HOMOLOGICAL LOCALISATION OF MODEL CATEGORIES

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Abstract. One of the most useful methods for studying the stable homotopy category is localising at some spectrum E. For an arbitrary stable model category we introduce a candidate for the E–localisation of this model category. We study the properties of this new construction and relate it to some well–known categories.

Introduction

The stable homotopy category is spectacularly complicated and yet of fundamental importance to homotopy theorists. A standard and highly successful method of dealing with this complexity is to "filter out" some of this information via a Bousfield localisation. In return we obtain a more structured category with useful and interesting patterns.

More precisely, we choose some homology theory E_* and replace the stable homotopy category $\mathrm{Ho}(\mathcal{S})$ with $\mathrm{Ho}(L_E\mathcal{S})$, the full subcategory of $\mathrm{Ho}(\mathcal{S})$ with objects the E–local spectra. This means that in the passage from $\mathrm{Ho}(\mathcal{S})$ to $\mathrm{Ho}(L_E\mathcal{S})$, the E_*–isomorphisms are formally inverted. Bousfield's paper [Bou79] is the original source of this idea.

There are a number of other model categories whose homotopy categories share many of the properties of $\mathrm{Ho}(\mathcal{S})$, namely stable model categories. It would be advantageous if we could generalise the notion of E–localisation to this class of categories. Thus we are interested in the construction of a *homological localisation* of a *stable* model category, one that is the analogue of forming $\mathrm{Ho}(L_E\mathcal{S})$ from $\mathrm{Ho}(\mathcal{S})$.

The main motivation comes again from the study of the stable homotopy category. In order to understand spectra, $\mathrm{Ho}(\mathcal{S})$ and its various E–localisations it is necessary to relate $\mathcal{S}$ and $L_E\mathcal{S}$ to other stable model categories $\mathcal{C}$. For example, one can study to what extent there is a stable model category $\mathcal{C}$ whose homotopy category "models" $\mathrm{Ho}(L_E\mathcal{S})$ and how similar $\mathcal{C}$ is to $L_E\mathcal{S}$ in terms of higher homotopy behaviour. To make those links it would be a desirable tool to have the corresponding E–localisations of $\mathcal{C}$ in order to compare E–local spectra to other counterparts related to $\mathcal{C}$.

A stable model category $\mathcal{C}$ is a model category whose associated homotopy category $\mathrm{Ho}(\mathcal{C})$ is triangulated via the construction of [Hov99, Section 7]. Lenhardt proved in [Len12] that $\mathrm{Ho}(\mathcal{C})$ is a module over $\mathrm{Ho}(\mathcal{S})$ whenever $\mathcal{C}$ is a stable model category. Hence we have a tensor product

$$- \wedge^L -\colon \mathrm{Ho}(\mathcal{C}) \times \mathrm{Ho}(\mathcal{S}) \longrightarrow \mathrm{Ho}(\mathcal{C})$$

and an enrichment of $\mathrm{Ho}(\mathcal{C})$ in $\mathrm{Ho}(\mathcal{S})$. This technique is called *stable frames*.

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Using this action on the homotopy category of a stable model category one could try to make a new model structure on $\mathcal{C}$ such that the weak equivalences are the "E_*–isomorphisms": those maps $f\colon X \to Y$ in $\mathcal{C}$ such that

$$f \wedge^L E\colon X \wedge^L E \to Y \wedge^L E$$

is an isomorphism in $\mathrm{Ho}(\mathcal{C})$. Such a model structure would deserve the name $L_E\mathcal{C}$. The machinery that allows one to create new model structures with larger collections of weak equivalences is Bousfield localisation, see [Hir03, Part I]. But it seems particularly difficult to check that $L_E\mathcal{C}$ exists for general $\mathcal{C}$. For spectra, the argument appears in [EKMM97, Section VIII.1] and requires numerous unpleasant cardinality arguments.

For well–behaved stable model categories $\mathcal{C}$ we are going to produce a new model structure $\mathcal{C}_E$ that avoids such set–theoretic awkwardness. This $\mathcal{C}_E$ is a good candidate for the E–localisation of $\mathcal{C}$ because of the following universal property: $\mathcal{C}_E$ is the "closest" model category to $\mathcal{C}$ such that any Quillen adjunction from spectra to $\mathcal{C}$

$$\mathcal{S} \rightleftarrows \mathcal{C}$$

gives rise to a Quillen adjunction

$$L_E\mathcal{S} \rightleftarrows \mathcal{C}_E$$

from E–local spectra to $\mathcal{C}_E$. We are also able to give another description of $\mathcal{C}_E$ in terms of pushouts of model categories, which shows how strong the universal property of this new model structure is.

We are also able to give an improvement of [BR11, Theorem 9.5]: we can show that for all E, the homotopy information of E–local spectra is entirely encoded in the $\mathrm{Ho}(\mathcal{S})$–module structure on the E–local stable homotopy category. This was previously only possible with the strong restriction that E is smashing. Hence we have the following, which appears as Theorem 7.1.

Theorem. *Let $\mathcal{C}$ be a stable model category. Assume we have an equivalence of triangulated categories*

$$\Phi : \mathrm{Ho}(L_E\mathcal{S}) \longrightarrow \mathrm{Ho}(\mathcal{C})$$

then $L_E\mathcal{S}$ and $\mathcal{C}$ are Quillen equivalent if and only if Φ is an equivalence of $\mathrm{Ho}(\mathcal{S})$–module categories.

Organisation. Firstly, we recall some definitions and conventions regarding Bousfield localisation and stable frames. We also re–introduce the concept of *stably E–familiar model categories*: in [BR11] we studied those $\mathcal{C}$ such that the action of $\mathrm{Ho}(\mathcal{S})$ factors over the functor $\mathrm{Ho}(\mathcal{S}) \to \mathrm{Ho}(L_E\mathcal{S})$. In particular the homotopy category of such a model category has an enrichment in the more structured category $\mathrm{Ho}(L_E\mathcal{S})$. We called such categories stably E–familiar.

We then turn to the question of altering a model structure on a given category so as to obtain a stably E–familiar model category. In Section 3 we consider the simpler case of spectral model categories: such a model category is defined in a similar way to a simplicial model category, but with simplicial sets replaced by the model category of symmetric spectra. We construct the *stable E–familiarisation* of a spectral model category in this section.

In Section 4 we extend our results to more general stable model categories. We prove that the *stable E–familiarisation* of a model category $\mathcal{C}$ is the closest stably E–familiar model category to $\mathcal{C}$ in the following sense. The result below also implies that our construction has the universal property we described earlier.

Theorem. *Let $\mathcal{C}$ be a stable, proper and cellular model category such that the domains of the generating cofibrations of $\mathcal{C}$ are cofibrant. Then there is a model structure $\mathcal{C}_E$ on $\mathcal{C}$ such that*

(1) *$\mathcal{C}_E$ is stably E–familiar,*
(2) *if $F: \mathcal{C} \to \mathcal{E}$ is a left Quillen functor and $\mathcal{E}$ is stably E–familiar, then F factors over $\mathcal{C} \to \mathcal{C}_E$.*

Section 5 consists of several *examples* of $\mathcal{C}_E$ for some E and $\mathcal{C}$ involving algebraic model categories, chromatic localisations and module categories over a ringoid spectrum.

In Section 6 we rephrase the universal property of $\mathcal{C}_E$ in terms of *homotopy pushouts* of model categories.

Finally, we prove a full version of the *modular rigidity* theorem that all homotopy information of E–local spectra is governed by the $\mathrm{Ho}(\mathcal{S})$–action on $\mathrm{Ho}(L_E\mathcal{S})$ given by framings.

1. Bousfield localisation

We begin with an introduction to Bousfield localisation at a homology theory E. Throughout the paper when we refer to spectra, we mean symmetric spectra equipped with the stable model structure [HSS00] unless stated otherwise.

