State and Parameter Estimation for Nonlinear Delay Systems Using Sliding Mode Techniques

Xing-Gang Yan, Sarah K. Spurgeon and Christopher Edwards

***Abstract*—In this paper, a class of time varying delay nonlinear systems is considered where both parametric uncertainty and structural uncertainty are involved. The uncertain parameters are embedded in the system nonlinearly. The bound on the structural uncertainty takes nonlinear form and is time delayed. A sliding mode observer is proposed to estimate the system state and an adaptive law is proposed to estimate the unknown parameters simultaneously. Using the Lyapunov-Razuminkhin approach, sufficient conditions are developed such that the error system is uniformly ultimately bounded. A simulation on a bioreactor system shows the effectiveness of the approach.**

***Index Terms*—adaptive estimation, nonlinear system, time delay, sliding mode, state estimation.**

I. Introduction

Time delay widely exists in reality and is frequently a source of instability. Sometimes even a small delay may affect the system performance greatly. The study of time delay systems is mainly based on the Lyapunov Razumikhin approach and the Laypunov Krasovskii approach [3], [15]. As pointed out in [3], there is perhaps a general preference to use Lyapunov Krasovskii functionals for delay-independent criteria and Lyapunov Razumikhin functions for delay dependent results. However, the Lyapunov Razumikhin approach does not impose restrictions on the derivative of the time delay [6] and is a powerful tool for systems involving time-varying delay, specifically when the time-varying delay is nondifferentiable or uncertain [15], although the approach usually leads to slightly conservative results [6], [9]. For a delay system, imposing a sliding mode dynamics without considering delay effects may lead to unstable or chaotic behaviour or a high level of chattering [8]. Therefore, the study of time delay system is very important.

Recently, sliding mode approaches have been successfully applied for control of time delay systems [12], [15], [20]. However, the application of sliding mode techniques to the observer problem is much less mature – especially for time delay systems [17]. Results concerning observer design for time delay systems using sliding mode techniques are very few. Niu *et al* proposed a sliding mode observer for a class of linear systems with matched nonlinear uncertainties [12] where the sliding mode observers are mainly designed for control purposes, and thus strong limitations are unavoidably required to guarantee that the theoretical proofs are tractable and the closed-loop controlled systems have the desired performance. A high order sliding mode observer was given for a class of systems with special structure in [4] but time delay is not considered. Jafarov proposed a sliding mode observer for both delayed and non-delayed systems in [8] where only matched uncertainty and matched nonlinearities are considered. In the very limited literature available for sliding mode observer design for time delay systems, it is usually required that the distribution matrix of the uncertainty satisfies a strong structural condition (see, e.g.[12], [8]) and the uncertain parameters appear linearly or affinely (see, e.g.[21], [18]).

In this paper, a robust observer is designed for nonlinear time delay systems based on sliding mode techniques. The unknown parameters are embedded in the system in a completely nonlinear way, and are estimated using an adaptive law. The only limitation on the structural uncertainty is that a bound on the uncertainty is known, which is employed in the design to reduce the effects of the uncertainty. A variable structure dynamical system is designed to estimate the system states. Then, a sliding surface is proposed for the error system between the system considered and the dynamical system which forms the observer. The associated sliding mode dynamics, which are time delayed and nonlinear, are studied using a Razuminkhin Lyapunov approach, and a reachability condition is given under which the error system is driven to the sliding surface. Mild conditions are developed to guarantee that the error system is uniformly ultimately bounded. Unlike the existing work, it is not required that the uncertain parameters appear linearly or affinely, and strong structural conditions are not required. The bound on the uncertainty has general nonlinear form and is time delayed. Simulation results reflect the effectiveness of the approach proposed.

Notation: For a square matrix A, $A > 0$ denotes a symmetric positive definite matrix, and $\lambda_{\min}(A)$ ($\lambda_{\max}(A)$) denotes the minimum (maximum) eigenvalue of A. For matrices $A_1, \ldots, A_m$, the symbol $\text{diag}\{A_1, \ldots, A_m\}$ denotes a block diagonal matrix. The symbol I_n represents the nth order unit matrix and $\mathcal{R}^+$ represents the set of nonnegative real numbers. The set of $n \times m$ real matrices will be denoted by $\mathcal{R}^{n\times m}$. The Lipschitz constant of a function f will be written as $\mathcal{L}_f$. Finally, $\|\cdot\|$ denotes the Euclidean norm or its induced norm.

II. System Description and Assumptions

Consider a nonlinear time delay system described, in suitable coordinates, by

$$\begin{aligned}\dot{x}(t) &= Ax(t) + f(x(t), x(t-d), u(t)) + g(y(t), u(t), \theta) \quad (1)\\ &\quad +\Delta f(t, x(t), x(t-d))\\ y(t) &= Cx(t), \quad (2)\end{aligned}$$

where $x \in \mathcal{R}^n$, $u \subset \mathcal{U} \in \mathcal{R}^m$ ($\mathcal{U}$ is an admissible control set) and $y \in \mathcal{Y} \subset \mathcal{R}^p$ ($\mathcal{Y}$ is the output space) are the state variables, inputs and outputs respectively; $A \in \mathcal{R}^{n\times n}$ and $C \in \mathcal{R}^{p\times n}$ are constant matrices of appropriate dimension with C being of full row rank, and $\theta \in \mathcal{R}^r$ are unknown constants. The nonlinear function $f(\cdot)$ is known, and the term $\Delta f(\cdot)$ represents all the structural uncertainty including modeling errors and external disturbances which satisfy

$$\|\Delta f(t, x(t), x(t-d))\| \le \rho(t, x(t), x(t-d)) \quad (3)$$

where $\rho(\cdot)$ is known and Lipschitz with respect to $x(t)$ and $x(t-d)$ for all $t \in \mathcal{R}^+$; $d := d(t)$ is the time-varying delay which is assumed to be known, nonnegative and bounded in $\mathcal{R}^+$, and thus $\overline{d} := \sup_{t\in\mathcal{R}^+}\{d(t)\} < \infty$. The initial condition associated with the delay is given by

$$x(t) = \phi(t), \qquad t \in [-\overline{d}, 0], \qquad \phi \in \mathcal{C} \quad (4)$$

where $\mathcal{C}$ denotes the space of continuous functions mapping $[-\overline{d}, 0]$ into R^n. Note that system (2) explicitly involves the output y and is described by a linear term plus nonlinear terms. Such a class of systems has been widely studied (see, e.g. [14], [4], [16]).

