Sampled-Data Sliding Mode Observer for Robust Fault Reconstruction: A Time-Delay Approach

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Abstract

A sliding mode observer in the presence of sampled output information and its application to robust fault reconstruction is studied. The observer is designed by using the delayed continuous-time representation of the sampled-data system, for which sufficient conditions are given in the form of Linear Matrix Inequalities (LMIs) to guarantee the ultimate boundedness of the error dynamics. Though an ideal sliding motion cannot be achieved in the observer when the outputs are sampled, ultimately bounded solutions can be obtained provided the sampling frequency is fast enough. The bound on the solution is proportional to the sampling interval and the magnitude of the switching gain. The proposed observer design is applied to the problem of fault reconstruction under sampled outputs and system uncertainties. It is shown that actuator or sensor faults can be reconstructed reliably from the output error dynamics. An example of observer design for an inverted pendulum system is used to demonstrate the merit of the proposed methodology compared to existing sliding mode observer design approaches.

Keywords:
Sliding mode observer, sampled output, delay, fault reconstruction, LMI.

1. Introduction

A sliding mode observer is a category of robust observer which facilitates the complete rejection of a class of uncertainty between the system and observer [27].

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In most cases, the sliding surface is set to be the difference between the observer outputs and system outputs, which is therefore forced to zero [3], [28]. A discontinuous injection term is designed and applied to drive the observer so that the error between the outputs of the observer and the outputs of the plant will move onto this surface within the error space and then remain there. In terms of implementation, delays exist in many applications, for example those caused by transmission delay and computational delay. If performance levels are to be optimised in the presence of such delays it is necessary to consider the development of methodologies which incorporate knowledge of the delay in the design framework. There have been many results that investigate the effect of state delay on observer design [1], [2]. However very little work has considered the effect of delays in the output measurement on observer performance. In terms of work that considers the effect of time-delay in sliding mode observers, the literature is very sparse [15] and is strongly aligned to observer based control rather than fault detection and estimation with an emphasis on state delay rather than measurement delay [23], [24]. Since the switching term in a sliding mode observer depends on the output measurement, which may be subject to delay in practice, the resulting discontinuous injection applied to the observer has the potential to cause chattering of large amplitude. This behaviour may limit the magnitude of the discontinuous signal that it is possible to apply with the observer.

There has been a great deal of interest in the application of sliding mode observers to the problem of model based fault detection and isolation [5], [12], [29]. The merit of the approach lies in the application of the so-called *equivalent output injection* to explicitly reconstruct fault signals. The results obtained to date mostly require that an ideal sliding motion is attained in finite time before the appearance of faults, and that no delay is present on the output measurement used to drive the observer. It is clear that in the presence of sampled outputs, the ideal sliding mode cannot be achieved. Indeed the error dynamics in the observer can become unstable as the sampling interval is increased. It is important to note that uncertain sampled-data systems have received significant interest in recent years [20], [22], [25]. Sampled-data models, in which the conventional continuous states and discrete observations interact, have proved useful for capturing many real world engineering phenomena [7], [17], [26] and [31]. Motivated by recent results in the area of relay delay control in [9], [11], this paper will consider the effects of sampled output measurements when designing sliding mode observers for fault reconstruction.

It has been shown in [10], [21] that a sampled-data output can be represented as a continuous one with fast varying delay. From this representation, the aim in this paper is to develop a general framework for sliding mode observer design and fault reconstruction under multiple sampled outputs. The error dynamics is forced to exhibit a bound proportional to the sampling period of the outputs and the magnitude of the discontinuous switching gain employed in the observer. The observer, which is designed using a singular perturbation approach, possesses a sufficiently small perturbation parameter μ such that faults are reliably constructed despite the presence of the sampled output. The observer synthesis is formulated in terms of LMIs, the feasibility of which is guaranteed for small enough μ. The effect of uncertainties on the fault reconstruction is minimized by incorporating H_∞ concepts within the observer design framework. In section 2, the problem of sliding mode observer design with sampled outputs is formulated in terms of a system representation with known fast varying delay. Section 3 develops a constructive observer design approach which ensures the ultimate boundedness of the error dynamics. By using the singular perturbation method, section 4 shows that approximate fault reconstruction can be achieved. The sensor fault reconstruction is demonstrated in section 5, where new measurable states are augmented to the original faulty system so that the the results developed in the previous sections can be applied as the sensor fault is now transformed to be an input fault. In section 6 the effectiveness of the result is demonstrated using a linearized model of the inverted pendulum. Some preliminary results from this paper in the context of the input delay problem were presented in [13].

Notation: Throughout the paper, the superscript "T" stands for matrix transposition, $\mathbb{R}^n$ denotes the n-dimensional Euclidean space with vector norm $\|\cdot\|$, $\mathbb{R}^{n\times m}$ is the set of all $n\times m$ real matrices, and the notation $P>0$, for $P\in\mathbb{R}^{n\times n}$ means that P is symmetric and positive definite. The symmetric elements of the symmetric matrix are denoted by $*$. The symbol $\|\cdot\|_\infty$ stands for essential supremum.

2. Problem statement

Consider the linear, uncertain system with sampled outputs

$$\begin{aligned} \dot{x}(t) &= Ax(t)+Bu(t)+Df_i(t)+M\zeta(t,y,u) \\ y(t) &= Cx_d(t_k), \;\; t_k \le t < t_{k+1} \end{aligned} \tag{1}$$

where $x\in\mathbb{R}^n$, $u\in\mathbb{R}^m$ are the state and the input vector respectively, A, B, C, D and M are constant and known system matrices of appropriate dimensions. The discrete-time measurements $y\in\mathbb{R}^p$ are taken at the discrete sampling instants $0=t_0<t_1\cdots<t_k<\cdots$ with $\lim_{k\to\infty}t_k=\infty$. The unknown actuator faults $f_i\in$

$\mathbb{R}^q$ are supposed to be bounded $\|f_i(t)\| \leq \Delta$, where Δ is known. The signal ζ : $\mathbb{R}_+ \times \mathbb{R}^p \times \mathbb{R}^m \to \mathbb{R}^k$ encapsulates the uncertainty in the system. It is assumed to be known but bounded subject to $\|\zeta(t,y,u)\| \leq \beta$ where the positive scalar β is known. For real systems methods to establish an appropriate fixed matrix M so that $M\zeta$ captures the uncertainty are developed in [16]. It is assumed $q \leq p < n$ and A, B, C, D are constant matrices of appropriate dimensions. Following the approach in [10], [21], system (1) with sampled output can be presented as a continuous-time system with a known output measurement delay

$$\begin{array}{l} \dot{x}(t) = Ax(t) + Bu(t) + Df_i(t) + M\zeta(t,y,u) \\ y(t) = Cx(t-\tau(t)), \;\; t \in [t_k, t_{k+1}), \;\; \tau(t) = t - t_k \end{array} \tag{2}$$

Sampling may be variable but subject to $t_{k+1} - t_k \leq h, \; \forall \; k \geq 0$, i.e. the time between any two sequential sampling instants is not greater than some pre-chosen $h > 0$. Then $\tau(t) \in (0,h]$ with $\dot{\tau}(t) = 1$ for $t \neq t_k$ is known with the known sampling instants t_k. It is assumed that

1. rank $(CD) = q$;
2. any invariant zeros of (A,D,C) lie in the left half plane.

Under these assumptions there exists a linear change of coordinates $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = T_0 x$, where T_0 is non-singular, such that the transformed system has the following form (see [4]):

$$\begin{array}{l} \dot{x}_1(t) = A_{11}x_1(t) + A_{12}x_2(t) + B_1u(t) + M_1\zeta(t,y,u) \\ \dot{x}_2(t) = A_{21}x_1(t) + A_{22}x_2(t) + B_2u(t) + D_1f_i(t) + M_2\zeta(t,y,u) \\ y(t) = Tx_2(t-\tau(t)) \end{array} \tag{3}$$

where $x_1 \in \mathbb{R}^{n-p}$, $x_2 \in \mathbb{R}^p$, $D_1 = \begin{bmatrix} 0 \\ \bar{D}_1 \end{bmatrix}$, $\bar{D}_1 \in \mathbb{R}^{q\times q}$, T is an orthogonal matrix. An observer will be designed which, for sufficiently large t, induces motion in the $h(\Delta+\beta)$-neighbourhood of the surface

$$\mathscr{E} = \{x_2, \hat{x}_2 \in \mathbb{R}^p \; : \; s_e(t) = T\big(x_2(t-\tau(t)) - \hat{x}_2(t-\tau(t))\big) = 0\}$$

where $\hat{x}_2(t-\tau(t))$ is the corresponding component of the estimated states from an observer to be designed. An ideal sliding mode can be achieved with $h = 0$ under assumptions 1, 2.

