(U'_{n+1})^{-1}, \qquad \text{where} \qquad (\mathcal{S}^3-1)(\mathcal{S}^2-1)(\mathcal{S}+1)\log\alpha_n = 0 \mod 2\pi\mathrm{i},$$

with $\mathcal{S}$ denoting the shift operator and the coefficient β being constant. (The autonomous case α_n =constant is a QRT map, explicitly solved in [28].) Observe that this choice of gauge introduces period 3 behaviour which is not present in $\mathcal{Z}_n$.

Example 5.4 (A particular case of Somos-6)**.** For $N = 6$, the T_z-system (5.6) is

$$x_{n+6}\,x_n = \mathcal{Z}_n\,(x_{n+5}\,x_{n+1} + x_{n+4}\,x_{n+2}), \qquad \frac{\mathcal{Z}_{n+4}\,\mathcal{Z}_n}{\mathcal{Z}_{n+3}\,\mathcal{Z}_{n+1}} = 1,$$

which corresponds to the exchange matrix (3.11), and gives the Y-system

$$y_{n+6}\, y_n = \frac{(1+y_{n+5})(1+y_{n+1})}{(1+y_{n+4}^{-1})(1+y_{n+2}^{-1})}.$$

From Example 3.10, the matrix B has rank 4, and setting $U_n = x_n x_{n+2}/(x_{n+1})^2$ in the T_z-system and solving the Z-system produces the U_z-system

$$U_{n+4}(U_{n+3}U_{n+2}U_{n+1})^2 U_n = \beta_n\, \mathfrak{q}^n\, (1 + U_{n+3}U_{n+2}U_{n+1}), \quad \text{with} \quad \beta_{n+3} = \beta_n.$$

Each iteration of the latter is a symplectic map, preserving the nondegenerate Poisson bracket in four dimensions defined by

$$\{U_n, U_{n+1}\} = U_n U_{n+1}, \quad \{U_n, U_{n+2}\} = -U_n U_{n+2}, \quad \{U_n, U_{n+1}\} = U_n U_{n+3}.$$

In the autonomous case ($\mathfrak{q} = 1$ and $\beta_n =$constant) this is equivalent to one of the Liouville integrable maps in [33].

The preceding two examples can be generalized to cover this whole family, depending on whether N is even/odd. For each N, the solution of the Z-system in (5.6) can be written as

$$\mathcal{Z}_n = \beta_n\, \mathfrak{q}^n, \qquad \beta_{n+N-3} = \beta_n;$$

when N is even, the substitution $U_n = x_n x_{n+2}/(x_{n+1})^2$ gives the U_z-system

$$U_{n+N-2} \left(\prod_{j=1}^{N-3} U_{n+j} \right)^2 U_n = \mathcal{Z}_n \left(1 + \prod_{k=1}^{N-3} U_{n+k} \right), \tag{5.8}$$

while for N odd, setting $U_n = x_n x_{n+3}/(x_{n+1}x_{n+2})$ yields the U_z-system

$$\prod_{j=0}^{N-3} U_{n+j} = \mathcal{Z}_n \left(1 + \prod_{k=0}^{(N-5)/2} U_{n+2k+1} \right). \tag{5.9}$$

We consider one further Somos-type example, which is outside the family (5.6).

Example 5.5 (A special case of Somos-7)**.** One of the examples considered in section 6 of [18] corresponds to the T-system for a 7×7 exchange matrix with rank $B = 2$, reducing to a symplectic map of the plane. The associated T_z-system is

$$x_{n+7}\, x_n = \mathcal{Z}_n (x_{n+6}\, x_{n+1} + x_{n+4}\, x_{n+3}), \qquad \frac{\mathcal{Z}_{n+5}\, \mathcal{Z}_n}{\mathcal{Z}_{n+3}\, \mathcal{Z}_{n+2}} = 1, \tag{5.10}$$