In this paper, the objective is to design a dynamical system and an update law such that the corresponding error dynamical systems are uniformly ultimately bounded by using sliding mode techniques.

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The local case will be treated in this paper but the results developed are straightforward to extend to the global case.

Assumption 1. The matrix pair (A, C) is observable.

Assumption 2. The known term $f(x(t), x(t-d), u(t))$ is Lipschitz with respect to the variables $x(t)$ and $x(t-d)$ uniformly for $u \in \mathcal{U}$, and $g(y, u, \theta)$ is differentiable and Lipschitz with respect to θ for $y \in \mathcal{Y}$ and $u \in \mathcal{U}$.

It is assumed that all the functions in this paper are continuous in their arguments to guarantee the existence and uniqueness of the system solution for any $u \in \mathcal{U}$.

From Lemma 1 in the Appendix, it follows that under Assumption 1, there exists a nonsingular matrix T_1 such that in the new coordinates $w := T_1 x$, system (2)–(2) can be described by

$$\begin{bmatrix} \dot{w}_1(t) \\ \dot{w}_2(t) \end{bmatrix} = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} \begin{bmatrix} w_1(t) \\ w_2(t) \end{bmatrix} + T_1 g(y(t), u, \theta)$$
$$+T_1\Big[f(x(t), x(t-d), u) + \Delta f(t, x(t), x(t-d))\Big]_{x=T_1^{-1}w} \quad (5)$$
$$y(t) = C_2 w_2(t) \quad (6)$$

where $w = \text{col}(w_1, w_2)$ with $w_1 \in \mathcal{R}^{n-p}$, the square matrix C_2 is nonsingular and the pair (A_1, A_3) is observable. Thus there exists a matrix $L \in \mathcal{R}^{(n-p)\times p}$ such that $A_1 - LA_3$ is Hurwitz stable, which implies that for any $Q > 0$, the Lyapunov equation

$$(A_1 - LA_3)^T P + P(A_1 - LA_3) = -Q \quad (7)$$

has a unique solution $P > 0$. For system (5)–(6), introduce a linear nonsingular coordinate transformation $T_2 : w \mapsto z = T_2 w$ as follows

$$z := \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \underbrace{\begin{bmatrix} I_{n-p} & -L \\ 0 & I_p \end{bmatrix}}_{T_2} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \quad (8)$$

where $z_1 \in \mathcal{R}^{n-p}$. Let $T = T_2 T_1$. Then system (2)–(2) and system (5)–(6) in the new coordinates $z = Tx$, can be described by

$$\dot{z}_1(t) = (A_1 - LA_3)z_1(t) + \big(A_2 - LA_4 + (A_1 - LA_3)L\big)z_2(t)$$
$$+f_1(z(t), z(t-d), u) + g_1(y, u, \theta) + \Delta f_1(\cdot) \quad (9)$$
$$\dot{z}_2(t) = A_3 z_1(t) + (A_4 + A_3 L)\, z_2(t) + f_2(z(t), z(t-d), u)$$
$$+g_2(y, u, \theta) + \Delta f_2(t, z(t), z(t-d)) \quad (10)$$
$$y(t) = C_2 z_2(t) \quad (11)$$

where $z := \text{col}(z_1, z_2)$ with $z_1 \in \mathcal{R}^{n-p}$ and $z_2 \in \mathcal{R}^p$, and

$$\begin{bmatrix} f_1(\cdot) \\ f_2(\cdot) \end{bmatrix} := T\,[f(\cdot)\,]_{x=T^{-1}z}, \quad \begin{bmatrix} g_1(\cdot) \\ g_2(\cdot) \end{bmatrix} := Tg(y, u, \theta) \quad (12)$$

$$\begin{bmatrix} \Delta f_1(\cdot) \\ \Delta f_2(\cdot) \end{bmatrix} := T\Big[\Delta f(t, x(t), x(t-d))\,\Big]_{x=T^{-1}z} \quad (13)$$

where $f_1(\cdot) \in \mathcal{R}^{n-p}$, $g_1(\cdot) \in \mathcal{R}^{n-p}$ and $\Delta f_1(\cdot) \in \mathcal{R}^{n-p}$.

Remark 1. Since T_1 can be obtained using matrix elementary operations and T_2 has been given in (8), the transformation $z = Tx$ can be obtained directly. Thus the transformed system (9)–(11) is well defined with a structure to facilitate sliding mode observer design. A similar transformation is also used in [10], [19]. Note there is no structural requirement on the uncertainty $\Delta f(\cdot)$ and only its bound as shown in (3) is required to be known.

From (3), it is straightforward to find a continuous function $\rho_0(\cdot)$ such that

$$\|\Delta f_2(t, z(t), z(t-d))\| \le \rho_0(t, z(t), z(t-d)) \quad (14)$$

where $\rho_0(t, z(t), z(t-d))$ is Lipschitz about $z(t)$ and $z(t-d)$ for all $t \in \mathcal{R}^+$.

Remark 2. Like much of the existing work [4], [16], [2], it is required in this paper that the nonlinear terms satisfy the Lipschitz condition. From (11), in the new coordinates z, the bound $\rho(\cdot)$ defined in (3) can be expressed as $\rho(t, z_1(t), C_2^{-1}y(t), z_1(t-d), C_2^{-1}y(t-d))$. Thus the limitation that $\rho(\cdot)$ is Lipschitz with respect to $x(t)$ and $x(t-d)$ can be relaxed to that of $\rho(\cdot)$ is Lipschitz with respect to z_1 and $z_1(t-d)$. This is also true for the nonlinear term $f(\cdot)$.

Let $g_2(\cdot) := (g_{21}, \cdots, g_{2p})^T$ and $\theta := \text{col}(\theta_1, \cdots, \theta_r)$. From the notation in Lemma 2 in the Appendix, the term $\frac{\bar{\partial} g_2(y,u,\theta_{g_{21}},\cdots,\theta_{g_{2p}})}{\bar{\partial}\theta} := \frac{\bar{\partial} g_2(\cdot)}{\bar{\partial}\theta}$ is a $p \times r$ matrix defined in $\mathcal{Y} \times \mathcal{U} \times \underbrace{\mathcal{R}^p \times \cdots \times R^p}_{r}$.

