Continuous adaptive finite reaching time control and second order sliding modes

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This paper focuses on the design of adaptive finite reaching time control for first and second order dynamic systems with perturbation terms given in a regressive form. The uncertainties considered here are assumed to be bounded with unknown bounds. The proposed adaptive finite reaching time controllers not only retain robustness to these disturbances, but also are continuous. The proposed finite reaching time adaptive control algorithms are interpreted as continuous second order sliding mode control laws. Simulation results demonstrate the efficacy of the proposed algorithms.

Keywords: adaptive control, finite reaching time, second order sliding modes.

1. Introduction

Control in the presence of uncertainty is one of the main topics of modern control theory. The design of high performance controllers often requires knowledge of the plant dynamics. The most desirable control systems are those that perform well amidst modeling inaccuracies, parametric uncertainties and external disturbances. Sliding mode control (SMC) (Edwards *et al*., 1998; Utkin *et al*., 1999; Utkin & Lee, 2007) and robust adaptive control (Sastry & Bodson, 1989; Astolfi *et al*., 2008) remain, probably, the most popular methods for handling bounded uncertainties/disturbances and unmodeled dynamics with known (SMC) and unknown (adaptive control) bounds. Classical sliding mode control (that is applicable to systems of relative degree 1) drives the state variables to the sliding surface in finite time and keep it there thereafter in the presence of bounded (with known bounds) matched uncertainties and disturbances. Thus, the system's dynamics, compensated by SMC, are invariant to matched bounded disturbances/uncertainties at the price of high frequency switching control. Adaptive classical and second order sliding mode control (that is applicable to systems of relative degree 2 with dynamically varying gains, achieve finite time stabilization of the sliding variable in the presence of bounded disturbances/uncertainties with unknown bounds. However, the control function still involves high frequency switching (see, for instance, Plestan *et al*, 2010) or the high frequency

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switching control action is hidden behind an integral term (see, for instance, Bartolini *et al*, 1999; Shtessel *et al*, 2010c). Classical adaptive control algorithms are robust to disturbances/uncertainties presented in a regressive format with unknown bounds (Sastry & Bodson, 1989; Astolfi *et al*., 2008). Note that classical adaptive control is continuous and usually does not contain discontinuous control terms and provides asymptotic convergence only. The distinctive feature of second order sliding mode control (2-SMC) is its ability to provide finite time convergence to zero not only to the sliding variable, but also to its derivative in the presence of bounded disturbances/uncertainties. The main advantage of 2-SMC becomes clear when it is implemented in discrete time: the accuracy of the sliding variable stabilization is enhanced and is proportional to τ^2, where τ is the time increment. It is worth noting that 2-SMC, including twisting and prescribed convergence law control algorithms (Levant, 2003)), generates high frequency switching control or continuous control with high frequency terms hidden behind the integral as in the super-twisting control algorithm (Levant, 2003; Bartolini *et al*, 1999). Thus, 2-SMC super-twisting control *alleviates* chattering.

In this paper, we consider the problem of designing continuous adaptive control for first and second order systems that drives the state variable and its derivative to zero in finite time in the presence of bounded disturbances presented in a regressive form with unknown bounds (the preliminary results are presented in the conference proceedings: (Shtessel et al, 2009, Shtessel et al, 2010a, 2010b)) in order to enhance the stabilization accuracy. The control action is not supposed to contain any discontinuous terms and *eliminates* chattering. The designed control law can be interpreted as continuous second order sliding mode control.

The paper is organized as follows. In Section 2, we define the problem statement. Then in Section 3, we discuss a theorem for generating continuous control and its modified version and application with adaptation to an arbitrary order system. In Section 4, we present the main results for a class of first order systems. In Section 5, we present the results for finite time convergence for a variety of second order systems. In Section 6, we illustrate the design methodology for all cases via numerical examples, and finally in Section 7, some concluding remarks are outlined.

2. Problem statement

Consider a single-input-single-output (SISO) uncertain n[th] order nonlinear system

$$x^{(n)} = f(\bar{x},t) + u \tag{2.1}$$

where $x \in \mathbb{R}$ is the output, $u \in \mathbb{R}$ is the control function, $\bar{x} = \left[x_1, x_2, ..., x_n\right]^T$ is a state vector with $x = x_1,\ \dot{x} = x_2, ..., x^{(n-1)} = x_n$, and $f(\bar{x},t) \in \mathbb{R}$ is a differentiable, partially known drift function. The partially known function $f(\bar{x},t)$ is assumed to be presented in a regressive form

$$f(\bar{x},t) = \theta^T \varphi(\bar{x},t) \tag{2.2}$$

where $\theta \in \mathbb{R}^m$ is an unknown bounded constant vector of parameters with unknown bounds, and $\varphi(\bar{x},t) \in \mathbb{R}^m$ is a known vector-function.

