Theory Propagation and Rational-Trees

Ed Robbins
School of Computing
University of Kent
er209@kent.ac.uk

Jacob M. Howe
School of Informatics
City University London
jacob@soi.city.ac.uk

Andy King
School of Computing
University of Kent
a.m.king@kent.ac.uk

ABSTRACT
SAT Modulo Theories (SMT) is the problem of determining the satisfiability of a formula in which constraints, drawn from a given constraint theory $T$, are composed with logical connectives. The DPLL($T$) approach to SMT has risen to prominence as a technique for solving these quantifier-free problems. The key idea in DPLL($T$) is to closely couple unit propagation in the propositional part of the problem with theory propagation in the constraint component. In this paper, it is demonstrated how reification provides a natural way for orchestrating this in the setting of logic programming. This allows an elegant implementation of DPLL($T$) solvers in Prolog. The work is motivated by a problem in reverse engineering, that of type recovery from binaries. The solution to this problem requires an SMT solver where the theory is that of rational-trees. This theory is not supported in off-the-shelf SMT solvers, but realised as unification in many Prolog systems. The solver is benchmarked against a number of type recovery problems, and compared against a lazy-basic SMT solver built on PicoSAT.

Categories and Subject Descriptors
[Software notations and tools]: Constraint and logic languages; [Software and application security]: Software reverse engineering; [Semantics and reasoning]: Program analysis

General Terms
Theory of Computation, Software and its Engineering

Keywords
SAT solving, reverse engineering

1. INTRODUCTION
DPLL-based SAT solvers have advanced to the point where they can rapidly decide the satisfiability of structured problems that involve thousands of variables. SAT Modulo Theories (SMT) seeks to extend these ideas beyond propositional formulae to formulae that are constructed from logical connectives that combine constraints drawn from a given underlying theory. This section introduces the motivating problem of type recovery and explains why it leads to work on theory propagation in a Prolog SMT solver.

1.1 Type recovery with SMT
The current work is motivated by reverse engineering and the problem of type recovery from binaries. Reversing executable code is of increasing relevance for a range of applications:

- Exposing flaws and vulnerabilities in commercial software, especially prior to deployment in government or industry [9, 13];
- Reuse of legacy software without source code for guaranteed compliance with hardware IO and/or timing behaviour, for example, for hardware drivers [7] or control systems [4];
- Understanding the operation of, and threat posed by, viruses and other malicious code by anti-virus companies [40].

An important problem in reverse engineering is that of type recovery [35]. A fragment of binary code will almost certainly have multiple source code equivalents, will contain a variety of complex addressing schemes, and during compilation will have lost most, if not all, of the type information explicit in the original source code. Additionally, container-like entities, analogous to high level source code variables and objects, cannot be readily extracted from binary code. The recovery of variables and their types is an essential component of reverse engineering, which makes understanding the semantics of the program considerably easier.

This paper observes that type recovery can be formulated as an SMT problem over rational-trees, a theory that in the context of type checking is referred to as circular unification [31]. Circular unification allows recursive types to be discovered in which a type variable can be unified with a term containing it. The use of rational-trees for type inference is not a new idea [31], but its application to the recovery of recursive types from an executable is far from straightforward because each instruction can be assigned many different types. Many SMT solvers include the theory of equality logic over uninterpreted functors [24, 37] which is strictly weaker than circular unification and cannot capture recursive types. Unfortunately the theory of rational-trees is not
1.2 SMT solving with lazy-basic

One straightforward approach to SMT solving is to apply the so-called lazy-basic technique which decouples SAT solving from theory solving. To illustrate, consider the SAT formula \( f = (x \leq -1 \lor -x \leq -1) \land (y \leq -1 \lor -y \leq -1) \) and the SAT formula \( g = (p \lor q) \land (r \lor s) \) that corresponds to its propositional skeleton. In the skeleton, the propositional variables \( p, q, r \) and \( s \), respectively, indicate whether the theory constraints \((x \leq -1), (-x \leq -1), (y \leq -1)\) and \((-y \leq -1)\) hold. In this approach, a model is found for \((p \lor q) \land (r \lor s)\), for instance, \( \{ p \Rightarrow true, q \Rightarrow true, r \Rightarrow true, s \Rightarrow false \} \).

Then, from the model, a conjunction of theory constraints \((x \leq -1) \land (-x \leq -1) \land (y \leq -1) \land (-y \leq -1)\) is constructed, with the polarity of the constraints reflecting the truth assignment. This conjunction is then tested for satisfiability in the theory component. In this case it is unsatisfiable, which triggers a diagnostic stage. This amounts to finding a conjunct, in this case \((x \leq -1) \land (-x \leq -1)\), which is also unsatisfiable, that identifies a source of the inconsistency. From this conjunct, a blocking clause \((\neg p \lor \neg q)\) is added to \( g \) to give \( g' \) which ensures that conflict between the theory constraints is never encountered again. Then, solving the augmented propositional formula \( g' \) might, for example, yield the model \( \{ p \Rightarrow false, q \Rightarrow true, r \Rightarrow true, s \Rightarrow true \} \), from which a second clause \((\neg r \lor \neg s)\) is added to \( g' \). Any model subsequently found, for instance, \( \{ p \Rightarrow false, q \Rightarrow true, r \Rightarrow true, s \Rightarrow false \} \), will give a conjunction that is satisfiable in the theory component, thereby solving the SMT problem.

The lazy-basic approach is particularly attractive when combining an existing SAT solver with an existing decision procedure, for instance, a solver provided by a constraint library. By using a foreign language interface a SAT solver can be invoked from Prolog [8] and a constraint library can be used to check satisfiability of the conjunction of theory constraints. A layer of code can then be added to diagnose the source of any inconsistency. This provides a simple way to construct an SMT solver that compares very favourably with the coding effort required to integrate a new theory into an existing open source SAT solver. The latter is normally a major undertaking and often can only be achieved in conjunction with the expert who is responsible for maintaining the solver. Furthermore, few open source solvers are actively maintained. Thus, although one might expect implementing a new theory to be merely an engineering task, it is actually far from straightforward.