Assumption 3. There exists a continuous function matrix $\Xi(u, y) \in \mathcal{R}^{r\times p}$ such that for any $\theta_{g_{2i}} \in \mathcal{R}^p$ for $i = 1, 2, \ldots, r$ and $(y, u) \in \mathcal{Y} \times \mathcal{U}$,

$$G(\cdot) := -\left(\Xi(y, u)\frac{\bar{\partial} g_2(\cdot)}{\bar{\partial}\theta}\right)^T - \Xi(y, u)\frac{\bar{\partial} g_2(\cdot)}{\bar{\partial}\theta} > 0 \quad (15)$$

Remark 3. The unknown parameters appear in a nonlinear way as $g(y, u, \theta)$ and the Assumption 3 is an extension of the condition used in [19]. The formulation considered here includes the existing work: for example [18], [21], [19] as special cases, where it is required that the unknown parameters appear affinely in the form $g(\cdot)\theta$ [18], [19] or linearly as $B(t)\theta$ [21].

III. Sliding Mode Observer Design

Section II has shown that a nonsingular transformation $z = Tx$ is available to transfer system (2)–(2) to (9)–(11). This structure will be used as a basis for the analysis which follows.

A. Error dynamical system formulation

For system (9)–(11), construct the following dynamical system

$$\dot{\hat{z}}_1(t) = (A_1 - LA_3)\hat{z}_1(t) + (A_2 - LA_4 + (A_1 - LA_3)L)$$
$$\cdot C_2^{-1}y(t) + f_1(\hat{z}_1(t), C_2^{-1}y(t), \hat{z}_1(t-d),$$
$$C_2^{-1}y(t-d), u) + g_1(y, u, \hat{\theta}) \quad (16)$$
$$\dot{\hat{z}}_2(t) = A_3\hat{z}_1(t) + (A_4 + A_3 L)\,\hat{z}_2(t) - K\,(y(t) - \hat{y}(t))$$
$$+f_2(\hat{z}_1(t), C_2^{-1}y(t), \hat{z}_1(t-d), C_2^{-1}y(t-d), u)$$
$$+g_2(y, u, \hat{\theta}) + \nu(t, u, y, \hat{z}) \quad (17)$$
$$\hat{y}(t) = C_2\hat{z}_2(t) \quad (18)$$

where $\hat{z} := \text{col}\,(\hat{z}_1, \hat{z}_2)$. The gain matrix K is chosen such that $A_4 + A_3 L + KC_2$ is symmetric negative definite (clearly this is always possible because C_2 is nonsingular). The function ν is defined by

$$\nu(\cdot) := k(\cdot)\frac{C_2^{-1}(y(t) - \hat{y}(t))}{\|C_2^{-1}(y(t) - \hat{y}(t))\|}, \quad \text{if} \quad y(t) - \hat{y}(t) \ne 0 \quad (19)$$

where $k(\cdot)$ is a positive scalar function to be determined later, and the vector $\hat{\theta}$ is given by the following adaptive law

$$\dot{\hat{\theta}} = \Xi(y, u)(A_3\hat{z}_1(t) + (A_4 + A_3 L)C_2^{-1}y + f_2(\hat{z}_1(t), C_2^{-1}y,$$
$$\hat{z}_1(t-d), C_2^{-1}y(t-d), u) + g_2(y, u, \hat{\theta}) - C_2^{-1}\dot{y}(t)) \quad (20)$$

where $\Xi(\cdot) \in \mathcal{R}^{r\times p}$ is a design parameter which satisfies Assumption 3. Obviously, both the dynamical system (16)–(17) and the update law (19) are time delayed. For any given constant $\mu > 0$, the initial condition associated with the time delay d for system (16)–(20) is chosen as any continuous function $\hat{\phi}(\cdot)$ defined in $[-\bar{d},\ 0]$ such that

$$\|T\phi(t) - \hat{\phi}(t)\| \le \mu \quad (21)$$

where $\phi(\cdot)$ is given in (4).

Let $e_1 = z_1 - \hat{z}_1$, $e_\theta = \theta - \hat{\theta}$ and $e_2 = z_2 - \hat{z}_2$. Since θ is constant, then, from (20) and by comparing system (9)–(11) with system (16)–(18), it follows that

$$\begin{aligned}\dot{e}_1(t) &= (A_1 - LA_3)e_1(t) + \delta(f_1, \hat{f}_1) + g_1(y,u,\theta) \\ &\quad -g_1(y,u,\hat{\theta}) + \Delta f_1(t, z(t), z(t-d)) \qquad (22)\end{aligned}$$

$$\begin{aligned}\dot{e}_\theta(t) &= \Xi(y,u)(A_3 e_1(t) + \delta(f_2, \hat{f}_2) + g_2(y,u,\theta) \\ &\quad -g_2(y,u,\hat{\theta}) + \Delta f_2(t, z(t), z(t-d))) \qquad (23)\\ \dot{e}_2(t) &= A_3 e_1(t) + (A_4 + A_3 L + KC_2)\, e_2(t) + \delta(f_2, \hat{f}_2) \\ &\quad +g_2(y,u,\theta) - g_2(y,u,\hat{\theta}) + \Delta f_2(\cdot) - \nu(\cdot) \qquad (24)\end{aligned}$$

where $\nu(\cdot)$ is defined by (19), and

$$\begin{aligned}\delta(f_1, \hat{f}_1) :&= f_1(z(t), z(t-d), u) \\ &\quad -f_1(\hat{z}_1(t), z_2(t), \hat{z}_1(t-d), z_2(t-d), u) \qquad (25)\\ \delta(f_2, \hat{f}_2) :&= f_2(z(t), z(t-d), u) \\ &\quad -f_2(\hat{z}_1(t), z_2(t), \hat{z}_1(t-d), z_2(t-d), u) \qquad (26)\end{aligned}$$

For the error system (22)–(24), consider a sliding surface

$$\mathcal{S} = \{(e_1, e_2, e_\theta) \mid \; e_2 = 0\} \qquad (27)$$

where $e_2 = C_2^{-1}(y(t) - \hat{y}(t)) = z_2(t) - \hat{z}_2(t)$. From the structure of system (22)–(24) and the definition of the sliding surface (27), it follows that the sliding motion associated with the sliding surface (27) is governed by system (22)–(23).