with the Y-system being

$$y_{n+7}\, y_n = \frac{(1+y_{n+6})(1+y_{n+1})}{(1+y_{n+4}^{-1})(1+y_{n+3}^{-1})}$$

while setting $U_n = x_{n+5}x_n/(x_{n+3}x_{n+2})$ yields the U_z-system

$$U_{n+2}\, U_n = \mathcal{Z}_n\, (U_{n+1} + 1).$$

In this case, it is convenient to choose a gauge transformation G_n, with $x_n = G_n x'_n$, which moves all of the non-autonomous part onto the coefficient of the second term on the right hand side (and the coefficient of the first term can be fixed to be 1). This gives an equation for $U'_n = x'_{n+5}x'_n/(x'_{n+3}x'_{n+2})$ which is of Lyness type, that is

$$U'_{n+2}\,U'_n = U'_{n+1} + a_n, \quad \text{with} \quad (\mathcal{S}-1)(\mathcal{S}^6-1)\log a_n = 0 \mod 2\pi\mathrm{i}. \tag{5.11}$$

The form of the above is consistent with the results in [4], which show that, in the case that a_n is periodic, the Lyness recurrence (5.11) is integrable when the period is a divisor of 6, and indicate that chaos is present for all other periods. However, the characteristic polynomial of the linear recurrence for $\log a_n$ has 1 as a double root, so $a_n = \alpha_n\,\mathfrak{q}^n$ with α_n being periodic with period 6, and (5.11) is yet another alternative q-Painlevé I equation.

5.3. A nonintegrable example

The examples considered so far in this section are all integrable in some sense. In the context of iteration of rational functions, one way to characterize integrability of discrete dynamical systems is in terms of the weak growth of degrees of iterates: defining the algebraic entropy as $\mathcal{E} := \lim_{n\to\infty} n^{-1}\log d_n$, where d_n is the degree of the nth iterate, discrete integrable systems should have zero entropy [3]. In [18], one of us used the growth of denominators of $\mathbf{x}$ variables to identify integrable maps among the T-systems (1.3); this growth is determined by the tropical version of the T-system (in the sense of the max-plus algebra). Integrability is a rare phenomenon: for the majority of these T-systems (unless the exponents a_j are of sufficiently small magnitude), the degrees grow exponentially with n, indicating nonintegrability. Likewise, the corresponding T_z-system also exhibits exponential growth of degrees in the generic case, and usually this can be detected by looking at the Z-system alone, which we illustrate here with one particular example.

For the exchange matrix

$$B = \begin{pmatrix} 0 & -2 & 6 & -4 & 6 & -2 \\ 2 & 0 & -14 & 6 & -16 & 6 \\ -6 & 14 & 0 & 10 & 6 & -4 \\ 4 & -6 & -10 & 0 & -14 & 6 \\ -6 & 16 & -6 & 14 & 0 & -2 \\ 2 & -6 & 4 & -6 & 2 & 0 \end{pmatrix}, \tag{5.12}$$

the T_z-system is

$$x_{n+6}\,x_n = \mathcal{Z}_n\,(x_{n+5}^2 x_{n+3}^4 x_{n+1}^2 + (x_{n+4}\,x_{n+2})^6). \tag{5.13}$$

with

$$\frac{\mathcal{Z}_{n+4}^2\,\mathcal{Z}_{n+2}^4\,\mathcal{Z}_n^2}{\mathcal{Z}_{n+3}^6\,\mathcal{Z}_{n+1}^6} = 1, \tag{5.14}$$

and the associated Y-system is

$$y_{n+6}\,y_n = \frac{(1+y_{n+5})^2(1+y_{n+1})^4(1+y_{n+1})^2}{(1+y_{n+4}^{-1})^6(1+y_{n+2}^{-1})^6}.$$

The autonomous version of (5.13) was presented as Example 2.6 in [17]: the matrix (5.12) has rank 2, with $\mathbf{v} = (1,-3,2-3,1,0)$ and $\mathrm{s}(\mathbf{v}) = (0,1,-3,2-3,1)$ providing a palindromic $\mathbb{Z}$-basis for $\mathrm{im}\, B_{\mathbb{Z}}$, which leads to the U_z-system

$$U_{n+2}\,U_n = \mathcal{Z}_n\,(U_{n+1}^{-1} + U_{n+1}^{-3}),$$

where $\mathcal{Z}_n$ satisfies (5.14).