3. Observer design

Noting that (A_{11}, A_{21}) is detectable (by assumptions 1, 2), choose a matrix $L \in \mathbb{R}^{(n-p)\times p}$, which has the form $L = \begin{bmatrix} \bar{L} & 0 \end{bmatrix}$ with $\bar{L} \in \mathbb{R}^{(n-p)\times(p-q)}$, such that $LD_1 = 0$ and $A_{11} + LA_{21}$ is stable. Consider the following observer of system (3):

$$\begin{array}{l} \dot{\hat{x}}_1(t) = A_{11}\hat{x}_1(t) + A_{12}\hat{x}_2(t) + B_1 u(t) \\ -(\frac{1}{\mu}L + A_{11}L)(x_2(t-\tau(t)) - \hat{x}_2(t-\tau(t))) + LT^T v(t-\tau(t)) \\ \dot{\hat{x}}_2(t) = A_{21}\hat{x}_1(t) + A_{22}\hat{x}_2(t) + B_2 u(t) \\ -(A_{21}L - \frac{1}{\mu}I_p)(x_2(t-\tau(t)) - \hat{x}_2(t-\tau(t)) - T^T v(t-\tau(t)) \\ \hat{y}(t) = T\hat{x}_2(t-\tau(t)) \end{array} \tag{4}$$

where $\mu > 0$ is a scalar. Here the discontinuous injection term v is given by

$$v(t-\tau(t)) = v(t_k) = -M_\beta[\operatorname{sign}\bar{e}_{2_1}(t_k), \ldots, \operatorname{sign}\bar{e}_{2_p}(t_k)]^T, \quad t \in [t_k, t_{k+1}) \tag{5}$$

where $M_\beta = \|TD_1\|\Delta + \delta_1\Delta + \delta_2\beta + \|TM_2\|\beta,\ \delta_1,\ \delta_2$ are positive scalars and subscript p denotes the p-th component of $e_2(t_k) = T(x(t_k) - \hat{x}(t_k))$.

Our objective is to find the appropriate design parameters $\delta_1,\ \delta_2,\ \mu$ such that $\|e_2(t_k)\|$ is minimized for a given sampling period h. The linear change of coordinates $\hat{x} = T_0^{-1}\begin{bmatrix} \hat{x}_1 & \hat{x}_2 \end{bmatrix}^T$ leads to the observer of (2) given by

$$\begin{array}{l} \dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) - G_l\bar{e}_2(t-\tau(t)) + G_n v(t-\tau(t)) \\ \hat{y}(t) = C\hat{x}_d(t_k), \ \ t_k \le t < t_{k+1} \end{array} \tag{6}$$

where observer gains $G_l \in \mathbb{R}^{n\times p}$, $G_n \in \mathbb{R}^{n\times p}$ are designed to be in the form

$$G_l = T_0^{-1}\begin{bmatrix} \frac{1}{\mu}L + A_{11}L \\ A_{21}L - \frac{1}{\mu}I_p \end{bmatrix}, \quad G_n = T_0^{-1}\begin{bmatrix} LT^T \\ -T^T \end{bmatrix} \tag{7}$$

where $\mu > 0$ is a scalar. Defining the state estimation error as $e_1(t) = x_1(t) - \hat{x}_1(t)$ and $e_2(t) = x_2(t) - \hat{x}_2(t)$, it is obtained that

$$\begin{array}{l} \dot{e}_1(t) = A_{11}e_1(t) + A_{12}e_2(t) + M_1\zeta(t,y,u) \\ +L\big(\frac{1}{\mu}e_2(t-\tau(t)) - T^T v(t-\tau(t))\big) + A_{11}Le_2(t-\tau(t)) \\ \dot{e}_2(t) = A_{21}e_1(t) + A_{22}e_2(t) + D_1 f_i(t) + M_2\zeta(t,y,u) \\ +T^T v(t-\tau(t)) - (\frac{1}{\mu}I_p - A_{21}L)e_2(t-\tau(t)) \end{array}$$

Changing variables $\begin{bmatrix} \bar{e}_1(t) \\ \bar{e}_2(t) \end{bmatrix} = T_L \begin{bmatrix} e_1(t) \\ e_2(t) \end{bmatrix}$ with $T_L = \begin{bmatrix} I_{n-q} & L \\ 0 & T \end{bmatrix}$, one obtains

$$\begin{array}{l} \dot{\bar{e}}_1(t) = (A_{11} + LA_{21})\bar{e}_1(t) - (A_{11}L + LA_{21}L - A_{12} - LA_{22})T^T\bar{e}_2(t) \\ +(A_{11} + LA_{21})LT^T\bar{e}_2(t-\tau(t)) + (M_1 + LM_2)\zeta(t,y,u) \end{array} \tag{8}$$

$$\begin{array}{l}\dot{\bar{e}}_2(t) = TA_{21}\bar{e}_1(t) - (TA_{21}LT^T - TA_{22}T^T)\bar{e}_2(t) + TA_{21}LT^T\bar{e}_2(t-\tau(t)) \\ -\frac{1}{\mu}\bar{e}_2(t-\tau(t)) + v(t-\tau(t)) + TD_1 f_i(t) + TM_2\zeta(t,y,u)\end{array} \qquad (9)$$

with initial condition

$$\bar{e}(t_0) = \bar{e}_0, \quad \bar{e}(t) = 0, \quad t < t_0 \qquad (10)$$

The dynamics of the switching manifold is governed by equation (8).

3.1. Input-to-state stability of the error dynamics: a singular perturbation approach

The closed-loop system (8), (9) can be expressed as

$$\dot{\bar{e}}_1(t) = \bar{A}_{11}\bar{e}_1(t) + \bar{A}_{12}\bar{e}_2(t) + \bar{A}_\tau \bar{e}_2(t-\mu\xi(t)) + \bar{M}_1\zeta(t,y,u) \qquad (11)$$

$$\mu\dot{\bar{e}}_2(t) = \mu\bar{A}_{21}\bar{e}_1(t) + \mu\bar{A}_{22}\bar{e}_2(t) + (\mu\bar{A}_{d22} - I_p)\bar{e}_2(t-\mu\xi(t)) + \mu\bar{f}_i(t) \qquad (12)$$

where $\mu\xi(t) = t - t_k$, $\xi(t) = (t-t_k)/\mu$ is the fast sawtooth delay corresponding to the fast sampling, $\bar{A}_{11} = A_{11} + LA_{21}$, $\bar{A}_{12} = -(A_{11}L + LA_{21}L - A_{12} - LA_{22})T^T$, $\bar{A}_\tau = (A_{11} + LA_{21})LT^T$, $\bar{M}_1 = (M_1 + LM_2)$, $\bar{A}_{21} = TA_{21}$, $\bar{A}_{22} = -(TA_{21}LT^T - TA_{22}T^T)$, $\bar{A}_{d22} = TA_{21}LT^T$, $\mu\xi(t) = \tau(t)$, $\mu\bar{\xi} = h$, $0 \le \xi(t) \le \bar{\xi}$ and $\bar{f}_i(t) = v(t-\mu\xi(t)) + TD_1 f_i(t) + TM_2\zeta(t,y,u)$, i.e. $\|\bar{f}_i(t)\| \le M_f = M_\beta\sqrt{p} + \|TD_1\|\Delta + \|TM_2\|\beta$. Let $P_\mu \in \mathbb{R}^{n\times n}$ be a positive definite matrix with the following structure [18]

$$P_\mu = \begin{bmatrix} P_1 & \mu P_3^T \\ * & \mu P_2 \end{bmatrix} > 0 \qquad (13)$$

where $P_1 \in \mathbb{R}^{n-p}$, and choose the Lyapunov-Krasovskii functional designed for sampled data system [7]:

$$V(t) = \bar{e}(t)^T P_\mu \bar{e}(t) + (\mu\bar{\xi} - \mu\xi(t))\int_{t-\mu\xi(t)}^{t} \mu\bar{\xi} e^{\bar{\alpha}(s-t)}\dot{\bar{e}}_2(s)U\dot{\bar{e}}_2(s)ds \qquad (14)$$

with respect to the error dynamics (11), (12), where $U \in \mathbb{R}^p$ is a positive matrix, then the following lemma can be stated:

Lemma 1. *Given positive tuning scalars μ, $\bar{\xi}$, $\bar{\alpha}$, $\bar{b}$ and $\bar{b}_1$, let there exist a $n\times n$ matrix $P_\mu > 0$ in (13), $p\times p$ matrices $U > 0$, P_4, P_5 and $(n-p)\times(n-p)$ matrices P_6, P_7 such that the following LMIs*

$$\Theta_{\mu 0} = \begin{bmatrix} \theta_{11} & \theta_{12} & \theta_{13} & \mu\bar{A}_{21}^T P_5 + \mu P_3^T & 0 & P_6^T\bar{M}_1 \\ * & \theta_{22} & \theta_{23} & \theta_{24} & P_4^T & 0 \\ * & * & -P_7^T - P_7 & 0 & 0 & P_7^T\bar{M}_1 \\ * & * & * & -P_5^T - P_5 + \bar{\xi}^2 U & P_5^T & 0 \\ * & * & * & * & -\bar{b}I & 0 \\ * & * & * & * & * & -\bar{b}_1 I \end{bmatrix} < 0 \qquad (15)$$

$$
\Theta_{\mu 1}=\begin{bmatrix}
\theta_{11} & \theta_{12} & \theta_{13} & \mu\bar{A}_{21}^TP_5+\mu P_3^T & -\bar{\xi}P_6^T\bar{A}_\tau & 0 & P_6^T\bar{M}_1\\
* & \theta_{22} & \theta_{23} & \theta_{24} & -\bar{\xi}P_4^T(\mu\bar{A}_{d22}-I_p) & P_4^T & 0\\
* & * & -P_7^T-P_7 & 0 & -\bar{\xi}P_7^T\bar{A}_\tau & 0 & P_7^T\bar{M}_1\\
* & * & * & -P_5^T-P_5 & -\bar{\xi}P_5^T(\mu\bar{A}_{d22}-I_p) & P_5^T & 0\\
* & * & * & * & -\bar{\xi}^2e^{-\bar{\alpha}\mu\bar{\xi}}U & 0 & 0\\
* & * & * & * & * & -\bar{b}I & 0\\
* & * & * & * & * & * & -\bar{b}_1I
\end{bmatrix}<0 \tag{16}
$$

where

$$
\begin{aligned}
&\theta_{11}=P_6^T\bar{A}_{11}+\bar{A}_{11}^TP_6+\bar{\alpha}P_1,\\
&\theta_{12}=\mu\bar{A}_{21}^TP_4+\bar{\alpha}\mu P_3^T+P_6^T(\bar{A}_{12}+\bar{A}_\tau)\\
&\theta_{13}=-P_6^T+P_1+\bar{A}_{11}^TP_7\\
&\theta_{22}=\mu P_4^T\bar{A}_{22}+\mu\bar{A}_{22}^TP_4+P_4^T(\mu\bar{A}_{d22}-I_p)\\
&+(\mu\bar{A}_{d22}^T-I_p)P_4+\mu\bar{\alpha}P_2,\\
&\theta_{23}=\mu P_3+(\bar{A}_{12}+\bar{A}_\tau)^TP_7\\
&\theta_{24}=\mu\bar{A}_{22}^TP_5+(\mu\bar{A}_{d22}^T-I_p)P_5+P_2-P_4^T
\end{aligned}
$$

are feasible, then solutions of (8)-(9) with initial condition (10) satisfy the bound

$$
\bar{e}^T(t)P_\mu\bar{e}(t)<e^{-\bar{\alpha}(t-t_0)}\bar{e}^T(t_0)P_\mu\bar{e}(t_0)+\frac{\mu^2\bar{b}}{\bar{\alpha}}\|\bar{f}_{i[t_0,t]}\|_\infty^2+\frac{\bar{b}_1}{\bar{\alpha}}\|\zeta_{[t_0,t]}\|_\infty^2 \tag{17}
$$

for all $\mu\xi(t)\in[0,\ h]$ with $\mu\dot{\xi}(t)=1$, thus (8)-(9) is input-to-state stable.

Proof 1. *The following inequality*

$$
W(t)=\frac{d}{dt}V(t)+\bar{\alpha}V(t)-\mu^2\bar{b}\bar{f}_i^T(t)\bar{f}_i(t)-\bar{b}_1\zeta^T(t,y,u)\zeta(t,y,u)<0
$$

along the trajectories of (8), (9) for $\|\bar{e}_0\|^2+\|\bar{f}_{i[t_0,t]}\|_\infty^2+\|\zeta_{[t_0,t]}\|_\infty^2>0$ guarantees (17) [8]. Differentiating V of the structure (13), (14) along (11), (12), and analogously to [7], it follows

$$
\begin{aligned}
&W(t)\le 2\big(\bar{e}_1^T(t)P_1\dot{\bar{e}}_1(t)+\bar{e}_2^T(t)\mu P_3\dot{\bar{e}}_1(t)+\mu\bar{e}_1^T(t)P_3^T\dot{\bar{e}}_2(t)\\
&+\mu\bar{e}_2^T(t)P_2\dot{\bar{e}}_2(t)\big)+\bar{\alpha}\big(\bar{e}_1^T(t)P_1\bar{e}_1(t)+\bar{e}_2^T(t)\mu P_3\bar{e}_1(t)\\
&+\mu\bar{e}_1^T(t)P_3^T\bar{e}_2(t)+\mu\bar{e}_2^T(t)P_2\bar{e}_2(t)\big)-\mu^2\bar{\xi}\xi(t)e^{-\bar{\alpha}\mu\bar{\xi}}v_2^TUv_2\\
&+\mu\bar{\xi}(\mu\bar{\xi}-\mu\xi(t))\dot{\bar{e}}_2^T(t)U\dot{\bar{e}}_2(t)-\mu^2\bar{b}\bar{f}_i^T(t)\bar{f}_i(t)-\bar{b}_1\zeta^T(t,y,u)\zeta(t,y,u)
\end{aligned} \tag{18}
$$

where $v_2(t)=\frac{1}{\mu\xi(t)}\int_{t-\mu\xi(t)}^t\dot{\bar{e}}_2(s)ds$. Apply the descriptor method [6], the right-hand sides of the expressions

$$
\begin{aligned}
&0=2[\bar{e}_2^T(t)P_4^T+\mu\dot{\bar{e}}_2^T(t)P_5^T][\mu\bar{A}_{21}\bar{e}_1(t)+\mu\bar{A}_{22}\bar{e}_2(t)+(\mu\bar{A}_{d22}\\
&-I_p)\bar{e}_2(t)-\mu\xi(t)(\mu\bar{A}_{d22}-I_p)v_2(t)+\mu\bar{f}_i(t)-\mu\dot{\bar{e}}_2(t)]\\
&0=2[\bar{e}_1^T(t)P_6^T+\dot{\bar{e}}_1^T(t)P_7^T][\bar{A}_{11}\bar{e}_1(t)+(\bar{A}_{12}+\bar{A}_\tau)\bar{e}_2(t)\\
&-\mu\xi(t)\bar{A}_\tau v_2(t)+\bar{M}_1\zeta(t,y,u)-\dot{\bar{e}}_1(t)]
\end{aligned}
$$

with some $p \times p$-matrices P_4, P_5 and $(n-p)\times(n-p)$ matrices P_6, P_7 are added into the right-hand side of (18). Setting $\eta_1(t) = col\{\bar{e}_1(t),\ \bar{e}_2(t),\ \dot{\bar{e}}_1(t),\ \mu\dot{\bar{e}}_2(t),\ \mu v_2(t),\ \mu\bar{f}_i(t),\ \zeta(t,y,u)\}$, it follows

$$\dot{V}(t) + \bar{\alpha}V(t) - \mu^2\bar{b}\bar{f}_i^T(t)\bar{f}_i(t) - \bar{b}_1\zeta^T(t,y,u)\zeta(t,y,u) \le \eta_1^T(t)\Theta_\mu\eta_1(t) < 0$$

if the following matrix inequality is feasible:

$$\Theta_\mu = \begin{bmatrix} \theta_{11} & \theta_{12} & \theta_{13} & \mu\bar{A}_{21}^T P_5 + P_3^T & -\xi(t)P_6^T\bar{A}_\tau & 0 & P_6^T\bar{M}_1 \\ * & \theta_{22} & \theta_{23} & \theta_{24} & -\xi(t)P_4^T(\mu\bar{A}_{d22} - I_p) & P_4^T & 0 \\ * & * & -P_7^T - P_7 & 0 & -\xi(t)P_7^T\bar{A}_\tau & 0 & P_7^T\bar{M}_1 \\ * & * & * & -P_5^T - P_5 + \bar{\xi}(\bar{\xi} - \xi(t))U & -\xi(t)P_5^T(\mu\bar{A}_{d22} - I_p) & P_5^T & 0 \\ * & * & * & * & -\bar{\xi}\xi(t)e^{-\bar{\alpha}\mu\bar{\xi}}U & 0 & 0 \\ * & * & * & * & * & -\bar{b}I & 0 \\ * & * & * & * & * & * & -\bar{b}_1 I \end{bmatrix} < 0 \tag{19}$$