Consider an augmented system defined as

$$\begin{cases} x^{(n)} = \theta^T \varphi(\bar{x},t) + u \\ \dot{\hat{\theta}} = \omega \end{cases} \tag{2.3}$$

The problem is to design an adaptive continuous state feedback control law

$$u = u(\bar{x},\hat{\theta}),\ \ \omega = \omega(\bar{x},\hat{\theta}) \tag{2.4}$$

that drives $x(t),\ \dot{x}(t), ..., x^{(n-1)}(t) \to 0$ or $x_1(t),\ x_2(t), ..., x_n(t) \to 0$ in finite time.

3. Finite reaching time continuous control

In the work of (Bhat & Bernstein, 2005), a continuous finite convergence time nonlinear control was developed for non-perturbed arbitrary order systems. This result is formulated in the following theorem:

THEOREM 1: Let $k_1, k_2 .... k_n > 0$ be such that the polynomial $\Delta(s) = s^n + k_n s^{n-1} + ... k_2 s + k_1$ is Hurwitz, and consider the system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ &\cdots\cdots\cdots \\ \dot{x}_n &= u \end{aligned} \tag{3.1}$$

then there exists a constant $\varepsilon_0 \in (0,1)$ such that for every $\alpha_i \in (1-\varepsilon_0, 1)$, the origin is a globally finite time stable equilibrium for the system (3.1) under the feedback law

$$u = -k_1 |x_1|^{\alpha_1} \operatorname{sgn} x_1 - ..... - k_n |x_n|^{\alpha_n} \operatorname{sgn} x_n \tag{3.2}$$

where the coefficients $\alpha_1, \alpha_2 .... \alpha_n$ satisfy

$$\alpha_{i-1} = \frac{\alpha_i \alpha_{i+1}}{2\alpha_{i+1} - \alpha_i}, i = 2, ..., n \tag{3.3}$$

with $\alpha_{n+1} = 1$ and $\alpha_n = \alpha$.

COROLLARY 1: Let $k_1, k_2 ..., k_n > 0$ and $\gamma_1, \gamma_2 ..., \gamma_n > 0$ be such that the polynomial

$$\tilde{\Delta}(s) = s^n + \mu_n s^{n-1} + ... + \mu_2 s + \mu_1, \quad \mu_i = k_i |\tilde{x}_i|^{1-\alpha_i} + \gamma_i \tag{3.4}$$

where $|\tilde{x}_i| > \varepsilon_i > 0, \ i = 1, 2, ..., n$, is Hurwitz, then the control law

$$u = -k_1 |x_1|^{\alpha_1} \operatorname{sgn} x_1 - ..... - k_n |x_n|^{\alpha_n} \operatorname{sgn} x_n - \gamma_1 x_1 - \gamma_2 x_2 - ... - \gamma_n x_n \tag{3.5}$$

provides finite time stabilization to the origin of system (3.1).

Sketch of the proof. The rational behind replacing the finite time convergent control law (3.2), (3.3) by the the control law (3.3), (3.5) is that the control law (3.5) is more receptive to the large initial conditions of the system's (3.1) states. The advantage of the proposed control format is that the linear control terms $\gamma_i x_i$ will dominate over the corresponding nonlinear control terms $-k_i |x_i|^{\alpha_i} \operatorname{sgn} x_i$ for $|x_i| > 1$ that yields faster asymptotic convergence of the states x_i driven by the control law (3.5) rather then by the control law (3.2). On the other hand, the nonlinear control terms $-k_i |x_i|^{\alpha_i} \operatorname{sgn} x_i$ will dominate over the linear ones $\gamma_i x_i$ for $|x_i| < 1$ that yields finite time convergence in accordance with Theorem 1. Analyzing system's (3.1), (3.3), (3.5) dynamics the 2 cases are considered.

Case 1. Assume $|x_i| \le \varepsilon_i << 1, \ \varepsilon_i > 0, \ i = 1, 2, ..., n$. The control law (3.5) can be presented in a form $u = -\sum_{i=1}^{n} (k_i |x_i|^{\alpha_i - 1} + \gamma_i) x_i$. Then, the choice of the gains $k_i >> \gamma_i \varepsilon_i^{1-\alpha_i}$ (these conditions are not conservative and are easy to fulfill, since ε_i are small positive numbers) provides the domination of the nonlinear terms $-k_i |x_i|^{\alpha_i} \operatorname{sgn} x_i$ that yields finite time convergence in the domain $|x_i| \le \varepsilon_i << 1, \ \varepsilon_i > 0, \ i = 1, 2, ..., n$ in accordance with Theorem 1.