The first thing to note about this example is that the Z-system involves indeterminacy due to the leading exponent 2: to iterate (5.14) requires taking a square root to solve for $\mathcal{Z}_{n+4}$, so that a $\pm$ sign must be chosen coherently at each step. The second main observation is that, up to sign choices, the explicit solution to the Z-system can be obtained by solving the linear recurrence

$$(\mathcal{S}^2+1)(\mathcal{S}^2-3\mathcal{S}+1)\log\mathcal{Z}_n = 0 \mod \pi\mathrm{i}. \tag{5.15}$$

The nonintegrable nature of the system can be seen from the second quadratic factor above, which has a characteristic root $\lambda_{max} = (3+\sqrt{5})/2$, of magnitude greater than 1; this is in accordance with the positive value $\mathcal{E} = \log\lambda_{max}$ for the entropy of the corresponding T-system, as found in Example 3.2 of [17]. Taking $\mathcal{Z}_0, \mathcal{Z}_1, \mathcal{Z}_2, \mathcal{Z}_3$ as initial data, the Z-system (5.14) generates monomials of the form

$$\mathcal{Z}_n = \pm\frac{\mathcal{Z}_3^{d_n^{(3)}}\,\mathcal{Z}_1^{d_n^{(1)}}}{\mathcal{Z}_2^{d_n^{(2)}}\,\mathcal{Z}_0^{d_n^{(0)}}},$$

where each exponent $d_n^{(j)}$ for $j = 0,1,2,3$ satisfies the same homogeneous linear recurrence that is defined by the difference operator in (5.15); hence the degrees of these monomials grow like a constant times λ_{max}^n.

Our experience with other T_z-systems obtained from cluster mutation-periodic quivers with periods 1 and 2 suggests that if the Z-system displays exponential growth, and/or it has a leading exponent greater than 1, then the underlying T-system should have positive entropy. However, in general (unlike the preceding example) the degrees of $\mathbf{x}$ variables and $\mathcal{Z}$ variables need not grow with the same rate.

6. Conclusions

We have shown that there is a link between periodicity in cluster algebras, Y-systems and discrete Painlevé equations. This link also produces higher-order analogues of q-Painlevé equations, such as the families (5.8) and (5.9), which to the best of our knowledge are new. That there is such a link should not be entirely surprising, bearing in mind that there appears to be a close connection between singularity confinement and the Laurent phenomenon [29, 39], and the singularity confinement test was one of the first tools used to obtain discrete Painlevé equations [23]. On the other hand, it is known that singularity confinement is not sufficient for integrability [25], and we have seen that nonintegrable examples can arise from Y-systems as well - indeed, integrability should be the exception rather than the rule.

In the future it would be interesting to connect this construction of discrete Painlevé equations to the geometrical approach pioneered by Sakai [51], which has subsequently been extended to certain higher-order equations [38, 53].

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Appendix: Proof of Proposition 3.9