Let E be a spectrum and let $[-,-]_*$ denote maps in the stable homotopy category. Then E corepresents a homology functor E_* on the category of spectra via

$$E_*(X) = [S^0, E \wedge X]_*$$

where S^0 denotes the sphere spectrum. Bousfield used this to construct a homotopy category of spectra where maps which induce isomorphisms on E_*–homology are isomorphisms [Bou79]. We recap some of the definitions from this work.

Definition 1.1. *A map $f: X \to Y$ of spectra is an E–*equivalence *if $E_*(f)$ is an isomorphism. A spectrum Z is E–*local *if $f^*: [Y,Z] \to [X,Z]$ is an isomorphism for all E–equivalences $f: X \to Y$. A spectrum A is E–*acyclic *if $[A,Z] = 0$ for all E–acyclic Z. An E–equivalence from X to an E–local object Z is called an E–*localisation*.*

Bousfield localisation of spectra gives a homotopy theory that is particularly sensitive towards E_* and E–local phenomena. The E–local homotopy theory is obtained from the category of spectra by formally inverting the E–equivalences.

This can be seen as a special case of a more general result by Hirschhorn. Let $\mathcal{C}$ be a model category. For $X, Y \in \mathcal{C}$, we let $\mathrm{map}_{\mathcal{C}}(X,Y) \in \mathrm{sSet}$ denote the homotopy function object, see [Hir03, Chapter 17] and Section 2.

Definition 1.2. *Let S be a class of maps in $\mathcal{C}$. Then an object $Z \in \mathcal{C}$ is S–*local *if*

$$\mathrm{map}_{\mathcal{C}}(s,Z) : \mathrm{map}_{\mathcal{C}}(B,Z) \longrightarrow \mathrm{map}_{\mathcal{C}}(A,Z)$$

is a weak equivalence in simplicial sets for any $s : A \longrightarrow B$ *in* S. *A map* $f : X \longrightarrow Y \in \mathcal{C}$ *is an* S–equivalence *if*

$$\mathrm{map}_{\mathcal{C}}(f, Z) : \mathrm{map}_{\mathcal{C}}(Y, Z) \longrightarrow \mathrm{map}_{\mathcal{C}}(X, Z)$$

is a weak equivalence for any S*–local* $Z \in \mathcal{C}$. *An object* $W \in \mathcal{C}$ *is* S–acyclic *if*

$$\mathrm{map}_{\mathcal{C}}(W, Z) \simeq *$$

for all S*–local* $Z \in \mathcal{C}$.

A *left Bousfield localisation* of a model category $\mathcal{C}$ with respect to a class of maps S is a new model structure $L_S\mathcal{C}$ on $\mathcal{C}$ such that

- the weak equivalences of $L_S\mathcal{C}$ are the S–equivalences,
- the cofibrations of $L_S\mathcal{C}$ are the cofibrations of $\mathcal{C}$,
- the fibrations of $L_S\mathcal{C}$ are those maps that have the right lifting property with respect to cofibrations that are also S–equivalences.

Hirschhorn proves that if S is a set and $\mathcal{C}$ is left proper and cellular then $L_S\mathcal{C}$ exists. (We will give rough definitions of these two terms below.) Note that an object is fibrant in $L_S\mathcal{C}$ if and only it is fibrant in $\mathcal{C}$ and S–local.

In the case of localising spectra at a homology theory one wants to invert the class of E_*–isomorphisms, i.e. those maps of spectra that induce isomorphisms in the homology theory E_*. Since this is not a set, one cannot use Hirschhorn's result directly. In [EKMM97, Section VIII.1] it is shown that there is a set S whose S–equivalences are exactly the E_*–isomorphisms. Hence, the key to proving the existence of homological localisations is to find a set giving the correct notion of equivalence. We shall encounter this idea again when constructing $\mathcal{C}_E$.

A model category is *left proper* if the pushout of a weak equivalence along a cofibration is a weak equivalence. A model category is *right proper* if the pullback of a weak equivalence along a fibration is a weak equivalence. If a model category is both left and right proper, we say that it is *proper*.

We also a need a stronger version of "cofibrantly generation", one which forces cell complexes to be better behaved. The actual definition is technical and not particularly illuminating, so we shall simply say that a model category is *cellular* if it is cofibrantly generated by sets I and J, and the domains and codomains of I and J satisfy some nice cardinality conditions. We leave the details to [Hir03, Definition 12.1.1].

2. Stable framings

Framings are a powerful tool that describe and classify Quillen functors from simplicial sets or spectra to arbitrary model categories. They were first developed by Hovey in [Hov99, Section 5.2]. For a model category $\mathcal{C}$, he investigates cosimplicial and simplicial resolutions of objects in $\mathcal{C}$. These are called "frames". In more detail, a frame of an object $A \in \mathcal{C}$ is a cofibrant replacement of the constant cosimplicial object $A \in \mathcal{C}^{\Delta}$ in the Reedy model category of cosimplicial objects in $\mathcal{C}$. From these notions one obtains bifunctors

$$\begin{aligned} - \otimes - &: \mathcal{C} \times \mathrm{sSet} \longrightarrow \mathcal{C}, \\ \mathrm{map}_l(-,-) &: \mathcal{C}^{op} \times \mathcal{C} \longrightarrow \mathrm{sSet}, \\ (-)^{(-)} &: \mathcal{C} \times \mathrm{sSet}^{op} \longrightarrow \mathcal{C}, \\ \mathrm{map}_r(-,-) &: \mathcal{C}^{op} \times \mathcal{C} \longrightarrow \mathrm{sSet} \end{aligned}$$

satisfying certain adjunction properties. The notation $\otimes$ stems from the fact that

$$A \otimes \Delta[0] \simeq A.$$

However, this set-up does not equip $\mathcal{C}$ with the structure of a simplicial model category because the "mapping spaces" $\mathrm{map}_l(X,Y)$ and $\mathrm{map}_r(X,Y)$ only agree up to a zig-zag of weak equivalences for cofibrant $X \in \mathcal{C}$ and fibrant $Y \in \mathcal{C}$ [Hov99, Proposition 5.4.7]. But their derived functors agree, leaving us with the following [Hov99, Theorem 5.5.3].

Theorem 2.1 (Hovey). *Let $\mathcal{C}$ be any model category. Then its homotopy category* $\mathrm{Ho}(\mathcal{C})$ *is a closed* $\mathrm{Ho}(\mathrm{sSet})$*–module category.*

In particular, this equips any model category with the notion of a homotopy mapping space. Moreover, framings satisfy the following important properties.

- If $\mathcal{C}$ carries the structure of a simplicial model category [Hov99, Definition 4.2.18], then the two Ho(sSet)–module structures coming from either framings or the simplicial structure agree [Hov99, Theorem 5.6.2].
- If $F : \mathrm{sSet} \longrightarrow \mathcal{C}$ is a left Quillen functor with $F(\Delta[0]) = A$, then the left derived functors of F and of the framing functor $A \otimes - : \mathrm{sSet} \longrightarrow \mathcal{C}$ agree. Thus, every left Quillen functor from simplicial sets to any model category can be described, up to homotopy, by a frame.

The second property follows from the fact that the category of cosimplicial objects $\mathcal{C}^\Delta$ is equivalent to the category of adjunctions $\mathrm{sSet} \rightleftarrows \mathcal{C}$ [Hov99, Proposition 3.1.5]. A cosimplicial object $A^\bullet$ corresponds to a Quillen adjunction under this equivalence if and only if it is a frame, that is $A^\bullet$ is cofibrant and homotopically constant, [BR11, Proposition 3.2].

In [Len12] Fabian Lenhardt described an analogous set-up for spectra and stable model categories. Now let $\mathcal{C}$ be a stable model category. First, Lenhardt shows that the category of adjunctions between spectra and a stable model category $\mathcal{C}$ is equivalent to the category of "Σ–cospectra" $\mathcal{C}^\Delta(\Sigma)$. An object in $\mathcal{C}^\Delta(\Sigma)$ consists of a sequence of cosimplicial objects $X_n \in \mathcal{C}^\Delta$ together with structure maps

$$\Sigma X_{n+1} \longrightarrow X_n.$$

The suspension of cosimplicial objects is described in [Len12, Section 3.3]. He then characterises those Σ–cospectra that give rise to Quillen adjunctions under this equivalence, calling them stable frames. These give rise to bifunctors $- \wedge -$ and $\mathrm{Map}(-,-)$ satisfying the expected adjunction properties.