Case 2. Assume $|x_i| > \varepsilon_i$. Linearizing system (3.1) in the points $x_i = \tilde{x}_i$, $|\tilde{x}_i| > \varepsilon_i$ we obtain a characteristic polynomial of compensated system (3.1) as

$$\tilde{\Delta}(s) = s^n + \mu_n s^{n-1} + ... + \mu_2 s + \mu_1, \quad \mu_i = k_i |\tilde{x}_i|^{1-\alpha_i} + \gamma_i$$

Let the gains of this polynomial $\mu_i(k_i, \gamma_i) > 0$ be such that the polynomial is Hurwitz. Then the corresponding gains k_i, γ_i are to be selected accordingly. The conditions $k_i >> \gamma_i \varepsilon_i^{1-\alpha_i}$ also must be taken into account. It was stated before that these conditions are easy to fulfill, since ε_i are small positive numbers. Then $x_i \to 0$ as time increases. As soon as $|x_i| \le \varepsilon_i$ the nonlinear terms start dominating and the finite time convergence is provided (see Case 1). It is worth noting that for $\alpha_i = 1,\ i = 1, 2, ..., n$ the polynomial (3.4) becomes $\tilde{\Delta}(s) = s^n + (k_n + \gamma_n) s^{n-1} + ... + (k_2 + \gamma_2) s + k_1 + \gamma_1$. Then the control law (3.5) provides only asymptotic convergence. The detailed study of the control law (3.5) for $n = 1$ using the Lyapunov function technique is provided in the proof of Theorem 2 that is presented in Section 4.

3.1 The known-parameter finite reaching time controller design

Assuming that the parameter vector θ is initially known, and then the control law can be designed for system (2.3) in the form

$$u = -k_1 |x_1|^{\alpha_1} \operatorname{sgn} x_1 - ... - k_n |x_n|^{\alpha_n} \operatorname{sgn} x_n - \gamma_1 x_1 - \gamma_2 x_2 - ... - \gamma_n x_n - \theta^T \varphi(\bar{x}, t) \qquad (3.6)$$

It is easy to see that the compensated dynamics of system (2.3) in an n^{th} order differential equation format becomes

$$x^{(n)} = -k_1 |x|^{\alpha_1} \operatorname{sgn} x - ..... - k_n \left|x^{(n-1)}\right|^{\alpha_n} \operatorname{sgn} x^{(n-1)} - \gamma_1 x - \gamma_2 \dot{x} - ... - \gamma_n x^{(n-1)} \qquad (3.7)$$

and the origin is a globally finite time stable equilibrium in accordance with Corollary 1 of Theorem 1 if the coefficients $k_1, k_2 ..., k_n > 0$ and $\gamma_1, \gamma_2 ..., \gamma_n > 0$ are selected in such a way that the polynomial (3.4) is Hurwitz.

3.2 Adaptive nonlinear control law with finite time convergence

Assume that the known function $\varphi(\bar{x}, t) \in \mathbb{R}^m$ does not depend explicitly on time, i.e. $\varphi(\bar{x}, t) = \varphi(\bar{x})$ and can be factorized as $\varphi(\bar{x}) = |x_n|^{1/2} \psi(x_n)$, where $\psi(x_n) \in \mathbb{R}^m$ is a known vector-function. Introduce an estimated parameter $\hat{\theta}$, and the adaptation vector function $\beta(x_n) \in \mathbb{R}^m$, such that the adaptive control law is constructed as

$$u(\bar{x}, \hat{\theta}) = -\left[\hat{\theta} + \beta(x_n)\right]^T \psi(x_n) |x_n|^{1/2} - k_1 |x_1|^{\alpha_1} \operatorname{sgn} x_1 - ..... - k_n |x_n|^{\alpha_n} \operatorname{sgn} x_n - \gamma_1 x_1 - \gamma_2 x_2 - ... - \gamma_n x_n \qquad (3.8)$$

Introduce an auxiliary variable $z \in \mathbb{R}^m$ according to

$$z = \hat{\theta} - \theta + \beta(x_n) \qquad (3.9)$$

where $\beta = \beta(x_n)$ is a vector-function to be determined. Its dynamics are given by

$$\dot{z} = \dot{\hat{\theta}} + \frac{d\beta(x_n)}{dx_n}\dot{x}_n = \dot{\hat{\theta}} + \frac{d\beta(x_n)}{dx_n}[\theta^T\psi(x_n)|x_n|^{1/2} - \left(\hat{\theta}+\beta(x_n)\right)^T\psi(x_n)|x_n|^{1/2}$$
$$-k_1|x_1|^{\alpha_1}\operatorname{sgn} x_1 - .....- k_n|x_n|^{\alpha_n}\operatorname{sgn} x_n - \gamma_1 x_1 - \gamma_2 x_2 - ... - \gamma_n x_n] \tag{3.10}$$
$$= \dot{\hat{\theta}} + \frac{d\beta(x_n)}{dx_n}[-z^T\psi(x_n)|x_n|^{1/2} - k_1|x_1|^{\alpha_1}\operatorname{sgn} x_1 - .....- k_n|x_n|^{\alpha_n}\operatorname{sgn} x_n -$$
$$\gamma_1 x_1 - \gamma_2 x_2 - ... - \gamma_n x_n]$$