B. Stability analysis of sliding motion

Theorem 1. Suppose that Assumptions 1-3 hold and $\sup\{\rho(\cdot)\|\Xi(\cdot)\|\} < +\infty$. Then, system (22)– (23) is uniformly ultimately bounded if $\inf_{\mathcal{Y}\times\mathcal{U}} \{\lambda_{\min}(W(\cdot))\} > 0$ where the symmetric function matrix $W(y,u)$ is defined by

$$W(\cdot) = \begin{bmatrix} \lambda_{\min}(Q) - 2\ell\mathcal{L}_{Pf_1} & \Gamma(\cdot) \\ \Gamma(\cdot) & \lambda_{\min}(G(\cdot)) - 2\ell_2\mathcal{L}_{\Xi f_2} \end{bmatrix} \qquad (28)$$

where $\ell_1 := 1 + \sqrt{\gamma_4\lambda_{\max}(P)/\lambda_{\min}(P)}$ and $\ell_2 := \sqrt{\gamma_4/\lambda_{\min}(P)}$ for some constant $\gamma_4 > 0$, $\Gamma(\cdot) := -\mathcal{L}_{Pg_1} - \|\Xi(y,u)A_3\| - \ell_1\mathcal{L}_{\Xi f_2} - \ell_2\mathcal{L}_{Pf_1}$ and the matrices P and Q satisfy (7).

Proof: For system (22)-(23), consider a candidate Lyapunov function

$$V(e_1(t), e_\theta(t)) = e_1^T(t)Pe_1(t) + e_\theta^T(t)e_\theta(t)$$

where $P > 0$ satisfies (7). The time derivative of V along the trajectories of (22)–(23) is given by

$$\begin{aligned}\dot{V} &= e_1^T(t)\left((A_1 - LA_3)^T P + P(A_1 - LA_3)\right)e_1(t) + 2e_1^T P \\ &\quad \cdot\delta(f_1, \hat{f}_1) + 2e_1^T P\big(g_1(y,u,\theta) - g_1(y,u,\hat{\theta})\big) + 2e_1^T P\Delta f_1(\cdot) \\ &\quad +2e_\theta^T(t)\Xi(y,u)A_3 e_1(t) + 2e_\theta^T(t)\Xi(y,u)\delta(f_2, \hat{f}_2) \\ &\quad +e_\theta^T(t)\Big(\Xi(y,u)\big(g_2(y,u,\theta) - g_2(y,u,\hat{\theta})\big)\Big) \\ &\quad +\Big(\Xi(y,u)\,\big(g_2(y,u,\theta) - g_2(y,u,\hat{\theta})\big)\Big)^T e_\theta(t) \\ &\quad +2e_\theta^T(t)\Xi(y,u)\Delta f_2(\cdot) \qquad (29)\end{aligned}$$

where $\delta(f_1, \hat{f}_1)$ and $\delta(f_2, \hat{f}_2)$ are defined by (25) and (26) respectively. Since $z = Tx$ is nonsingular, and $f(x(t), x(t-d), u)$ is Lipschitz about $x(t)$ and $x(t-d)$ for all $u \in \mathcal{U}$, it follows from (25) and (26) that

$$\begin{aligned}\left\|P\delta(f_1, \hat{f}_1)\right\| &\leq \mathcal{L}_{Pf_1}\left(\|e_1(t)\| + \|e_1(t-d)\|\right) \qquad (30)\\ \left\|\Xi(y,u)\delta(f_2, \hat{f}_2)\right\| &\leq \mathcal{L}_{\Xi f_2}\left(\|e_1(t)\| + \|e_1(t-d)\|\right) \qquad (31)\end{aligned}$$

where $\mathcal{L}_{Pf_1}$ and $\mathcal{L}_{\Xi f_2}$ are functions of y and u. The fact that the Lipschitz coefficient is a function has been employed in [7], [20], and enlarges the classes of allowed functions.

From Lemma 2 in the Appendix, there exist $\theta^*_{g_{2i}} \in \mathcal{R}^p$ for $i = 1, 2, \ldots, p$ such that

$$\begin{aligned}& e_\theta^T(t)(\Xi(y,u)(g_2(y,u,\theta) - g_2(y,u,\hat{\theta}))) \\ &\qquad +(\Xi(y,u)(g_2(y,u,\theta) - g_2(y,u,\hat{\theta})))^T e_\theta(t) \\ &= e_\theta^T(t)\Xi(y,u)\frac{\bar{\partial} g_2(y,u,\theta^*_{g_{21}}, \theta^*_{g_{22}}, \ldots, \theta^*_{g_{2p}})}{\bar{\partial}\theta}e_\theta \\ &\quad +e_\theta^T\left(\Xi(y,u)\frac{\bar{\partial} g_2(y,u,\theta^*_{g_{21}}, \theta^*_{g_{22}}, \ldots, \theta^*_{g_{2p}})}{\bar{\partial}\theta}\right)^T e_\theta(t) \\ &= -e_\theta^T(t)G(\cdot)e_\theta(t) \leq -\lambda_{\min}(G(\cdot))\|e_\theta\|^2 \qquad (32)\end{aligned}$$

where $G(\cdot)$ is defined in (15) which, from Assumption 3, is negative definite. It is clear that

$$\begin{aligned}& 2e_1^T P\Delta f_1(\cdot) + 2e_\theta^T(t)\Xi(y,u)\Delta f_2(\cdot) \\ &= 2\begin{bmatrix} e_1^T & e_\theta^T \end{bmatrix}\mathrm{diag}\{P, \Xi(y,u)\}\begin{bmatrix} \Delta f_1(\cdot) \\ \Delta f_2(\cdot) \end{bmatrix} \qquad (33)\end{aligned}$$

Substituting (30)–(33) into (29),

$$\begin{aligned}\dot{V} &\leq -\lambda_{\min}(Q)\|e_1(t)\|^2 + 2\mathcal{L}_{Pf_1}\left(\|e_1(t)\|^2 + \|e_1(t)\|\|e_1(t-d)\|\right) \\ &\quad +2\mathcal{L}_{Pg_1}\|e_1(t)\|\,\|e_\theta(t)\| + 2\|\Xi(\cdot)A_3\|\,\|e_\theta(t)\|\,\|e_1(t)\| \\ &\quad +2\mathcal{L}_{\Xi f_2}\left(\|e_1(t)\|\,\|e_\theta(t)\| + \|e_\theta(t)\|\,\|e_1(t-d)\|\right) - \lambda_{\min}(G(\cdot)) \\ &\quad \cdot\|e_\theta(t)\|^2 + 2\rho(\cdot)\left\|\begin{bmatrix} e_1^T(t) & e_\theta^T(t) \end{bmatrix}\right\|\,\|\mathrm{diag}\{P, \Xi(\cdot)\}T\| \qquad (34)\end{aligned}$$

where (3) and (12) have been employed to obtain inequality (34).