As in the unstable case, this is not rigid enough to equip any stable model category $\mathcal{C}$ with the structure of a spectral model category. However, the above bifunctors give rise to the following [Len12, Theorem 6.3].

Theorem 2.2 (Lenhardt). *Let $\mathcal{C}$ be a stable model category. Then* $\mathrm{Ho}(\mathcal{C})$ *is a closed* $\mathrm{Ho}(\mathcal{S})$*–module category.*

As expected, this satisfies the following key properties.

- If $\mathcal{C}$ is already a spectral model category, then the $\mathrm{Ho}(\mathcal{S})$–module structure derived from the spectral structure agrees with the $\mathrm{Ho}(\mathcal{S})$–module structure coming from stable frames [BR11, Example 6.7].

- By construction, every left Quillen functor $F : \mathcal{S} \longrightarrow \mathcal{C}$ is, up to homotopy, of the form $X \wedge - : \mathcal{S} \longrightarrow \mathcal{C}$ for some fibrant–cofibrant $X \in \mathcal{C}$.
- In particular, for any fibrant–cofibrant $X \in \mathcal{C}$ there is a left Quillen functor $\mathcal{S} \longrightarrow \mathcal{C}$ that sends the sphere spectrum to X.
- Any stable frame and thus any Quillen functor $\mathcal{S} \longrightarrow \mathcal{C}$ is, up to homotopy, entirely determined by its image on the sphere.

As we have already mentioned, the homotopy theory of $L_E\mathcal{S}$ is often much better understood than $\mathcal{S}$. So it is worth asking if some stable model categories have more in common with $L_E\mathcal{S}$ than $\mathcal{S}$. We answer this question and obtain several useful results using this idea in [BR11]. We give the fundamental definitions below.

Definition 2.3. *We say that a stable frame $X \in \mathcal{C}^{\Delta}(\Sigma)$ is an E–*local frame *if it gives rise to a Quillen functor pair*

$$X \wedge - : L_E\mathcal{S} \rightleftarrows \mathcal{C} : \mathrm{Map}(X, -).$$

A stable model category $\mathcal{C}$ is stably E–familiar *if every stable frame is an E–local frame.*

This is [BR11, Definition 7.1]. This generalises the notion of an $L_E\mathcal{S}$–model category in the following sense: if $\mathcal{C}$ is already an $L_E\mathcal{S}$–model category, then the $\mathrm{Ho}(L_E\mathcal{S})$–module structure on $\mathrm{Ho}(\mathcal{C})$ agrees with the $\mathrm{Ho}(L_E\mathcal{S})$–module structure given by E–local frames [BR11, Proposition 7.6]. We can further characterise stably E–familiar model categories as follows [BR11, Theorem 7.8].

Theorem 2.4. *Let $\mathcal{C}$ be a stable model category. Then $\mathcal{C}$ is stably E–familiar if and only if the homotopy mapping spectrum $\mathbb{R}\,\mathrm{Map}_{\mathcal{C}}(X, Y)$ is an E–local spectrum for all $X, Y \in \mathcal{C}$.*

We can use the theory of E–local framings to study *algebraic model categories.* An algebraic model category is a $\mathrm{Ch}(\mathbb{Z})$–model category in the sense of [Hov99, Definition 4.2.18]. Thus a $\mathrm{Ch}(\mathbb{Z})$–model category is enriched, tensored and cotensored over chain complexes and satisfies the $\mathrm{Ch}(\mathbb{Z})$–analogue of the compatibility axiom (SM7). This implies that the homomorphism spectra obtained via framings are products of Eilenberg–Mac Lane spectra [GJ99, Proposition III.2.20], [DS07, Proposition 1.6]. Using the computations of Gutiérrez in [Gut10] one can draw the following conclusions [BR11, Section 9].

- For $n \geq 1$ there are no algebraic stably $K(n)$–familiar model categories, where n denotes the n^{th} Morava–K–theory.
- Let $E(n)$ denote the n^{th} chromatic Johnson–Wilson spectrum. An algebraic model category is stably $E(n)$–familiar if and only if it is rational.

Now we turn to the question of whether any model category can be made stably E–familiar in some natural way.

3. E–Familiarisation of spectral model categories

For any homology theory E we can consider the category of spectra with the E–local model structure, $L_E\mathcal{S}$. Hence we would like to know if a reasonable notion of E–localisation exists for an arbitrary stable model category $\mathcal{C}$.

Intuitively, a promising definition would be a Bousfield localisation $L_E\mathcal{C}$ of $\mathcal{C}$ where one localises at the class of "E–equivalences" given by

$$\{f : X \longrightarrow Y \in \mathcal{C} \mid f \wedge^L E : X \wedge^L E \longrightarrow Y \wedge^L E \text{ is an isomorphism in } \mathrm{Ho}(\mathcal{C})\},$$

where the action $\wedge$ of a spectrum on an element of $\mathcal{C}$ is defined via stable frames. However, showing the existence of Bousfield localisations at a class of maps is set–theoretically awkward. The standard method to circumvent this difficulty is to find a set of maps S such that the S–equivalences are precisely the E–equivalences. This is an extremely difficult task, see [EKMM97, Section VIII.1], so it is not clear if a good notion of E–localisation exists for general model categories.

Instead, we will construct the *stable E–familiarisation* $\mathcal{C}_E$ of $\mathcal{C}$ which is the "closest" stably E–familiar model category to $\mathcal{C}$. We will then draw some conclusions about its properties which will show that this construction is the right choice for an analogue of E–localisation for general stable $\mathcal{C}$. For example, the first theorem will show that every Quillen adjunction

$$\mathcal{S} \rightleftarrows \mathcal{C}$$

will give rise to a Quillen adjunction

$$L_E\mathcal{S} \rightleftarrows \mathcal{C}_E.$$

The first question to answer is: what kind of maps do we want to invert in order to construct $\mathcal{C}_E$? In a stably E–familiar model category $\mathcal{D}$ any map of the form

$$X \wedge^L j\colon X \wedge^L A \to X \wedge^L B$$

for $j\colon A \to B$ an E–equivalence of spectra and $X \in \mathcal{D}$ is a weak equivalence. Hence we could try to localise $\mathcal{C}$ at this class of maps. So we must find some set of maps S such that the S–equivalences equals this class.

We need a couple of technical results first. For this section we shall work with $\mathcal{S}$–model categories in the sense of [Hov99, Definition 4.2.18], where $\mathcal{S}$ again denotes the model category of symmetric spectra. Such a model category $\mathcal{D}$ is enriched, tensored and cotensored over symmetric spectra in simplicial sets and satisfies the appropriate analogue of Quillen's (SM7) axiom for simplicial model categories. We shall refer to $\mathcal{D}$ as being a *spectral model category*. We may also talk about $L_E\mathcal{S}$–model categories, where we use the E–local model structure on $\mathcal{S}$. A spectral model category is in particular stable and simplicial, see [SS03, Lemma 3.5.2]. We will see later that the restriction to spectral model categories is not as big a restriction as it might seem at first.

We denote the pushout–product of two maps by $\square$, so for $f : X \longrightarrow Y$ and $g : A \longrightarrow B$ the pushout–product of f and g is

$$f\square g : X \wedge B \coprod_{X\wedge A} Y \wedge A \longrightarrow Y \wedge B.$$

Recall that a set of maps S in a stable model category $\mathcal{D}$ is said to be *stable* if the class of S–local objects is closed under suspension. By [BR13, Proposition 4.6] if $\mathcal{D}$ and S are stable then so is $L_S\mathcal{D}$.