Define the following adaptation law for the parameter θ

$$\dot{\hat{\theta}} = \frac{d\beta(x_n)}{dx_n}[k_1|x_1|^{\alpha_1}\operatorname{sgn} x_1 + ...+ k_n|x_n|^{\alpha_n}\operatorname{sgn} x_n + \gamma_1 x_1 + \gamma_2 x_2 + ... + \gamma_n x_n] \tag{3.11}$$

Then the compensated dynamics of the auxiliary variable z are

$$\dot{z} = -\left[\frac{d\beta(x_n)}{dx_n}\psi^T(x_n)|x_n|^{1/2}\right]z \tag{3.12}$$

Stability of the auxiliary variable compensated dynamics (3.12) can be achieved by selecting $\beta = \beta(x_n)$ so that

$$\frac{d\beta(x_n)}{dx_n}\varphi^T(x) = Q(x) \geq 0 \tag{3.13}$$

where $Q(x) \in \mathbb{R}^{m\times m}$ is a positive semi-definite matrix. For instance, if the vector-function $\beta(x_n)$ is defined as

$$\frac{d\beta(x_n)}{dx_n} = \frac{1}{4\gamma_0}\cdot\frac{\varphi(x_n)}{|x_n|} \tag{3.14}$$

then the semi-definite matrix $Q \in \mathbb{R}^{m\times m}$ in eq. (3.13) has the form

$$Q(x_n) = \frac{1}{4\gamma_0}\cdot\psi(x_n)\cdot\psi^T(x_n) \tag{3.15}$$

Finally, the dynamics of the adaptive system in (2.3), governed by the adaptive control law in (3.8), (3.11), (3.14) can be written as

$$\begin{aligned} x_1^{(n)} &= -z^T\psi(x_n)|x_n|^{1/2} - k_1|x_1|^{\alpha_1}\operatorname{sgn} x_1 - .....- k_n|x_n|^{\alpha_n}\operatorname{sgn} x_n - \gamma_1 x_1 - \gamma_2 x_2 - ... - \gamma_n x_n] \\ \dot{z} &= -Q(x_n)z \end{aligned} \tag{3.16}$$

4. Adaptive control law with finite time convergence for first order system

In this subsection we study system (2.3) in the special case when $n = 1$. The adaptive control law in (3.8), (3.11), (3.15) can be written in the form

$$\begin{cases} u(x_1,\hat{\theta}) & = & -\left[\hat{\theta}+\beta(x_1)\right]^T \psi(x_1)|x_1|^{1/2} - k_1|x_1|^{\alpha_1}\operatorname{sgn} x_1 - \gamma_1 x_1 \\ \dot{\hat{\theta}} & = & \dfrac{d\beta(x_1)}{dx}[k_1|x_1|^{\alpha_1}\operatorname{sgn} x_1 + \gamma_1 x_1] \\ \dfrac{d\beta(x_1)}{dx} & = & \dfrac{1}{4\gamma_0}\cdot\dfrac{\psi(x_1)}{|x_1|^{1/2}}, \quad \gamma_0 = \gamma_1 \end{cases} \tag{4.1}$$

Therefore, the dynamics of the adaptive system in (3.16), with $n=1$ and with the adaptive control law in (4.1) becomes

$$\begin{aligned} \dot{x}_1 &= -z^T\psi(x_1)|x_1|^{1/2} - \gamma_1 x_1 - k_1|x_1|^{\alpha_1}\operatorname{sgn} x_1 \\ \dot{z} &= -Q(x_1)z \end{aligned} \tag{4.2}$$

THEOREM 2: For the system given in (4.2) $x(t), \dot{x}(t) \to 0$ in finite time, while $z(t) \to 0$ as time increases.

