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DOI

Link to record in KAR

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Document Version

UNSPECIFIED
Abstract
Proofs of the mathematical foundations and propositions and theorems stated and used in (Kriener and King, 2011).
1 Introduction

In (Kriener and King, 2011) we present a determinacy inference for Prolog including cut. This inference is developed in as a static analysis, comprising of a number of concrete semantics and abstractions thereof. Three theorems and a number of propositions formalise the connections that hold between these. We present here formal justification for the mathematical backdrop used in (Kriener and King, 2011), as well as proofs of the propositions and theorems stated there.

2 Appendix - Proofs

2.1 $\text{Con}_{seq}^\top$ is a complete lattice

2.1.1 Relation on $\text{Con}_{seq}^\top$ is a partial order

The relation is reflexive: $\Theta \sqsubseteq \Theta$

Observe that: $\forall \Theta \in \text{Con}_{seq}^\top (\Theta \subseteq_{pw} \Theta \wedge \Theta \in \text{Sub}_|\Theta|)$

hence $\forall \Theta \in \text{Con}_{seq}^\top (\Theta \sqsubseteq \Theta)$

by selecting $\Phi = \Theta$

The relation is transitive: $\Theta_1 \sqsubseteq \Theta_2 \wedge \Theta_2 \sqsubseteq \Theta_3 \rightarrow \Theta_1 \sqsubseteq \Theta_3$

$\forall \Theta_1, \Theta_2, \Theta_3 \in \text{Con}_{seq}^\top ((\Theta_1 \sqsubseteq \Theta_2 \wedge \Theta_2 \sqsubseteq \Theta_3) \rightarrow (\Theta_1 \sqsubseteq \Theta_3))$

let $|\Theta_1| = l$, $|\Theta_2| = m$, $|\Theta_3| = n$,

$l \leq m \leq n$

$(\Theta_1 \sqsubseteq \Theta_2) \rightarrow \exists \Phi_1 \in \text{Sub}_l(\Theta_2). (\Theta_1 \subseteq_{pw} \Phi_1)$

$(\Theta_2 \sqsubseteq \Theta_3) \rightarrow \exists \Phi_2 \in \text{Sub}_m(\Theta_3). (\Theta_2 \subseteq_{pw} \Phi_2)$

since $\Theta_2 \subseteq_{pw} \Phi_2$ and $\exists \Phi_1 \in \text{Sub}_l(\Theta_2). (\Theta_1 \subseteq_{pw} \Phi_1) : \exists \Phi_3 \in \text{Sub}_n(\Phi_2). (\Theta_1 \subseteq_{pw} \Phi_3)$

$\text{Sub}_l(\Phi_2) \subseteq \text{Sub}_l(\Theta_3)$

hence $\exists \Phi_3 \in \text{Sub}_l(\Theta_3). (\Theta_1 \subseteq_{pw} \Phi_3)$

therefore $\Theta_1 \sqsubseteq \Theta_3$

The relation is anti-symmetric: $\forall \Theta_1, \Theta_2 \in \text{Con}_{seq}^\top (\Theta_1 \sqsubseteq \Theta_2 \wedge \Theta_2 \sqsubseteq \Theta_1 \rightarrow \Theta_1 = \Theta_2)$

let $|\Theta_1| = m$, $|\Theta_2| = n$

$(\Theta_1 \sqsubseteq \Theta_2) \rightarrow \exists \Phi_1 \in \text{Sub}_m(\Theta_2)$ such that $\Theta_1 \subseteq_{pw} \Phi_1$

$(\Theta_2 \sqsubseteq \Theta_1) \rightarrow \exists \Phi_2 \in \text{Sub}_n(\Theta_1)$ such that $\Theta_2 \subseteq_{pw} \Phi_2$

$|\Phi_1| = m$ and $|\Phi_1| \leq n$ hence $m \leq n$

$|\Phi_2| = n$ and $|\Phi_2| \leq m$ hence $n \leq m$

hence $m = n$ (by anti-symmetry of $\subseteq$)

hence $\Phi_1 = \Theta_2$ and $\Phi_2 = \Theta_1$

hence $\Theta_1 \subseteq_{pw} \Theta_2$ and $\Theta_2 \subseteq_{pw} \Theta_1$

therefore:

$\Theta_1 = \Theta_2$ (by anti-symmetry of $\subseteq_{pw}$)

2.1.2 The meet of two sequences is unique and therefore well defined:

First note that by the definition of $\cap$, $\Theta \cap \Psi \sqsubseteq \Theta$ and $\Theta \cap \Psi \sqsubseteq \Psi$.

Then show: $\forall \Theta, \Psi, \Gamma \in \text{Con}_{seq}^\top : \Gamma \subseteq \Theta \wedge \Gamma \subseteq \Psi \rightarrow \Gamma \subseteq (\Theta \cap \Psi)$
\[ \vert \vec{\Theta} \vert = n, \quad \vert \vec{\Psi} \vert = m, \quad \vert \vec{\Gamma} \vert = k \]
\[ \vec{\Gamma} \sqsubseteq \vec{\Theta} \rightarrow \exists \vec{\Theta}_1 \in \text{Sub}_k(\vec{\Theta}).(\vec{\Gamma} \sqsubseteq_{pw} \vec{\Theta}_1) \]
\[ \vec{\Gamma} \sqsubseteq \vec{\Psi} \rightarrow \exists \vec{\Psi}_1 \in \text{Sub}_k(\vec{\Psi}).(\vec{\Gamma} \sqsubseteq_{pw} \vec{\Psi}_1) \]
\[ \vert \vec{\Theta}_1 \vert = k, \quad \vert \vec{\Psi}_1 \vert = k \]

assume (without loss of generality): \( n \geq m \), then: \( \vert \vec{\Theta} \cap \vec{\Psi} \vert = l, \quad l \leq m \)

since \( \vec{\Gamma} \sqsubseteq \vec{\Theta} \) and \( \vec{\Gamma} \sqsubseteq \vec{\Psi}, \quad k \leq m \) (and \( k \leq n \))

since \( \vec{\Gamma} \subseteq_{pw} \vec{\Theta}_1 \) and \( \vec{\Gamma} \subseteq_{pw} \vec{\Psi}_1 \), \( \vec{\Gamma} \subseteq_{pw} (\vec{\Theta}_1 \cap_{pw} \vec{\Psi}_1) \)

hence \( \vec{\Gamma} \sqsubseteq (\vec{\Theta}_1 \cap_{pw} \vec{\Psi}_1) \)
\[ (\vec{\Psi}_1 \in \text{Sub}_k(\vec{\Psi})) \rightarrow (\vec{\Theta}_1 \sqsubseteq \vec{\Psi}) \]
\[ (\vec{\Theta}_1 \in \text{Sub}_k(\vec{\Theta})) \rightarrow (\vec{\Theta}_1 \sqsubseteq \vec{\Theta}) \]
\[ (\vec{\Theta}_1 \cap_{pw} \vec{\Psi}_1) \subseteq \{ \vec{X} \cap_{pw} \vec{\Psi}_1 \mid \vec{X} \in \text{Sub}_k(\vec{\Theta}) \} \]
since \( \vec{\Theta}_1 \in \text{Sub}_k(\vec{\Theta}) \)
\[ (\vec{\Theta}_1 \cap_{pw} \vec{\Psi}_1) \subseteq_{pw} \bigcup_{pw} \{ \vec{X} \cap_{pw} \vec{\Psi}_1 \mid \vec{X} \in \text{Sub}_k(\vec{\Theta}) \} \]

(note that since \( \vec{\Gamma} \in \text{Con}_{seq} \), \( \vec{\Gamma} \) does not contain \{false\}
and since \( \vec{\Gamma} \subseteq_{pw} (\vec{\Theta}_1 \cap_{pw} \vec{\Psi}_1) \), \( \vec{\Theta}_1 \cap_{pw} \vec{\Psi}_1 \) does not contain \{false\}

hence \( (\vec{\Theta}_1 \cap_{pw} \vec{\Psi}_1) = \text{trim}(\vec{\Theta}_1 \cap_{pw} \vec{\Psi}_1) \))
\[ (\vec{\Theta}_1 \cap_{pw} \vec{\Psi}_1) \sqsubseteq (\vec{\Theta} \cap \vec{\Psi}) \]
\[ (\vec{\Theta} \cap \vec{\Psi}) \sqsubseteq (\vec{\Theta} \cap \vec{\Psi}) \] (since \( \vec{\Psi}_1 \sqsubseteq \vec{\Psi} \) and \( \cap \) is monotonic)

therefore \( \vec{\Gamma} \sqsubseteq (\vec{\Theta} \cap \vec{\Psi}) \)

### 2.2 Cut-normal form

We transform Prolog predicates that are defined by any number of clauses, none of which contains a disjunction, into this form by constructing \( G_1 \), \( G_2 \), \( G_3 \) and \( G_4 \) as follows:

\[ \begin{align*}
G_1: \quad & \text{If no clause precedes the clause containing the first cut, set } G_1 \text{ to post(false).} \\
& \text{Else, if a single clause precedes the clause containing the first cut, set } G_1 \text{ to the body of this clause.} \\
& \text{Otherwise, define an auxiliary predicate to wrap up all clauses preceding the clause containing the first cut and set } G_1 \text{ to a call to that predicate.}
\end{align*} \]

\[ \begin{align*}
G_2: \quad & \text{If there is no cut in the predicate, set } G_2 \text{ to post(false).} \\
& \text{Else, if no atom precedes the first cut, set } G_2 \text{ to post(true).} \\
& \text{Otherwise, set } G_2 \text{ to the compound goal before the first cut.}
\end{align*} \]
$G_3$: If there is no cut in the predicate, set $G_3$ to any goal, e.g. $\text{post}(\text{true})$.

Else, if no goal follows the first cut, set $G_3$ to $\text{post}(\text{true})$.

Else, if the compound goal following the first cut does not contain another cut, set $G_3$ to that goal.

Otherwise, define an auxiliary predicate to wrap up the compound goal following the first cut and set $G_3$ to a call to that predicate.

$G_4$: If no clause follows the clause containing the first cut, set $G_4$ to $\text{post}(\text{false})$.

Else, if a single, cut-free clause follows the clause containing the first cut, set $G_4$ to the body of this clause.

Otherwise, define an auxiliary predicate to wrap up all clauses following the clause containing the first cut and set $G_4$ to a call to that predicate.