Prolog has rich support for implementing decision procedures for theories, for instance, attributed variables [14, 15]. (Attributed variables provide an interface between Prolog and a constraint solver by permitting logical variables to be associated with state, for instance, the range of values that a variable can possibly assume.) Several theories come prepackaged with many Prolog systems. This raises the questions of how to best integrate a theory solver with a SAT solver, and how powerful an SMT solver written in a declarative language can actually be. This motivates further study of the coupling between the theory and the propositional component of the SAT solver which goes beyond the lazy-basic approach, to the roots of logic programming itself.

The equation Algorithm = Logic + Control [26] expresses the idea that in logic programming algorithm design can be decoupled into two separate steps: specifying the logic of the problem, classically as Horn clauses, and orchestrating control of the sub-goals. The problem of satisfying a SAT formula is conceptually one of synchronising activity between a collection of processes where each process checks the satisfiability of a single clause. Therefore it is perhaps no surprise that control primitives such as delay declarations [36] can be used to succinctly specify the watched literal technique [34]. In this technique, a process is set up to monitor two variables of each clause. To illustrate, consider the clause \((x \lor y \lor \neg z)\). The process for this clause will suspend on two of its variables, say \( x \) and \( y \), until one of them is bound to a truth-value. Suppose \( x \) is bound. If \( x \) is bound to \( true \) then the clause is satisfied, and the process terminates; if \( x \) is bound to \( false \), then the process suspends until either \( y \) or \( z \) is bound. Suppose \( z \) is subsequently bound, either by another process or by labelling. If \( z \) is true then \( y \) is bound to \( true \) since otherwise the clause is not satisfied; if \( z \) is \( false \) then the clause is satisfied and the process closes down without inferring any value for \( y \). Note that in these steps the process only waits on two variables at any one time. Unit propagation is at the heart of SAT solving and when implemented by watched literals combined with backtracking, the resulting solver is efficient enough to solve some non-trivial propositional formulae [16, 17, 19]. In addition to issues of performance the correctness of this approach has been examined [12]. To summarise, Prolog not only provides constraint libraries, but also the facility to implement a succinct SAT solver [19]. The resulting solver can be regarded as a glass box, as opposed to a black one, which allows a solver to be extended to support, among other things, new theories and theory propagation.

1.3 SMT solving with theory propagation

The lazy-basic approach to SMT alternates between SAT solving and checking whether a conjunction of theory constraints is satisfiable which, though having conceptual and implementation advantages, is potentially inefficient. With a glass box solver it is possible to refine this interaction by applying theory propagation. In theory propagation, the SAT solving and theory checking are interleaved. The solver not only checks the satisfiability of a conjunction of theory constraints, but decides whether a conjunction of some constraints entails or disentails others. Returning to the earlier example, observe that \((x \leq -1) \land (-x \leq -1)\) is unsatisfiable, hence for the partial assignment \( \{ p \Rightarrow true \} \) it follows that \((x \leq -1)\) holds in the theory component, therefore \((-x \leq -1)\) is disentailed and the assignment can be extended to \( \{ p \Rightarrow true, q \Rightarrow false \} \). Theory propagation is essentially the coordination problem of scheduling unit propagation with the simultaneous checking of whether theory constraints are entailed or disentailed. This paper shows how this synchronisation can be realised straightforwardly in Prolog, again using control primitives. The resulting solver is capable of solving some non-trivial problems and outperforms an SMT solver constructed from PicoSAT [3] and a Prolog coded theory solver using the lazy-basic approach.

1.4 Contributions

This paper shows how to integrate theory propagation and unit propagation in Prolog using reification and thereby realise an SMT solver in Prolog which can solve type recovery...
problems. Reification is a constraint handling mechanism in which a constraint is augmented with a boolean variable that indicates whether the constraint is entailed (implied by the store) or disentailed (is inconsistent with the store). Building on this mechanism, the paper makes the following contributions:

- A framework for using reification as a mechanism to realise theory propagation is presented. The idea is simple in hindsight and can be realised straightforwardly in Prolog. The simplicity of the code contrasts with the investment required to integrate a theory into an existing open source SMT solver.

- This framework is realised for two theories. The first theory is that of rational-trees [32], where the control provided by block and when-declarations can realise reification. Efficient rational-tree unification [21] is integral to many Prolog systems, hence the theory part of the solver is provided essentially for free. The second theory is that of quantifier-free linear real arithmetic, where CLP(R) provides a decision procedure for the theory part of the solver; reification is achieved using a combination of delay declarations and entailment checking.

- Theory propagation for rational-trees provides the key motivation for the paper. Standard SMT packages do not include the theory of rational-trees, but SMT problems over rational-trees arise in reverse engineering, in particular type recovery. It is demonstrated that an elegant Prolog-based solver is capable of recovering types for a range of binaries. It is also shown how the failed literal technique [29] is simply realised in Prolog to optimise the search. The solver is benchmarked on these type recovery problems and also compared against an SMT solver constructed from PicoSAT using the lazy-basic approach.

- Cutting through all of these contributions, the paper also argues that SMT has a role in type recovery, indeed an SMT formula is a natural medium for expressing the disjunctive nature of the types that arise in reverse engineering.

2. MOTIVATION: APPLICATION IN TYPE RECOVERY

During compilation code is translated to low level operations on registers and memory addresses, and all type information is lost. When source code is not available, type information is of great use to reverse engineers in determining the operation of a program, and tooling for recovery of high level types is of significant utility. The problem is hard, since the typing of most assembly instructions can be interpreted in multiple ways, and progress on the problem has been comparatively slow [2, 6, 28, 30, 35, 39], stopping short of recovering recursive types.