Proof. The proof is split into two steps. In the first step we will show that $z^T\psi(x) \to 0,\ x \to 0$ as time increases. In order to do this the following Lyapunov function candidate is introduced.

$$V_1 = V_2 + \frac{1}{2}z^T z, \quad V_2 = |x_1| \tag{4.3}$$

Its derivative is calculated on the trajectory of eq. (4.2) as

$$\begin{aligned} \dot{V}_1 &= \dot{x}_1 \cdot sign(x_1) + z^T\dot{z} \\ &= -z^T\psi(x_1)|x_1|^{1/2} sign(x_1) - \gamma_1|x_1| - k_1|x_1|^{\alpha_1} - \frac{1}{4\gamma_1}z^T\psi(x_1)\psi^T(x_1)z \\ &= -z^T\psi(x_1)|x_1|^{1/2} sign(x_1) - \gamma_1|x_1| - \frac{1}{4\gamma_1}\left(z^T\psi(x_1)\right)^2 - k_1|x_1|^{\alpha_1} \\ &= -\frac{1}{\gamma_1}\left(\frac{1}{2}z^T\psi(x_1) + \gamma_1|x_1|^{1/2} sign(x_1)\right)^2 - k_1|x_1|^{\alpha_1} \end{aligned} \tag{4.4}$$

Based on (4.3) and (4.4), it can be observed that $z^T\psi(x_1) \to 0,\ x_1 \to 0$ as time increases. Furthermore, $z(t)$ remains bounded due to (3.12) and (4.2). Also, it is not imperative that $\psi(x_1) \to 0$ as $x_1 \to 0$. However if we assume the entries of the vector-function $\psi(x_1)$ are linearly independent, then $z^T\psi(x_1) \to 0,\ x_1 \to 0$ yields $\dot{V}_1 < 0$, and hence $z \to 0$ as time increases.

Next, we will prove that $x_1 \to 0$ in finite time. Indeed, since $z^T\psi(x_1) \to 0$ asymptotically, there exists a time instant t_1 such that $\left|z^T\psi\right| \le k_1 - \eta \quad \forall t \ge t_1$ where η is a positive scalar satisfying $\eta < k_1$. Consider $V_2 = |x_1|$ as a candidate Lyapunov function to demonstrate finite time convergence of x_1 in eq. (4.2) to the origin.

$$\begin{aligned} \dot{V}_2 &= \dot{x}_1 \cdot sign(x_1) = -z^T\psi(x_1)|x_1|^{1/2} sign(x_1) - \gamma_1|x|_1 - k_1|x_1|^{\alpha_1} \\ &\le \left|z^T\psi(x_1)\right||x_1|^{1/2} - \gamma_1|x_1| - k_1|x_1|^{\alpha_1} \le \left|z^T\psi(x_1)\right||x_1|^{1/2} - k_1|x_1|^{\alpha_1} \end{aligned} \tag{4.5}$$

Since it is proven that $x_1 \to 0$ as time increases, then there exists a time $t = \tau$ such that $|x_1| \le 1\ \forall t \ge \tau$. Therefore, bearing in mind that $0 < \alpha_1 \le 0.5$, it follows that $-k_1|x_1|^{\alpha_1} \le -k_1|x_1|^{1/2} \quad \forall |x_1| \le 1$

Now inequality (4.5) yields $\forall t \ge t_1 > \tau$

$$\dot{V}_2 \le \left|z^T\psi(x_1)\right|\left|x_1\right|^{1/2} - k_1\left|x_1\right|^{1/2} \le \left(\left|z^T\psi(x_1)\right| - k_1\right)\left|x_1\right|^{1/2} \le -\eta\left|x_1\right|^{1/2} \le -\eta V_2^{1/2} \tag{4.6}$$

It follows that $x_1(t)$ converges to the origin in finite time t_r that is estimated as:

$$t_r \le \frac{2}{\eta} V_2^{1/2}(0) \tag{4.7}$$

From eq. (4.2), as soon as $x_1(t)$ reaches zero, $x_2(t) = \dot{x}_1(t)$ reaches zero as well. Consequently Theorem 2 is proven.

5. Adaptive control law with finite time convergence for second order system: Case 1

In this subsection, system (2.3) is studied with $n = 2$. For such a system, the target finite convergent time compensated dynamics are obtained from eq. (3.7) as

$$\begin{cases} \ddot{x}_1 = -\gamma_1 x_1 - \gamma_2 x_2 - k_1\left|x_1\right|^{\alpha_1}\operatorname{sgn} x_1 - k_2\left|x_2\right|^{\alpha_2}\operatorname{sgn} x_2 \\ x_2 = \dot{x}_1 \end{cases} \tag{5.1}$$

The adaptive control law in (3.8), (3.11), (3.14) can be written in the form

$$\begin{cases} u(x_1, x_2, \hat{\theta}) = -\left[\hat{\theta} + \beta(x_2)\right]^T \psi(x_2)\left|x_2\right|^{1/2} - k_1\left|x_1\right|^{\alpha_1}\operatorname{sgn} x_1 - k_2\left|x_2\right|^{\alpha_2}\operatorname{sgn} x_2 - \gamma_1 x_1 - \gamma_2 x_2 \\ \quad\dot{\hat{\theta}} \quad = \dfrac{d\beta(x_2)}{dx_2}[k_1\left|x_1\right|^{\alpha_1}\operatorname{sgn} x_1 + k_2\left|x_2\right|^{\alpha_2}\operatorname{sgn} x_2 + \gamma_1 x_1 + \gamma_2 x_2] \\ \dfrac{d\beta(x_2)}{dx_2} = \dfrac{1}{4\gamma_0}\cdot\dfrac{\psi(x_2)}{\left|x_2\right|^{1/2}} \end{cases} \tag{5.2}$$