### 2.3 Theorem 1: $\bigcup (F_G[G]_{\mu_P\tilde{\Theta}}) \subseteq \bigcup (\Theta) \cap S_G[G]$

Notice first that the following things hold:

| $\bigcup (\tilde{\Phi}) \subseteq \bigcup (\text{trim}(\tilde{\Phi}))$ |
| $\downarrow (\Theta \cup \tilde{\Phi}) = \downarrow \Theta \cup \downarrow \tilde{\Phi}$ |
| $\downarrow (\Theta \cap \tilde{\Phi}) = \downarrow \Theta \cap \downarrow \tilde{\Phi}$ |
| $\exists y (\Theta \cup \tilde{\Phi}) = \exists y (\Theta) \cup \exists y (\tilde{\Phi})$ |
| $\exists y (\Theta \cap \tilde{\Phi}) \subseteq \exists y (\Theta) \cap \exists y (\tilde{\Phi})$ |
| $\rho_x, y (\Theta \cup \tilde{\Phi}) = \rho_x, y (\Theta) \cup \rho_x, y (\tilde{\Phi})$ |
| $\rho_x, y (\Theta \cap \tilde{\Phi}) \subseteq \rho_x, y (\Theta) \cap \rho_x, y (\tilde{\Phi})$ |
| $\exists y (\exists y (\Theta)) = \exists y (\Theta)$ |

Proof by induction on length of $\tilde{\Theta}$:

Base Case: $\tilde{\Theta} = \emptyset$

$\bigcup (F_G[G]_{\mu_P[]} ) = \bigcup (\emptyset) = \emptyset$

$\bigcup (\emptyset) \cap S_G[G] = \emptyset \cap S_G[G] = \emptyset$

$\emptyset \subseteq \emptyset$

therefore: $\bigcup (F_G[G]_{\mu_P[]} ) \subseteq \bigcup (\emptyset) \cap S_G[G]$

Induction Step:

Assume: $\bigcup (F_G[G]_{\mu_P\tilde{\Theta}} ) \subseteq \bigcup (\Theta) \cap S_G[G]$  

Show: $\bigcup (F_G[G]_{\mu_P(\Theta : \tilde{\Theta})} ) \subseteq \bigcup (\Theta : \tilde{\Theta}) \cap S_G[G]$

Induction on structure of $G$:

Two base cases: (1) $G = \text{post}(\phi)$, (2) $G = p(\vec{x})$

(1) $G = \text{post}(\phi)$

Assume: $\bigcup (F_G[\text{post}(\phi)]_{\mu_P\tilde{\Theta}} ) \subseteq \bigcup (\tilde{\Theta}) \cap S_G[\text{post}(\phi)]$

Show: $\bigcup (F_G[\text{post}(\phi)]_{\mu_P(\Theta : \tilde{\Theta})} ) \subseteq \bigcup (\Theta : \tilde{\Theta}) \cap S_G[\text{post}(\phi)]$

$\bigcup (\Theta : \tilde{\Theta}) \cap S_G[\text{post}(\phi)]$

$= (\Theta \cap S_G[\text{post}(\phi)]) \cup (\bigcup (\tilde{\Theta}) \cap S_G[\text{post}(\phi)])$

$= (\Theta \cap \downarrow (\phi)) \cup (\bigcup (\tilde{\Theta}) \cap S_G[\text{post}(\phi)])$
To be on the safe side, consider the sequence resulting from appending both possibilities for $\Psi$, the union of which is certainly a superset of the above:

$$
\subseteq (\bigcup \{ \varphi_{\vec{y},\vec{x}} \spadesuit_{\vec{y}}(F_G[G_1]_\mu[\Theta':F_G[G_3]_\mu[\Theta'] : F_G[G_4]_\mu[\Theta']) \} \cap \Theta) \cup \bigcup (F_G[p(\vec{x})]_\mu \Theta)
$$

where $\Phi = F_G[G_2]_\mu[\Theta']$

Again, changing this to include all, rather than only the first, possibilities for $F_G[G_2]_\mu[\Theta']$ will result in a safe over-approximation, i.e. a superset of the above: $\subseteq (\bigcup \{ \varphi_{\vec{y},\vec{x}} \spadesuit_{\vec{y}}(F_G[G_1]_\mu[\Theta'] : F_G[G_2]_\mu[\Theta'] : F_G[G_3]_\mu[\Theta'] : F_G[G_4]_\mu[\Theta']) \} \cap \Theta) \cup \bigcup (F_G[p(\vec{x})]_\mu \Theta)$

since: $\subseteq (\bigcup (F_G[G_1]_\mu[\Theta']) \subseteq S_G[G_1] \cap \Theta'$

and: $\subseteq (\bigcup (F_G[G_2]_\mu[\Theta']) \subseteq S_G[G_2] \cap \Theta'$

hence: $\subseteq (\bigcup (F_G[G_3]_\mu[\Theta'] \subseteq S_G[G_3] \cap (S_G[G_2] \cap \Theta'))$

hence: $\subseteq (\bigcup (F_G[G_4]_\mu[\Theta'] \subseteq (S_G[G_3] \cap S_G[G_2]) \cap \Theta'$
hence: \( \bigcup (F_G[G_3] \mu (F_G[G_2] \mu (\Theta'))) \subseteq S_G[G_2, G_3] \cap \Theta' \)
and: \( \bigcup (F_G[G_1] \mu (\Theta')) \subseteq S_G[G_4] \cap \Theta' \)
using these, therefore, the above superset of \( \bigcup (F_G[p(\bar{x})] \mu (p : \Theta)) \) is a subset of:
\[
\subseteq ((\rho_{\bar{y}, \bar{x}} \bar{y}(S_\bar{G}[G_1] \cap \Theta') \cup \rho_{\bar{y}, \bar{x}} \bar{y}(S_\bar{G}[G_2, G_3] \cap \Theta')) \cup \rho_{\bar{y}, \bar{x}} \bar{y}(S_\bar{G}[G_4] \cap \Theta')) \cap \Theta
\]
\[
\cup \bigcup (F_G[p(\bar{x})] \mu (\Theta))
\]
since: \( \Theta' = \rho_{\bar{y}, \bar{x}} \bar{y}(\Theta) \), the following holds: \( \rho_{\bar{y}, \bar{x}} \bar{y}(\Theta) = (\rho_{\bar{y}, \bar{x}} \bar{y}(\Theta)) \)
intersecting this with \( \Theta \) therefore gives \( \Theta \) itself: \( \rho_{\bar{y}, \bar{x}} \bar{y}(\Theta) \cap \Theta = \Theta \) distributing the projections and collecting and intersection the occurrences of \( \rho_{\bar{y}, \bar{x}} \bar{y}(\Theta') \) and \( \Theta \) above therefore gives:
\[
\subseteq ((\rho_{\bar{y}, \bar{x}} \bar{y}(S_\bar{G}[G_1]) \cup \rho_{\bar{y}, \bar{x}} \bar{y}(S_\bar{G}[G_2, G_3]) \cup \rho_{\bar{y}, \bar{x}} \bar{y}(S_\bar{G}[G_4]))) \cap \Theta) \cup \bigcup (F_G[p(\bar{x})] \mu (\Theta))
\]
\[
\subseteq ((\rho_{\bar{y}, \bar{x}} \bar{y}(S_\bar{G}[G_1]) \cup \rho_{\bar{y}, \bar{x}} \bar{y}(S_\bar{G}[G_2, G_3]) \cup \rho_{\bar{y}, \bar{x}} \bar{y}(S_\bar{G}[G_4]))) \cap \Theta)
\]
\[
\cup (\bigcup (\Theta) \cap S_G[p(\bar{x})])
\]
\[
= \bigcup (\Theta) \cap S_G[p(\bar{x})]
\]
Induction Step: \( G = G_1, G_2 \)
Assume: \( \bigcup (F_G[G_1, G_2] \mu (p : \Theta)) \subseteq \bigcup (\Theta) \cap S_G[G_1, G_2] \)
And: \( \bigcup (F_G[G_1] \mu (p : \Theta)) \subseteq (\bigcup (\Theta) \cap S_G[G_1]) \)
And: \( \bigcup (F_G[G_2] \mu (p : \Theta)) \subseteq (\bigcup (\Theta) \cap S_G[G_2]) \)
Show: \( \bigcup (F_G[G_1, G_2] \mu (p : \Theta)) \subseteq (\bigcup (\Theta) \cap S_G[G_1, G_2]) \)

\[
\bigcup (\Theta) \cap S_G[G_1, G_2]
\]
\[
= (\Theta \cap S_G[G_1, G_2]) \cup (\bigcup (\Theta) \cap S_G[G_1, G_2])
\]
\[
= (\Theta \cap S_G[G_1] \cap S_G[G_2]) \cup (\bigcup (\Theta) \cap S_G[G_1, G_2])
\]
\[
\bigcup (F_G[G_1, G_2] \mu (p : \Theta))
\]
\[
= \bigcup (F_G[G_2] \mu (F_G[G_1] \mu (p : \Theta)) \cap S_G[G_2])
\]
\[
\subseteq \bigcup (\Theta) \cap S_G[G_1] \cap S_G[G_2]
\]
\[
= \bigcup (\Theta) \cap S_G[G_1, G_2]
\]
QED

2.4 **Theorem 2:** For \( \Theta \in Con^+ \) and stratified \( P = P_0 \cup \ldots \cup P_n: \Theta \subseteq D_G[G] \delta_P \Rightarrow |F_G[G] \mu (p : \Theta)| \leq 1. \)

2.4.1 **Lemma 1:** \((F_G[G] \mu (\bar{\Theta})) \cap \Psi = F_G[G] \mu (\bar{\Theta} \cap \Psi)\)

Proof by nested induction on:
1. \( \mu \),
2. \( |ar{\Theta}| \),
3. structure of \( G \)

1 Base Case: \( \mu = \mu_\perp \)
Show: \((F_G[G] \mu_\perp (\bar{\Theta})) \cap \Psi = F_G[G] \mu_\perp (\bar{\Theta} \cap \Psi)\)

1.1 Base Case: \( \bar{\Theta} = [] \)
Show: \((F_G[G] \mu_\perp []) \cap \Psi = F_G[G] \mu_\perp ([] \cap \Psi)\)
\(([]) \cap \Psi = F_G[G] \mu_\perp ([])\)
\([] = []\)
1.2 Induction Step: \((\Theta : \tilde{\Theta})\)