Consider the problem of inferring types for the registers in the following x86 assembly code function for summing the elements in a linked list of type struct A { int value; struct A *next}. Note this function is based on Mycroft’s Register Transfer Language (RTL) example [35].

```
1   mov edx, [esp+0x4]
2   mov eax, 0x0
3   loop: test edx, edx
4       jz end
5       add eax, [edx]
6       mov edx, [edx+0x4]
7       jmp loop
8   end:  ret
```

The function is simple: first edx is set to point at the first list item (from the argument carried at [esp + 0x4]) and eax, the accumulator, is initialised to 0 (lines 1 and 2). In the loop body the value of the item is added to eax (line 5) and edx is set to point to the next item by dereferencing the next field from [edx + 0x4] (line 6). This repeats until a NULL pointer is found by the test on line 3, whereupon execution jumps to end and the function returns.

Before typing the function, indirect addressing is simplified by introducing new operations on fresh intermediate variables. This reduction ensures that indirect addressing only ever occurs on mov instructions, thus simplifies the constraints on all other instructions. Registers are then broken into live ranges by transforming into Single Static Assignment (SSA) form. This gives each variable a new index whenever it is written to, and joins variables at control flow merge points with $\phi$ functions [10].

The listing below shows the result of applying these transformations:

```
1   mov A1, esp0
2   add A2, 0x4
3   mov edx1, [A2]
4   mov eax1, 0x0
5   loop: mov (eax2, edx2),
6          $\phi(\langle eax1, edx1, (eax3, edx3)\rangle)$
7       test edx2, edx2
8       jz end
9       mov B1, [edx2]
10      add edx3, B1
11      mov C1, edx2
12      add C2, 0x4
13      mov edx3, [C2]
14      jmp loop
15      end:  ret
```

Rational-tree expressions [20], constraints describing unification of terms and type variables, are now derived for each instruction. These are similar to the disjunctive constraints described by [35] for RTL, but include a memory model that tracks pointer manipulation by representing memory in ‘pointed to’ locations as a 3-tuple. The type of the specific location being pointed to is the middle element, the first element is a list of types for the types preceding the location, and the last the types for the bytes succeeding.

The lists are open, as indicated by the ellipsis (…), since the areas of memory extending to either side are unknown. For example, consider add on line 11. This gives rise to two constraints, one for each possible meaning of the code:

\[
(TC_2 = \text{basic}(_2, \text{int}, 4) \land TC_1 = TC_2) \\
\lor (TC_1 = \text{ptr}([\ldots, \beta_0, \beta_1, \beta_2, \beta_3, \beta_4, \ldots]) \land \\
TC_2 = \text{ptr}([\ldots, \beta_0, \beta_1, \beta_2, \beta_3, \ldots]))
\]

The first clause of the disjunction states that $C_2$ is of basic type, specifically a four byte integer (derived from the register size) with unknown signedness (as indicated by a sign parameter that is an uninstantiated variable), the result of
adding 4 to C₁, which has the same type. This is disjoint from the second clause, that asserts that C₁ is a pointer to an unknown type β₀, whose address is incremented by 4 by the **add** operation so that its new instance, C₂, points to another location of type β₄. Observe how Tₑ₁ prescribes types of objects that follow the object of type β₀ in memory whereas Tₑ₂ details types of objects that precede the object of type β₄. If further information is later added to Tₑ₃ due to unification it will propagate into Tₑ₁, and vice-versa, and thus aggregate types analogous to C structs are derived.

The table below shows all constraints generated for the program. Note that some type variables have been relaxed to α, indicating an uninstantiated variable, so as to simplify the presentation of the types. The complete problem is described by the conjunction of these constraints. Type recovery then amounts to solving the constraints such that the type equations remain consistent, whilst also ensuring that the propositional skeleton of the problem is satisfied.

<table>
<thead>
<tr>
<th>Line</th>
<th>Generated Constraints</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( Tₐ₁ = Tₑ₅₀ )</td>
</tr>
<tr>
<td>2</td>
<td>( (Tₑ₂ = \text{basic}([\text{int}, 4]) \land Tₐ₁ = Tₑ₄₂) \lor ) ( Tₐ₁ = \text{ptr}([\text{...}, α₀, [...α₁,..., α₄]]) \land ) ( Tₑ₄₂ = \text{ptr}([\text{...}, α₀, [...α₁,..., α₄]]) )</td>
</tr>
<tr>
<td>3</td>
<td>( Tₑ₃₀ = \text{ptr}([\text{...}, Tₑ₃₈, [...α₄,...]]) )</td>
</tr>
<tr>
<td>4</td>
<td>( Tₑ₃₈ = \text{basic}([\text{int}, 4]) \lor Tₑ₃₈ = \text{ptr}([\text{...}, α₂,...]) )</td>
</tr>
<tr>
<td>5</td>
<td>( Tₑ₃₉ = Tₑ₃₈ \land Tₑ₃₂₀ = Tₑ₃₂₁ \lor ) ( Tₑ₃₂₁ = \text{basic}([\text{int}, 4]) \land ) ( Tₑ₃₉ = \text{ptr}([\text{...}, α₃,...]) \land ) ( Tₑ₃₂₀ = \text{basic}([\text{int}, 4]) \land ) ( Tₑ₃₈ = \text{ptr}([\text{...}, α₄,...]) \land ) ( Tₑ₃₀ = \text{ptr}([\text{...}, α₀,...]) )</td>
</tr>
<tr>
<td>6</td>
<td>( Tₑ₃₂₁ = Tₑ₃₂₀ )</td>
</tr>
<tr>
<td>11</td>
<td>( (Tₑ₃₂ = \text{basic}([\text{int}, 4]) \land Tₑ₃₁ = Tₑ₃₂) \lor ) ( Tₑ₃₁ = \text{ptr}([\text{...}, α₅,...]) \land ) ( Tₑ₃₂ = \text{ptr}([\text{...}, α₇,...]) \land ) ( Tₑ₃₀ = \text{ptr}([\text{...}, α₈,...]) )</td>
</tr>
</tbody>
</table>

For the register corresponding to **struct** A, constraint solving will derive a recursive type:

\( Tₑ₃₈ = \text{ptr}([\text{...}, \text{basic}([\text{int}, 4]), [...α₇,..., Tₑ₃₉, [...α₄,...]]) \) which requires rational-tree unification.