Proposition 3.1. *Let $\mathcal{D}$ be a left proper, cellular and spectral model category. Let S be a stable set of maps in $\mathcal{D}$. Then $L_S\mathcal{D}$ is also a spectral model category.*

Proof. Since $\mathcal{D}$ is left proper and cellular, $L_S\mathcal{D}$ exists by [Hir03, Theorem 4.1.1]. We must prove that if i is a cofibration of $L_S\mathcal{D}$ and j is a cofibration of $\mathcal{S}$ then $i\square j$ is a cofibration of $L_S\mathcal{D}$ that is a weak equivalence (in $L_S\mathcal{D}$) if either of i or j is. Since $\mathcal{D}$ is spectral and the cofibrations are unchanged by left Bousfield localisation, we know that $i\square j$ is a cofibration whenever i and j are. Furthermore if j is an acyclic cofibration of symmetric spectra, then $i\Box j$ is a weak equivalence in $\mathcal{D}$ and hence it is also an S–equivalence.

The third case is where i is an acyclic cofibration of $L_S\mathcal{D}$ and j is a cofibration of symmetric spectra. We must show that $i\Box j$ is an S–equivalence. By [Hov99, Lemma 4.2.4] it suffices to prove this for j a generating cofibration of symmetric spectra and i a generating acyclic cofibration of $L_S\mathcal{D}$. By [HSS00, Proposition 3.4.2] we may assume that j is of the form

$$F_nK \to F_nL$$

where F_n is the left adjoint to evaluation at level n, and K and L are simplicial sets. By [Hir03, Proposition 4.5.1] the domain of i is cofibrant, so it follows that both the domain and codomain of $i\Box j$ are cofibrant. The set S is stable, so the class of S–equivalences in $\mathrm{Ho}(\mathcal{D})$ is closed under suspension and desuspension. Thus $i\Box j$ is an S–equivalence if and only if

$$\Sigma^n(i\Box j) \cong i\Box\Sigma^n j$$

is an S–equivalence for all n.

We know that $\Sigma^n F_nK$ is weakly equivalent to F_0K in $\mathcal{S}$. Hence for any cofibrant $X \in \mathcal{D}$,

$$X \wedge \Sigma^n F_nK \to X \wedge F_0K$$

is a weak equivalence of $\mathcal{D}$. We also know that the domains of the maps $i\Box\Sigma^n j$ and $i\Box(F_0K \to F_0L)$ are pushouts of cofibrations between cofibrant objects. It follows that $i\Box\Sigma^n j$ is weakly equivalent to the map $i\Box(F_0K \to F_0L)$. The bifunctor

$$- \wedge F_0- \colon \mathcal{D} \times \mathrm{sSet} \to \mathcal{D}$$

gives $\mathcal{D}$ the structure of a simplicial model category. We may now use [Hir03, Theorem 4.1.1], which states that since $\mathcal{D}$ is simplicial, so is $L_S\mathcal{D}$. Consequently we see that $i\Box(F_0K \to F_0L)$ is an S–equivalence. Hence $i\Box j$ is also an S–equivalence and $L_S\mathcal{D}$ is a spectral model category. □

Proposition 3.2. *Let $\mathcal{D}$ be a left proper, cellular and spectral model category with generating cofibrations $I_\mathcal{D}$ and generating acyclic cofibrations $J_\mathcal{D}$. Let J_E be the set of generating acyclic cofibrations for $L_E\mathcal{S}$. Define*

$$S = I_\mathcal{D}\Box J_E = \{i\Box j \mid i \in I_\mathcal{D}, \quad j \in J_E\}.$$

Then $L_S\mathcal{D}$ is an $L_E\mathcal{S}$–model category and hence is stably E–familiar.

Proof. The set J_E is closed under desuspension in the sense that for any element $j \in J_E$ there is an element j' with $\Sigma j' \simeq j$. It follows that the same holds for S, so it is stable in the sense of [BR13, Definition 3.2]. Thus $L_S\mathcal{D}$ is also a stable model category. By Lemma 3.1 it is also an $\mathcal{S}$–model category.

To see that it is an $L_E\mathcal{S}$–model category we only need to check that if i is a cofibration of $L_S\mathcal{D}$ and j is an acyclic cofibration of $L_E\mathcal{S}$ then $i\Box j$ is an S–equivalence. By [Hov99, Lemma 4.2.4] it suffices to prove this for $i \in I_\mathcal{D}$ and $j \in J_E$. But then $i\Box j$ is an element of S and hence is an S–equivalence. □

Proposition 3.3. *Let $\mathcal{D}$ be a left proper, cellular and spectral model category and S as in Proposition 3.2. Assume that the domains of the generating cofibrations of $\mathcal{D}$ are cofibrant. Then if $\mathcal{D}$ is a monoidal model category so is $L_S\mathcal{D}$.*

Proof. Since $\mathcal{D}$ is spectral, the maps in S are cofibrations between cofibrant objects. Thus by [BR13, Lemma 6.1] $L_S\mathcal{D}$ is monoidal if and only if

$$I_{\mathcal{D}} \square S = I_{\mathcal{D}} \square (I_{\mathcal{D}} \square J_E) \cong (I_{\mathcal{D}} \square I_{\mathcal{D}}) \square J_E$$

lies in the class of S–equivalences. As $\mathcal{D}$ is monoidal, $I_{\mathcal{D}} \square I_{\mathcal{D}}$ consists of cofibrations. By Proposition 3.2, $L_S\mathcal{D}$ is an $L_E\mathcal{S}$–model category. Hence the pushout product of a cofibration of $\mathcal{D}$ and an acyclic cofibration of $L_E\mathcal{S}$ is an S–equivalence as required. □

We now show that this set S has the correct homotopical behaviour in terms of E–familiarity by giving another description of the weak equivalences of $L_S\mathcal{D}$.

Proposition 3.4. *Let $\mathcal{D}$ be a left proper, cellular spectral model category, such that the domains of the generating cofibrations of $\mathcal{D}$ are cofibrant. Let T be the class of maps*

$$T = \{X \wedge^L f \mid X \in \mathcal{D},\ f \text{ is an } E\text{–equivalence of spectra}\}.$$

Then the class of T–equivalences is equal to the class of S–equivalences.

Proof. Take some cofibrant $X \in \mathcal{D}$. Then the functor

$$X \wedge - : L_E\mathcal{S} \to L_S\mathcal{D}$$

is a left Quillen functor by Proposition 3.2. Hence $X \wedge -$ takes E–equivalences between cofibrant spectra to S–equivalences. Thus every element of T is a weak equivalence in $L_S\mathcal{D}$.

Now we will show that every element of S is also a T–equivalence. Consider $i \square j \in S$ for $i: X \to Y$ a generating cofibration of $\mathcal{D}$ and $j: A \to B$ a generating acyclic cofibration for $L_E\mathcal{S}$. Since X, Y, A and B are all cofibrant, the maps $X \wedge j$ and $Y \wedge j$ are in the class T. Let P be the domain of $i \square j$, then by [Hir03, Lemma 3.4.2], the map from $Y \wedge A \to P$ is also a T–equivalence. It follows by the two–out–of–three property that $i \square j$ is a T–equivalence. □

If the category $\mathcal{D}$ is already stably E–familiar then the class T is already contained in the category of weak equivalences. Hence so is the set S, and $\mathcal{D}$ is in fact an $L_E\mathcal{S}$–model category.

Corollary 3.5. *Let $\mathcal{D}$ be a left proper cellular spectral model category that is stably E–familiar. Assume that the domains of the generating cofibrations of $\mathcal{D}$ are cofibrant. Then $\mathcal{D}$ is an $L_E\mathcal{S}$–model category.* □

4. E–Familiarisation of stable model categories

We now want to consider model categories that are not necessarily spectral. Consider a proper and cellular stable model category $\mathcal{C}$. By [BR13, Theorem 8.2] $\mathcal{C}$ is Quillen equivalent to a spectral model category, namely the category $\mathcal{D} = \mathcal{S}^\Sigma(s\mathcal{C})$ of symmetric spectra in simplicial objects in $\mathcal{C}$ equipped with a non–standard model structure. Hence there is a Quillen equivalence which by abuse of notation we call

$$\Sigma^\infty : \mathcal{C} \rightleftarrows \mathcal{D} = \mathcal{S}^\Sigma(s\mathcal{C}) : \Omega^\infty$$

This model category $\mathcal{D}$ is also proper and cellular. Furthermore, if the generating cofibrations for $\mathcal{C}$ have cofibrant domains, then so do the generating cofibrations for $\mathcal{D}$.