Assume: \((F_G[H][\mu_1 \Theta]) \cap \Psi = F_G[H][\mu_1 (\tilde{\Theta} \cap \Psi))\)

Show: \((F_G[G][\mu_1 (\Theta : \tilde{\Theta})) \cap \Psi = F_G[G][\mu_1 ((\Theta : \tilde{\Theta}) \cap \Psi))\)

1.2.1 Two Base Cases: (1) \(G = \text{post}(\phi)\), (2) \(G = p(x)\)

(1) \(G = \text{post}(\phi)\)
Show: \((F_G[\text{post}(\phi)][\mu_1 (\Theta : \tilde{\Theta})) \cap \Psi = F_G[\text{post}(\phi)][\mu_1 ((\Theta : \tilde{\Theta}) \cap \Psi))\)

\(trim([\Theta]) \cap \Theta : F_G[\text{post}(\phi)] \cap \Psi = F_G[\text{post}(\phi)]((\Theta \cap \Psi))\)

by assumption: \((F_G[\text{post}(\phi)][\mu_1 \Theta]) \cap \Psi = F_G[\text{post}(\phi)][\mu_1 (\tilde{\Theta} \cap \Psi))\)

\(trim(F_G[\text{post}(\phi)][\mu_1 \Theta]) \cap \Psi = trim(F_G[\text{post}(\phi)][\mu_1 (\tilde{\Theta} \cap \Psi))\)

by assumption: \((F_G[p(x)][\mu_1 \Theta]) \cap \Psi = F_G[p(x)][\mu_1 (\tilde{\Theta} \cap \Psi))\)

\(F_G[p(x)][\mu_1 ((\Theta : \tilde{\Theta}) \cap \Psi))\)

= \(F_G[p(x)][\mu_1 ((\Theta \cap \Psi))\)

= \(F_G[p(x)][(\tilde{\Theta} \cap \Psi)]\)

by assumption: \((F_G[p(x)][\mu_1 \Theta]) \cap \Psi = F_G[p(x)][\mu_1 (\tilde{\Theta} \cap \Psi))\)

\(F_G[p(x)][\mu_1 ((\Theta : \tilde{\Theta}) \cap \Psi))\)

= \(F_G[p(x)][\mu_1 ((\Theta \cap \Psi))\)

= \(F_G[p(x)][\mu_1 (\tilde{\Theta} \cap \Psi)]\)

hence: \((F_G[p(x)][\mu_1 (\Theta : \tilde{\Theta})) \cap \Psi = F_G[p(x)][\mu_1 ((\Theta : \tilde{\Theta}) \cap \Psi))\)

1.2.2 Induction Step: \(G = G_1, G_2\)

Assume: \((F_G[G_1][\mu_1 (\Theta : \tilde{\Theta})) \cap \Psi = F_G[G_1][\mu_1 ((\Theta : \tilde{\Theta}) \cap \Psi))\)

And: \((F_G[G_2][\mu_1 (\Theta : \tilde{\Theta})) \cap \Psi = F_G[G_2][\mu_1 ((\Theta : \tilde{\Theta}) \cap \Psi))\)

Show: \((F_G[G_1, G_2][\mu_1 (\Theta : \tilde{\Theta})) \cap \Psi = F_G[G_1, G_2][\mu_1 ((\Theta : \tilde{\Theta}) \cap \Psi))\)

\(F_G[G_1, G_2][\mu_1 ((\Theta : \tilde{\Theta}) \cap \Psi))\)

\(= (F_G[G_2][\mu_1 (F_G[G_1][\mu_1 (\Theta : \tilde{\Theta})) \cap \Psi))\)

\(= (F_G[G_2][\mu_1 (F_G[G_1][\mu_1 (\Theta : \tilde{\Theta})) \cap \Psi))\)

\(= F_G[G_2][\mu_1 (F_G[G_1][\mu_1 ((\Theta : \tilde{\Theta}) \cap \Psi))]\)

\(= F_G[G_2][\mu_1 ((\Theta : \tilde{\Theta}) \cap \Psi)]\)

2 Induction Step: \(\mu = \mu_{k+1}\)
Assume: \((F_G[H][\mu_\Lambda]) \cap \Lambda = F_G[H][\mu_k (\tilde{\Lambda} \cap \Lambda))\)
2.2.1 Two Base Cases: (1) $G = \text{post}(\phi)$, (2) $G = p(\vec{x})$

(1) $G = \text{post}(\phi)$

Show: $(\mathcal{F}_G[G]_{\mu_k+1}(\emptyset)) \cap \Psi = \mathcal{F}_G[G]_{\mu_k+1}(\emptyset) \cap \Psi$

\[
\mathcal{F}_G[G]_{\mu_k+1}(\emptyset) \cap \Psi = \mathcal{F}_G[G]_{\mu_k+1}(\emptyset) \cap \Psi
\]

(2) $G = p(\vec{x})$

Assume (without loss of generality): $p(\vec{y}) \leftarrow G_1; G_2, \ldots, G_4 \in P$

Show: $(\mathcal{F}_G[G](\vec{x})_{\mu_k+1}(\emptyset)) \cap \Psi = \mathcal{F}_G[G](\vec{x})_{\mu_k+1}(\emptyset) \cap \Psi$

\[
\left(\mathcal{F}_G[G](\vec{x})_{\mu_k+1}(\emptyset) \cap \Psi\right) = \left(\mathcal{F}_G[G](\vec{x})_{\mu_k+1}(\emptyset) \cap \Psi\right)
\]
Observe that for any $F$: $\varphi_{\vec{g},\vec{x},\vec{y}}(\rho_{\vec{x},\vec{y}, \vec{z}}(\Theta)) = \vec{z}_x((\rho_{\vec{x},\vec{y}, \vec{z}}(\Theta)))$

hence: $(\varphi_{\vec{g},\vec{x},\vec{y}}(\rho_{\vec{x},\vec{y}, \vec{z}}(\Theta))) = (\vec{z}_x((\rho_{\vec{x},\vec{y}, \vec{z}}(\Theta)))) \cap \Theta \cap \Psi$

$(\vec{z}_x(F_G[\rho_{\vec{x},\vec{y}, \vec{z}}(\Theta)])) \cap \Theta \cap \Psi$

which by assumption is equal to:

$(\vec{z}_x(F_G[\rho_{\vec{x},\vec{y}, \vec{z}}(\Theta)])) \cap \Theta \cap \Psi$

by parallel reasoning:

$(\varphi_{\vec{g},\vec{x},\vec{y}}(\rho_{\vec{x},\vec{y}, \vec{z}}(\Theta))) \cap \Theta \cap \Psi$

also by parallel reasoning:

$(\varphi_{\vec{g},\vec{x},\vec{y}}(\rho_{\vec{x},\vec{y}, \vec{z}}(\Theta))) \cap \Theta \cap \Psi$

hence if $(\varphi_{\vec{g},\vec{x},\vec{y}}(\rho_{\vec{x},\vec{y}, \vec{z}}(\Theta))) \cap \Theta \cap \Psi = \Lambda : \tilde{A}$

and $(\varphi_{\vec{g},\vec{x},\vec{y}}(\rho_{\vec{x},\vec{y}, \vec{z}}(\Theta))) \cap \Theta \cap \Psi = \Phi : \tilde{B}$

then $\Lambda = \Phi$

hence $(\varphi_{\vec{g},\vec{x},\vec{y}}(\rho_{\vec{x},\vec{y}, \vec{z}}(\Theta))) \cap \Theta \cap \Psi = (\varphi_{\vec{g},\vec{x},\vec{y}}(\rho_{\vec{x},\vec{y}, \vec{z}}(\Theta))) \cap \Theta \cap \Psi$

now say $\vec{F} = \begin{cases} F_G[\rho_{\vec{x},\vec{y}, \vec{z}}(\Theta)] & \text{if } F_G[\rho_{\vec{x},\vec{y}, \vec{z}}(\Theta)] = \Phi : \tilde{B} \\ F_G[\rho_{\vec{x},\vec{y}, \vec{z}}(\Theta)] & \text{if } F_G[\rho_{\vec{x},\vec{y}, \vec{z}}(\Theta)] = \Lambda : \tilde{A} \end{cases}$

then $\vec{F} = \vec{F}$

hence: $\varphi_{\vec{g},\vec{x},\vec{y}}(\rho_{\vec{x},\vec{y}, \vec{z}}(\Theta)) \cap \Theta \cap \Psi$

hence: $(\varphi_{\vec{g},\vec{x},\vec{y}}(\rho_{\vec{x},\vec{y}, \vec{z}}(\Theta))) \cap \Theta \cap \Psi$

therefore: $(\varphi_{\vec{g},\vec{x},\vec{y}}(\rho_{\vec{x},\vec{y}, \vec{z}}(\Theta))) \cap \Theta \cap \Psi = F_G[p(\vec{x})]_{\mu_{k+1}(\Theta : \tilde{\Theta})} \cap \Psi = F_G[p(\vec{x})]_{\mu_{k+1}(\Theta : \tilde{\Theta})}$

2.2.2 Induction Step $G = G_1, G_2$

Assume: $(F_G[\rho_{\vec{x},\vec{y}, \vec{z}}(\Theta)]) \cap \Psi = F_G[\rho_{\vec{x},\vec{y}, \vec{z}}(\Theta)] \cap \Psi$

And: $(F_G[\rho_{\vec{x},\vec{y}, \vec{z}}(\Theta)]) \cap \Psi = F_G[\rho_{\vec{x},\vec{y}, \vec{z}}(\Theta)] \cap \Psi$

Show: $(F_G[\rho_{\vec{x},\vec{y}, \vec{z}}(\Theta)]) \cap \Psi = F_G[\rho_{\vec{x},\vec{y}, \vec{z}}(\Theta)] \cap \Psi$

$(F_G[\rho_{\vec{x},\vec{y}, \vec{z}}(\Theta)]) \cap \Psi$

QED
2.4.2 Lemma 2: \( F_G[G][\mu(\Theta)] = F_G[G][\mu(\Theta \cap S_G[G])] \)

Proof in two stages:
(a) \( F_G[G][\mu(\Theta \cap S_G[G])] \subseteq F_G[G][\mu(\Theta)] \)
(b) \( F_G[G][\mu(\Theta)] \subseteq F_G[G][\mu(\Theta \cap S_G[G])] \)