The latter matrix inequality for $\xi(t) \to 0$ and $\xi(t) \to \bar{\xi}$, leads to the LMIs (15), (16). Setting $\eta_0(t) = col\{\bar{e}_1(t),\ \bar{e}_2(t),\ \dot{\bar{e}}_1(t),\ \mu\dot{\bar{e}}_2,\ \mu\bar{f}_i(t),\ \zeta(t,y,u)\}$, then the following holds

$$\frac{h-\tau(t)}{h}\eta_0^T\Theta_{\mu 0}\eta_0 + \frac{\tau(t)}{h}\eta_1^T\Theta_{\mu 1}\eta_1 = \eta_1^T\Theta_\mu\eta_1 < 0,\ \ \forall\, \eta_1 \ne 0$$

3.2. LMIs for switching gain design

Conditions will now be derived that guarantee the following bound for the solutions of (9):

$$\begin{aligned} &\limsup_{t\to\infty} \| \begin{bmatrix} \bar{A}_{21} & \bar{A}_{22} \end{bmatrix} \bar{e}(t) \| \le k_1(\delta_1\Delta + \delta_2\beta), \\ &\limsup_{t\to\infty} \| \begin{bmatrix} 0 & \bar{A}_{d22} \end{bmatrix} \bar{e}(t-\tau(t)) \| \le k_2(\delta_1\Delta + \delta_2\beta) \end{aligned} \tag{20}$$

with some $k_1, k_2 \ge 0$ such that $k_1 + k_2 = 1$. Taking into account (17) it can be concluded that (20) holds if the following inequalities are satisfied for $t \to \infty$:

$$\bar{e}^T(t)[\bar{A}_{21}\ \bar{A}_{22}]^T[\bar{A}_{21}\ \bar{A}_{22}]\bar{e}(t) < \frac{\bar{\alpha}k_1^2(\delta_1\Delta+\delta_2\beta)^2\bar{e}^T(t)P_\mu\bar{e}(t)}{\mu^2\bar{b}M_f^2+\bar{b}_1\beta^2} \tag{21}$$

$$\bar{e}^T(t-\mu\xi(t))[0\ \bar{A}_{d22}]^T[0\ \bar{A}_{d22}]\bar{e}(t-\mu\xi(t)) < \frac{\bar{\alpha}k_2^2(\delta_1\Delta+\delta_2\beta)^2\bar{e}^T(t-\mu\xi(t))P_\mu\bar{e}(t-\mu\xi(t))}{\mu^2\bar{b}M_f^2+\bar{b}_1\beta^2} \tag{22}$$

Hence, the inequalities

$$\begin{aligned} &\begin{bmatrix} -k_1^2M_\Omega P_1 & -\mu k_1^2 M_\Omega P_3^T & \bar{A}_{21}^T \\ * & -\mu k_1^2 M_\Omega P_2 & \bar{A}_{22}^T \\ * & * & -I_p \end{bmatrix} < 0, \\ &\begin{bmatrix} -k_2^2M_\Omega P_1 & -\mu k_2^2 M_\Omega P_3^T & 0 \\ * & -\mu k_2^2 M_\Omega P_2 & \bar{A}_{d22}^T \\ * & * & -I_p \end{bmatrix} < 0 \end{aligned} \tag{23}$$

where $M_\Omega = \frac{\bar{\alpha}(\delta_1\Delta+\delta_2\beta)^2}{\mu^2\bar{b}M_f^2+\bar{b}_1\beta^2}$, guarantee that the solutions of (8), (9) satisfy the bound (20).

Matrix inequalities (13), (15), (16) and (23) have been derived for finding parameters μ and δ_1, δ_2 of the observer (6), it will now be shown that if the μ-dependent LMIs

$$\Theta_{\mu 0|\mu=0} < 0, \ \ \Theta_{\mu 1|\mu=0} < 0 \tag{24}$$

are feasible, then for big enough $\delta_1 > 0$ and $\delta_2 > 0$ inequalities (13), (15), (16) and (23) are feasible for small enough μ and $\bar{\xi}$. Let P_1, P_2, P_3 satisfy the above inequality, then for small enough $\mu > 0$ and $\bar{\xi} > 0$ (13), (15), (16) are feasible for the same μ-independent matrices P_1, P_2, P_3. Hence, given big enough $\delta_1 > 0$ and $\delta_2 > 0$, (21) and (22) are feasible for small enough μ and $\bar{\xi}$.

Note that feasibility of (24) guarantees exponential stability with decay rate $\bar{\alpha}/2$ of the slow subsystem

$$\dot{\bar{e}}_1(t) = \bar{A}_{11}\bar{e}_1(t)$$

and asymptotic stability of the fast subsystem

$$\mu\dot{\bar{e}}_2(t) = -\bar{e}_2(t_k), \ \ t \in [t_k,\ t_{k+1})$$

Since by design $\bar{A}_{11}$ is Hurwitz, there exists a $P_6 > 0$ satisfying $P_6^T\bar{A}_{11} + A_{11}^TP_6 + \bar{\alpha}P_1 < 0$ for small enough $\bar{\alpha} > 0$. Choose next $P_1 = P_6^T$, $P_2 = P_4^T$, $P_3 = 0$, $P_4 = p_4I$, $P_5 = p_5I$ and $P_7 = p_7I$ for big enough p_4, $U > 0$ and small enough p_5, p_7, $\bar{\xi} > 0$. By using Schur complements, it can be shown that (24) holds for big enough $\bar{b}$, $\bar{b}_1$.

Proposition 1. *(i) Given positive tuning scalars $\bar{\alpha}$, $\mu, \bar{\xi}$, $\bar{b}$, $\bar{b}_1$, let there exist an $(n-p)\times(n-p)$ matrix $P_1 > 0$, a $p\times p$ matrix $P_2 > 0$, a $p\times p$ matrix $U > 0$, a*

$p\times(n-p)$ matrix P_3, $p\times p$ matrices P_4, P_5 and $(n-p)\times(n-p)$ matrices P_6, P_7 such that LMIs in (24) are feasible. Then, for big enough $\delta_1,\ \delta_2>0$ there exist small enough $\bar\alpha,\ \mu,\ \bar\xi>0$ such that LMIs (13), (15), (16) and (23) are feasible and, thus, solutions of (8), (9) satisfy the bound (20).

(ii) LMIs in (24) are feasible for small enough $\bar\alpha,\ \mu,\ \bar\xi$ and big enough $\bar b,\ \bar b_1$.

3.3. Ultimate boundedness of the error dynamics

Let $\phi(t,t_0,\mu)$ be the fundamental solution of the equation

$$\mu\dot z(t) = -z(t_k), \quad z(t)\in\mathbb{R},\ t\in[t_k,\ t_{k+1}) \tag{25}$$

with $\phi(t_0,t_0,\mu)=1$ and $\phi(t,t_0,\mu)=0$ for $t<t_0$. It is shown in [7] that (25) remains exponentially stable for all variable delays $\mu\xi(t)\le 1.99$. Then the following bound holds

$$\|\phi(t,t_0,\mu)\|\le e^{-\frac{\alpha_2(t-t_0)}{\mu}} \tag{26}$$

for small enough $\alpha_2>0$ and $\forall\ \mu>0,\ \mu\xi(t)\le h,\ \mu\dot\xi(t)=1$. Main results may now be stated (proof follows the arguments of [13] as shown in Appendix):

Theorem 1. *Given positive tuning scalars $\mu,\ \bar\xi,\ \bar\alpha,\ \bar b,\ \bar b_1,\ \delta_1,\ \delta_2$ and $k_1,\ k_2$, let there exist a $n\times n$-matrix $P_\mu>0$, positive $p\times p$-matrices $U>0$ and $p\times p$ matrices P_4, P_5, $(n-p)\times(n-p)$ matrices P_6, P_7 such that LMIs (13), (15), (16) and (23) are feasible. Let $\bar e(t)$ be a solution to (8), (9), then every component of $\bar e_2(t)$ satisfies the bound*

$$\limsup_{t\to\infty}|\bar e_{2_i}(t)|\le 2M_0\mu\bar\xi \tag{27}$$

where $M_0=2(\delta_1\Delta+\delta_2\beta+\|TD_1\|\Delta+\|TM_2\|\beta)$ with Δ and β denoting the known bound of f_i and ζ respectively, $i=1,\dots,p$ denotes the i-th component of $\bar e_2$ for all $\mu\xi(t)\in[0,\ h]$ with $\mu\dot\xi(t)=1$.

Remark 1. *In the case the condition on the disturbance $\|\zeta\|\le k_0+k_1\|y\|+k_2\|u\|$ is preferred then it is always possible to find a large enough r_0 such that the terms in (20) are bounded by*

$$\limsup_{t\to\infty}(\|[\bar A_{21}\ \bar A_{22}]\bar e(t)\|+\|[0\ \bar A_{d22}]\bar e(t-\tau(t))\|)\le r_0 \tag{28}$$

Choosing switching gain in (5) as $M_\beta=r_o+\|TD_1\|\Delta+\|TM_2\|(k_0+k_1\|y\|+k_2\|u\|)$ Theorem 1 will still hold with $M_0=2M_\beta$.