If the inequality $V(e_1(t-d), e_\theta(t-d)) \leq \gamma_4 V(e_1(t), e_\theta(t))$ holds for any $d \in [0, \; \bar{d}]$ and some $\gamma_4 > 1$, then,

$$\|e_1(t-d)\| \leq (\ell_1 - 1)\|e_1(t)\| + \ell_2\|e_\theta\| \qquad (35)$$

where $\ell_1 := 1 + \sqrt{\gamma_4\lambda_{\max}(P)/\lambda_{\min}(P)}$ and $\ell_2 := \sqrt{\gamma_4/\lambda_{\min}(P)}$. Substituting (35) into (34), it follows that

$$\begin{aligned}\dot{V} &\leq -\left(\lambda_{\min}(Q) - 2\ell_1\mathcal{L}_{Pf_1}\right)\|e_1(t)\|^2 + 2\big(\mathcal{L}_{Pg_1} + \|\Xi(\cdot)A_3\| \\ &\quad +\ell_1\mathcal{L}_{\Xi f_2} + \ell_2\mathcal{L}_{Pf_1}\big)\|e_1(t)\|\,\|e_\theta\| - \big(\lambda_{\min}(G(\cdot)) - \\ &\quad 2\ell_2\mathcal{L}_{\Xi f_2}\big)\|e_\theta(t)\|^2 + 2\rho(\cdot)\;\|\mathrm{diag}\{P, \Xi(\cdot)\}T\|\begin{bmatrix} \|e_1(t)\| \\ \|e_\theta(t)\| \end{bmatrix} \\ &= -X^T W(\cdot)X + 2\gamma_0\|X\| \\ &\leq -\Big(\inf_{\mathcal{Y}\times\mathcal{U}}\{\lambda_{\min}(W(\cdot))\}\|X\| - 2\gamma_0\Big)\|X\| \\ &= -\Big(\|X\| - \frac{2\gamma_0}{\inf_{\mathcal{Y}\times\mathcal{U}}\{\lambda_{\min}(W)\}}\Big)\inf_{\mathcal{Y}\times\mathcal{U}}\{\lambda_{\min}(W)\}\|X\|\end{aligned}$$

where $X := [\|e_1\| \; \|e_\theta\|]^T$, $\gamma_0 := \sup\{\rho(\cdot)\|\mathrm{diag}\{P, \Xi(\cdot)\}T\|\}$. Hence, the conclusion follows from $\inf_{\mathcal{Y}\times\mathcal{U}}\{\lambda_{\min}(W(\cdot))\} > 0$. #

Remark 4. A sufficient condition is presented in Theorem 1, where the design parameters Q and L which affect P, the design function matrix $\Xi(y,u)$ and constant $l > 0$ appear in the matrix W defined in (28). There is no general constructive approach to choose these parameters such that W is positive definite due to the complex nature of W. Note that $G(\cdot)$ defined in (15) is independent of L and Q and $\mathcal{L}_{Pg_1} \leq \lambda_{\max}(P)\mathcal{L}_{g_1}$. Considering the structure of the matrix W, one choice is to find L and the matrix Q to minimise $\frac{\lambda_{\min}(Q)}{\lambda_{\max}(P)}$. The associated discussion is available in [13].

Remark 5. Theorem 1 shows that system (22)–(23) is uniformly ultimately bounded under certain conditions, which implies that there exist positive constants ϖ_1 and ϖ_2 such that for any $t \in \mathcal{R}^+$

$$\|e_1(t)\| \leq \varpi_1 \qquad \text{and} \qquad \|e_\theta(t)\| \leq \varpi_2 \qquad (36)$$

To estimate the upper bounds ϖ_1 and ϖ_2 is not trivial, but one possibility is to use the Gronwall-Bellman inequality (see [19]). By applying (21), it follows that

$$\|e_1(t-d)\| \leq \max\{\mu, \varpi_1\} := \mu_0 \quad (37)$$

Inequalities (36) and (37) will be used in the reachability analysis described later in the paper.

Remark 6. It should be noted that $\Xi(u, y)$ in (20) is a design parameter and Assumption 3 provides a limitation on the parameter which is necessary to guarantee that $W > 0$ in (28). For linear systems, the developed condition is usually attributed to a series of LMIs. In this paper, the condition that $W > 0$ implies that the condition (15) holds. To some extent, the role of the matrix inequality (15) for system (22)–(24) is similar to that observed in the corresponding LMIs for linear systems.

C. Reachability analysis

In this section, the $k(\cdot)$ in (19) will be designed such that the reachability condition holds.

Theorem 2. Under Assumptions 1-3, system (22)–(24) is driven to the sliding surface (27) in finite time and maintains a sliding motion on it if the gain $k(\cdot)$ in (19) satisfies

$$\begin{aligned} k(\cdot) \geq\ & \varpi_1\|A_3\| + (\mathcal{L}_{f_2} + \mathcal{L}_{\rho_0})(\varpi_1 + \mu_0) + \varpi_2\mathcal{L}_{g_2} + \\ & \rho_0(t, \hat{z}_1(t), C_2^{-1}y(t), \hat{z}_1(t-d), C_2^{-1}y(t-d)) + \eta \quad (38) \end{aligned}$$

where $\rho_0(\cdot)$ satisfies (14) and η is a positive constant.