Observe that there may be multiple solutions; in fact the problem outlined above has two solutions, which differ in typing **eax₁**, **eax₂** and **eax₃**. The first correctly infers that they are (like B₁) integers of size 4 bytes, while the second defines them as pointers to an unknown type, \( \text{ptr}([\text{...}, α₅,...]) \). Both solutions have the following typings in common:

\( B₁ = \text{basic}([\text{int}, 4]) \)
\( Tₑ₃₁ = Tₑ₃₂ = Tₑ₃₉ = Tₑ₃₁ = \) \( \text{ptr}([\text{...}, \text{basic}([\text{int}, 4]), [...α₇,..., Tₑ₃₉, [...α₄,...]]) \)
\( Tₑ₃₂ = \text{ptr}([\text{...}, α₅,...]) \land \) \( Tₑ₃₀ = \text{ptr}([\text{...}, α₈,...]) \land \) \( Tₑ₃₈ = \text{ptr}([\text{...}, Tₑ₃₉, [...α₄,...]]) \)

The second solution is equivalent to typing **eax** as **void** and performing addition using pointer arithmetic. In the wider context of this program, this solution is removed by constraints derived from the **main()** function.

### 3. SMT AND THEORY PROPAGATION

#### 3.1 SAT solving and unit propagation

The Boolean satisfiability problem (SAT) is the problem of determining whether for a given Boolean formula, there is a truth assignment to the variables of the formula under which the formula evaluates to true. Most recent SAT solvers are based on the Davis, Putnam, Logemann, Loveland (DPLL) algorithm [11] with watched literals [34]; this includes the solver in [18] that this paper extends.

At the heart of the DPLL approach is unit propagation. Let \( f \) be a propositional formula in CNF over a set of propositional variables \( X \). Let \( \theta : X \rightarrow \{ \text{true}, \text{false} \} \) be a partial (truth) function. Unit propagation examines each clause in \( f \) to deduce a truth assignment \( \theta' \) that extends \( \theta \) and necessarily holds for \( f \) to be satisfiable. For example, suppose \( f = (\neg v \lor z) \land (u \lor w \land v) \land (\neg w \lor y \lor \neg z) \) so that \( X = \{ u, v, w, x, y, z \} \) and \( \theta \) is the partial function \( \theta = \{ x \rightarrow \text{true}, y \rightarrow \text{false} \} \). In this instance for the clause \((\neg v \lor z) \) to be satisfiable, hence \( f \) as a whole, it is necessary that \( z \rightarrow \text{true} \). Moreover, for \((\neg w \lor y \lor \neg z) \) to be satisfiable, it follows that \( w \rightarrow \text{false} \). The satisfiability of \((u \lor \neg v \lor w) \) depends on two unknowns, \( u \) and \( v \), hence no further information can be deduced from this clause. Therefore \( \theta' = \theta \cup \{ w \rightarrow \text{false}, z \rightarrow \text{true} \} \).

Searching for a satisfying assignment proceeds as follows: starting from an empty truth function \( \theta \), an unassigned variable occurring in \( f \), \( x \), is selected and \( x \rightarrow \text{true} \) is added to \( \theta \). Unit propagation extends \( \theta \) until either no further propagation is possible or a contradiction is established. In the first case, if all clauses are satisfied then \( f \) is satisfied, else another unassigned variable is selected. In the second case, \( x \rightarrow \text{false} \) is added to \( \theta \); if this fails search backtracks to a previous assignment. Further details can be found in [18, 44].

#### 3.2 SMT solving, the lazy-basic approach

SAT modulo theories (SMT) gives a general scheme for determining the satisfiability of problems consisting of a formula over atomic constraints in some theory \( T \), whose set of literals is denoted \( \Sigma \) [38, 43]. The scheme separates the propositional skeleton – the logical structure of combinations of theory literals – and the meaning of the literals. A bijective encoder mapping \( e : \Sigma \rightarrow X \) associates each literal with a unique propositional variable. Then the encoder mapping \( e \) is lifted to theory formulae, using \( e(\phi) \) to denote the propositional skeleton of a theory formula \( \phi \).

Consider the theory of quantifier-free linear real arithmetic where the constants are numbers, the functions are interpreted as addition and subtraction, and the predicates include equality, disequality and both strict and non-strict inequalities. The problem of checking the entailment \((a < b) \land (a = 0 \lor a = 1) \land (b = 0 \lor b = 1) \equiv (a + b = 1) \) amounts to determining that the theory formula \( \phi = (a < b) \land (a = 0 \lor a = 1) \land (b = 0 \lor b = 1) \land (a + b = 1) \) is not satisfiable. For this problem, the set of literals is \( \Sigma = \{ a < b, ..., a + b = 1 \} \).
Suppose, in addition, that the encoder mapping is defined:
\[
e(a < b) = x, \quad e(a = 0) = y, \quad e(a = 1) = z, \\
e(b = 0) = u, \quad e(b = 1) = v, \quad e(a + b = 1) = w
\]

Then the propositional skeleton of \( \phi \), given \( e \), is \( e(\phi) = x \land (y \lor z) \land (u \lor v) \land \neg w \). A SAT solver gives a truth assignment \( \theta \) satisfying the propositional skeleton. From this, a conjunction of theory literals, \( T h_2(\theta, e) \), is constructed. Where \( \ell \in \Sigma \), a conjunct if \( \theta(e(\ell)) = \text{true} \) and \( \neg \ell \) if \( \theta(e(\ell)) = \text{false} \). The subscript will be omitted when \( \Sigma \) refers to all literals in a problem. This problem is passed to a solver for the theory that can determine satisfiability of conjunctions of constraints. Either satisfiability or unsatisfiability is determined, in the latter case the SAT solver is asked for further satisfying truth assignments. This formulation is known as the lazy-basic approach and details on its Prolog implementation can be found in [19].