4. Input fault reconstruction in the presence of uncertainty

The fault reconstruction properties of the observer designed above are now considered. Effectively this extends the presentation in [29] to consider the effect of sampled outputs. For sufficiently small μ, (11) and (12) become

$$\dot{\bar{e}}_1(t) = (A_{11} + LA_{21})\bar{e}_1(t) + (M_1 + LM_2)\zeta(t, y, u) \tag{29}$$

$$0 \approx \bar{A}_{21}\bar{e}_1(t) - \frac{1}{\mu}\bar{e}_2(t - \tau(t)) + v(t - \tau(t)) + TD_1 f_i(t) + TM_2\zeta(t, y, u) \tag{30}$$

Define

$$W := \begin{bmatrix} W_1 & \bar{D}_1^{-1} \end{bmatrix}$$

where $W_1 \in \mathbb{R}^{q\times(p-q)}$ is a tuning matrix, and

$$\hat{f}_i(t) = WT^T(v(t - \tau(t)) - \frac{1}{\mu}\bar{e}_2(t - \tau(t))) \tag{31}$$

Then equation (30) can be rewritten as

$$0 \approx WA_{21}\bar{e}_1(t) + \hat{f}_i(t) + WD_1 f_i(t) + WM_2\zeta(t, y, u)$$

or equivalently

$$-\hat{f}_i(t) = f_i(t) + \hat{G}(s)\zeta(t, y, u) \tag{32}$$

where $G(s) = WA_{21}(sI - (A_{11} + LA_{21}))^{-1}(M_1 + LM_2) + WM_2$. The effect of $\zeta(t, y, u)$ on fault reconstruction can be minimized by minimizing the H_∞ norm of the transfer function $\hat{G}(s)$ from $\zeta(t, y, u)$ to $\hat{f}_i(t)$. This is equivalent to ensuring the following inequality is satisfied, whereby the H_∞ norm of the transfer function is less than some positive γ.

$$\begin{bmatrix} P(A_{11} + LA_{21}) + (A_{11} + LA_{21})^T P & -P(M_1 + LM_2) & -(WA_{21})^T \\ * & -\gamma I & (WM_2)^T \\ * & * & -\gamma I \end{bmatrix} < 0 \tag{33}$$

where $P \in \mathbb{R}^{(n-p)\times(n-p)} > 0$. The above inequality can be reduced to LMI by taking $Y = PL$.

Remark 2. *Fault reconstruction using sliding mode technique usually requires an ideal sliding motion to be attained in finite time [5], [30]. Practically, due to model uncertainties and sampled output effects for example, an ideal sliding* *motion in the observer does not usually appear. Instead, the motion is bounded within a region of the sliding surface. This paper uses a singular perturbation approach for fault reconstruction under sampled outputs, for which by choosing a sufficiently small μ the fault can be approximated depending only on the outputs error.*

5. Sensor fault reconstruction in the presence of uncertainty

In this case, the system under consideration is the following

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) + M\zeta(t,y,u) \\ y(t) &= Cx(t-\tau(t)) + Nf_0(t), \ \ t \in [t_k,,t_{k+1}), \ \ \tau(t) = t - t_k \end{aligned} \tag{34}$$

where $f_0 \in \mathbb{R}^r$ with $\|f_0\| \leq \Delta$ is the vector of sensor faults, and $N \in \mathbb{R}^{p\times r}$ and $r \leq p$. Consider new measurable states $z_f \in \mathbb{R}^p$ that satisfies

$$\dot{z}_f(t) = -A_f z_f(t) + A_f Cx(t-\tau(t)) + A_f N f_0(t), \tag{35}$$

where $-A_f$ is a user-defined stable matrix. The augmented system of (34) and (35) is

$$\begin{aligned} \begin{bmatrix} \dot{x}(t) \\ \dot{z}_f(t) \end{bmatrix} &= \underbrace{\begin{bmatrix} A & 0 \\ 0 & -A_f \end{bmatrix}}_{\tilde{A}} \underbrace{\begin{bmatrix} x(t) \\ z_f(t) \end{bmatrix}}_{x_z(t)} + \underbrace{\begin{bmatrix} 0 & 0 \\ A_f C & 0 \end{bmatrix}}_{\tilde{A}_d} \begin{bmatrix} x(t-\tau(t)) \\ z_f(t-\tau(t)) \end{bmatrix} \\ &+ \underbrace{\begin{bmatrix} B \\ 0 \end{bmatrix}}_{\tilde{B}} u(t) + \underbrace{\begin{bmatrix} 0 \\ A_f N \end{bmatrix}}_{\tilde{D}} f_0(t) + \underbrace{\begin{bmatrix} M \\ 0 \end{bmatrix}}_{\tilde{M}} \zeta(t,y,u) \\ z_f(t) &= \underbrace{\begin{bmatrix} 0 & I_p \end{bmatrix}}_{\tilde{C}} x_z(t-\tau(t)) \end{aligned} \tag{36}$$

Since equation (36) has a similar structure as (2) apart from the term $\tilde{A}_d x_z(t-\tau(t))$, the observer structure for system (36) will be similar to (6), where

$$v(t_k) = -(\|TD_1\|\Delta + \delta_1\Delta + \delta_2\beta)[\operatorname{sign}\bar{e}_{2_1}(t_k), \ldots, \operatorname{sign}\bar{e}_{2_p}(t_k)]^T$$

and $\hat{x}_z$ represents the observer states with an additional term $\tilde{A}_d\hat{x}_z(t-\tau(t))$. Denoting A, $-A_f$ in $\tilde{A}$ and A_fC in $\tilde{A}_d$ in (36) as A_{11}, A_{22} and a new term A_{d21} in (3) respectively, and $A_f N$ and M in (36) as D_1 and M_1 in (3), then the error dynamics (11) thus becomes

$$\dot{\bar{e}}_1(t) = \bar{A}_{11}\bar{e}_1(t) + \bar{A}_{12}\bar{e}_2(t) + \bar{A}_{d11}\bar{e}_1(t-\mu\xi(t)) + \bar{A}_\tau\bar{e}_2(t-\mu\xi(t)) + \bar{M}_1\zeta(t,y,u) \tag{37}$$

$$\mu\dot{\bar{e}}_2(t) = \mu\bar{A}_{22}\bar{e}_2(t) + \mu TA_{d21}\bar{e}_1(t-\mu\xi(t)) + (\mu\bar{A}_{d22} - I_p)\bar{e}_2(t-\mu\xi(t)) + \mu\bar{f}_i(t) \tag{38}$$

where $\bar{A}_{11} = A_{11}, \bar{A}_{d11} = LA_{d21}, \bar{A}_{12} = -(A_{11}L - LA_{22})T^T, \bar{A}_\tau = (A_{11} - LA_{d21})LT^T$, $\bar{M}_1 = M_1, \bar{A}_{22} = TA_{22}T^T, \bar{A}_{d22} = -TA_{d21}LT^T$, $\mu\xi(t) = \tau(t)$, $\mu\bar{\xi} = h$, $0 \le \xi(t) \le \bar{\xi}$ and $\bar{f}_i(t) = v(t-\mu\xi(t)) + TD_1 f_i(t)$, i.e. $\|\bar{f}_i(t)\| \le M_f = (\|TD_1\|\Delta + \delta_1\Delta + \delta_2\beta)\sqrt{p} + \|TD_1\|\Delta$. Choose the Lyapunov-Krasovskii functional as

$$\begin{aligned} V(t) &= \bar{e}(t)^T P_\mu \bar{e}(t) + (\mu\bar{\xi} - \mu\xi(t))\int_{t-\mu\xi(t)}^{t} \mu\bar{\xi} e^{\bar{\alpha}(s-t)}\dot{\bar{e}}_1(s)V\dot{\bar{e}}_1(s)ds \\ &+ (\mu\bar{\xi} - \mu\xi(t))\int_{t-\mu\xi(t)}^{t} \mu\bar{\xi} e^{\bar{\alpha}(s-t)}\dot{\bar{e}}_2(s)U\dot{\bar{e}}_2(s)ds \end{aligned}$$

where $P_\mu \in \mathbb{R}^{n+p}$, $V \in \mathbb{R}^n$, and $U \in \mathbb{R}^p$ are positive matrices, then some modifications on inequality (19), i.e. denoting