Theorem 4.1. *Let $\mathcal{C}$ be a stable, proper and cellular model category, such that the domains of the generating cofibrations of $\mathcal{C}$ are cofibrant. Let $\mathcal{D} = \mathcal{S}^\Sigma(s\mathcal{C})$, with generating cofibrations $I_\mathcal{D}$ and fibrant replacement $\hat{f}$. We set $S = I_\mathcal{D} \Box J_E$ as in Proposition* 3.2.

Define $\mathcal{C}_E$ to be the left Bousfield localisation of $\mathcal{C}$ at the set of maps $\Omega^\infty \hat{f} S$. Then

(1) $\mathcal{C}_E$ is stably E–familiar,
(2) the weak equivalences of $\mathcal{C}_E$ are the T'–equivalences, for T' the class below

$$T' = \{X \wedge^L f \mid X \in \mathcal{C},\ f \text{ is an } E\text{–equivalence of spectra}\}$$

(3) if $F : \mathcal{C} \to \mathcal{E}$ is a left Quillen functor and $\mathcal{E}$ is stably E–familiar, then F factors over $\mathcal{C} \to \mathcal{C}_E$, i.e. $F : \mathcal{C}_E \longrightarrow \mathcal{E}$ is also a left Quillen functor.

Proof. The model categories $\mathcal{C}$ and $\mathcal{D} = \mathcal{S}^\Sigma(s\mathcal{C})$ are Quillen equivalent. Hence [Hir03, Theorem 3.3.20] tell us that the adjunction

$$\Sigma^\infty : \mathcal{C} \rightleftarrows \mathcal{D} : \Omega^\infty.$$

induces a Quillen equivalence between $L_{\Omega^\infty \hat{f} S}\mathcal{C}$ and $L_{\Sigma^\infty \hat{c} \Omega^\infty \hat{f} S}\mathcal{D}$. (Here, $\hat{c}$ denotes the cofibrant replacement in $\mathcal{C}$.) The model category $L_{\Sigma^\infty \hat{c} \Omega^\infty \hat{f} S}\mathcal{D}$ is equal to $L_S\mathcal{D}$ since $(\Sigma^\infty, \Omega^\infty)$ is a Quillen equivalence. Thus we have a Quillen equivalence between $\mathcal{C}_E = L_{\Omega^\infty \hat{f} S}\mathcal{C}$ and $L_S\mathcal{D}$. The second category is stably E–familiar by Proposition 3.2. Hence so is $\mathcal{C}_E$ by [BR11, Lemma 7.10].

We may also conclude that the left derived functor of Σ^∞ induces an bijection between the weak equivalences of $\mathcal{C}_E$ (considered as a class in $\operatorname{Ho}\mathcal{C}$) and the S–equivalences of $\operatorname{Ho}\mathcal{D}$. Proposition 3.4 tells us that the class of S–equivalences in $\mathcal{D}$ is equal to the class of T–equivalences where

$$T = \{X \wedge^L f \mid X \in \mathcal{D},\ f \text{ is an } E\text{–equivalence of spectra}\}.$$

Consider the class of maps

$$T' = \{X \wedge^L f \mid X \in \mathcal{C},\ f \text{ is an } E\text{–equivalence of spectra}\}.$$

Let $\mathbb{L}\Sigma^\infty$ and $\mathbb{R}\Omega^\infty$ denote the left and right derived functors of Σ^∞ and Ω^∞ respectively. By [Len12, Theorem 6.3]

$$\mathbb{L}\Sigma^\infty(X \wedge^L f) = (\mathbb{L}\Sigma^\infty X) \wedge^L f.$$

Hence $\mathbb{L}\Sigma^\infty$ takes elements of T' to elements of T. Consider some element $Y \wedge^L f$ of T. This is weakly equivalent to

$$(\mathbb{L}\Sigma^\infty \mathbb{R}\Omega^\infty Y) \wedge^L f$$

and hence is in $\mathbb{L}\Sigma^\infty T'$. Thus the derived functor of Σ^∞ induces a bijection between the class T' and the class T up to weak equivalence. As a consequence the derived functor of Σ^∞ induces a bijection between the class of T'–equivalences and the class of T–equivalences. It follows that the T'–equivalences must be the class of weak equivalences of $\mathcal{C}_E$.

For the final point, let $F : \mathcal{C} \to \mathcal{E}$ be a left Quillen functor. If $\mathcal{E}$ is stably E–familiar, then the left derived functor of F takes the T'–equivalences to weak equivalences of $\mathcal{E}$. Hence $F : \mathcal{C}_E \to \mathcal{E}$ is also a left Quillen functor. □

Remark 4.2. *Let $\mathcal{C}$ be a stable, proper and cellular model category, such that the domains of the generating cofibrations of $\mathcal{C}$ are cofibrant. Then the above result says that $\mathcal{C}_E$ is the "closest" stably E–familiar model category to $\mathcal{C}$.*

In particular a model category $\mathcal{C}$ is stably E–familiar if and only if $\mathcal{C}_E = \mathcal{C}$.

Remark 4.3. *The assumptions on $\mathcal{C}$ are more reasonable than they might seem in practice. Since we want to perform a left Bousfield localisation, we will have to assume that $\mathcal{C}$ is left proper and cellular. To assume that $\mathcal{C}$ is also right proper is not too much of a restriction.*

We also need another assumption: that the domains of the generating cofibrations of $\mathcal{C}$ are cofibrant. This is a subtle assumption that occurs elsewhere in the literature, for example in [Hov01]. *We note that this assumption holds for almost all of the cofibrantly generated model categories that arise naturally.*

It is easy to check that the homotopy mapping spectra for $\mathcal{C}_E$ are given by the formula below, where Y_E is the fibrant replacement of Y in $\mathcal{C}_E$.

$$\mathbb{R}\operatorname{Map}_{\mathcal{C}_E}(X,Y) = \mathbb{R}\operatorname{Map}_{\mathcal{C}}(X,Y_E)$$

In particular, this mapping spectrum is E–local. We can use this to draw some immediate consequences of E–familiarisation.

For example, the chromatic Johnson–Wilson theories $E(n)$ satisfy

$$L_{E(n-1)}L_{E(n)} = L_{E(n-1)}$$

[Rav92, 7.5.3]. Thus,

Corollary 4.4. *For a proper and cellular stable model category $\mathcal{C}$ we have*

$$(\mathcal{C}_{E(n)})_{E(n-1)} = \mathcal{C}_{E(n-1)}.$$

□

We can further use our knowledge of stably E–familiar algebraic model categories described at the end of Section 2 to read off the following corollaries.

Corollary 4.5. *Let $\mathcal{C}$ be an algebraic model category and $K(n)$ the n^{th} Morava–K–theory for $n \geq 1$. Then $\mathcal{C}_{K(n)}$ is trivial.* □

Corollary 4.6. *Let $\mathcal{C}$ be an algebraic model category and let $E(n)$ denote the n^{th} chromatic Johnson–Wilson spectrum. Then $\mathcal{C}_{E(n)} = \mathcal{C}_{H\mathbb{Q}}$.* □

If we assume that localisation at E is smashing, we can obtain a nicer description of the weak equivalences of $\mathcal{C}_E$: in the smashing case $\mathcal{C}_E$ is precisely the "naive" localisation of $\mathcal{C}$ at L_ES^0 as described in the introduction of Section 3. That is, the left Bousfield localisation of $\mathcal{C}$ at the class of L_ES^0–equivalences (which we denote as $L_{L_ES^0}\mathcal{C}$) exists and is equal to $\mathcal{C}_E$. With this extra assumption we also see that

$$\mathcal{C}_E = \mathcal{C}_{L_ES^0}.$$

However, for a general model category $\mathcal{C}$ and smashing E it is unclear whether the model category $L_E\mathcal{C}$ exists and if it would be Quillen equivalent to $L_{L_ES^0}\mathcal{C}$.