and the dynamics of the adaptive system becomes

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -z^T\psi(x_2)\left|x_2\right|^{1/2} - k_1\left|x_1\right|^{\alpha_1}\operatorname{sgn} x_1 - k_2\left|x_2\right|^{\alpha_2}\operatorname{sgn} x_2 - \gamma_1 x_1 - \gamma_2 x_2 \\ \dot{z} = -Q(x)z \end{cases} \tag{5.3}$$

THEOREM 3: For the system given in (5.3) $x_1(t), x_2(t) \to 0$ in finite time, while $z(t) \to 0$ as time increases.

*Sketch of the proof***.** The proof largely follows the steps of the proof for Theorem 2. In the first step we can show that $z^T\psi(x_2) \to 0,\ x_1 \to 0,\ x_2 \to 0$ as time increases using the Lyapunov function $V_1 = c_1\left|x_1\right| + c_2\left|x_2\right| + \frac{1}{2}z^T z,\quad c_1, c_2 > 0$. In the second step, it can be proven that $x_1 \to 0,\ x_2 \to 0$ in finite time by using another Lyapunov function candidate $V_2 = \left|x_1\right| + \left|x_2\right|$.

Remark 1. Note that the adaptive control law (5.2) can be interpreted as adaptive continuous second order sliding mode control since $x_1(t), \dot{x}_1(t) \to 0$ in finite time in the presence unknown bounded disturbance term $\theta^T\varphi(x_2)$ with unknown bound.

5.1 Adaptive non linear control law for a second order system: Case 2

In this subsection, we study a special class of second order systems, where the single control u is split into two controls u_1 and u_2, each appearing separately in the equations governing the system.

Consider the following second order system

$$\begin{cases} \dot{x}_1 = x_2 + u_1 \\ \dot{x}_2 = \theta^T \varphi(x_2) + u_2 \end{cases} \tag{5.4}$$

where $\tilde{x} = [x_1, x_2]^T$ is the state vector, $\tilde{u} = [u_1, u_2]^T$ is the control vector, $\theta \in \mathbb{R}^m$ is a bounded unknown vector of parameters with unknown bounds, and $\varphi(x_2) \in \mathbb{R}^m$ is a known vector-function. Assume $\varphi(x_2)$ can be factorized as $\varphi(x_2) = |x_2|^{1/2} \psi(x_2)$, where $\psi(x_2) \in \mathbb{R}^m$ is a known vector-function.

5.1.1 The known-parameter nonlinear controller design.

Consider the cascade dynamics

$$\begin{cases} \dot{x}_1 = x_2 - \alpha_1 |x_1|^{a_1} sgn(x_1) \\ \dot{x}_2 = -\alpha_2 |x_2|^{a_2} sgn(x_2) - \gamma x_2 \end{cases} \tag{5.5}$$

where $\alpha_1, \alpha_2 > 0$, $a_1 \in (0,1)$, $a_2 \in (0, 0.5]$.

It is easy to see that equation $\dot{x}_2 = -\alpha_1 |x_2|^{a_2} sgn(x_2) - \gamma x_2$ is finite time convergent, i.e. $x_2, \dot{x}_2 \to 0$ in finite time t_1. Then equation $\dot{x}_1 = x_2 - \alpha_1 |x_1|^{a_1} sgn(x_1)$ becomes $\dot{x}_1 = -\alpha_1 |x_1|^{a_1} sgn(x_1)$ $\forall t \geq t_1$ which is also finite time convergent, i.e. $x_1, \dot{x}_1 \to 0$ in finite time $t_f = t_1 + t_2$. Therefore, $x_1, x_2, \dot{x}_1, \dot{x}_2, \to 0$ and system (5.5) can be considered as a finite time convergent compensated target system for (5.4).

If the parameter θ is known, the corresponding control law $u = u(x_1, x_2, \theta)$ can be designed as

$$u = \begin{bmatrix} u_1(x_1) \\ u_2(x_1, x_2, \theta) \end{bmatrix} = \begin{bmatrix} -\alpha_1 |x_1|^{a_1} sgn(x_1) \\ -\theta^T \varphi(x_2) - \gamma x_2 - \alpha_2 |x_2|^{a_2} sgn(x_2) \end{bmatrix}, \ \alpha_1, \alpha_2, \gamma > 0 \tag{5.6}$$