$$\begin{aligned} v_1(t) &= \tfrac{1}{\mu\xi(t)}\int_{t-\mu\xi(t)}^{t} \dot{\bar{e}}_1(s)ds \\ \theta_{11} &= P_6^T(\bar{A}_{11} + \bar{A}_{d11}) + (\bar{A}_{11} + \bar{A}_{d11})^T P_6 + \bar{\alpha}P_1 \\ \theta_{12} &= \mu A_{d21}^T T^T P_4 + \bar{\alpha}\mu P_3^T + P_6^T(\bar{A}_{12} + \bar{A}_\tau) \\ \theta_{13} &= -P_6^T + P_1 + (\bar{A}_{11} + \bar{A}_{d11})^T P_7 \\ \theta_{14} &= \mu A_{d21}^T T^T P_5 + \mu P_3^T \\ \theta_{18} &= -\xi(t)P_6^T\bar{A}_{d11} \\ \theta_{28} &= -\mu\xi(t)P_4^T TA_{d21} \\ \theta_{33} &= -P_7^T - P_7 + \mu^2\bar{\xi}(\bar{\xi} - \xi(t))V \\ \theta_{38} &= -\xi(t)P_7^T\bar{A}_{d11} \\ \theta_{48} &= -\mu\xi(t)P_5^T TA_{d21} \\ \theta_{88} &= -\bar{\xi}\xi(t)e^{-\bar{\alpha}h}V \end{aligned} \tag{39}$$

while keeping all the other terms unchanged where $\bar{A}_{21} = 0$, can be made so that the error dynamics (37) and (38) satisfy the bound (17). In switching gain design, (20) becomes

$$\begin{aligned} &\limsup_{t\to\infty}\|\begin{bmatrix} \bar{A}_{21} & \bar{A}_{22} \end{bmatrix}\bar{e}(t)\| \le k_1(\delta_1\Delta + \delta_2\beta), \\ &\limsup_{t\to\infty}\|\begin{bmatrix} A_{d21} & \bar{A}_{d22} \end{bmatrix}\bar{e}(t-\tau(t))\| \le k_2(\delta_1\Delta + \delta_2\beta) \end{aligned} \tag{40}$$

Proposition 2. *(i) Given positive tuning scalars $\bar{\alpha}$, $\mu, \bar{\xi}$, $\bar{b}$, $\bar{b}_1$, δ_1, δ_2, let there exist an $n \times n$ matrix $P_1 > 0$, a $p \times p$ matrix $P_2 > 0$, a $p \times p$ matrix $U > 0$, a $n \times n$*

matrix $V>0$, a $p\times n$ matrix P_3, $p\times p$ matrices P_4, P_5 and $n\times n$ matrices P_6, P_7 such that inequality (13), (15), (16) with entries in (39) and $\mu=0$ is feasible. Then for big enough δ_1, $\delta_2>0$ there exist small enough $\bar{\alpha}$, μ, $\bar{\xi}>0$ such that inequalities (13), (15), (16) with both $\xi(t)=0$ and $\xi(t)=\bar{\xi}$, and (40) are feasible. Then the solutions of (37), (38) satisfy the bound (27), where $M_0=2(\delta_1\Delta+\delta_2\beta+\|TD_1\|)\Delta$.

(ii) LMIs (19) with both $\xi(t)=0$ and $\xi(t)=\bar{\xi}$ are feasible for small enough $\bar{\alpha}$, μ, $\bar{\xi}$ and big enough $\bar{b}$, $\bar{b}_1$.

Similarly to section 4 for sufficiently small μ, equations (29) and (32) become

$$\begin{array}{l}\dot{\bar{e}}_1(t)=\bar{A}_{11}\bar{e}_1(t)+\bar{A}_{d11}\bar{e}_1(t-\mu\xi(t))+\bar{M}_1\zeta(t)\\ -\hat{f}_i(t)=f_i+WA_{d21}\bar{e}_1(t-\mu\xi(t))\end{array}$$

Applying the result from [32] to the above equation, the H_∞ norm from $\zeta(t,y,u)$ to $\hat{f}_i(t)$ will be less than a positive number γ if the following inequality is feasible

$$\begin{bmatrix} \hat{P}_2^T\bar{A}_{11}+\bar{A}_{11}^T\hat{P}_2+YA_{d21}+A_{d21}^TY^T & \hat{P}_1-\hat{P}_2^T+\varepsilon(\bar{A}_{11}^T\hat{P}_2+A_{d21}^TY^T) & \mu\bar{\xi}YA_{d21} & \hat{P}_2^T\bar{M}_1 & A_{d21}^TW^T \\ * & -\varepsilon(\hat{P}_2+\hat{P}_2^T)+\mu\bar{\xi}R & \mu\bar{\xi}\varepsilon YA_{d21} & \varepsilon\hat{P}_2^T\bar{M}_1 & 0 \\ * & * & -\mu\bar{\xi}R & 0 & \mu\bar{\xi}A_{d21}^TW^T \\ * & * & * & -\gamma^2 I_k & 0 \\ * & * & * & * & -I_r \end{bmatrix}<0 \tag{41}$$

where $\hat{P}_1,\ R\in\mathbb{R}^{n\times n}>0,\ \hat{P}_3=\varepsilon\hat{P}_2\in\mathbb{R}^{n\times n},\ Y=\hat{P}_2^TL\in\mathbb{R}^{n\times p}$, ε is a nonzero scalar and $\gamma>0$. To reconstruct the fault signals, continuous approximation of the discontinuous component $v(t)$, i.e.

$$v_r=-(\|TD_1\|+\delta)\Delta[\frac{\bar{e}_{2_1}}{|\bar{e}_{2_1}|+r},\ldots,\frac{\bar{e}_{2_p}}{|\bar{e}_{2_p}|+r}]^T \tag{42}$$

where $r\geq 0$ as proposed in [5], will be adopted.

6. Example

An inverted pendulum system is considered as in [5] which is linearized about the equilibrium at the origin

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -1.9333 & -1.9872 & 0.0091 \\ 0 & 36.9771 & 6.2589 & -0.1738 \end{bmatrix}, \quad B = D = \begin{bmatrix} 0 \\ 0 \\ 0.3205 \\ -1.0095 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad M = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad \zeta = \begin{bmatrix} 0 & 2.4 & 2.4 \end{bmatrix} y$$

The sensor fault distribution matrix $N = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$ in (34). A compensator approach from [14] is designed to stabilize the pendulum. It is assumed that an input fault is bounded by $\|f_i\| \leq \Delta = 0.6$.

6.1. Reconstruction of an actuator fault

The partitioned system (3) can be obtained by choosing the transformation matrix $T_0 = \begin{bmatrix} 0 & -3.86 & 3.15 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$, which yields $T = I_3$ an identity matrix. In deriving observer gain, LMI (33) is feasible under tuning parameters $\gamma = 0.02$, $W_1 = \begin{bmatrix} 0.1 & 0.001 \end{bmatrix}$, which leads to the observer gain $L = [0\ 0.43\ 0]$. LMIs (15) and (16) are feasible with $\bar{\alpha} = 3.9$, $\mu = 0.028$, sampling time $\mu\bar{\xi} = 0.018s$ and $P_1 = 0.0007$, $P_2 = \begin{bmatrix} 0.5029 & 0.0006 & -0.0024 \\ 0.0006 & 0.5989 & 0.0052 \\ -0.0024 & 0.0052 & 0.6391 \end{bmatrix}$, $P_3 = \begin{bmatrix} -0.0001 & -0.0002 & 0.0022 \end{bmatrix}^T$. LMI (23) is feasible with $\delta_1 = 2$, $\delta_2 = 19$ and $k_1 = 0.9$, $k_2 = 0.1$. Hence the observer (6) with gains in (5) and (7) has been designed. This ensures the error variable is bounded in the range $|e_{2_i}(t)| \leq 0.51$ according to the estimate (27). To verify the estimation Figure 1 is plotted using the sign function. It can be seen that every error variable is stabilized into a bound $|e_{2_i}| \leq 0.12$. Note that the high degree of switching is acceptable for an observer error signal; this is avoided in the reconstruction of the fault signals.