To construct a palindromic basis for $\operatorname{im} B$, working over $\mathbb{Q}$ to begin with, one can start with any vector $\mathbf{v} \in \operatorname{im} B$. If $\mathbf{v}$ does not have palindromic support, then apply (positive or negative) powers of s to $\mathrm{r}(\mathbf{v})$ to obtain a vector $\mathrm{s}^k(\mathrm{r}(\mathbf{v}))$ for k such that $\mathrm{supp}(\mathbf{v}) = \mathrm{supp}(\mathrm{s}^k(\mathrm{r}(\mathbf{v})))$; then make the replacement $\mathbf{v} \to \mathbf{v} + \mathrm{s}^k(\mathrm{r}(\mathbf{v}))$, where the latter vector does have palindromic support. Now given the length $\ell = |\mathrm{supp}(\mathbf{v})|$, if necessary one can apply s^{-1} sufficiently many times, to replace $\mathbf{v} \to \mathrm{s}^{-k^*}(\mathbf{v})$, for $k^* \geq 0$ such that the latter vector has support $[1, \ell]$. Due to Lemma 3.6, having prepared $\mathbf{v}$ suitably, the set $S = \{\mathrm{s}^j(\mathbf{v}) \,|\, j \in [0, N - \ell]\}$ is a palindromic basis for a subspace $V \subset \operatorname{im} B$, having dimension $N - \ell + 1$, and if $N - r + 1 = \ell$ then V coincides with $\operatorname{im} B$. Otherwise there is some row $\mathbf{b}_i$ of B which is not in the span of the basis generated by $\mathbf{v}$. It may be that $|\mathrm{supp}(\mathbf{b}_i)| < \ell$, but if not one can always subtract multiples of the vectors in S to obtain a new vector $\mathbf{v}'$ whose support has length less than ℓ, and by applying the same process as for $\mathbf{v}$ before one obtains a palindromic basis S' for a subspace $V' \subset \operatorname{im} B$ of dimension larger than that of V. Repeating the same steps sufficiently many times produces a palindromic basis for $\operatorname{im} B$.

To work over $\mathbb{Z}$, assume that a palindromic basis $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_r\}$ for $\operatorname{im} B$ is given; then without loss of generality (by suitable rescaling if necessary)

$$\mathbf{v}_1 = (a_1^*, a_2^*, \ldots, a_{N-r+1}^*, 0, \ldots, 0),$$

where a_j^* are integers (with $a_1^* \neq 0$ and $a_j^* = a_{N-r-j+2}^*$) such that the highest common factor of the non-zero entries is 1. Thus $\mathbf{v}_j \in \operatorname{im} B_{\mathbb{Z}}$ for $j \in [1, r]$, but it is not immediately clear that these vectors provide a $\mathbb{Z}$-basis. To see this, pick any $\mathbb{Z}$-basis for $\mathrm{im}\, B_{\mathbb{Z}}$, and use it to form the rows of an $r \times N$ matrix $A = (a_{ij})$. By applying elementary row operations, equivalent to premultiplying by a unit matrix in $\mathrm{Mat}_r(\mathbb{Z})$, one can arrange it so that A has the upper-triangular form

$$A = \begin{pmatrix} a_{11} & \cdots & \cdots & \cdots & \cdots & a_{1N} \\ 0 & a_{22} & \cdots & \cdots & \cdots & a_{2N} \\ \vdots & \ddots & \ddots & & & \vdots \\ 0 & \cdots & 0 & a_{rr} & \cdots & a_{rN} \end{pmatrix},$$

where $a_{jj} \neq 0$ for $j \in [1, r]$, and the highest common factor of the non-zero entries in each row is 1. Now consider the last row of A: this vector must have support $[r, N]$, since otherwise it would be a vector in $\mathrm{im}\, B$ of length less than $N - r + 1$, which would generate a palindromic basis for a space of dimension greater than r; and similarly, this vector must be a multiple of $\mathbf{v}_r$, since otherwise one could subtract $(a_1^*)^{-1} a_{rr} \mathbf{v}_r$ from the last row to obtain a vector in $\mathrm{im}\, B$ having support of smaller length. Then since both $\mathbf{v}_r$ and the last row are both integer vectors whose non-zero entries have highest common factor 1, this last row must equal $\pm \mathbf{v}_r$; with out loss of generality take the plus sign. Proceeding by induction, assume that rows $j+1$ to r of A can be taken to be the vectors $\mathbf{v}_{j+1}, \ldots, \mathbf{v}_r$, and let $\mathbf{v}'_j$ denote row j. The vector $\mathbf{v}_j$ belongs to the span of rows j to N (which span the vectors with support in $[j, N]$), so it can be written as a $\mathbb{Z}$-linear combination