Proof: From the error system in (24),

$$\begin{aligned} e_2^T(t)\dot{e}_2(t) =\ & e_2^T(t)\big(A_3e_1(t) + (A_4 + A_3L + KC_2)e_2(t) + \\ & \delta(f_2, \hat{f}_2) + g_2(y, u, \theta) - g_2(y, u, \hat{\theta}) + \Delta f_2(\cdot) - \nu(\cdot)\big) \quad (39) \end{aligned}$$

Then, from the definition of $\delta(f_2, \hat{f}_2)$ in (26), it follows that

$$\left\|\delta(f_2, \hat{f}_2)\right\| \leq \mathcal{L}_{f_2}(\|e_1(t)\| + \|e_1(t-d)\|) \quad (40)$$

Further, from Assumption 2,

$$\|g_2(y, u, \theta) - g_2(y, u, \hat{\theta})\| \leq \mathcal{L}_{g_2}\|e_\theta\| \quad (41)$$

where $\mathcal{L}_{g_2}$ is a function of y and u.

By applying inequalities (40) and (41) to equation (39), it follows from (19), (36) and (37) that

$$\begin{aligned} e_2^T(t)\dot{e}_2(t) \leq\ & \varpi_1\|A_3\|\,\|e_2(t)\| + e_2^T(t)(A_4 + A_3L + KC_2)e_2(t) \\ & + \mathcal{L}_{f_2}(\varpi_1 + \mu_0)\|e_2(t)\| + \varpi_2\mathcal{L}_{g_2}\|e_2(t)\| \\ & + \rho_0(t, z(t), z(t-d))\|e_2(t)\| - k(\cdot)\frac{e_2^T(t)e_2(t)}{\|e_2(t)\|} \end{aligned}$$

Since, by design, $A_4 + A_3L + KC_2$ is symmetric negative definite, it follows that $e_2^T(t)(A_4 + A_3L + KC_2)e_2(t) \leq 0$. Therefore,

$$\begin{aligned} e_2^T(t)\dot{e}_2(t) \leq\ & \big(\varpi_1\|A_3\| + \mathcal{L}_{f_2}(\varpi_1 + \mu_0) + \varpi_2\mathcal{L}_{g_2}\| + \rho_0(\cdot)\big) \\ & \cdot\|e_2(t)\| - k(\cdot)\|e_2(t)\| \quad (42) \end{aligned}$$

Since $\rho_0(\cdot)$ is Lipschitz, it follows that

$$\begin{aligned} & \|\rho_0(t, z(t), z(t-d)) - \rho_0(t, \hat{z}_1(t), C_2^{-1}y(t), \hat{z}_1(t-d), \\ & \quad C_2^{-1}y(t-d))\| \\ & \leq \mathcal{L}_{\rho_0}(\|z_1(t) - \hat{z}_1(t)\| + \|z_1(t-d) - \hat{z}_1(t-d)\|) \\ & \leq \mathcal{L}_{\rho_0}(\varpi_1 + \mu_0) \quad (43) \end{aligned}$$

Then, applying (38) to (42), it follows from (43) that $e_2^T(t)\dot{e}_2(t) \leq -\eta\|e_2(t)\|$. This shows that the reachability condition holds and thus the error system is driven to the sliding surface in infinite time. Hence the conclusion follows. #

By combining Theorems 1 and 2, it follows from sliding mode theory that system (22)–(23) is uniformly ultimately bounded. Since $z = Tx$ is a nonsingular coordinate transformation, it is easy to see that $\hat{x} = T\hat{z}$ gives an estimate of the states x of system (2)–(2) where $\hat{z}$ is given by (16)–(17).

IV. An Application Example—Bioreactor

Consider a simple model of a bioreactor described in [11] which is based on classical mass balances for biomass, sulphate (substrate) and sulphide (product) concentration as follows:

$$\begin{aligned} \dot{x}_1 &= -x_1u + h(x_2)x_1 && (44) \\ \dot{x}_2 &= (\gamma_{in} - x_2)u - h(x_2)\frac{x_1}{\xi_1} && (45) \\ \dot{x}_3 &= -x_3u + h(x_2)\frac{x_1}{\xi_2} && (46) \end{aligned}$$

where x_1, x_2 and x_3 represent biomass concentration (g/l), sulphate concentration (g/l) and sulphide concentration (g/l) respectively. Following the well known Monod model [11], $h(x_2) = \frac{h_1x_2}{h_2+x_2}$ where h_1 and h_2 are constants. The control u is the dilution rate (1/hour), ξ_1 and ξ_2 are yield coefficients and γ_{in} is the influent sulphate concentration. It is assumed that the sulphate concentration x_2 and sulphide concentration x_3 can be measured by sensors.

In order to illustrate the approach developed in the paper, the influent sulphate concentration $\gamma_{in} := \theta$ is assumed to be an unknown constant. Similar to the work in [1], assume that there exist delay effects on the sulphide concentration x_3. Then system (44)–(46) is described by

$$\begin{aligned} \dot{x} =\ & \underbrace{\begin{bmatrix} h_1 & 0 & 0 \\ -\frac{h_1}{\xi_1} & 0 & 0 \\ \frac{h_1}{\xi_2} & 0 & 0 \end{bmatrix}}_{A} x + \underbrace{\begin{bmatrix} 0 \\ \theta u - y_1u \\ -ay_2u \end{bmatrix}^T}_{g(\cdot)} \\ & + \underbrace{\begin{bmatrix} -x_1u - \frac{h_1h_2x_1}{h_2+x_2} \\ \frac{h_1h_2x_1}{\xi_1(h_2+x_2)} \\ -(1-a)x_3(t-d)u - \frac{h_1h_2x_1}{\xi_2(h_2+x_2)} \end{bmatrix}}_{f(\cdot)} + \Delta f(\cdot) \quad (47) \\ y =\ & \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{C} x(t) \quad (48) \end{aligned}$$

where the parameter $a \in [0, 1]$ is the retarded coefficient and the limits $a = 1$ and $a = 0$ correspond to no-delay and full delay terms respectively. The term

$$\Delta f(\cdot) := [\Delta f_{01}(\cdot) \quad \underbrace{\Delta f_{02}(\cdot) \quad \Delta f_{03}(\cdot)}_{\Delta f_2}]^T$$

includes all the disturbances which componentwise satisfy $|\Delta f_{01}(\cdot)| \leq 0.05\cos^2(x_2(t-d))$, $|\Delta f_{02}(\cdot)| \leq 0.05\sin^2(x_2(t-d))$ and $|\Delta f_{01}(\cdot)| \leq 0.1\exp\{-t\}|\sin(x_1(t))|$. The parameters are chosen as in [11]: $h_1 = 0.035$, $h_2 = 0.90$, $\xi_1 = 0.25$ and $\xi_2 = 0.26$ while $a = 0.75$ is as in [1]. Importantly, concentration can never be negative. It is straightforward to check that Assumptions 1 and 2 hold. It is clear that (47)–(48) is in the form of (5)–(6) with

$$\begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} = \left[\begin{array}{c|cc} 0.0350 & 0 & 0 \\ \hline -0.1346 & 0 & 0 \\ 0.1400 & 0 & 0 \end{array}\right], \quad [0 \quad C_2] = \left[\begin{array}{c|cc} 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]$$