Suppose the input fault is $f_i(t) = 0.6sin(5t)$. This is reconstructed in Figure 2 with different values of γ and h. The smoothing parameter $r = 0.1$ was chosen in (42). It can be seen that in the top sub-figure, where $\gamma = 0.02$ and with

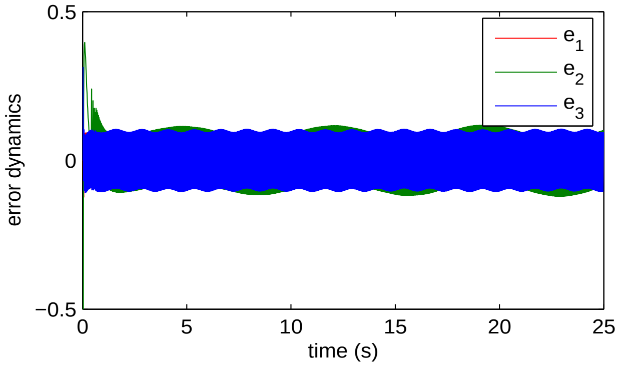

Figure 1: Error response e_2 with $\gamma = 0.02$ and sampling period $h = 18ms$ at outputs.

constant sampling interval $t_{k+1} - t_k = h = 18ms$, the fault is reconstructed reliably in the presence of uncertainty and sampling. In the second sub-figure, the reconstruction is obtained by choosing a larger $\gamma = 3.2$, leading to the observer gain as $L = \begin{bmatrix} 0 & -0.5 & 0 \end{bmatrix}$. Comparing to the top sub-figure the reconstruction is not accurate enough due to the larger γ chosen.

6.2. Reconstruction of sensor fault

By choosing tuning parameters $\varepsilon = 0.4$, $h = 8ms$, $\gamma = 0.39$, $W_1 = \begin{bmatrix} 0.1 & 0.5 \end{bmatrix}$ and $A_f = 0.1I_p$ in (41) and taking into account that the augmented system (36) is already in the required partitioned form (3), the observer gain is obtained as

$$L = \begin{bmatrix} -7.08 & -3.85 & 0 \\ -1.89 & -41.36 & 0 \\ -62.28 & 7.46 & 0 \\ -51.18 & -476.43 & 0 \end{bmatrix}$$

LMIs (15), (16) and (23), where replacements for their entries are made in (39), are feasible with $\mu = 0.036$, $\mu\bar{\xi} = 8ms$, $\bar{\alpha} = 2.2$, $\bar{b} = 0.51$, $\bar{b}_1 = 0.05$, $k_1 = 0.1$, $k_2 = 0.9$, $\delta_1 = 14$ and $\delta_2 = 4$. The values of matrix (13) are

$$P_1 = \begin{bmatrix} 0.68 & 0.39 & 0.22 & -0.07 \\ 0.39 & 0.22 & 0.12 & -0.04 \\ 0.22 & 0.12 & 0.07 & -0.02 \\ -0.07 & -0.04 & -0.02 & 0.01 \end{bmatrix} \times 10^{-4},\ P_2 = \begin{bmatrix} 0.377 & 0.145 & 0.022 \\ 0.145 & 0.501 & -0.005 \\ 0.022 & -0.005 & 0.367 \end{bmatrix},$$

$$P_3 = \begin{bmatrix} -0.53 & -0.02 & -0.11 & 0.009 \\ -0.167 & -0.212 & -0.039 & 0.008 \\ -0.091 & -0.056 & -0.161 & 0.011 \end{bmatrix} \times 10^{-3}$$

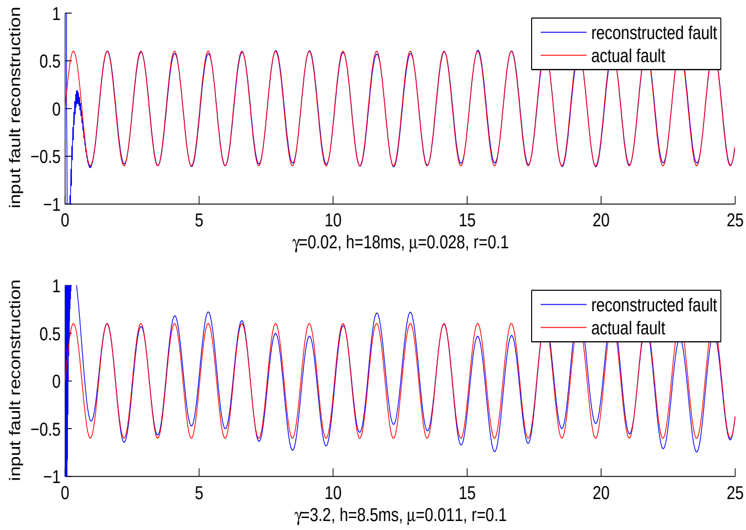

Figure 2: Input fault reconstruction in the presence of uncertainty and with different γ, h and r values

The observer obtained leads to a bound on the errors of $|e_{2_i}| \leq 0.31$. A different parameter setting can increase the allowable sample time as shown in Figure 3. Under the fault $f_o = 0.6sin(t)$ while uncertainties are remained as before, Figure 3 shows fault reconstruction under different values of γ, h and r. It suggests that a smaller γ corresponds to a smaller sample time that can be attained. But better disturbance rejection can be achieved under smaller sample time, while larger sample time can deteriorate the disturbance rejection properties. Nevertheless, in both cases the fault is reconstructed faithfully.

6.3. Comparison with a classical fault reconstruction scheme

The classical observer design in [5], which utilizes the equivalent injection term (42) only to reconstruct the fault, but does not consider the effect of output sampling, will now be used to benchmark the proposed design. Comparing the fault reconstruction in Figure 4 to the previous results, the proposed method produces much higher precision in the construction. Decreasing r further below its given values in Figure 4 will not filter out the high switching terms in simulations. It is observed that the precision of the fault reconstruction in (42) depends heavily on the value of the smoothing term r in the equivalent injection. Larger r causes larger reconstruction error. On the other hand this error is compensated in our re-

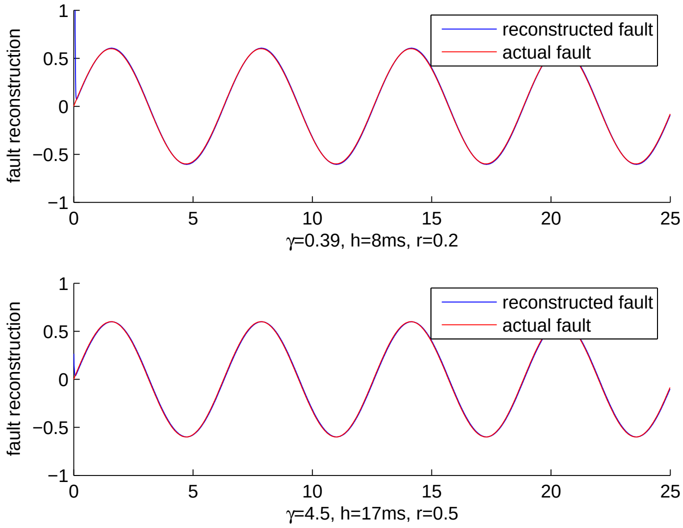

Figure 3: Output fault reconstructions under uncertainties and different γ, h and r

sult by using singular perturbation method, despite using larger value of r. Hence, the proposed method of observer design is shown to have significant advantages when compared with the classical approach if the output is sampled. It should be noted that the $\bar{e}_2$ term in (31) is pertinent to the reconstruction accuracy. For the observer in [5], the equivalent term is assumed to be zero.

7. Conclusion

This paper develops an observer design framework for systems with multiple outputs where the outputs are sampled and thus the output error signals used to drive the observer are subject to delay. A singular perturbation approach is employed for the analysis and ensures the ultimate bound on the error dynamics is proportional to the sampling time and the switching gain. A corresponding robust fault reconstruction technique is proposed utilizing a robust optimization technique and finds a sufficiently small value of the singular perturbation parameter, μ. It is demonstrated that the faults can be reconstructed reliably even if the measured outputs are subject to sampling and system uncertainties are present.

Acknowledgements

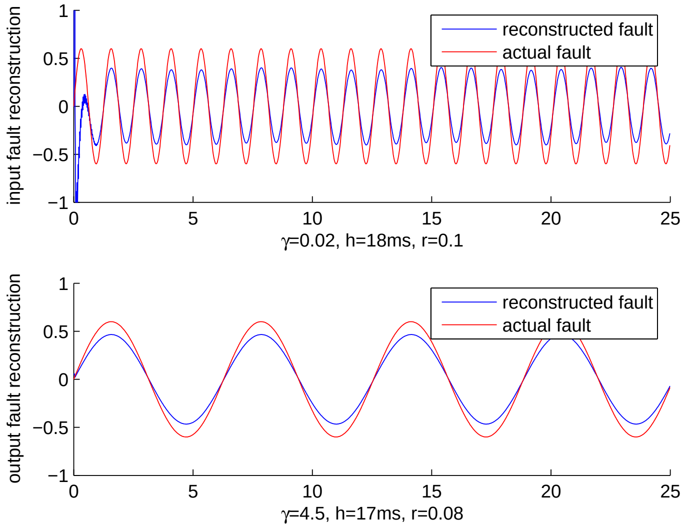

Figure 4: Fault reconstructions using the equivalent injection term only [5]

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Appendix A. Proof of Theorem 1