(a) by monotonicity of \( F_G[G] \):
\[ [\Theta \cap S_G[G]] \subseteq [\Theta] \Rightarrow F_G[G][\mu(\Theta \cap S_G[G])] \subseteq F_G[G][\mu(\Theta)] \]

(b) \( F_G[G][\mu(\Theta)] \subseteq F_G[G][\mu(\Theta \cap S_G[G])] \)
Proof by nested induction on:
1. \( \mu \)
2. structure of \( G \):

1 Base Case: \( F_G[G][\mu(\Theta)] \subseteq F_G[G][\mu(\Theta \cap S_G[G])] \)

induction on structure of \( G \):
1.1 Two Base Cases: (1) \( G = \text{post}(\phi) \), (2) \( G = p(\vec{x}) \)
(1) \( G = \text{post}(\phi) \)
Show: \( F_G[\text{post}(\phi)][\mu(\Theta)] \subseteq F_G[\text{post}(\phi)][\mu(\Theta \cap S_G[\text{post}(\phi)])] \)

\[
\begin{align*}
F_G[\text{post}(\phi)][\mu(\Theta)] &= \text{trim}([\Theta \cap \downarrow(\phi)]) \\
F_G[\text{post}(\phi)][\mu(\Theta \cap S_G[\text{post}(\phi)])] &= \text{trim}([\Theta \cap S_G[\text{post}(\phi)] \cap \downarrow(\phi)]) \\
&= \text{trim}([\Theta \cap \downarrow(\phi) \cap \downarrow(\phi)]) \\
&= \text{trim}([\Theta \cap \downarrow(\phi)])
\end{align*}
\]

(2) \( G = p(\vec{x}) \)
Show: \( F_G[p(\vec{x})][\mu(\Theta)] \subseteq F_G[p(\vec{x})][\mu(\Theta \cap S_G[p(\vec{x})])] \)

\[
\begin{align*}
F_G[p(\vec{x})][\mu(\Theta)] &= [] \\
F_G[p(\vec{x})][\mu(\Theta \cap S_G[p(\vec{x})])] &= []
\end{align*}
\]

1.2 Induction Step: \( G = G_1, G_2 \)
Assume: \( F_G[G_1][\mu(\Theta)] \subseteq F_G[G_1][\mu(\Theta \cap S_G[G_1])] \)
And: \( F_G[G_2][\mu(\Theta)] \subseteq F_G[G_2][\mu(\Theta \cap S_G[G_2])] \)
Show: \( F_G[G_1, G_2][\mu(\Theta)] \subseteq F_G[G_1, G_2][\mu(\Theta \cap S_G[G_1, G_2])] \)

\[
\begin{align*}
F_G[G_1, G_2][\mu(\Theta)] &= F_G[G_2][\mu(\Theta)] \\
&= F_G[G_2][\mu(\Theta \cap S_G[G_2])] \\
&\subseteq F_G[G_2][\mu(\Theta \cap S_G[G_2]) \cap S_G[G_2]] \\
&\subseteq F_G[G_2][\mu(\Theta \cap S_G[G_2]) \cap S_G[G_2]] \\
&\subseteq F_G[G_2][\mu(\Theta \cap S_G[G_2]) \cap S_G[G_2]]
\end{align*}
\]

by assumption: \( F_G[G_2][\mu(\Theta)] \subseteq F_G[G_2][\mu(\Theta \cap S_G[G_2])] \)

by Lemma 1: \( (F_G[G_1][\mu(\Theta \cap S_G[G_1])) \subseteq F_G[G_2][\mu(\Theta \cap S_G[G_2))) \)

\[
\begin{align*}
F_G[G_1, G_2][\mu(\Theta)] &= F_G[G_2][\mu(\Theta \cap S_G[G_2])] \\
&= F_G[G_2][\mu(\Theta \cap S_G[G_2]) \cap S_G[G_2]] \\
&= F_G[G_2][\mu(\Theta \cap S_G[G_2]) \cap S_G[G_2]]
\end{align*}
\]
\[ F_G[G_1, G_2] \subseteq F_G[G_1, G_2] \]

therefore: \[ F_G[G_1, G_2] \subseteq F_G[G_1, G_2] \]

2 Induction Step:
Assume: \[ F_G[H] \subseteq F_G[H] \]
Show: \[ F_G[G][\mu_{k+1}] \subseteq F_G[G][\mu_{k+1}] \]
where \( \mu_{k+1} = F_P[P] \mu_k \)

induction on structure of G:
2.1 Two Base Cases: (1) \( G = \text{post}(\phi) \), (2) \( G = p(\vec{x}) \)
(1) \( G = \text{post}(\phi) \)
Show: \[ F_G[\text{post}(\phi)] \mu_{k+1} \subseteq F_G[\text{post}(\phi)] \mu_{k+1} \]
where \( \mu_{k+1} = F_P[P] \mu_k \)

\[ F_G[\text{post}(\phi)] \mu_{k+1} = \text{trim}((\Theta \cap \{\phi\})) \]
\[ F_G[\text{post}(\phi)] \mu_{k+1} = \text{trim}((\Theta \cap \{\phi\})) \]
\[ F_G[\text{post}(\phi)] \mu_{k+1} = \text{trim}((\Theta \cap \{\phi\})) \]

(2) \( G = p(\vec{x}) \)
Assume (without loss of generality): \( p(\vec{y}) \leftarrow G_1; G_2, !, G_3; G_4 \in P \)
Show: \[ F_G[p(\vec{x})] \mu_{k+1} \subseteq F_G[p(\vec{x})] \mu_{k+1} \]
where: \( \mu_{k+1} = F_P[P] \mu_k \)

\[ F_G[p(\vec{x})] \mu_{k+1} = \text{trim}((\Theta \cap \{\phi\})) \]
\[ F_G[p(\vec{x})] \mu_{k+1} = \text{trim}((\Theta \cap \{\phi\})) \]
\[ F_G[p(\vec{x})] \mu_{k+1} = \text{trim}((\Theta \cap \{\phi\})) \]

now: \( S_G[p(\vec{x})] \subseteq \text{post}(\phi) \)
and: \( S_G[p(\vec{x})] \subseteq \text{post}(\phi) \)
and: \( S_G[p(\vec{x})] \subseteq \text{post}(\phi) \)

Intersecting each side with \( \Theta \) preserves the order, hence: \( \Theta \cap S_G[p(\vec{x})] \subseteq \Theta \cap S_G[p(\vec{x})] \)
and: \( \Theta \cap S_G[p(\vec{x})] \subseteq \Theta \cap S_G[p(\vec{x})] \)
and: \( \Theta \cap S_G[p(\vec{x})] \subseteq \Theta \cap S_G[p(\vec{x})] \)

Again, projecting and renaming both sides in the same way preserves the order, hence: \( \text{post}(\phi) \cap S_G[p(\vec{x})] \subseteq \text{post}(\phi) \cap S_G[p(\vec{x})] \)
and: \( \text{post}(\phi) \cap S_G[p(\vec{x})] \subseteq \text{post}(\phi) \cap S_G[p(\vec{x})] \)
and: \( \text{post}(\phi) \cap S_G[p(\vec{x})] \subseteq \text{post}(\phi) \cap S_G[p(\vec{x})] \)

Now, since the following holds in general:
performing the same transformation on the above still preserves the order,

hence: \( \rho_{x,y}(\Theta \cap S_G[p(\bar{x})]) \supseteq \rho_{x,y}(\Theta) \cap S_G[G_1] \)

and: \( \rho_{x,y}(\Theta \cap S_G[p(\bar{x})]) \supseteq \rho_{x,y}(\Theta) \cap S_G[G_2, G_3] \)

by monotonicity of \( F_G \), therefore:

by assumption: \( F_G[G_1][\mu_k \rho_{x,y}(\Theta) \cap S_G[G_1]] \supseteq F_G[G_1][\mu_k \rho_{x,y}(\Theta)] \cap S_G[G_1] \)

hence the following holds of the first part of the sequence:

by assumption: \( F_G[G_1][\mu_k \rho_{x,y}(\Theta) \cap S_G[G_1]] \supseteq F_G[G_1][\mu_k \rho_{x,y}(\Theta)] \cap S_G[G_1] \)

hence the parallel thing holds for the second possibility of the second part of the sequence:

As for the first possibility for the second part of the sequence, consider this:

by monotonicity of \( F_G \):

by assumption:

by Lemma 1: \( F[G_2][\mu_k \rho_{x,y}(\Theta) \cap S_G[G_2] \cap S_G[G_3]] \supseteq F[G_2][\mu_k \rho_{x,y}(\Theta) \cap S_G[G_3]] \)

then: \( \Phi : \tilde{\Phi} \supseteq (\Lambda : \tilde{\Lambda}) \cap S_G[G_3] \)

hence: \( \Phi \supseteq [\Lambda \cap S_G[G_3]] \)

by assumption: \( F[G_3][\mu_k [\Lambda \cap S_G[G_3]]] \supseteq F[G_3][\mu_k [\Lambda]] \)

hence: \( F[G_3][\mu_k [\Phi] \supseteq F[G_3][\mu_k [\Lambda]] \)

These last few lines show that each part of the sequence we are considering is greater than the sequence we are aiming for. Pulling these together, we arrive at:

where

and

therefore:

applying the same renaming and projection to both sides preserves the order:

now name these two sequences:

and notice the following two facts:

(1) \( \Delta' \cap \Theta = F_G[p(\bar{x})][\mu_{k+1}(\Theta)] \)

(2) \( \Psi' \cap S_G[p(\bar{x})] \cap \Theta = F_G[p(\bar{x})][\mu_{k+1}(\Theta) \cap S_G[p(\bar{x})]] \)
then from above we have: $\Delta' \subseteq \Psi'$

hence: $\Delta' \cap \Theta \subseteq \Psi'$

by (1) and Theorem 1, therefore: $\bigcup(\Delta') \cap \Theta = \bigcup(\Delta' \cap \Theta) \subseteq \Theta \cap S_G[p(\bar{x})]$

hence: $\bigcup(\Delta') \subseteq S_G[p(\bar{x})]$.

therefore for each $\Delta'$ in $\Delta'$: $\Delta' \subseteq S_G[p(\bar{x})]$

hence for each $\Delta'$ in $\Delta'$: $\Delta' \cap S_G[p(\bar{x})] = \Delta'$

hence: $\hat{\Delta}' \cap S_G[p(\bar{x})] = \Delta'$

hence: $\hat{\Delta}' \cap \Theta = \Delta' \cap S_G[p(\bar{x})] \cap \Theta \subseteq \Psi' \cap S_G[p(\bar{x})] \cap \Theta$

substituting using (2), we therefore arrive at:

$$F_G[p(\bar{x})]\mu_{k+1}[\Theta] \sqsubseteq F_G[p(\bar{x})]\mu_{k+1}[\Theta \cap S_G[p(\bar{x})]]$$