5.1.2 Adaptive nonlinear control law with finite time convergence.

Now suppose the parameter θ is unknown. Introduce the estimated parameter $\hat{\theta}$, and the adaptation vector-function $\beta(x_2) \in \mathbb{R}^m$ so that the control law $u_2 = u_2(x_1, x_2, \theta)$ in eq. (5.6) is constructed as

$$u_2(x_1, x_2, \hat{\theta}) = -\left[\hat{\theta} + \beta(x_2)\right]^T \varphi(x_2) - \gamma x_2 - \alpha_2 |x_2|^{a_2} sgn(x_2) \tag{5.7}$$

As before, introduce an auxiliary variable $z \in \mathbb{R}^m$, where

$$z = \hat{\theta} - \theta + \beta(x_2) \tag{5.8}$$

and thus

$$\dot{z} = \dot{\hat{\theta}} + \frac{d\beta(x_2)}{dx_2} \dot{x}_2 = \dot{\hat{\theta}} + \frac{d\beta(x_2)}{dx_2} \left[-z^T \varphi(x_2) - \gamma x_2 - \alpha_2 |x_2|^{a_2} sgn(x_2) \right] \tag{5.9}$$

Introduce the following adaptation law for the parameter θ

$$\dot{\hat{\theta}} = \frac{d\beta(x_2)}{dx_2}\left(\alpha_2 \left|x_2\right|^{a_2} sgn(x_2) + \gamma x_2\right) \tag{5.10}$$

The compensated dynamics of the auxiliary variable z are given by

$$\dot{z} = -\left[\frac{d\beta(x_2)}{dx_2}\varphi^T(x_2)\right]z \tag{5.11}$$

The stability of the auxiliary variable compensated dynamics in (5.11) can be achieved by selecting $\beta = \beta(x)$ so that $\frac{d\beta(x_2)}{dx}\varphi^T(x_2) = Q(x_2) \geq 0$, where $Q(x_2) \in \mathbb{R}^{m\times m}$ is a positive semi-definite matrix. For example, if

$$\frac{d\beta(x_2)}{dx_2} = \frac{1}{4\gamma}\cdot\frac{\varphi(x_2)}{\left|x_2\right|}$$

then

$$Q = \frac{1}{4\gamma}\cdot\psi(x_2)\cdot\psi^T(x_2)\,.$$

Finally, the dynamics of the adaptive system in (5.4) becomes

$$\dot{x}_1 = -\alpha_1\left|x_1\right|^{a_1} sgn(x_1) + x_2 \tag{5.12}$$

$$\dot{x}_2 = -z^T\psi(x_2)\left|x_2\right|^{1/2} - \gamma x_2 - \alpha_2\left|x_2\right|^{a_2} sgn(x_2) \tag{5.13}$$

$$\dot{z} = -Q(x)z \tag{5.14}$$

It is easy to see that the adaptive system dynamics in (5.13), (5.14) coincide with the dynamics of adaptive system (4.2). Therefore, the following theorem is formulated by analogy to Theorem 1:

THEOREM 4: For the system given in (5.12), (5.13) and (5.14) $x_1(t), x_2(t), \dot{x}_1(t), \dot{x}_2(t) \to 0$ in finite time while $z(t) \to 0$ as time increases.

*Proof***.** The finite time convergence $x_2(t),\ \dot{x}_2(t) \to 0$ and the convergence $z(t) \to 0$ as time increases, can be proven by analogy with the proof of Theorem 1. As soon as $x_2(t) = 0 \quad \forall t \geq t_r$, (5.12) becomes $\dot{x}_1 = -\alpha_1\left|x_1\right|^{a_1} sgn(x_1)$ and $x_1(t), \dot{x}_1(t) \to 0$ in finite time. Theorem 4 is proven.

It is worth noting that the cascade dynamics in (5.5) are not the only possible smooth second order dynamics with finite convergence time. Different smooth 2nd order finite time convergent dynamics are used next as the target system and this yields another adaptive finite time convergent control algorithm.

Remark 2. Note that the adaptive control law (5.7) can be interpreted as adaptive continuous second order sliding mode control since $x_1(t), \dot{x}_1(t) \to 0$ in finite time in the presence unknown bounded disturbance term $\theta^T\varphi(x_2)$ with unknown bound.

5.2 Finite reaching time adaptive nonlinear control for the second order system: case 3

The dynamics in system (5.4) is modified using a relative degree approach while introducing a scalar control function u instead of the vector control function $\left[u_1, u_2\right]^T$. A presentation of the system's dynamics in this way makes sense, since it can be easily generalized for uncertain input-output dynamics with an arbitrary relative degree. The following uncertain second order input-output dynamics are obtained by differentiating first equation in eq. (5.4):

$$\ddot{x}_1 = \theta^T \varphi(x_1, \dot{x}_1) + u \tag{5.15}$$

where $x_1 \in \mathbb{R}$ is the state, $u = \dot{u}_1 + u_2 \in \mathbb{R}$ is the control function, $\theta \in \mathbb{R}^m$ is a bounded unknown vector of parameters with unknown bounds, and $\varphi(x_1,\dot{x}_1) \in \mathbb{R}^m$ is a known vector-function. Assume $\varphi(x_1,\dot{x}_1) \in \mathbb{R}^m$ can be factorized as $\varphi(x_1,\dot{x}_1)=\left|\dot{x}_1\right|^{1/2} \psi(\dot{x}_1)$, where $\psi(\dot{x}_1) \in \mathbb{R}^m$ is a known vector-function.