Lemma 4.7. *In addition to the assumptions of Theorem 4.1, assume that localisation at E is smashing. Then a map f in $\mathcal{C}_E$ is a weak equivalence if and only if $f \wedge^L L_ES^0$ is a weak equivalence in $\mathcal{C}$. Hence the weak equivalences of $\mathcal{C}_E$ are precisely the L_ES^0–equivalences.*

Proof. We first show the statement for a spectral model category $\mathcal{D}$. Recall the model category $L_S\mathcal{D}$ for S the set $I_{\mathcal{D}}\Box J_E$ from the previous section. We will show that the S–equivalences are precisely the $L_E S^0$–equivalences of $\mathcal{D}$.

Every map in the set S is an $L_E S^0$–equivalence, hence every S–equivalence is a $L_E S^0$–equivalence. Now take some $L_E S^0$–equivalence $f\colon X \to Y$ in $\mathcal{D}$. The map

$$X \to X \wedge^L L_E S^0$$

is a T–equivalence with

$$T = \{X \wedge^L f \mid X \in \mathcal{D},\ f \text{ is an } E\text{–equivalence of spectra}\}$$

defined earlier in this section. Hence it is an S–equivalence. Thus the commutative square

$$\begin{array}{ccc} X & \xrightarrow{\quad f \quad} & Y \\ \downarrow & & \downarrow \\ X \wedge^L L_E S^0 & \xrightarrow{f \wedge^L L_E S^0} & Y \wedge^L L_E S^0 \end{array}$$

shows that f is S–equivalent to a weak equivalence in $\mathcal{D}$. Weak equivalences in $\mathcal{D}$ are in particular S–equivalences, so by the 2–out–of–3 axiom of model categories, f must be an S–equivalence.

To move this result from a spectral $\mathcal{D}$ to a general $\mathcal{C}$ we use a similar argument to that of the second point of Theorem 4.1. The Quillen equivalence $(\Sigma^\infty, \Omega^\infty)$ takes the $L_E S^0$–equivalences of $\mathcal{D} = \mathcal{S}^\Sigma(s\mathcal{C})$ bijectively to the $L_E S^0$–equivalences of $\mathcal{C}$. It follows that the $L_E S^0$–equivalences of $\mathcal{C}$ are precisely the weak equivalences of $\mathcal{C}_E$. □

The following corollary shows that stable E–familiarisation restricts to E–localisation in the case of spectra. This shows that the notion of $\mathcal{C}_E$ is indeed a good candidate for an analogue of E–localisation of a general $\mathcal{C}$.

Corollary 4.8. *Consider the category of modules over a ring spectrum R. Then*

$$(R\text{–mod})_E = L_E(R\text{–mod})$$

where the right hand side is the naive localisation of R–mod. *It has weak equivalences those maps of R–modules which forget to E–equivalences of spectra and the same cofibrations as* R–mod. *In particular*

$$\mathcal{S}_E = L_E\mathcal{S}.$$

Proof. We start by noting that $L_E(R\text{–mod})$ is equal to the model structure of R-modules in $L_E\mathcal{S}$ from [SS00, Theorem 4.1]. Hence this model structure has generating sets of cofibrations and acyclic cofibrations $R \wedge I_{\mathcal{S}}$ and $R \wedge J_E$ [SS00, Lemma 2.3].

We claim that every map in $R \wedge J_E$ is a T'–equivalence, where T' is from Theorem 4.1. We know that the domains of J_E are cofibrant, hence $R \wedge j$ is weakly equivalent to $R \wedge^L j$ for any $j \in J_E$. Thus the claim holds. It follows that every acyclic cofibration of $L_E(R\text{–mod})$ is a weak equivalence (and also a cofibration) of $(R\text{–mod})_E$.

We must now show the converse. Every acyclic cofibration of $(R\text{–mod})_E$ is an E–equivalence of underlying spectra and hence is an acyclic cofibration of $L_E(R\text{–mod})$. Thus the two model structures agree. □

We can now give a simple proof that stable E–familiarisation preserves Quillen equivalences.

Proposition 4.9. *Let $\mathcal{C}$ and $\mathcal{E}$ be proper, cellular and stable model categories such that the domains of their generating cofibrations are cofibrant. Let*

$$F : \mathcal{C} \rightleftarrows \mathcal{E} : G$$

be a Quillen equivalence. Then there is a Quillen equivalence between the E–familiarised model categories

$$F : \mathcal{C}_E \rightleftarrows \mathcal{E}_E : G.$$

Proof. Composing F with the identity on $\mathcal{E}$ gives us a left Quillen functor

$$F : \mathcal{C} \to \mathcal{E}_E$$

and $\mathcal{E}_E$ is of course stably E–familiar. Hence by the universal property of $\mathcal{C}_E$ proved in Theorem 4.1 we have a left Quillen functor $F : \mathcal{C}_E \to \mathcal{E}_E$. We now need to show that gives us a Quillen equivalence. We do so using Proposition 3.4 and the method of the second part of the proof of Theorem 4.1.

Let T be the class of maps

$$T = \{A \wedge^L f \mid A \in \mathcal{C},\ f \text{ is an } E\text{–equivalence of spectra}\}.$$

Similarly, let T' be the class of maps

$$T' = \{B \wedge^L f \mid B \in \mathcal{E},\ f \text{ is an } E\text{–equivalence of spectra}\}.$$

Then $\mathcal{C}_E = L_T\mathcal{C}$ and $\mathcal{E}_E = L_{T'}\mathcal{E}$. Let $\mathbb{L}F$ and $\mathbb{R}G$ denote the left and right derived functors of F and G respectively. By [Hir03, Theorem 3.3.20], the adjunction (F, G) induces a Quillen equivalence between $L_T\mathcal{C}$ and $L_{\mathbb{L}F(T)}\mathcal{E}$. But the set $\mathbb{L}F(T)$ is isomorphic in $\operatorname{Ho}\mathcal{E}$ to the set T' because Quillen equivalences induce equivalences of $\operatorname{Ho}(\mathcal{S})$–module categories [Len12, Theorem 6.3]. □

Remark 4.10. *One could try to prove an analogue of Proposition 3.3 and show that if $\mathcal{C}$ is monoidal then so is $\mathcal{C}_E$. This would require the adjunction $(\Sigma^\infty, \Omega^\infty)$ at the beginning of this section to be monoidal. However we do not know if this is the case.*

5. Examples

Let $\mathcal{C}$ be a spectral model category, such that the domains of its generating cofibrations are cofibrant. (Recall from [BR13, Theorem 7.2] that any stable, proper and cellular model category is Quillen equivalent to a spectral one.) Assume that $\mathcal{C}$ has a set of compact generators for its homotopy category, [SS03, Definition 2.1.2]. Schwede and Shipley prove in the above–mentioned paper that any such $\mathcal{C}$ is Quillen equivalent to a category mod–$\mathcal{E}$ where $\mathcal{E}$ can be thought of as a "ring spectrum with several objects". In the case of $\mathcal{C}$ having a single compact generator, $\mathcal{E}$ is simply a ring spectrum.

Let us briefly recap some of the definitions and constructions of that result. Let $\mathcal{G}$ denote the set of generators of $\mathcal{C}$. Then the $\mathcal{S}$–enriched category $\mathcal{E}$ is simply defined as the full $\mathcal{S}$–enriched subcategory of $\mathcal{C}$ with objects $\mathcal{G}$. An object $M \in$ mod–$\mathcal{E}$ consists of a spectrum $M(G)$ for each $G \in \mathcal{G}$ plus morphisms of spectra

$$\mathcal{E}(G', G) \wedge M(G) \longrightarrow M(G') \ \text{ for } \ G, G' \in \mathcal{G}$$

satisfying certain coherence conditions. By adjunction, such an M is the same as a contravariant spectral functor from $\mathcal{E}$ to $\mathcal{S}$. A standard example of an object of mod–$\mathcal{E}$ is given by the spectral functor $\mathcal{E}(-,G)$ for fixed $G \in \mathcal{G}$. The model structure on mod–$\mathcal{E}$ is has weak equivalences and fibrations defined objectwise [SS03, Theorem A.1.1]. This means that a natural transformation $f : M \longrightarrow N$ is a weak equivalence or a fibration if and only if

$$f_G : M(G) \longrightarrow N(G)$$

is so for each $G \in \mathcal{G}$. Theorem 3.9.3 of [SS03] then describes a Quillen equivalence

$$\mathrm{Hom}(\mathcal{G}, -) : \mathcal{C} \leftrightarrows \text{mod–}\mathcal{E} : - \wedge_{\mathcal{E}} \mathcal{G}$$

for spectral $\mathcal{C}$.