2.4 Induction Step: $G = G_1, G_2$

Assume: $F_G[G_1]\mu_{k+1}[\Theta_1] \sqsubseteq F_G[G_1]\mu_{k+1}[\Theta_1 \cap S_G[G_1]]$

And: $F_G[G_2]\mu_{k+1}[\Theta_2] \sqsubseteq F_G[G_2]\mu_{k+1}[\Theta_2 \cap S_G[G_2]]$

Show: $F_G[G_1, G_2]\mu_{k+1}[\Theta] \sqsubseteq F_G[G_1, G_2]\mu_{k+1}[\Theta \cap S_G[G_1, G_2]]$

$$F_G[G_1, G_2]\mu_{k+1}[\Theta] = F_G[G_2]\mu_{k+1}(F_G[G_1]\mu_{k+1}[\Theta])$$

by assumption: $F_G[G_1]\mu_{k+1}[\Theta] \sqsubseteq F_G[G_1]\mu_{k+1}[\Theta \cap S_G[G_1]]$

hence: $F_G[G_2]\mu_{k+1}(F_G[G_1]\mu_{k+1}[\Theta]) \sqsubseteq F_G[G_2]\mu_{k+1}(F_G[G_1]\mu_{k+1}[\Theta \cap S_G[G_1]])$

by assumption:

$$F_G[G_2]\mu_{k+1}(F_G[G_1]\mu_{k+1}[\Theta \cap S_G[G_1]]) \sqsubseteq F_G[G_2]\mu_{k+1}((F_G[G_1]\mu_{k+1}[\Theta \cap S_G[G_1]]) \cap S_G[G_2])$$

by Lemma 1: $F_G[G_1]\mu_{k+1}[\Theta \cap S_G[G_1]] \cap S_G[G_2] = F_G[G_1]\mu_{k+1}[\Theta \cap S_G[G_1] \cap S_G[G_2]]$

hence: $F_G[G_2]\mu_{k+1}((F_G[G_1]\mu_{k+1}[\Theta \cap S_G[G_1]]) \cap S_G[G_2])$

$$= F_G[G_2]\mu_{k+1}(F_G[G_1]\mu_{k+1}[\Theta \cap S_G[G_1]]) \cap S_G[G_2])$$

$$= F_G[G_2]\mu_{k+1}[\Theta \cap S_G[G_1, G_2]]$$

therefore: $F_G[G_1, G_2]\mu_{k+1}[\Theta] \sqsubseteq F_G[G_1, G_2]\mu_{k+1}[\Theta \cap S_G[G_1, G_2]]$

Therefore since: (a) $F_G[G]\mu[\Theta \cap S_G[G]] \sqsubseteq F_G[G]\mu[\Theta]$

and (b) $F_G[G]\mu[\Theta] \sqsubseteq F_G[G]\mu[\Theta \cap S_G[G]]$,


QED

2.4.3 Proof of Theorem 2: For $\Theta \in Con^+$ and stratified $P = P_0 \cup \ldots \cup P_n$: $\Theta \subseteq D_G[G]\delta_P \Rightarrow |F_G[G]\mu_P[\Theta]| \leq 1$.

First notice that the following things hold:

1. $\Theta \subseteq (\Phi \rightarrow \Psi) \Rightarrow \Theta \cap \Phi \subseteq \Psi$
2. $\Theta \subseteq \cup \Theta \cap S_G[G]$
3. $F_G[G]\mu(\Theta) \subseteq \bigcup \Theta \cap S_G[G]$
4. $\overline{\Phi}(\Theta) \cap S_G[G]$
5. $\Theta \subseteq \overline{\Psi}(\Phi)$

This holds due to the following few lines of reasoning:

$$\Theta \subseteq \overline{\Phi}(\Theta) \text{ (since } \overline{\Phi} \text{ is extensive)}$$

if $\Theta \subseteq \overline{\Psi}(\Phi)$

then $\overline{\Psi}(\Theta) \subseteq \overline{\Psi}(\overline{\Psi}(\Phi))$

(by monotonicity of $\overline{\Phi}$)

then $\overline{\Psi}(\overline{\Phi}(\Theta)) = \overline{\Psi}(\overline{\Psi}(\overline{\Phi}(\Theta)))$

QED
(by monotonicity of $\downarrow \rho$)

\[ \downarrow \rho_{\bar{y}, \bar{y}} \bar{\rho}_{\bar{y}, \bar{x}} \text{ cancel out and } \forall \text{ is reductive, hence:} \]

\[ \downarrow \rho_{\bar{y}, \bar{y}} \bar{\rho}_{\bar{y}, \bar{x}} (\Theta) \subseteq \Phi \]

(6) $F_G[G][\mu(\Theta)] \subseteq F_G[G][\mu(\Theta) : \Theta])$

again for any $\mu$ constructed by application of $F_P[P]$ to $\mu_1$

(7) $\Theta_1 \subseteq \Theta_2 \Rightarrow |\Theta_1| \leq |\Theta_2|

Proof by nested induction on:

1. $\mu$,
2. structure of $G$:

1 Base Case: $\mu = \mu_1$

show: $\Theta \subseteq D_G[G] \delta_P \Rightarrow |F_G[G][\mu_1[\Theta]]| \leq 1$

Induction on structure of $G$:

1.1 Two Base Cases: (1) $G = \text{post}(\phi)$, (2) $G = \rho(\bar{x})$

(1) $G = \text{post}(\phi)$:

Show: $\Theta \subseteq D_G[G][\text{post}(\phi)] \delta_P \Rightarrow |F_G[G][\text{post}(\phi)] [\mu_1[\Theta]]| \leq 1$

$F_G[G][\text{post}(\phi)][\mu_1[\Theta]] = \text{trim}(\{\Phi \cap \Theta\})$

hence: $|F_G[G][\text{post}(\phi)][\mu_1[\Theta]]| = |\text{trim}(\{\Phi \cap \Theta\})| \leq 1$

(2) $G = \rho(\bar{x})$

Show: $\Theta \subseteq D_G[G][\rho(\bar{y})] \delta_P \Rightarrow |F_G[G][\rho(\bar{y})][\mu_1[\Theta]]| \leq 1$

$F_G[G][\rho(\bar{y})][\mu_1[\Theta]] = \downarrow \rho_{\bar{y}, \bar{y}} \bar{\rho}_{\bar{y}, \bar{x}} (\mu_1(\rho(\bar{y}) \downarrow \rho_{\bar{y}, \bar{y}} \bar{\rho}_{\bar{y}, \bar{x}} (\Theta))) \cap \Theta : []$

$\mu_1(\rho(\bar{y})) \downarrow \rho_{\bar{y}, \bar{y}} \bar{\rho}_{\bar{y}, \bar{x}} (\Theta) = []$

hence: $F_G[G][\rho(\bar{y})][\mu_1[\Theta]] = []$

hence: $|F_G[G][\rho(\bar{y})][\mu_1[\Theta]]| = |[]| = 0$

1.2 Induction Step:

$G = G_1, G_2$

Assume: $\Theta_1 \subseteq D_G[G_1] \delta_P \Rightarrow |F_G[G_1][\mu_1[\Theta_1]]| \leq 1$

And: $\Theta_2 \subseteq D_G[G_2] \delta_P \Rightarrow |F_G[G_2][\mu_1[\Theta_2]]| \leq 1$

Show: $\Theta \subseteq D_G[G] \delta_P \Rightarrow |F_G[G][\mu_1[\Theta]]| \leq 1$

$D_G[G] \delta_P = (S_G[G_2] \rightarrow D_G[G_1] \delta_P) \cap (S_G[G_1] \rightarrow D_G[G_2] \delta_P)$

$\Theta \subseteq D_G[G] \delta_P \Rightarrow \Theta \subseteq (S_G[G_1] \rightarrow D_G[G_2] \delta_P) \Rightarrow \Theta \cap S_G[G_1] \subseteq D_G[G_2] \delta_P$

$\Theta \subseteq D_G[G] \delta_P \Rightarrow \Theta \subseteq (S_G[G_2] \rightarrow D_G[G_1] \delta_P) \Rightarrow \Theta \cap S_G[G_2] \subseteq D_G[G_1] \delta_P$

$F_G[G][\mu_1(\Theta)] = F_G[G][\mu_1(\Theta \cap S_G[G])] \text{ (by Lemma 2)}$

$F_G[G][\mu_1(\Theta \cap S_G[G])] = F_G[G][\mu_1(\Theta \cap S_G[G])]

$\Theta \cap S_G[G_1, G_2] = \Theta \cap S_G[G_1] \cap S_G[G_2] \subseteq \Theta \cap S_G[G_2] \subseteq D_G[G_1] \delta_P$

hence by assumption: $|F_G[G_1][\mu_1(\Theta \cap S_G[G])]| \leq 1$

distinguish two cases:

(a) $|F_G[G_1][\mu_1(\Theta \cap S_G[G_1, G_2])| = 0$,

(b) $|F_G[G_1][\mu_1(\Theta \cap S_G[G_1, G_2])| = 1$
(a) $|F_G[G_1]|_{\mu_1}[\Theta \cap S_G[G_1, G_2]] = 0$
$F_G[G_1]_{\mu_1}[\Theta \cap S_G[G_1, G_2]] = \emptyset$
$F_G[G_2]_{\mu_1}[\Theta \cap S_G[G_1, G_2]] = F_G[G_2]_{\mu_1}[] = \emptyset$

hence: $|F_G[G_2]_{\mu_1}[\Theta \cap S_G[G_1, G_2]]| \leq 1$

by Lemma 2 (remembering $G = G_1, G_2$): $|F_G[G]_{\mu_1}[\Theta]| \leq 1$

(b) $|F_G[G_1]|_{\mu_1}[\Theta \cap S_G[G_1, G_2]] = 1$
$F_G[G_1]_{\mu_1}[\Theta \cap S_G[G_1, G_2]] = [\Psi]$

by Theorem 1: $\bigcup(F_G[G_1]_{\mu_1}[\Theta \cap S_G[G_1, G_2]])$

$
\subseteq \Theta \cap S_G[G_1, G_2] \cap S_G[G_1]$

$
\subseteq \Theta \cap S_G[G_1]$


hence: $\Psi \subseteq \Theta \cap S_G[G_1]$

hence by assumption: $|F_G[G_2]_{\mu_1}[\Psi]| \leq 1$

hence (again by Lemma 2): $|F_G[G_2]_{\mu_1}(F_G[G_1]_{\mu_1}[\Theta \cap S_G[G_1, G_2]])|$