$$\mathbf{v}_j = c_j \mathbf{v}'_j + c_{j+1} \mathbf{v}_{j+1} + \ldots + c_N \mathbf{v}_N,$$

with $c_j \neq 0$. The non-zero entries of the latter give a sequence of equations beginning

$$a_1^* = c_j a_{jj}, \quad a_2^* = c_j a_{j,j+1} + c_{j+1} a_1^*, \quad a_3^* = c_j a_{j,j+2} + c_{j+1} a_2^* + c_{j+2} a_1^*, \quad \ldots .$$

If $p | c_j$ then the first equation in this sequence implies that $p | a_1^*$, whence the second equation gives $p | a_2^*$, and then the third implies $p | a_3^*$, and so on, meaning that p is a common factor of the non-zero entries in $\mathbf{v}_1$. Hence $c_j = \pm 1$, and so $\mathbf{v}_j$ can be taken to replace $\mathbf{v}'_j$ in row j of A. By induction, the palindromic basis provides a $\mathbb{Z}$-basis for $\mathrm{im}\, B_{\mathbb{Z}}$, unique up to multiplying all basis vectors by -1. This completes the proof.

References

[1] J. Alman, C. Cuenca and J. Huang, `arXiv:1309.0751v2`
[2] I. Assem, C. Reutenauer and D. Smith, Adv. Math. **225** (2010) 3134–3165.
[3] M.P. Bellon and C.M. Viallet, Commun. Math. Phys. **204** (1999) 425–437.
[4] A. Cima, A. Gasull and V. Mañosa, Dynamical Systems **28** (2013) 518–538.
[5] P. Di Francesco and R. Kedem, Lett. Math. Phys. **89** (2009) 183–216.
[6] P. Di Francesco, Lett. Math. Phys. **96** (2011) 299–324.
[7] J.J. Duistermaat, *Discrete Integrable Systems: QRT Maps and Elliptic Surfaces*, Springer (2010).
[8] R. Eager, S. Franco and K. Schaeffer, JHEP **06** (2012) 106.
[9] R. Eager and S. Franco, JHEP **09** (2012) 038.
[10] L. Faybusovich and M. Gekhtman, J. Math. Phys. **41** (2000) 2905–2921.
[11] V.V. Fock and A.B. Goncharov, Ann. Sci. Éc. Norm. Supér. **42** (2009) 865–930.