Choose $L = [-1 \ \ 0]$ and $Q = 1$. Then, the solution to the Lyapunov equation (7) is $P = 5.0193$. Under the coordinate transformation

$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} := \begin{bmatrix} z_1 \\ \hline z_{21} \\ z_{22} \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{T} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \tag{49}$$

system (47)–(48) has the same form as (9)–(11) with

$$\begin{bmatrix} f_1(\cdot) \\ f_2(\cdot) \end{bmatrix} = \begin{bmatrix} -(z_1 - z_{21})u + \frac{0.0897}{0.9+z_{21}})(z_1 - z_{21}) \\ \hline \frac{0.1212(z_1-2z_{21})}{0.9+z_{21}} \\ -0.25z_{22}(t-d)u - \frac{0.126(z_1-2z_{21})}{0.9+z_{21}} \end{bmatrix}$$

$$\begin{bmatrix} g_1(\cdot) \\ g_2(\cdot) \end{bmatrix} = \begin{bmatrix} \theta u - 2y_1 u \\ \hline \theta u - y_1 u \\ -0.75 y_2 u \end{bmatrix}$$

$$\begin{bmatrix} \Delta f_1(\cdot) \\ \Delta f_2(\cdot) \end{bmatrix} = \begin{bmatrix} \Delta f_{01}(\cdot) + \Delta f_{02}(\cdot) \\ \hline \Delta f_{02}(\cdot) \\ \Delta f_{03}(\cdot) \end{bmatrix}_{x=T^{-1}z}$$

It is straightforward to verify that Assumption 3 is satisfied with $\Xi(\cdot) = -[0.14 \;\; 0.1346](1 + \frac{1}{u})$ for all $u \in \mathcal{U} := \{u \mid 0.0001 \le u \le 0.1840\}$. By direct computation, it follows that

$$\mathcal{L}_{Pf_1}(\cdot) = \left| -u + \frac{0.0897}{0.9+y_1} \right|, \; \mathcal{L}_{g_1}(\cdot) = \mathcal{L}_{g_2}(\cdot) = u, \; \Xi(\cdot)A_3 = 0$$
$$G(\cdot) = 0.1400(1+u), \; \ell_1 = 2.0005, \; \ell_2 = 0.4585, \; \mathcal{L}_{\Xi f_2}(\cdot) = 0$$
$$\rho_2(\cdot) = \sqrt{0.0025\sin^4(z_{22}(t-d)) + 0.01\exp\{-2t\}\sin^2(z_1(t) - z_{21}(t))}$$

and the conditions in Theorem 1 hold. Finally, choose $k(\cdot)$ to satisfy (38). Then system (16)–(17) with the adaptive law (20) is well defined, and is an observer of system (47)–(48) in z coordinates.

For simulation purposes, the control signal is chosen as $u = 0.004(2y_1 + y_2)$ and the parameter $\theta = 0.15$. The initial conditions are chosen as $x_0 = (0.12, 5, 0.16)$ or $z_0 = (5.16, 5, 0.12)$ as in [11] and $\theta_0 = -5$. The delay is chosen as $d(t) = 2 + \sin t$ and the initial value associated with the delay is chosen as $\phi(t) = [2 - \cos t \;\; 1 \;\; 2 + \sin t]^T$. Figure 1 shows the estimates for the

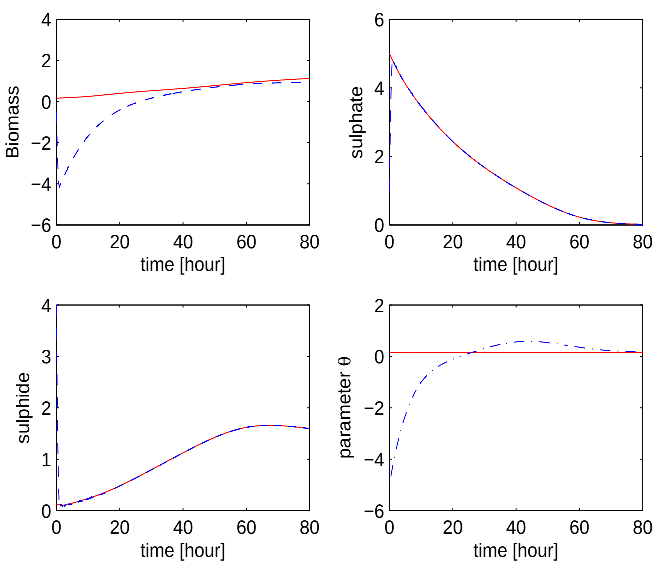

Fig. 1. The evolution of system states x and parameter θ (solid line) and the estimates $\hat{x}$ and $\hat{\theta}$ (dashed line)

system states and the parameter. The simulation results show the effectiveness of the proposed approach.

Remark 7. Note that the term (19) appearing in the observer (16)–(18) is discontinuous which may result in chattering. In the simulation, the discontinuous function (19) has been replaced by the saturation function to avoid chattering [5]. It should be noted that in the simulation example, the matrix $\Xi(\cdot)$ is chosen as $[0.14 \;\; 0.1346](1 + \frac{1}{u})$ due to the term θu in (47), which may result in large gain in the adaptive law (20) when u is very small. As is usual for practical implementation, physical limits on the available control would need to be incorporated.

V. Conclusions

In this paper, a sliding mode observer with an update law has been proposed to estimate the system states and unknown parameters. Coordinate transformations are used to explore the system structure and the features of the sliding mode approach are fully used to reduce conservatism. Both parametric uncertainty and structural uncertainty are considered. The system plant and the derived error dynamical system are time delayed and nonlinear. Furthermore, the unknown parameters are embedded in the system in a nonlinear fashion. The developed results is applicable to a wide class of systems.