This is a highly useful description of a stable model category and we would like to obtain a description of the E–familiarisation $\mathcal{C}_E$ of $\mathcal{C}$ in terms of mod–$\mathcal{E}$. We note that this is a rather special case as not every stable model category has a set of compact generators [HS99, Corollary B.13].

By Proposition 4.9 we know that $\mathcal{C}_E$ and $(\text{mod–}\mathcal{E})_E$ are Quillen equivalent, so we shall find another description of $(\text{mod–}\mathcal{E})_E$.

Since mod–$\mathcal{E}$ is a spectral model category, it is easily seen that $(\text{mod–}\mathcal{E})_E$ is given by $L_S\,\text{mod–}\mathcal{E}$ as in Pro position 3.2. Recall that $S = I \square J_E$ for I the set of generating cofibrations for mod–$\mathcal{E}$. Hence in $(\text{mod–}\mathcal{E})_E$ any map of the form below is a weak equivalence.

$$\mathcal{E}(-,G) \wedge (i \square j)$$

In the above, G is a cofibrant and fibrant replacement of one of the compact generators for $\mathcal{C}$, i is a generating cofibration for $\mathcal{S}$ and j is a generating acyclic cofibration for $L_E\mathcal{S}$.

We can make another model structure on mod–$\mathcal{E}$ by taking the same cofibrations as before, but taking the generating set of acyclic cofibrations to be those maps of the form

$$\mathcal{E}(-,G) \wedge j$$

for G a generator and j a generating acyclic cofibration for $L_E\mathcal{S}$. We shall call this set of maps K and let mod–$\mathcal{E}_K$ denote the corresponding model structure. One can either check directly that these sets give a model structure or one can alter [SS03, Theorem A.1.1] to use $L_E\mathcal{S}$ instead of $\mathcal{S}$.

We claim that this model structure equals the model structure of $(\text{mod–}\mathcal{E})_E$. An element of K can be described as

$$\mathcal{E}(-,G) \wedge \big((* \to S^0) \square j\big).$$

Hence every element of K is an acyclic cofibration of $(\text{mod–}\mathcal{E})_E$. Conversely, mod–$\mathcal{E}$ equipped with this new model structure is stably E–familiar. Hence the identity functor $(\text{mod–}\mathcal{E})_E \to \text{mod–}\mathcal{E}_K$ is a left Quillen functor. Hence every acyclic cofibration of $(\text{mod–}\mathcal{E})_E$ is an acyclic cofibration of mod–$\mathcal{E}_K$. Thus these two model structures have the same cofibrations and acyclic cofibrations. We have therefore shown the following.

Proposition 5.1. *The model category* $(\text{mod–}\mathcal{E})_E$ *is the category of contravariant spectral functors from* $\mathcal{E}$ *to* $L_E\mathcal{S}$*, equipped with the model structure where fibrations and weak equivalences are defined objectwise. Thus the fibrant objects are those functors* M *such that* $M(G)$ *is fibrant in* $\mathcal{S}$ *and* E*–local for all* $G \in \mathcal{E}$. □

Consider the case where $\mathcal{C}$ has a single compact generator. Following the above we can replace this by a category of functors to $\mathcal{S}$. Indeed, [SS03, Theorem 3.1.1] states that $\mathcal{C}$ is Quillen equivalent to the category of R–modules, mod–R, for some ring spectrum R. In this case, the above proposition recovers the result of Corollary 4.8.

6. E–familiarisation and homotopy pushouts

We want to give another description of $\mathcal{C}_E$ via a universal property. We will relate $\mathcal{C}_E$ to a pushout of model categories. While the pullback of model categories is well–understood, [Ber11], the pushout is more complicated and is not often used. Roughly speaking, the homotopy pushout of a corner diagram of Quillen adjunctions

$$\mathcal{C} \leftrightarrows \mathcal{D} \rightleftarrows \mathcal{E}$$

is supposed to be a model category $\mathcal{P}$ that satisfies a universal property analogous to the pushout of a diagram within a category. Unfortunately, the homotopy–theoretic pushout construction is rather delicate and its existence and description not always clear.

However there is a special case where we can construct pushouts of model categories and verify that they have the correct universal property. By working in a particular context, we avoid the general question of whether homotopy pushouts of model categories exist in general.

Let $\mathcal{M}_2$ be a left Bousfield localisation of $\mathcal{M}_1$ at a class of maps W. Without loss of generality we assume that the maps in W are morphisms between cofibrant objects. (If the elements of W did not satisfy this, one can replace them with weakly equivalent morphisms between cofibrant objects. This would then give rise to the same Bousfield localisations.) In particular, this gives us a Quillen pair

$$\mathrm{Id}\colon \mathcal{M}_1 \rightleftarrows \mathcal{M}_2 = L_W\mathcal{M}_1 : \mathrm{Id}$$

Assume that we have a Quillen adjunction

$$F : \mathcal{M}_1 \rightleftarrows \mathcal{N}_1 : G.$$

We are now going to discuss the homotopy pushout of the corner diagram below for this special case

$$L_W\mathcal{M}_1 = \mathcal{M}_2 \leftrightarrows \mathcal{M}_1 \rightleftarrows \mathcal{N}_1.$$

Definition 6.1. *The homotopy pushout of the above diagram is defined, if it exists, as the Bousfield localisation $L_{\mathbb{L}FW}\mathcal{N}_1$ of $\mathcal{N}_1$. Here, $\mathbb{L}F$ denotes the left derived functor of F.*

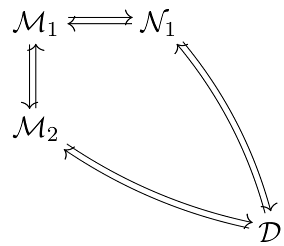
To justify this definition we need to see that $\mathcal{N}_2 = L_{\mathbb{L}FW}\mathcal{N}_1$ (provided it exists) satisfies the desired properties that a homotopy pushout is supposed to have. First we note that by [Hir03, Theorem 3.3.20] F and G induce a Quillen adjunction

$$F : \mathcal{M}_2 \rightleftarrows \mathcal{N}_2 : G$$

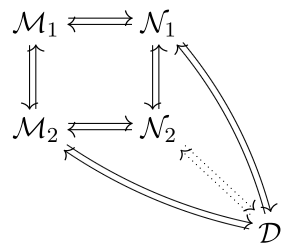
Assume that there is a model category $\mathcal{D}$ with Quillen adjunctions

$$\begin{array}{rcl} \mathcal{M}_2 & \rightleftarrows & \mathcal{D} \\ F' : \mathcal{N}_1 & \rightleftarrows & \mathcal{D} : G' \end{array}$$

such that in the diagram below, the two different composites of left adjoints from $\mathcal{M}_1$ to $\mathcal{D}$ agree up to natural isomorphism.



Because the vertical functors in the square below are simply identity functors it follows immediately that we may add $\mathcal{N}_2$ and obtain a commutative diagram of adjoint pairs.