$= |F_G[G]_{\mu_1}[\Theta \cap S_G[G_1, G_2]]|$

$= |F_G[G]_{\mu_1}[\Theta]| \leq 1$

2 Induction Step:
Assume: $X \subseteq D_G[H]_{\delta_P} \Rightarrow |F_G[H]_{\mu_k}[X]| \leq 1$
Show: $\Theta \subseteq D_G[G]_{\delta_P} \Rightarrow |F_G[G]_{\mu_{k+1}}[\Theta]| \leq 1$
where $\mu_{k+1} = F_P[P]_{\mu_k}$

Induction on structure of $G$:
2.1 Two base cases: (1) $G = \text{post}(\phi)$, (2) $G = p(\bar{x})$

(1) $G = \text{post}(\phi)$:
Show: $\Theta \subseteq D_G[\text{post}(\phi)]_{\delta_P} \Rightarrow |F_G[\text{post}(\phi)]_{\mu_{k+1}}[\Theta]| \leq 1$

$F_G[\text{post}(\phi)]_{\mu_{k+1}}[\Theta] = \text{trim}([\Psi \cap \Theta])$

hence: $|F_G[\text{post}(\phi)]_{\mu_{k+1}}[\Theta]| = |\text{trim}([\Psi \cap \Theta])| \leq 1$

(2) $G = p(\bar{x})$
Assume (without loss of generality): $p(\bar{y}) \leftrightarrow G_1; G_2, !, G_3; G_4 \in P$
Show: $\Theta \subseteq D_G[p(\bar{x})]_{\delta_P} \Rightarrow |F_G[p(\bar{x})]_{\mu_{k+1}}[\Theta]| \leq 1$

$F_G[p(\bar{x})]_{\mu_{k+1}}[\Theta] = \{F_G[G_3]_{\mu_k}[\Psi] \text{ if } F_G[G_2]_{\mu_k}[\rho_{\bar{y}, \bar{z}} \bar{g}([\theta])] = \Phi : \bar{\Psi}\}$

where $\bar{\Psi} = \left\{ F_G[G_3]_{\mu_k}[\Psi] \right\} \text{ if } F_G[G_2]_{\mu_k}[\rho_{\bar{y}, \bar{z}} \bar{g}([\theta])] = \Phi : \bar{\Psi}$

and: $\mu_{k+1}(p(\bar{y})) \rho_{\bar{y}, \bar{z}} \bar{g}([\theta]) = \bar{\Psi}(F_G[G_1]_{\mu_k}[\rho_{\bar{y}, \bar{z}} \bar{g}([\theta])] : \Psi)$

where $\bar{\Psi} = \left\{ F_G[G_3]_{\mu_k}[\Psi] \right\} \text{ if } F_G[G_2]_{\mu_k}[\rho_{\bar{y}, \bar{z}} \bar{g}([\theta])] = \Phi : \bar{\Psi}$

$|\bar{\Psi}(F_G[G_1]_{\mu_k}[\rho_{\bar{y}, \bar{z}} \bar{g}([\theta])] : \Psi)| = |F_G[G_1]_{\mu_k}[\rho_{\bar{y}, \bar{z}} \bar{g}([\theta])] : \bar{\Psi}|$

Show $\Theta \subseteq D_G[p(\bar{x})]_{\delta_P} \Rightarrow |F_G[G_1]_{\mu_k}[\rho_{\bar{y}, \bar{z}} \bar{g}([\theta])] : \bar{\Psi}| \leq 1$ in two steps:

1 Show that each component cannot be longer than 1:
1a Show: $\Theta \subseteq D_G[p(\bar{x})]_{\delta_P} \Rightarrow |F_G[G_1]_{\mu_k}[\rho_{\bar{y}, \bar{z}} \bar{g}([\theta])]| \leq 1$
1b Show: \( \Theta \subseteq D_G[p(\tilde{x})]_\delta p \Rightarrow |F_G[G_1]_{\mu_k} \Psi_{x,\tilde{y}}(\Theta)| \leq 1 \)

1c Show: \( \Theta \subseteq D_G[p(\tilde{x})]_\delta p \Rightarrow |F_G[G_3]_{\mu_k} \Phi| \leq 1 \)
where \( F_G[G_2]_{\mu_k} \Psi_{x,\tilde{y}}(\Theta) = \Phi : \tilde{\Phi} \)

2 Show that only one component can be longer than 0;
\( \Theta \subseteq D_G[p(\tilde{x})]_\delta p \Rightarrow -(|F_G[G_1]_{\mu_k} \Psi_{x,\tilde{y}}(\Theta)|) \neq 0 \land |\tilde{\Psi}| \neq 0 \)
This is done thus:

2a Show:
\( \Theta \subseteq D_G[p(\tilde{x})]_\delta p \Rightarrow -(|F_G[G_1]_{\mu_k} \Psi_{x,\tilde{y}}(\Theta)|) \neq 0 \land |F_G[G_4]_{\mu_k} \Psi_{x,\tilde{y}}(\Theta)| \neq 0 \)

2b Show:
where \( F_G[G_2]_{\mu_k} \Psi_{x,\tilde{y}}(\Theta) = \Phi : \tilde{\Phi} \)

1a Show: \( \Theta \subseteq D_G[p(\tilde{x})]_\delta p \Rightarrow |F_G[G_1]_{\mu_k} \Psi_{x,\tilde{y}}(\Theta)| \leq 1 \)
\( \Theta \subseteq D_G[p(\tilde{x})]_\delta p \Rightarrow \Theta \subseteq \Psi_{x,\tilde{y}}(D_G[G_1])_\delta p \)
hence (by (5) stated above): \( \Psi_{x,\tilde{y}}(\Theta) \subseteq D_G[G_1]_\delta p \)
hence by assumption: \( |F_G[G_1]_{\mu_k} \Psi_{x,\tilde{y}}(\Theta)| \leq 1 \)

1b Show: \( \Theta \subseteq D_G[p(\tilde{x})]_\delta p \Rightarrow |F_G[G_1]_{\mu_k} \Psi_{x,\tilde{y}}(\Theta)| \leq 1 \)
\( \Theta \subseteq D_G[p(\tilde{x})]_\delta p \Rightarrow \Theta \subseteq \Psi_{x,\tilde{y}}(D_G[G_1])_\delta p \)
hence (again by (5) above): \( \Psi_{x,\tilde{y}}(\Theta) \subseteq D_G[G_1]_\delta p \)
hence by assumption: \( |F_G[G_1]_{\mu_k} \Psi_{x,\tilde{y}}(\Theta)| \leq 1 \)

1c Show: \( \Theta \subseteq D_G[p(\tilde{x})]_\delta p \Rightarrow |F_G[G_3]_{\mu_k} \Phi| \leq 1 \)
where \( F_G[G_2]_{\mu_k} \Psi_{x,\tilde{y}}(\Theta) = \Phi : \tilde{\Phi} \)
\( \Theta \subseteq D_G[p(\tilde{x})]_\delta p \Rightarrow \Theta \subseteq \Psi_{x,\tilde{y}}(S_G[G_2])_\delta p \)
hence (again by (5) above): \( \Psi_{x,\tilde{y}}(\Theta) \subseteq (S_G[G_2])_\delta p \)
hence (by (1) stated above): \( \Psi_{x,\tilde{y}}(\Theta) \subseteq S_G[G_2] \)
by Theorem 1: \( \bigcup(\Phi : \tilde{\Phi}) = \bigcup(F_G[G_2]_{\mu_k} \Psi_{x,\tilde{y}}(\Theta)) \subseteq \Psi_{x,\tilde{y}}(\Theta) \subseteq S_G[G_2] \)
therefore (since \( \Phi \subseteq \bigcup(\Phi : \tilde{\Phi}) \)): \( \Psi_{x,\tilde{y}}(\Theta) \subseteq S_G[G_2] \)
by assumption: \( |F_G[G_3]_{\mu_k} \Phi| \leq 1 \)

2a Show:
\( \Theta \subseteq D_G[p(\tilde{x})]_\delta p \Rightarrow -(|F_G[G_1]_{\mu_k} \Psi_{x,\tilde{y}}(\Theta)|) \neq 0 \land |F_G[G_4]_{\mu_k} \Psi_{x,\tilde{y}}(\Theta)| \neq 0 \)
\( \Theta \subseteq D_G[p(\tilde{x})]_\delta p \Rightarrow \Theta \subseteq \Psi_{x,\tilde{y}}(S_G[G_1], S_G[G_4]) \)
hence (by (5) stated above): \( \Psi_{x,\tilde{y}}(\Theta) \subseteq \Psi_{x,\tilde{y}}(\Theta) \subseteq S_G[G_1], S_G[G_4] \)
hence (by (2) stated above):

\((\varphi_{x,y} \varphi_{x,y}(\Theta) \land S_G[G_1] = false) \lor (\varphi_{x,y} \varphi_{x,y}(\Theta) \land S_G[G_4] = false)\)
by Theorem 1: \(\varphi_{x,y} \varphi_{x,y}(\Theta) \land S_G[G_1] = false \Rightarrow \mathcal{F}_G[G_1][\mu_k \varphi_{x,y} \varphi_{x,y}([\Theta])] = \square\)
hence: \(\varphi_{x,y} \varphi_{x,y}(\Theta) \land S_G[G_1] = false \Rightarrow [\mathcal{F}_G[G_1][\mu_k \varphi_{x,y} \varphi_{x,y}([\Theta])] = 0\)
similarly: \(\varphi_{x,y} \varphi_{x,y}(\Theta) \land S_G[G_4] = false \Rightarrow \mathcal{F}_G[G_4][\mu_k \varphi_{x,y} \varphi_{x,y}([\Theta])] = \square\)
hence: \(\varphi_{x,y} \varphi_{x,y}(\Theta) \land S_G[G_1] = false \Rightarrow \mathcal{F}_G[G_1][\mu_k \varphi_{x,y} \varphi_{x,y}([\Theta])] = 0\)
therefore: \((\mathcal{F}_G[G_1][\mu_k \varphi_{x,y} \varphi_{x,y}([\Theta])] = 0) \lor ((\mathcal{F}_G[G_4][\mu_k \varphi_{x,y} \varphi_{x,y}([\Theta])] = 0)\)
hence: \(-((\mathcal{F}_G[G_1][\mu_k \varphi_{x,y} \varphi_{x,y}([\Theta])] = 0) \land (\mathcal{F}_G[G_4][\mu_k \varphi_{x,y} \varphi_{x,y}([\Theta])] = 0))\)