Appendix

Consider a matrix pair (A, C) where $A \in \mathcal{R}^{n\times n}$, and $C \in \mathcal{R}^{p\times n}$ is full row rank with $p < n$. The following result can be obtained.

Lemma 1 If a matrix pair (A, C) is observable and the matrix C is full row rank, then there exists a nonsingular matrix T_1 such that

$$T_1 A T_1^{-1} = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}, \qquad CT_1^{-1} = [0 \;\; C_2] \tag{50}$$

where $A_1 \in \mathcal{R}^{(n-p)\times(n-p)}$, the matrix $C_2 \in \mathcal{R}^{p\times p}$ is nonsingular and (A_1, A_3) is observable.

Proof: From the fact that C is full row rank, there exists a nonsingular matrix T_1 such that

$$CT_1^{-1} = [0 \;\; C_2] \tag{51}$$

where $C_2 \in \mathcal{R}^{p\times p}$ is nonsingular. Partition $T_1AT_1^{-1}$ in a compatible way with CT_1^{-1} in (51) as

$$T_1AT_1^{-1} = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} \tag{52}$$

where $A_1 \in \mathcal{R}^{(n-p)\times(n-p)}$. Then, it is only required to prove that (A_1, A_3) is observable. By direct verification, it follows from (51) and (52) that for any complex number s,

$$\begin{bmatrix} sI_n - A \\ C \end{bmatrix} = \left[\begin{array}{c|c} T_1^{-1}\begin{bmatrix} I_{n-p} & 0 \\ 0 & -I_p \end{bmatrix} & 0 \\ 0 & I_p \end{array}\right] \cdot \begin{bmatrix} sI_{n-p} - A_1 & -A_2 \\ A_3 & -sI_p + A_4 \\ \hline 0 & C_2 \end{bmatrix} T_1$$

and thus for any complex number s,

$$\text{rank}(\begin{bmatrix} sI_n - A \\ C \end{bmatrix}) = \text{rank}(\begin{bmatrix} sI_{n-p} - A_1 \\ A_3 \end{bmatrix}) + \text{rank}(C_2) \tag{53}$$

Since (A, C) is observable and $C_2 \in \mathcal{R}^{p\times p}$ is nonsingular, from the PHB rank test,

$$\text{rank}\left(\begin{bmatrix} sI_{n-p} - A_1 \\ A_3 \end{bmatrix}\right) = n - p \tag{54}$$

Again, from the PHB rank test, equation (54) implies that the matrix pair (A_1, A_3) is observable. Hence the conclusion follows. #

Let $\bar{x}, \hat{x} \in \mathcal{R}^n$. A convex set associated with $\bar{x}$ and $\hat{x}$ is defined by

$$D(\bar{x}, \hat{x}) := \{\mu\bar{x} + (1-\mu)\hat{x} \mid 0 \le \mu \le 1\}$$

which is a subset in $\mathcal{R}^n$.

Lemma 2 Let $f(x) = [f_1(x)\ f_2(x)\ \cdots\ f_p(x)]^T$ where $x \in \mathcal{R}^n$ and the function $f_i : \mathcal{R}^n \mapsto \mathcal{R}$ is differentiable in $\mathcal{R}^n$ for $i = 1, 2, \ldots, p$. Then for any $\bar{x} \in \mathcal{R}^n$ and $\hat{x} \in \mathcal{R}^n$, there exists $x^*_{f_i} \in D(\bar{x}, \hat{x}) \subset \mathcal{R}^n$ for $i = 1, 2, \cdots, p$ such that

$$f(\bar{x}) - f(\hat{x}) = \frac{\bar{\partial} f(x^*_{f_1}, x^*_{f_2}, \cdots, x^*_{f_p})}{\bar{\partial} x}(\bar{x} - \hat{x}) \tag{55}$$

where the notation $\frac{\bar{\partial} f(\cdot)}{\bar{\partial} x}$ represents a $p \times n$ function matrix defined by

$$\frac{\bar{\partial} f(x^*_{f_1}, x^*_{f_2}, \cdots, x^*_{f_p})}{\bar{\partial} x} := \begin{bmatrix} \frac{\partial f_1(x^*_{f_1})}{\partial x} & \ldots & \frac{\partial f_p(x^*_{f_p})}{\partial x} \end{bmatrix}^T \tag{56}$$

where $\frac{\partial f_i(x^*_{f_i})}{\partial x} := \begin{bmatrix} \frac{\partial f_i(x)}{\partial x_1} & \frac{\partial f_i(x)}{\partial x_2} & \cdots & \frac{\partial f_i(x)}{\partial x_n} \end{bmatrix}_{x = x^*_{f_i}}$ for $i = 1, 2, \ldots, p$.

Proof: From the multi-variable differential Mean Value Theorem, it follows that for any $\bar{x} \in \mathcal{R}^n$ and $\hat{x} \in \mathcal{R}^n$, there exists $x^*_{f_i} \in D(\bar{x}, \hat{x}) \subset \mathcal{R}^n$ such that

$$f_i(\bar{x}) - f_i(\hat{x}) = \frac{\partial f_i(x^*_{f_i})}{\partial x}(\bar{x} - \hat{x}), \qquad i = 1, 2, \ldots, p \tag{57}$$

where the point $x^*_{f_i}$ depends not only on $\bar{x} \in \mathcal{R}^n$ and $\hat{x} \in \mathcal{R}^n$, but on f_i as well.

Let $\alpha_i = [0\ \cdots\ 0\ \overset{(i)}{1}\ 0\ \cdots\ 0]^T$ which has the i-th entry as 1 and all other terms as 0. Then

$$f(\bar{x}) - f(\hat{x}) = \sum_{i=1}^{p} \alpha_i \Big(f_i(\bar{x}) - f_i(\hat{x}) \Big) \tag{58}$$

Substituting (57) into (58) yields

$$\begin{aligned} f(\bar{x}) - f(\hat{x}) &= \sum_{i=1}^{p} \alpha_i \frac{\partial f_i(x^*_{f_i})}{\partial x}(x - \hat{x}) \\ &= \frac{\bar{\partial} f(x^*_{f_1}, x^*_{f_2}, \cdots, x^*_{f_p})}{\bar{\partial} x}(\bar{x} - \hat{x}) \end{aligned}$$

where $\frac{\bar{\partial} f(\cdot)}{\bar{\partial} x}$ is defined in (55). Hence the conclusion follows. #

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