### 3.3 SMT, the DPLL(\( T \)) approach

The approach detailed in the previous section finds complete satisfying assignments to the SAT problem given by the propositional skeleton before computing the satisfiability of the theory problem \( T h(\theta, e) \). Another approach is to couple the SAT problem and the theory problem more tightly by determining constraints entailed by the theory and propagating the bindings back into the SAT problem. This is known as theory propagation and is encapsulated in the DPLL(\( T \)) approach. Figure 1 gives a recursive formulation of DPLL(\( T \)) deriving of Algorithm 11.2.3 from [27]. A more general formulation of DPLL(\( T \)) might replace lines (11)-(15) with a conflict analysis step that would encapsulate not just the approach presented, but also backjumping and clause learning heuristics. However, the key component of DPLL(\( T \)) is the interleaving of unit and theory propagation and the choice of conflict analysis is an orthogonal issue. The instantiation to chronological backtracking presented in Figure 1 was chosen to match the implementation work.

The first argument to the function DPLL(\( T \)) is a Boolean formula \( f \), its second a partial truth assignment, \( \theta \), and its third an encoder mapping, \( e \). In the initial call, \( f \) is the propositional skeleton of input the problem, \( e(\phi) \), and \( \theta \) is empty. DPLL(\( T \)) returns a truth assignment if the problem is satisfiable and constant \( \bot \) otherwise.

The call to propagate is the key operation. The function returns a pair consisting of a truth assignment and \( res \) taking value \( \top \) or \( \bot \) indicating the satisfiability of \( f \) and \( T h(\theta, e) \). The fourth argument to propagate is a set of theory literals, \( D \), and the function begins by extending the truth assignment by assigning propositional variables identified by the encoder mapping. Next, unit propagation as described in section 3.1 is applied. The deduction function then infers those literals that hold as a consequence of the extended truth assignment. The function returns a pair consisting of a set of theory literals entailed by \( T h(\theta_2, e) \) and a flag \( res \) whose value is \( \bot \) if \( T h(\theta_2, e) \) or \( \theta_2 \) is inconsistent and \( \top \) otherwise. The function propagate calls itself recursively until no further propagation is possible. After deduction returns, if \( f \) is not yet satisfied then a further truth assignment is made and DPLL(\( T \)) calls itself recursively.

The key difference between the lazy-basic approach and the DPLL(\( T \)) approach is that where the lazy-basic approach computes a complete satisfying assignment to the variables of the propositional skeleton before investigating the satisfiability of the corresponding theory formula, the DPLL(\( T \)) approach incrementally investigates the consistency of the posted constraints as propositional variables are assigned. Further, it identifies literals, \( \ell \), such that \( T h(\theta, e) \models \ell \), allowing \( e(\ell) \) to be assigned during propagation. It is the interplay between propositional satisfiability, posting constraints and the consistency of the store \( T h(\theta, e) \) that is at the heart of this investigation.

### 4. PROPAGATION AND REIFICATION

This section provides a framework for incorporating theory propagation into the propagation framework of the SAT solver from [19]. The approach is based on reifying theory literals with logical variables. As will be illustrated in subsequent sections, this allows the use of the control provided by delay declarations to realise theory propagation. The integration is almost seamless since the base SAT solver is also realised using logical variables and by exploiting the control provided by delay declarations.

#### 4.1 Theory propagation

There are three major steps in setting up a DPLL(\( T \)) solver for some problem \( \phi \): setting up the encoder map \( e \), linking each theory literal in a problem with a logical variable; posting theory propagators (adding constraints) that reify the theory literals with the logical variables provided by \( e \); posting the SAT problem defined by the propositional skeleton \( e(\phi) \), then solving this problem. The code in Figure 2 describes the high level call to the solver.

**Set up.**

Where \( \text{Prob} \) is an SMT formula over some theory, let \( \text{lit}(\text{Prob}) \) be the set of literals occurring in \( \text{Prob} \). TheoryLiteral is a list of pairs \( \ell \leftrightarrow e(\ell) \) (or rather, \( \ell \leftrightarrow e(\ell) \)), where \( \ell \in \text{lit}(\text{Prob}) \), that defines the encoder mapping \( e \). Skeleton represents the propositional skeleton of the problem, \( e(\phi) \). Vars represents the set of variables \( e(\ell) \), where \( \ell \in \text{lit}(\text{Prob}) \). The role of the predicate setup\((+,-,-,-)\) is, given \( \text{Prob} \), to instantiate the remaining variables.