[12] S. Fomin and A. Zelevinsky, J. Amer. Math. Soc. **15** (2002) 497–529.
[13] S. Fomin and A. Zelevinsky, Adv. Appl. Math. **28** (2002) 119–144.
[14] S. Fomin and A. Zelevinsky, Compositio Mathematica **143** (2007) 112–164.
[15] A.P. Fordy and R.J. Marsh, Journal of Algebraic Combinatorics, **34** (2011) 19–66.
[16] A.P. Fordy, Phil. Trans. Roy. Soc. A **369** (2011) 1264–1279
[17] A.P. Fordy and A.N.W. Hone, SIGMA **7** (2011) 091.
[18] A.P.Fordy and A.N.W.Hone, Commun. Math. Phys. **325** (2014) 527–584.
[19] D. Gale, Mathematical Intelligencer **13(1)** (1991), 40–42; Mathematical Intelligencer **13(4)** (1991), 49–50.
[20] M. Gekhtman, M. Shapiro and A. Vainshtein, Mosc. Math. J. **3** (2003) 899–934.
[21] M. Gekhtman, M. Shapiro and A. Vainshtein, Duke Math. J. **127** (2005) 291–311.
[22] A.B. Goncharov and R. Kenyon, `arXiv:1107.5588v2`
[23] B. Grammaticos, A. Ramani, V. Papageorgiou, Phys. Rev. Lett. **67** (1991) 1825.
[24] P. Heideman and E. Hogan, Elec. J. Comb. **15** (2008) #R54.
[25] J. Hietarinta and C.M. Viallet, Phys. Rev. Lett. **81** (1998) 325.
[26] R. Hirota, Phys. Rev. Lett. **27** (1972) 1192–1194.
[27] A.N.W. Hone, Bull. Lond. Math. Soc. **37** (2005) 161–171.
[28] A.N.W. Hone, Trans. Amer. Math. Soc. **359** (2007) 5019–5034.
[29] A.N.W. Hone, Physics Letters A **361** (2007) 341–345.
[30] A.N.W. Hone. SIGMA **3** (2007) 022.
[31] A.N.W. Hone, in J. McKee & C. Smyth (eds.), *Number Theory and Polynomials*, LMS Lect. Notes Series **352**, Cambridge (2008) 188–210.
[32] A.N.W. Hone, in A.V. Mikhailov (ed.), *Integrability*, Lect. Notes Phys. **767**, Springer, Berlin-Heidelberg (2009) 245–277.
[33] A.N.W. Hone, Applicable Analysis **89** (2010) 473–492.
[34] A.N.W. Hone, P.H. van der Kamp, G.R.W. Quispel and D.T. Tran, Proc. Roy. Soc. A **469** (2013) 20120747.
[35] A.N.W. Hone and C. Ward, Bull. Lond. Math. Soc. (2014) `doi:10.1112/blms/bdu004`
[36] R. Inoue, O. Iyama, A. Kuniba, T. Nakanishi, J. Suzuki, Nagoya Math. J. **197** (2010) 59–174.
[37] R. Inoue and T. Nakanishi, RIMS Kokyuroku Bessatsu **B28** (2011) 63–88.
[38] K.Kajiwara, T.Masuda, M.Noumi, Y.Ohta and Y.Yamada, Sémin. Congr. **14** (2006) 169–198.
[39] M. Kanki, J. Mada and T. Tokihiro, J. Phys. A: Math. Theor. **47** (2014) 065201.
[40] B. Keller and S. Scherotzke, Adv. Math. **228** (2011) 1842–1862.
[41] Y. Kodama and L. Williams, Invent. math. (2014) `doi:10.1007/s00222-014-0506-3`
[42] I. Krichever, O. Lipan, P. Wiegmann and A. Zabrodin, Commun. Math. Phys. **188** (1997) 267–304.
[43] V.B. Kuznetsov and E.K. Sklyanin, J. Phys. A: Math. Gen. **31** (1998) 2241.
[44] T. Lam and P. Pylyavskyy, `arXiv:1206.2611`
[45] S. Lobb and F. Nijhoff, J. Phys. A **42** (2009) 454013.
[46] S. Maeda, Proc. Japan Acad. **63**, Ser. A (1987) 198–200.
[47] T. Nakanishi, in A. Skowronski & K. Yamagata (eds.), *Representations of algebras and related topics*, EMS Series of Congress Reports, Eur. Math. Soc. (2011) 407–444.
[48] G.R.W. Quispel, J.A.G. Roberts and C.J. Thompson, Physics Letters A **126** (1988) 419–421; Physica D **34** (1989) 183–192.
[49] A. Ramani, B. Grammaticos and J. Satsuma, J. Phys. A: Math. Gen. **28** (1995) 4655–4665.
[50] S.N.M. Ruijsenaars, in B.A. Kupershmidt (ed.), *Integrable and Super-Integrable Systems*, World Scientific, Singapore (1990) 165–206.
[51] H. Sakai, Commun. Math. Phys. **220** (2001) 165–229.
[52] J.S. Scott, Proc. Lond. Math. Soc. **92** (2006) 345–380.
[53] T. Tsuda, Lett. Math. Phys. **77** (2006) 21–30.
[54] A.P. Veselov, Russ. Math. Surveys **46** (1991) 1–51.
[55] Al. B. Zamolodchikov, Phys. Lett. B **253** (1991) 391–394.