2b Show: \(\Theta \subseteq \mathcal{D}_G[p(x)] \delta_P \Rightarrow -((\mathcal{F}_G[G_1][\mu_k \varphi_{x,y} \varphi_{x,y}([\Theta]) = 0) \lor (\mathcal{F}_G[G_3][\mu_k \phi]\)) \neq 0)
where \(\mathcal{F}_G[G_2][\mu_k \varphi_{x,y} \varphi_{x,y}([\Theta]) = \Phi \land \Phi\)
\(\Theta \subseteq \mathcal{D}_G[p(x)] \delta_P \Rightarrow \theta \subseteq \varphi_{x,y} \varphi_{x,y}(\max(S_G[G_1], S_G[G_2, G_3]))\)
hence (again by (5) above): \(\varphi_{x,y} \varphi_{x,y}([\Theta]) \subseteq \max(S_G[G_1], S_G[G_2, G_3])\)

2.2 Induction Step:
\(G = G_1, G_2\):
Assume: \(\Theta_1 \subseteq \mathcal{D}_G[G_1] \delta_P \Rightarrow \mathcal{F}_G[G_1][\mu_{k+1} \Theta_1] \leq 1\)
And: \(\Theta_2 \subseteq \mathcal{D}_G[G_2] \delta_P \Rightarrow \mathcal{F}_G[G_2][\mu_{k+1} \Theta_2] \leq 1\)
Show: \(\Theta \subseteq \mathcal{D}_G[G] \Rightarrow \mathcal{F}_G[G][\mu_{k+1} \Theta] \leq 1\)
where \(\mu_{k+1} = \mathcal{F}_P[P][\mu_k\]

\(\mathcal{D}_G[G] \delta_P = (S_G[G_2] \rightarrow \mathcal{D}_G[G_1] \delta_P) \land (S_G[G_1] \rightarrow \mathcal{D}_G[G_2] \delta_P)\)
therefore if \(\Theta \subseteq \mathcal{D}_G[G] \delta_P\)
then \(\Theta \subseteq (S_G[G_1] \rightarrow \mathcal{D}_G[G_2] \delta_P)\)
and hence \(\Theta \subseteq S_G[G_1] \subseteq \mathcal{D}_G[G_2] \delta_P\)
similarly if \(\Theta \subseteq \mathcal{D}_G[G] \delta_P\)
then \(\Theta \subseteq (S_G[G_2] \rightarrow \mathcal{D}_G[G_1] \delta_P)\)
and hence \(\Theta \subseteq S_G[G_2] \subseteq \mathcal{D}_G[G_1] \delta_P\)
by Lemma 2: \(\mathcal{F}_G[G][\mu_{k+1} \Theta] = \mathcal{F}_G[G][\mu_{k+1} \Theta \land S_G[G]]\)
applying the definition of \(\mathcal{F}_G:\)
\(\mathcal{F}_G[G][\mu_{k+1} \Theta \land S_G[G]] = \mathcal{F}_G[G_2][\mu_{k+1} \mathcal{F}_G[G_1][\mu_{k+1} \Theta \land S_G[G_1, G_2]]\)
now notice that: \(\Theta \land S_G[G_1, G_2] = \Theta \land S_G[G_1] \land S_G[G_2] \subseteq S_G[G_2] \subseteq \mathcal{D}_G[G_1] \delta_P\)
hence by assumption: \(\mathcal{F}_G[G_1][\mu_{k+1} \Theta \land S_G[G_1, G_2]] \leq 1\)
distinguish two cases:
(a) $|\mathcal{F}_G[G_1][\mu_{k+1}(\Theta \cap S_G[G_1, G_2])| = 0,$
(b) $|\mathcal{F}_G[G_1][\mu_{k+1}(\Theta \cap S_G[G_1, G_2])| = 1$

(a) $|\mathcal{F}_G[G_1][\mu_{k+1}(\Theta \cap S_G[G_1, G_2])| = 0$
\[\mathcal{F}_G[G_1]\mu_{k+1}(\Theta \cap S_G[G_1, G_2]) \subseteq \]
\[\mathcal{F}_G[G_2]\mu_{k+1}(\Theta \cap S_G[G_1, G_2]) = \mathcal{F}_G[G_2]\mu_{k+1}[] = []
\]
hence: $|\mathcal{F}_G[G]\mu_{k+1}(\Theta \cap S_G[G_1, G_2])| \leq 1$
hence by Lemma 2 (remembering $G = G_1, G_2$): $|\mathcal{F}_G[G]\mu_{k+1}(\Theta)| \leq 1$

(b) $|\mathcal{F}_G[G_1]\mu_{k+1}(\Theta \cap S_G[G_1, G_2])| = 1$
$\mathcal{F}_G[G_1]\mu_{k+1}(\Theta \cap S_G[G_1, G_2]) = \Psi$
therefore: $\bigcup(\mathcal{F}_G[G_1]\mu_{k+1}(\Theta \cap S_G[G_1, G_2])) = \Psi$
by Theorem 1: $\Psi \subseteq \Theta \cap S_G[G_1, G_2] \cap S_G[G_1] \subseteq \Theta \cap S_G[G_1]$
hence since $\Theta \cap S_G[G_1] \subseteq D_G[G_2] \delta_P$ (see above): $\Psi \subseteq D_G[G_2] \delta_P$
hence by assumption: $|\mathcal{F}_G[G_2]\mu_{k+1}(\Psi)| \leq 1$
hence (again using Lemma 2): $|F_1[G_2]\mu_{k+1}(\mathcal{F}_G[G_1]\mu_{k+1}(\Theta \cap S_G[G_1, G_2]))|$
$= |\mathcal{F}_G[G]\mu_{k+1}(\Theta \cap S_G[G_1, G_2])|$
$= |\mathcal{F}_G[G]\mu_{k+1}(\Theta)| \leq 1$
QED

2.5 Abstraction Proofs

2.5.1 Proposition 1: If $\Theta_1 \subseteq \gamma_F(f_1)$ and $\gamma_F(f_1) \subseteq \Theta_2$ then $\gamma_F(f_1 \Rightarrow f_2) \subseteq \Theta_1 \rightarrow \Theta_2$

$\gamma_F(f_1 \Rightarrow f_2)$
$= \bigcup\{\gamma_F(f) \mid f \models f_1 \Rightarrow f_2\}$
$= \bigcup\{\Theta \mid \alpha_F(\Theta) \vDash f_1 \Rightarrow f_2\}$
$= \bigcup\{\Theta \mid (\alpha_F(\Theta) \vDash f_1) \Rightarrow (\alpha_F(\Theta) \vDash f_2)\}$
$= \bigcup\{\Theta \mid (\Theta \subseteq \gamma_F(f_1)) \Rightarrow (\Theta \subseteq \gamma_F(f_2))\}$
$\subseteq \bigcup\{\Theta \mid (\Theta \subseteq \Theta_1) \Rightarrow (\Theta \subseteq \Theta_2)\}$
$= \bigcup\{\Theta \mid (\Theta \subseteq \Theta_1 \cap \Theta_2) \vee (\Theta \subseteq \Theta_1)\}$
$= \bigcup\{\Theta \mid (\Theta \subseteq \Theta_1 \cap \Theta_2) \cup (\text{Con} \setminus \Theta_1)\}$
$= \bigcup\{\Theta \mid \Theta \cap \Theta_1 \subseteq \Theta_2\}$
$= \Theta_1 \rightarrow \Theta_2$

2.5.2 Proposition 2: $\gamma_F(\mu_{x\bar{\alpha}}(\Theta_1^{DK}, \Theta_2^{DK})) \subseteq \mu_{x\bar{\alpha}}(\Theta_1, \Theta_2)$

Proof:
First notice that by the definition of the Galois connection (i.e. of $\gamma()$ and $\alpha()$ the following:
$\gamma_F(\mu_{x\bar{\alpha}}(\Theta_1^{DK}, \Theta_2^{DK})) \subseteq \mu_{x\bar{\alpha}}(\Theta_1, \Theta_2)$
is equivalent to: $\alpha_F(\Psi) \models \mu_{x\bar{\alpha}}(\Theta_1^{DK}, \Theta_2^{DK})$ $\rightarrow \Psi \subseteq \mu_{x\bar{\alpha}}(\Theta_1, \Theta_2)$

Now: $\alpha_F(\Psi) \models \mu_{x\bar{\alpha}}(\Theta_1^{DK}, \Theta_2^{DK})$ iff for each clause in $\alpha_F(\Psi)$ there is a clause in $\mu_{x\bar{\alpha}}(\Theta_1^{DK}, \Theta_2^{DK})$ that
is entailed by it, ie:
$\forall \psi \in \Psi \exists Y \subseteq \text{vars}(\bar{x}), (\forall \theta_1 \in \Theta_1^{DK}, \forall \theta_2 \in \Theta_2^{DK}$
Theorem 3: \( \forall \psi \in \gamma(x)(\Theta_{1}^{DK}, \Theta_{2}^{DK}) \) contains only positive (i.e., non-negated) literals, only the positive literals entailed by \( \alpha_x(\psi) \) are relevant.

Now, the positive literals entailed by \( \alpha_x(\psi) \) are exactly \( \text{vars}(\bar{x}) \cap \text{fix}(\psi) \).