**Theory propagators.**

The role of post\(-\)theory is to set up predicates to reify each theory literal. The control on these predicates is key; the predicates need to be blocked until either \( e(\ell) \) is assigned, or the literal (or its negation) is entailed by the constraint store \( T h(\theta, e) \). That is, the predicate for \( \ell \leftrightarrow -e(\ell) \) will propagate in one of four ways:

- If \( T h(\theta, e) \models \ell \) then \( e(\ell) \to \text{true} \)
- If \( T h(\theta, e) \models \neg \ell \) then \( e(\ell) \to \text{false} \)
- If \( e(\ell) = \text{true} \) then the store is updated to \( T h(\theta \cup \{e(\ell) \to \text{true}\}, e) \)
- If \( e(\ell) = \text{false} \) then the store is updated to \( T h(\theta \cup \{e(\ell) \to \text{false}\}, e) \)

**Boolean propagators.**

The role of post\(-\)boolean is to set up propagators for the SAT part of the problem \( e(\phi) \). This is a call to prob\(-\)setup as described in [19]. Search is then driven by assignments to the variables using elim\_vars.
function DPLL(T)(f: CNF formula, θ : truth assignment, e : Σ → X)
begin
(θ3, res) := propagate(f, θ, e, ∅);
if (is-satisfied(f, θ3)) then
return θ3;
else if (res = ⊥) then
return ⊥;
else
    x := choose-free-variable(f, θ3);
    (θ4, res) := DPLL(T)(f, θ3 ∪ {x → true}, e);
    if (res = ⊤) then
        return θ4;
    else
        return DPLL(T)(f, θ3 ∪ {x → false}, e);
endif
end

function propagate(f: CNF formula, θ : truth assignment, e : Σ → X, D : set of theory literals)
begin
θ1 := θ ∪ {e(ℓ) → true | ℓ ∈ D ∩ Σ} ∪ {e(ℓ) → false | ¬ℓ ∈ D ∩ ℓ ∈ Σ};
θ2 := θ1 ∪ unit-propagation(f, θ1);
(D, res) := deduction(Th(θ2, e));
if (D = ∅ ∨ res = ⊥)
    return (θ2, res);
else
    return propagate(f, θ2, e, D);
endif
end

Figure 1: Recursive formulation of the DPLL(T) algorithm

dpll_t(Prob):-
    setup(Prob, TheoryLiterals, Skeleton, Vars),
    post_theory(TheoryLiterals),
    post_boolean(Skeleton),
    elim_var(Vars).

Figure 2: Interface to the DPLL(T) solver

Implementing the interface provided by predicates setup and post_theory, together with the SAT solver from [19] results in a DPLL(T) SMT solver. Note that the propagators posted for the theory and Boolean components are intended to capture the spirit of the function propagate from Figure 1. Indeed, the integration between theory and Boolean propagation is even tighter than the algorithm indicates. Rather than performing unit propagation to completion, then performing theory propagation, then repeating, here the assignment of a Boolean variable is immediately communicated to the theory. This tactic is known as immediate propagation and is a natural consequence of using Prolog’s control to implement propagators. Immediate propagation does away with the need to analyse failure to determine an unsatisfiable core when a set of theory constraints is unsatisfiable, but attracts a cost in monitoring the entailment status of the theory literals.

4.2 Labelling strategies

The solvers presented in [19] maintain Boolean variables in a list and elim_vars assigns them values in the order in which they occur; the list has typically been ordered by the number of occurrences of the variables in the SAT instance before the search begins, the most frequently occurring assigned first. This tactic is straightforward to accommodate into a solver coded in Prolog. The desire for improved performance motivates the adoption of more sophisticated heuristics for variable assignment. Although orthogonal to the theme of theory propagation, the description of the SMT solver would be incomplete without explanation of labelling.

One classic strategy for labelling that is also straightforward to incorporate into a solver written in a declarative language is to rank variables by their number of occurrences in clauses of minimal size [23]. This associates a weight to each unbound variable according to the number of its occurrences in the unsatisfied clauses of the (Boolean) problem. The ranking weights variables with fewer unbound literals less heavily than those in clauses with a greater number of unbound literals. A variable with greatest weight is selected for labelling, the aim being to assign one that is more likely to lead to propagation.

A refinement of this idea is to apply lookahead [29] in conjunction with this labelling tactic. Each variable with greatest weight, and therefore each candidate for labelling, is speculatively assigned a truth value. For example, if X is assigned true and this results in failure, then in order to satisfy the propositional formula (skeleton) then X must be assigned false. Likewise, if failure occurs when X is assigned false then X must be true. Moreover, if one variable can be
assigned using lookahead, then often so can others, hence this tactic is repeatedly applied until no further variables can be bound. Thus lookahead is tried before any variable is assigned by search.

Scoping this activity over the variables of greatest weight limits the overhead of lookahead. The net effect is to direct search away from variable assignments that will ultimately fail. Lookahead can be considered to be dual of clause learning since the former seeks to avoid inconsistency by considering assignments that are still to be made, whereas the latter diagnoses an inconsistency from an assignment that has previously been made. The case for lookahead versus learning has been studied [29], but in a declarative context, particularly one where backtracking is supported, lookahead has been studied [29], but in a declarative context, learning has been studied [29], but in a declarative context, lookahead has been studied [29], but in a declarative context.

The code in Figure 4 demonstrates the use of delay to realise theory propagation over rational-tree constraints via reification. An SMT problem over rational-trees consists of Boolean combinations of theory literals \( \ell \). The call to setup/4 will instantiate TheoryLiterals to a list of pairs of the form \( \ell - e(\ell) \); the propositional skeleton and a list of the \( e(\ell) \) variables are also produced. In the following, a labelled literal eqn(Term1, Term2)-X is discussed. post_theory sets up propagators for each theory literal in two steps. theory_wait propagates from the theory constraints into the Boolean variables.

theory_wait uses the built-in control predicate when/2, which blocks the goal in its second argument until the first argument evaluates to true. In this instance the condition ?-(Term1, Term2) is true either if Term1 and Term2 are identical, or if the terms cannot be unified. That is, if Term1=Term2 is entailed by the store then theory_prop is called and assigns X=true. Similarly, if the constraint is not consistent with the store, then Term1 and Term2 cannot be unified and again theory_prop reflects this by assigning X=false. In the opposite direction, bool_wait communicates assignments made to Boolean variables to the theory literals. The predicate is blocked on the instantiation of the logical variables, waking when they become true or false. When true the constraint must hold so Term1 and Term2 are unified. When false, it is not possible for the two terms to be unified, hence the constraint is discarded and the call to bool_wait succeeds. Note that it is not possible to post a constraint that asserts that two terms cannot be unified, since the control predicate dif/2 is defined as:

\[
dif(X, Y) :- \text{when} (?)=(X, Y), X \neq Y.
\]