The i-th component of differential equation (9) with the initial condition (10) can be represented in the form of an integral equation [19]

$$\begin{aligned}\bar{e}_{2_i}(t) = \phi(t,t_0,\mu)\bar{e}_{2_i}(t_0) + \textstyle\int_{t_0}^{t} \phi(t,s,\mu)\Big[[\bar{A}_{21_i}\ \bar{A}_{22_i}]\bar{e}(s) \\ +\bar{A}_{d22_i}\bar{e}_2(s-\mu\xi(t)) + (TD_1)_i f_i(s) + (TM_2)_i\zeta(s,y,u) \\ -(\|TD_1\|\Delta + \delta_1\Delta + \delta_2\beta + \|TM_2\|\beta)\,\mathrm{sign}\,\bar{e}_{2_i}(s-\mu\xi(s))\Big]ds\end{aligned} \tag{A.1}$$

The feasibility of (23) implies the bound (20), then the following inequality holds for $t \to \infty$:

$$\begin{aligned}|[\bar{A}_{21_i}\ \bar{A}_{22_i}]\bar{e}(s) + \bar{A}_{d22_i}\bar{e}_2(s-\mu\xi(t)) + (TD_1)_i f_i(s) \\ +(TM_2)_i\zeta(s,y,u) - (\|TD_1\|\Delta + \delta_1\Delta + \delta_2\beta \\ +\|TM_2\|\beta)\,\mathrm{sign}\,\bar{e}_{2_i}(s-\mu\xi(s))| < M_0\end{aligned} \tag{A.2}$$

Taking into account (26) and (A.2), it is established from (A.1) that for $t \to \infty$

$$
\begin{aligned}
&|\bar{e}_{2_i}(t+\theta) - \bar{e}_{2_i}(t)| \leq \big| \int_{t+\theta}^{t} \phi(t,s,\mu)\Big([\bar{A}_{21_i}\ \bar{A}_{22_i}]\bar{e}(s) \\
&+\bar{A}_{d22_i}\bar{e}_2(s-\mu\xi(t)) + (TD_1)_i f_i(s) + (TM_2)_i\zeta(s,y,u) \\
&-(\|TD_1\|\Delta + \delta_1\Delta + \delta_2\beta + \|TM_2\|\beta)\,\mathrm{sign}\,\bar{e}_{2_i}\big(s-\mu\xi(s)\big)\Big)ds\big| \\
&< M_0 \int_{t+\theta}^{t} e^{\frac{\alpha_2(t-s)}{\mu}} ds < \mu M_0 \tfrac{1-e^{2\alpha_2 h}}{\alpha_2} \\
&\leq 2M_0\mu\bar{\xi}
\end{aligned}
$$

where $\theta \in [-2\mu\bar{\xi},\ 0]$. Therefore,

$$
\bar{e}_{2_i}(t) - 2M_0\mu\bar{\xi} < \bar{e}_{2_i}(t+\theta) < \bar{e}_{2_i}(t) + 2M_0\mu\bar{\xi} \tag{A.3}
$$

for $t \to \infty$ and the following implication holds

$$
|\bar{e}_{2_i}(t)| \geq 2M_0\mu\bar{\xi} \Rightarrow \mathrm{sign}\,\bar{e}_{2_i}(t+\theta) = \mathrm{sign}\,\bar{e}_{2_i}(t) \tag{A.4}
$$

for large enough t. Thus, from (20), (A.2) and (A.4) for sufficiently large t the following holds:

$$
\begin{aligned}
&|\bar{e}_{2_i}(t)| \geq 2M_0\mu\bar{\xi} \Rightarrow \\
&\bar{e}_{2_i}^T(t)\big[[\bar{A}_{21_i}\ \ \bar{A}_{22_i}]\bar{e}(t+\theta) + \bar{A}_{d22_i}\bar{e}_2(t-\mu\xi(t)+\theta) \\
&+(TD_1)_i f_i(t+\theta) + (TM_2)_i\zeta(s+\theta,y,u) && \text{(A.5)} \\
&-(\|TD_1\|\Delta + \delta_1\Delta + \delta_2\beta + \|TM_2\|\beta)\,\mathrm{sign}\,\bar{e}_{2_i}(t+\theta)\big] \\
&< |\bar{e}_{2_i}(t)|\big(|[\bar{A}_{21_i}\ \ \bar{A}_{22_i}]\bar{e}(t+\theta)| + |[0\ \ \bar{A}_{d22_i}]\bar{e}(t-\mu\xi(t)+\theta)| \\
&+\|TD_1\|\Delta + \|TM_2\|\beta\big) - (\|TD_1\|\Delta + \delta_1\Delta + \delta_2\beta + \|TM_2\|\beta)|\bar{e}_{2_i}(t)| \\
&< 0 && \text{(A.6)}
\end{aligned}
$$

It will be shown next that the $\bar{e}_{2_i}$-component of the solutions to (9) exponentially converges to the ball (27). Moreover, for sufficiently large t, whenever $\bar{e}_{2_i}(t)$ achieves the ball (27), it will never leave it. Taking into account (A.6), for sufficiently large t it follows that

$$
\begin{aligned}
&|\bar{e}_{2_i}(t)| \geq 2M_0\mu\bar{\xi} \Rightarrow \\
&\tfrac{d}{dt}\mu\bar{e}^2_{2_i}(t) = 2\mu\bar{e}_{2_i}(t)\dot{\bar{e}}_{2_i}(t) \\
&= 2\bar{e}_{2_i}(t)\big[-\bar{e}_{2_i}(t-\mu\xi(t)) + \mu\big([\bar{A}_{21_i}\ \ \bar{A}_{22_i}]\bar{e}(t) + [0\ \ \bar{A}_{d22_i}]\bar{e}(t-\mu\xi(t)) \\
&+(TD_1)_i f_i(t) + (TM_2)_i\zeta(t,y,u) - (\|TD_1\|\Delta + \delta_1\Delta + \delta_2\beta + \|TM_2\|\beta)\,\mathrm{sign}\,\bar{e}_{2_i}(t)\big)\big] \\
&\leq -2\bar{e}_{2_i}(t)\big(\bar{e}_{2_i}(t) - \textstyle\int_{t-\mu\xi(t)}^{t}\dot{\bar{e}}_{2_i}(s)ds\big) \\
&= -2\bar{e}^2_{2_i}(t) + 2\bar{e}_{2_i}(t)\textstyle\int_{t-\mu\xi(t)}^{t}\Big[-\frac{\bar{e}_{2_i}(s-\mu\xi(t))}{\mu} + [\bar{A}_{21_i}\ \ \bar{A}_{22_i}]\bar{e}(s) \\
&+[0\ \ \bar{A}_{d22_i}]\bar{e}(s-\mu\xi(t)) + (TD_1)_i f_i(s) + (TM_2)_i\zeta(t,y,u) \\
&-(\|TD_1\|\Delta + \delta_1\Delta + \delta_2\beta + \|TM_2\|\beta)\,\mathrm{sign}\,\bar{e}_{2_i}(s)\Big]ds \\
&\leq -2\bar{e}^2_{2_i}(t) - 2\tfrac{\bar{e}_{2_i}(t)}{\mu}\textstyle\int_{t-\mu\xi(t)}^{t}\bar{e}_{2_i}(s-\mu\xi(t))ds
\end{aligned}
$$

Therefore, given (A.4) holds for large enough t, it follows that

$$
-\int_{t-\mu\xi(t)}^{t}\bar{e}_{2_i}(t)\bar{e}_{2_i}(s-\mu\xi(t))ds \leq 0
$$

Hence

$$
|\bar{e}_{2_i}(t)| \geq 2M_0\mu\bar{\xi} \Rightarrow \frac{d}{dt}\mu\bar{e}^2_{2_i}(t) \leq -2\bar{e}^2_{2_i}(t) \tag{A.7}
$$

Assume now that for large enough t_1 the $\bar{e}_{2_i}$ component of the solution to (1) is outside the ball (27). Then from (A.7) it follows that for all $t \geq t_1$ such that $|\bar{e}_{2_i}(t)| \geq 2M_0\mu\bar{\xi}$ then

$$
\bar{e}^2_{2_i}(t) \leq e^{-\frac{2}{\mu}(t-t_1)}\bar{e}^2_{2_i}(t_1) \tag{A.8}
$$

i.e. $\bar{e}_{2_i}$ exponentially converges to the ball (27). Let $t_2 > t_1$ is the time when $|\bar{e}_{2_i}(t_2)| = 2M_0\mu\bar{\xi}$. Then due to (A.7) $\bar{e}^2_{2_i}(t_2^+) < \bar{e}^2_{2_i}(t_2)$. Therefore, whenever $\bar{e}_{2_i}(t)$ attains the ball (27), it will never leave it.

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