Therefore: \( \psi \in \gamma(x)(\Theta_{1}^{DK}, \Theta_{2}^{DK}) \) if \( \exists \gamma \subseteq (\text{vars}(\bar{x}) \cap \text{fix}(\psi)), (\forall \theta_1 \in \Theta_{1}^{DK}, \forall \theta_2 \in \Theta_{2}^{DK}.(\exists \gamma(\theta_1) \land \exists \gamma(\theta_2) = \text{false})) \)

Now observe that the following three things hold:

1. \( \forall \phi \in \Phi, \exists \phi' \in \Phi^{DK}(\phi \models \phi') \)
2. \( ((f_1 \models f_1') \land (f_2 \models f_2') \land (f'_1 \land f'_2 = \text{false})) \rightarrow f_1 \land f_2 = \text{false} \)
3. \( \phi \models \phi' \rightarrow \exists \gamma(\phi) = \exists \gamma(\phi') \)

Therefore from \( \forall \theta_1' \in \Theta_{1}^{DK}, \forall \theta_2' \in \Theta_{2}^{DK}.(\exists \gamma(\theta_1') \land \exists \gamma(\theta_2') = \text{false}) \)

it follows: \( \forall \theta_1 \in \Theta_{1}, \forall \theta_2 \in \Theta_{2}.(\exists \gamma(\theta_1) \land \exists \gamma(\theta_2) = \text{false}) \)

And thus: \( \exists \gamma(\Theta_{1}) \land \exists \gamma(\Theta_{2}) = \{\text{false}\} \)

Hence the following entailment holds:

\( \forall \phi. (\forall \gamma(\Theta_{1}) \land \exists \gamma(\Theta_{2}) = \{\text{false}\}) \models \exists \gamma(\Theta_{1}) \land \exists \gamma(\Theta_{2}) = \{\text{false}\} \)

Therefore: \( \forall \phi \in \gamma(x)(\Theta_{1}^{DK}, \Theta_{2}^{DK}) \rightarrow \phi \in \text{max}(\Theta_{1}, \Theta_{2}) \)

From which it follows: \( \gamma(x)(\Theta_{1}^{DK}, \Theta_{2}^{DK}) \subseteq \text{max}(\Theta_{1}, \Theta_{2}) \)

2.6 Theorem 3: \( \forall i \in \mathbb{N} : \gamma_{\text{vars}(G)}(D_G^o[\bar{G}][\delta_i^o]) \subseteq D_G^o[\bar{G}][\delta_i] \) where \( \delta_i^o/\delta_i \) are the results of \( i \) applications of \( D_G^o[P]/D_P[P] \) to \( \delta_i^o/\delta_T \) respectively.

Proof by nested induction on:

1. \( i \)
2. the structure of \( G \):

notice first that: \( \gamma_{\text{vars}(\bar{x})}(\rho_{\bar{y}, \bar{x}}^o \exists \bar{y} \exists f(\bar{x})) \subseteq \rho_{\bar{y}, \bar{x}} \exists \bar{y} \exists f(\gamma_{\text{vars}(\bar{y})}(f)) \)

1 Base Case: \( i = 0 \)

\( \delta_0^o = \delta_0^T \)

Show: \( \gamma_{\text{vars}(G)}(D_G^o[\bar{G}][\delta_0^o]) \subseteq D_G^o[\bar{G}][\delta_T] \)

Induction on structure of \( G \):

1.1 Two base cases: (1) \( G = \text{post}(\phi) \), (2) \( G = p(\bar{x}) \)

(1) \( G = \text{post}(\phi) \)

\( \gamma_{\text{vars}(\phi)}(D_G^o[\text{post}(\phi)][\delta_T^o]) = \gamma_{\text{vars}(\phi)}(\text{true}) = \downarrow\{\text{true}\} = D_G^o[\text{post}(\phi)][\delta_T] \)

hence: \( \gamma_{\text{vars}(\phi)}(D_G^o[\text{post}(\phi)][\delta_T^o]) \subseteq D_G^o[\text{post}(\phi)][\delta_T] \)

(2) \( G = p(\bar{x}) \)

\( \gamma_{\text{vars}(\bar{x})}(D_G^o[p(\bar{x})][\delta_T^o]) = \gamma_{\text{vars}(\bar{x})}(\rho_{\bar{y}, \bar{x}} \exists \bar{y} \exists f(\text{true})) \subseteq \rho_{\bar{y}, \bar{x}} \exists \bar{y} \exists f(\gamma_{\text{vars}(\bar{y})}(f)) \)
\[ \gamma_{\text{vars}}(G) \subseteq \mathcal{D}_G[G] \]

1.2 Induction step: \( G = G_1, G_2 \)
Assume: \( \gamma_{\text{vars}}(G_{1/2})(\mathcal{D}_G^{a}[G_{1/2}]\delta^a_{G_{1/2}}) \subseteq \mathcal{D}_G[G_{1/2}] \delta_{G_{1/2}} \)

\[ \mathcal{D}_G[G_1, G_2 \delta^a_{G_1, G_2}] \]

Assume (without loss of generality): \( \gamma_{\text{vars}}(G_1, G_2)(\mathcal{D}_G^a[G_1, G_2] \delta^a_{G_1, G_2}) \)
Show: \( \mathcal{D}_G[G_1, G_2 \delta^a_{G_1, G_2}] \subseteq \mathcal{D}_G[G_1, G_2 \delta^a_{G_1, G_2}] \)

where \( \delta_{k+1} = \mathcal{D}_P[P_\delta_k] \) and \( \delta^a_{k+1} = \mathcal{D}_P[P_\delta^a_k] \)

Induction on structure of \( G \):
2.1 Two base cases: (1) \( G = \text{post}(\phi) \), (2) \( G = p(\bar{x}) \)

(1) \( G = \text{post}(\phi) \)
\[ \gamma_{\text{vars}}(\text{post}(\phi)) \subseteq \mathcal{D}_G[\text{post}(\phi)] \delta_{k+1} \]

(2) \( G = p(\bar{x}) \)
Assume (without loss of generality): \( p(\bar{y}) \leftarrow G_1; G_2, !, G_3; G_4 \in P \)

\[ \mathcal{D}_G[p(\bar{x})] \subseteq \mathcal{D}_G[p(\bar{x})] \delta^a_{k+1} \]

2 Induction step: \( i = k + 1 \)
Assume: \( \gamma_{\text{vars}}(G)(\mathcal{D}_G^a[G] \delta^a_k) \subseteq \mathcal{D}_G[G] \delta_k \)
Show: \( \gamma_{\text{vars}}(G)(\mathcal{D}_G^a[G] \delta^a_{k+1}) \subseteq \mathcal{D}_G[G] \delta_{k+1} \)

hence: \( \gamma_{\text{vars}}(G)(\mathcal{D}_G^a[G] \delta^a_{k+1}) \subseteq \mathcal{D}_G[G] \delta_{k+1} \)

Induction on structure of \( G \):
2.1 Two base cases: (1) \( G = \text{post}(\phi) \), (2) \( G = p(\bar{x}) \)

(1) \( G = \text{post}(\phi) \)
\[ \gamma_{\text{vars}}(\text{post}(\phi)) \subseteq \mathcal{D}_G[\text{post}(\phi)] \delta_{k+1} \]

(2) \( G = p(\bar{x}) \)
Assume (without loss of generality): \( p(\bar{y}) \leftarrow G_1; G_2, !, G_3; G_4 \in P \)
\[ \rho_{\gamma, \delta} \gamma(S_G[G_1] \delta_k \cap (S_G[G_2] \rightarrow D_G[G_3] \delta_k) \cap D_G[G_1] \delta_k \]
\[ \cap \mu(G[S_G[G_1], S_G[G_1]]) \]
\[ \cap \mu(G[S_G[G_1], S_G[G_2], G_3])) \]
\[ = D_G[p(\bar{x})] \delta_{k+1} \]

2.2 Induction step: \( G = G_1, G_2 \)

Assume: \( \gamma_{\text{vars}}(G_1/2)(D^\alpha_G[G_1, G_2] \delta_{k+1}^\alpha) \subseteq D_G[G_1/2] \)
again, notice that: (1) \( A \subseteq B \Rightarrow \gamma_B(f) \subseteq \gamma_A(f) \)
and: \( \text{vars}(G_1, G_2) = \text{vars}(G_1) \cup \text{vars}(G_2) \)
and hence: \( (2^1) \text{vars}(G_1) \subseteq \text{vars}(G_1, G_2) \)
and similarly: \( (2^2) \text{vars}(G_2) \subseteq \text{vars}(G_1, G_2) \)

\[ \gamma_{\text{vars}}(G_1, G_2) \left( (S^\alpha_G[G_1, G_2] \cap (S_G[G_1] \Rightarrow D^\alpha_G[G_1, G_2]) \right) \right) \cap (S_G[G_1] \Rightarrow D^\alpha_G[G_2] \delta_{k+1}^\alpha)) \]
\[ \subseteq \gamma_{\text{vars}}(G_1, G_2) \left( (S^\alpha_G[G_1, G_2] \cap (S_G[G_1] \Rightarrow D^\alpha_G[G_1, G_2])) \right) \cap \gamma_{\text{vars}}(G_1, G_2) \left( (S^\alpha_G[G_1] \Rightarrow D^\alpha_G[G_2] \delta_{k+1}^\alpha)) \right) \]
(by monotonicity i.e. \( \gamma_{\text{vars}}(G_1, G_2)(\bar{f}_1 \land \bar{f}_2) \subseteq \gamma_{\text{vars}}(G_1, G_2)(\bar{f}_1) \))
\[ \subseteq \gamma_{\text{vars}}(G_2) \left( (S^\alpha_G[G_2]) \cap \gamma_{\text{vars}}(G_1) \left( (S^\alpha_G[G_1]) \right) \right) \]
\[ \cap \gamma_{\text{vars}}(G_1) \left( (S^\alpha_G[G_1]) \right) \]
(by Proposition 1 and Proposition 3 and the induction assumption)
\[ \subseteq \gamma_{\text{vars}}(G_2) \left( (S^\alpha_G[G_2]) \cap \gamma_{\text{vars}}(G_1) \left( (S^\alpha_G[G_1]) \right) \right) \]
\[ \cap \gamma_{\text{vars}}(G_1) \left( (S^\alpha_G[G_1]) \right) \]
(by (1), (2^1) and (2^2) above)
\[ \subseteq S_G[G_2] \cap D_G[G_1] \delta_k \cap S_G[G_1] \rightarrow D_G[G_2] \delta_{k+1} \]
\[ = D_G[G_1, G_2] \delta_{k+1} \]
QED

**Acknowledgements**  This work was inspired by the cuts that are ravaging the UK, but funded by a ACM-W scholarship and a DTA bursary. We thank Lunjin Lu and Samir Genaim for discussions that provided the backdrop for this work. We thank Michel Billaud for sending us copies of his early work and for his comments on the wider literature. We also thank an anonymous reviewer for invaluable help with the proofs in the appendix.

**References**