That is, it blocks until either \( X \) and \( Y \) are identical or they cannot be unified, then tests whether or not they are identical. Hence dif/2 acts as a test, rather than a propagating constraint. Consistency of the store is maintained by theory_wait; if X=false and the constraint is discarded, then later it is determined that Term1=Term2, theory_wait will attempt to unify \( X \) with true, which will fail. Finally, post_boolean sets up the propositional skeleton for the solver from [18].

6. INSTANTIATION FOR LINEAR REAL ARITHMETIC

Many Prolog systems come with the CLP(R) constraints package, which can determine consistency of conjunctions of linear arithmetic constraints. This decision procedure makes quantifier-free linear real arithmetic a sensible theory for the solver. The challenge is to implement reification for the constraints, an operation not directly supported.

The code in Figure 5 demonstrates the integration of linear real arithmetic as realised by CLP(R) into the DPLL(T) scheme. It assumes that the input problem has been normalised so that all the constraint predicates are drawn from \( =, =< \) and \(<\). The propagators, theory_wait, are blocked on two variables. The first of these is the labelling variable \( e(C) \); if this is instantiated, the appropriate constraint is posted. To complete the reification, the propagators need
function findcore (e = [t₁ ↦ x₁, ..., tₙ ↦ xₙ] : Σ → X, f : CNF formula, c : int, core : Σ → X)
begin
  if (e = [])
    return core;
  else if (c = 0)
    core′ := [t₁ ↦ x₁, ..., tₙ ↦ xₙ] ∪ core;
    findcore([t₂ ↦ x₂, ..., tₙ₋₁ ↦ xₙ₋₁], f, ⌊ n−1 2 ⌋, core');
  else
    i := 1; j := n;
    if (¬DPLL(T)(f, ∅, [tᵢ₊₁ ↦ xᵢ₊₁, ..., tₙ ↦ xₙ] ∪ core))
      i := c + 1;
    if (¬DPLL(T)(f, ∅, [tᵢ ↦ xᵢ, ..., tₙ₋₁ ↦ xₙ₋₁] ∪ core))
      j := n − c;
    if (c = 1)
      c′ := 0;
    else
      c′ := ⌊ c+1 2 ⌋;
      endif
    endif
    findcore([tᵢ ↦ xᵢ, ..., tⱼ ↦ xⱼ], f, c′, core);
  endif
end

Figure 3: Finding an unsatisfiable core

post_theory([],).
post_theory([eqn(Term1,Term2)-X|Rest]) :-
  setup_reify(X, Term1, Term2),
  post_theory(Rest).

setup_reify(X, Term1, Term2) :-
  bool_wait(X, Term1, Term2),
  theory_wait(X, Term1, Term2).

:- block bool_wait(−, ?, ?).
bool_wait(true, Term1, Term2) :-
  Term1 = Term2, !.
bool_wait(false, _Term1, _Term2).

theory_wait(X, Term1, Term2) :-
  when(?=(Term1, Term2),
    theory_prop(X, Term1, Term2)).

theory_prop(X, Term1, Term2) :-
  Term1 =TERM 2 ->
    X = true
  ;
    X = false
  .

Figure 4: Theory propagation for rational-tree constraints

to detect the entailment of the linear constraint (or its negation). This can be achieved using the built-in entailed/1, however the control for ensuring that this is called at an appropriate time is less obvious.

Once a new constraint has been posted (or once the constraint store has changed) other constraints or their negations might be entailed and this needs to be detected and propagated. The communication between the propagators to capture this is achieved with the second argument to theory_wait. Each propagator is set with its second argument the same logical variable (Y in the code) and the propagators are blocked on this second argument. When a constraint is posted, Y is instantiated, Y = prop(_). This wakes all active propagators which either propagate or block again on the new variable. An alternative approach, which would invoke the propagators less frequently, would be to only wake up the active propagators for those constraints that share a variable with the posted constraint.

It should be emphasised, however, that although a linear solver is interesting for self-contained Prolog applications, this theory is supported by a number of off-the-shelf SMT solvers; the approach presented in this paper is primarily designed for constraint theories that are unavailable in standard SMT distributions.

7. EXPERIMENTAL RESULTS

The DPLL(T) solver for rational-trees has been coded in SICStus Prolog 4.2.1, as described in section 5. Henceforth this will be called the Prolog solver. To assess the solver it has been applied to a benchmark suite of 84 type recovery problems, its target application. The first eight benchmarks are drawn from compilations at different optimisation levels of three small programs manufactured to check their types against those derived by the solver. These benchmarks are designed to check that the inferred types match against those prescribed in the source file, and also assess the robustness of
post_theory(TheoryLiterals):-
    setup_reify(TheoryLiterals, _).

setup_reify([], _).
setup_reify([C-V|Cs], Y) :-
    negate(C, NegC),
    theory_wait(V, Y, C, NegC),
    setup_reify(Cs, Y).

negate(X =< Y, X > Y).
negate(X = Y, X =\= Y).

next_var(Y, Z) :-
    var(Y), !,
    Y = Z.
next_var(prop(Y), Z) :-
    next_var(Y, Z).