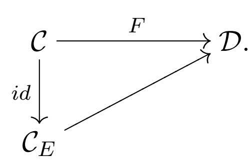


We must check that the adjunction below is a Quillen adjunction.

$$F' : \mathcal{N}_2 \rightleftarrows \mathcal{D} : G'$$

The model category $\mathcal{N}_2$ is the Bousfield localisation of $\mathcal{N}_1$ with respect to the class of maps Ff where f is a weak equivalence between cofibrant objects of $\mathcal{M}_2$. Thus $(F' \circ F)(f)$ is a weak equivalence in $\mathcal{D}$. This means that F' uniquely factors over $\mathcal{N}_2$. Furthermore, by construction, $\mathcal{N}_2$, if it exists, is unique up to Quillen equivalence.

Recall that the stable E–familiarisation $\mathcal{C}_E$ satisfies the following universal property. Given a left Quillen functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ with $\mathcal{D}$ stably E–familiar, F also gives rise to a left Quillen functor $C_E \longrightarrow \mathcal{D}$ via



This fact allows us to relate $\mathcal{C}_E$ and certain homotopy pushouts. Let $X \in \mathcal{C}$ be fibrant and cofibrant. Then we have a Quillen adjunction

$$X \wedge - : \mathcal{S} \rightleftarrows \mathcal{C} : \mathrm{Map}(X, -).$$

Using Definition 6.1 we can read off the following for a proper and cellular stable model category $\mathcal{C}$.

Lemma 6.2. *The homotopy pushout* $\mathcal{P}_X$ *of the diagram*

$$L_E\mathcal{S} \leftrightarrows \mathcal{S} \rightleftarrows \mathcal{C}$$

exists and is the Bousfield localisation of $\mathcal{C}$ with respect to the set of maps below, where J_E is the set of generating acyclic cofibrations of $L_E\mathcal{S}$.

$$X \wedge^L J_E = \{X \wedge^L j \mid j \in J_E\}$$

□

So in particular we know that this homotopy pushout exists. Because $\mathcal{C}_E$ is stably E–familiar we have a commutative square of Quillen adjunctions

$$\begin{array}{ccc} \mathcal{S} & \rightleftarrows & \mathcal{C} \\ \updownarrow & & \updownarrow \\ L_E\mathcal{S} & \rightleftarrows & \mathcal{C}_E \end{array}$$

By the universal property of $\mathcal{P}_X$, there is a Quillen adjunction $\mathcal{P}_X \rightleftarrows \mathcal{C}_E$ for each X. We can show that $\mathcal{C}_E$ is the "closest" model category to those pushouts in the following sense.

Theorem 6.3. *Let $\mathcal{C}$ be a stable, proper and cellular model category, such that the domains of the generating cofibrations of $\mathcal{C}$ are cofibrant. The Quillen adjunction*

$$\mathcal{C} \rightleftarrows \mathcal{C}_E$$

factors over

$$\mathcal{P}_X \rightleftarrows \mathcal{C}_E$$

for all fibrant–cofibrant $X \in \mathcal{C}$. If there is any other stable $\mathcal{D}$ with a Quillen adjunction

$$F : \mathcal{C} \rightleftarrows \mathcal{D} : G$$

that factors over

$$\mathcal{P}_X \rightleftarrows \mathcal{D}$$

for all fibrant–cofibrant X, then (F, G) also factors over $\mathcal{C}_E$.

Proof. The pushout $\mathcal{P}_X$ is defined as the Bousfield localisation of $\mathcal{C}$ at the set of maps $X \wedge^L j$ with $j \in J_E$. By Proposition 3.4 we know that $\mathcal{C}_E$ is the localisation of $\mathcal{C}$ at the class of maps of the form $X \wedge^L f$ for f an E–equivalence of spectra. Thus we see that for every $X \in \mathcal{C}$ the identity gives us a Quillen adjunction

$$\mathrm{Id} : \mathcal{P}_X \rightleftarrows \mathcal{C}_E : \mathrm{Id}$$

because every weak equivalence in $\mathcal{P}_X$ is also a weak equivalence in $\mathcal{C}_E$.

If the given Quillen adjunction (F, G) induces a Quillen adjunction

$$F : \mathcal{P}_X \rightleftarrows \mathcal{D} : G,$$

then F sends all morphisms of the form $X \wedge^L j$, for $j \in J_E$, to weak equivalences in $\mathcal{D}$. Hence F also sends all maps of the form $X \wedge^L f$, for f an E–equivalence of spectra, to weak equivalences in $\mathcal{D}$.

If (F, G) gives such Quillen adjunctions for all fibrant–cofibrant X then it must send any map of the form $X \wedge^L f$ with X fibrant–cofibrant and f an E–equivalence of spectra to a weak equivalence in $\mathcal{D}$. Thus it induces a Quillen adjunction

$$\mathcal{C}_E \rightleftarrows \mathcal{D}$$

by Theorem 4.1, which is what we wanted to prove. □

7. Modular rigidity for E–local spectra

We can show that stable frames encode all homotopical information of the E–local stable homotopy category. The triangulated structure of $\mathrm{Ho}(L_E\mathcal{S})$ alone is not sufficient for this: given just a triangulated equivalence

$$\Phi : \mathrm{Ho}(L_E\mathcal{S}) \longrightarrow \mathrm{Ho}(\mathcal{C})$$

for a stable model category $\mathcal{C}$ does not imply in general that $L_E\mathcal{S}$ and $\mathcal{C}$ are Quillen equivalent. In fact, Quillen equivalence can only be deduced from a triangulated equivalence of homotopy categories in some very special cases. To this date, the only nontrivial cases known of this 'rigidity' are the stable homotopy category itself [Sch07] and the case $E = K_{(2)}$ [Roi07]. However, if we do not only have a triangulated equivalence as above but also assume that this equivalence is a $\mathrm{Ho}(\mathcal{S})$–module equivalence, we can show that $L_E\mathcal{S}$ and $\mathcal{C}$ are Quillen equivalent.

We now give a more general version of [BR11, Theorem 9.5], in particular the assumption that E is smashing is no longer required.

Theorem 7.1. *Let $\mathcal{C}$ be a stable model category. Assume we have an equivalence of triangulated categories*

$$\Phi : \mathrm{Ho}(L_E\mathcal{S}) \longrightarrow \mathrm{Ho}(\mathcal{C})$$

then $L_E\mathcal{S}$ and $\mathcal{C}$ are Quillen equivalent if and only if Φ is an equivalence of $\mathrm{Ho}(\mathcal{S})$–module categories.

Proof. The "only if" part is true by [Len12, Theorem 6.3]: a Quillen equivalence induces a $\mathrm{Ho}(\mathcal{S})$–module equivalence.

Now let us assume that we have a $\mathrm{Ho}(\mathcal{S})$–module equivalence

$$\Phi : \mathrm{Ho}(L_E\mathcal{S}) \longrightarrow \mathrm{Ho}(\mathcal{C})$$

It follows that Φ^{-1} induces a weak equivalence of homotopy mapping spectra

$$\Phi^{-1} : \mathbb{R}\,\mathrm{Map}_{\mathcal{C}}(X, Y) \longrightarrow \mathbb{R}\,\mathrm{Map}_{L_E\mathcal{S}}(\Phi^{-1}X, \Phi^{-1}Y)$$

for $X, Y \in \mathcal{C}$. The right–hand–side is an E–local spectrum as $L_E\mathcal{S}$ is stably E–familiar. Hence every homotopy mapping spectrum of $\mathcal{C}$ is E–local, so $\mathcal{C}$ is stably E–familiar by [BR11, Theorem 7.8].

Thus for fibrant and cofibrant $X \in \mathcal{C}$, the Quillen functor

$$X \wedge - : \mathcal{S} \longrightarrow \mathcal{C}$$

factors over $L_E\mathcal{S}$ as a Quillen functor

$$X \wedge - : L_E\mathcal{S} \longrightarrow \mathcal{C}.$$

Now let X be a cofibrant–fibrant replacement of $\Phi(S^0)$. Because Φ is a $\mathrm{Ho}(\mathcal{S})$–module equivalence we see that

$$X \wedge^L (-) = \Phi(S^0) \wedge^L (-) = \Phi(S^0 \wedge^L -) = \Phi(-).$$

This means that Φ is derived from a Quillen functor. This Quillen functor must therefore be a Quillen equivalence, which is what we wanted to prove. ☐

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