:- block theory_wait(-, - , ?, ?).

theory_wait(V, Y, C, NegC),
    V = false, !,
    {NegC}, Y = prop(_).
theory_wait(V, Y, C, NegC),
    V = true, !,
    {C}, Y = prop(_).
theory_wait(V, Y, C, NegC),
    nonvar(Y),
    entailed(C), !,
    V = true.
theory_wait(V, Y, C, NegC),
    nonvar(Y),
    entailed(NegC), !,
    V = false.
theory_wait(V, Y, C, NegC),
    next_var(Y, U),
    theory_wait(V, U, C, NegC).

Figure 5: Theory propagation for linear real arithmetic

For the Prolog solver.

On no occasion is the hybrid solver faster than the Prolog solver, which suggests that a succinct implementation of theory propagation is more powerful than deploying an off-the-shelf SAT solver as a black box in combination with a handcrafted theory solver using the lazy-basic approach.

It can be observed in Table 1 that many of the problems are unsatisfiable. For these problems an explanation for a type conflict is returned rather than a satisfying type assignment. As a strength test of the solver these problems are good since the exhaustive search required to demonstrate unsatisfiability is more demanding than search for a first satisfying assignment. There are two results that require discussion. Benchmark 4 has an unsatisfiable core of 26 constraints, whereas most cores have less than 10 constraints.

This bug has been fixed in the forthcoming SICStus 4.3.
This explains why it is relatively slow. Benchmark 7 has timed out, a reminder that large SMT problems can be hard to solve.

Note that the time require for type recovery is sensitive to optimisation level, though it is not obvious why different optimisation levels impact on the difficulty of the SMT instance, apart from the obvious effect on code size.

For the unsatisfiable problems, a core of unsatisfiable constraints is calculated using multiple calls to the DPLL(T) solver as indicated. This core can be used to diagnose unsatisfiability, in turn allowing the analysis to be refined to return meaningful information despite the initial result.

In the benchmarks unsatisfiability is typically owing to no \texttt{nop} instructions such as \texttt{nop \{rax+rax+0x0\}}. This instruction does nothing, but has been generated by the compiler with an encoded operand in order to make it a specific size for optimal performance. The indirect addressing is broken down and constraints generated as follows:

\begin{verbatim}
mov $A_1, rax_1$ $T_{A_1} = T_{rax_1}$
add $A_2, rax_1$ \begin{align*}
T_{A_2} &= \text{basic}(_{\text{int,}4}) \land T_{A_1} = T_{A_2} \land T_{rax_1} = T_{A_2} \\
T_{A_1} &= \text{ptr}([\ldots, a_1, \ldots]) \land T_{A_1} = \text{basic}(_{\text{int,}4}) \\
T_{A_2} &= \text{ptr}([\ldots, a_3, \ldots]) \land T_{A_2} = \text{basic}(_{\text{int,}4}) \\
T_{rax_1} &= \text{ptr}([\ldots, a_4, \ldots])
\end{align*}
\end{verbatim}

\texttt{mov A_3, [A_2]} $T_{A_2} = \text{ptr}([\ldots, T_{A_3}, \ldots, \ldots])$

\texttt{nop A_3}
The final constraint states that $A_1$ must have pointer type, hence those for the `add` dictate that one of $A_1$ and `rax` must be of basic type, and the other a pointer; however, the first constraint says they have the same type, so the system is inconsistent.

Another unexpected source of inconsistency is the hard-coded pointer addresses sometimes found in `mov` instructions. These are often addresses of strings included in the binary, but also include constructor and destructor lists, added by the linker for construction and destruction of objects. For example, the instruction `mov ebx, 0x605e38` appears in the `cksum` binary, and moves the address of a string into `ebx`, resulting in the constraint $T_{ebx} = \text{basic}_{\text{int},4}$. Later however, `ebx` is dereferenced, which implies that it is a pointer, and conflicts with the earlier inference.

Quite apart from the disjunctive nature of constraints, the sheer number of x86 instructions pose an engineering challenge when writing a type recovery tool; indeed the constraint generator module has taken longer to develop than both SMT solvers together. Moreover, as the above two examples illustrate, type conflicts stem from type interactions between different instructions which makes the type conflicts difficult to anticipate. The result produced from the solver is either a successful recovery of types, or a core of inconsistent types, both of which can be achieved sufficiently quickly. Since the core is typically small, it is of great utility in pinpointing omissions in the type generation phase. It seems attractive to augment the solver with a domain specific language for expressing and editing the type constraints so that they can be refined, if necessary, by a user.

9. CONCLUSIONS AND FUTURE WORK

This paper has presented a DPLL($T$) SMT solver coded in Prolog for two theories – rational-tree unification and quantifier-free linear real arithmetic. The motivation for this work is the need for an SMT solver over rational-tree unification in order to recover types from x86 binaries; with Prolog providing a decision procedure for rational-tree unification the integration with the SAT solver in [19] is a natural development. The effectiveness of the approach has been demonstrated by the successful application of the solver to a suite of type recovery problems.

The solver can be extended by providing decision procedures for further theories. Finite domain solvers, such as SICStus CLP(FD), often allow reified constraints [5], hence finite domain constraints might appear a good candidate to incorporate into the DPLL($T$) framework. Unfortunately, finite domain constraint solvers typically maintain stores that are potentially inconsistent, hence without labelling (an unattractive step) a decision procedure for conjunctions of theory constraints is not readily available.

The approach to theory propagation described in this paper is not necessarily tied to DPLL-based SAT solvers and future work is to describe how to integrate it into a generalisation [42] of Stålmarck's proof procedure [41]. Other future work is to add certification, as in [1]. That is, for unsatisfiable instances not only is the result returned, but also a demonstration of unsatisfiability that can be determined by a small trusted computing base. Another line of inquiry will be to investigate how to systematically relax the SMT instance so that a type assignment can be always found, without manual intervention, even in the presence of conflicting constraints. MaxSMT techniques seem to be well-suited to this task.

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11. REFERENCES


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11. REFERENCES