5.2.1 Known-parameter nonlinear controller design.

Equation (5.5) represents the finite time-convergent compensated target system for eq. (5.4). Therefore, we use eq. (5.5) to obtain the target system for eq. (5.15) by differentiating first equation in eq (5.4). Finally, the cascade finite time-convergent dynamics (5.5) is transformed to a finite time-convergent $x_1, \dot{x}_1 \to 0$ second order differential equation

$$\begin{aligned} &\ddot{x}_1 + \left(\alpha_1 a_1 \left|x_1\right|^{a_1-1} sgn\left(x_1\right)\right)\dot{x}_1 + \alpha_2 \left|\dot{x}_1 + \alpha_1 \left|x_1\right|^{a_1} sgn\left(x_1\right)\right|^{a_2} sgn\left(\dot{x}_1 + \alpha_1 \left|x_1\right|^{a_1} sgn\left(x_1\right)\right) \\ &+ \gamma\left(\dot{x}_1 + \alpha_1 \left|x_1\right|^{a_1} sgn\left(x_1\right)\right) = 0 \end{aligned} \tag{5.16}$$

where $\alpha_1, \alpha_2 > 0,\ a_1 \in (0,1),\ a_2 \in (0, 0.5]$.

Given that the parameter θ is known, the corresponding control law $u = u(x_1, \dot{x}_1, \theta)$ can be designed for system (5.15) as

$$\begin{aligned} u = &-\theta^T \varphi(x_1, \dot{x}_1) - \left(\alpha_1 a_1 \left|x_1\right|^{a_1-1} sgn\left(x_1\right)\right)\dot{x}_1 \\ &- \alpha_2 \left|\dot{x}_1 + \alpha_1 \left|x_1\right|^{a_1} sgn\left(x_1\right)\right|^{a_2} sgn\left(\dot{x}_1 + \alpha_1 \left|x_1\right|^{a_1} sgn\left(x_1\right)\right) - \gamma\left(\dot{x}_1 + \alpha_1 \left|x_1\right|^{a_1} sgn\left(x_1\right)\right) \end{aligned} \tag{5.17}$$

Additional constraints needs to be imposed on the coefficients a_1 and a_2, otherwise the term $\left(\alpha_1 a_1 \left|x_1\right|^{a_1-1} sgn\left(x_1\right)\right)\dot{x}_1$ can become discontinuous at $x_1 = 0$. Thus, as soon as $x_2(t) = \dot{x}_1 + \alpha_1 \left|x_1\right|^{a_1} sgn\left(x_1\right) \to 0$ in finite time, the variable x_1 satisfies equation $\dot{x}_1 = -\alpha_1 \left|x_1\right|^{a_1} sgn\left(x_1\right)$ and hence $\left(\alpha_1 a_1 \left|x_1\right|^{a_1-1} sgn\left(x_1\right)\right)\dot{x}_1 = -\alpha_1^2 a_1 \left|x_1\right|^{2a_1-1} sgn\left(x_1\right)$ (5.18)

Ensuring the following inequality

$$\alpha_1, \alpha_2 > 0,\ a_1 \in (0.5,1),\ a_2 \in (0, 0.5] \tag{5.19}$$

retains the continuity of the control function (5.17) at $x_1 = 0$ while preserving the finite time convergence.

5.2.2 Adaptive nonlinear control law with finite time convergence

As before, introduce the estimate parameter $\hat{\theta}$ and the adaptation vector-function $\beta(\dot{x}_1) \in \mathbb{R}^m$ so that the control law $u = u(x_1, \dot{x}_1, \hat{\theta})$ in (5.15) is constructed as

$$u(x_1,\dot{x}_1,\hat{\theta}) = -\left[\hat{\theta}+\beta(\dot{x}_1)\right]^T \varphi(\dot{x}_1) - \left(\alpha_1 a_1 \left|x_1\right|^{a_1-1} sgn\left(x_1\right)\right)\dot{x}_1$$
$$-\alpha_2\left|\dot{x}_1+\alpha_1\left|x_1\right|^{a_1} sgn\left(x_1\right)\right|^{a_2} sgn\left(\dot{x}_1+\alpha_1\left|x_1\right|^{a_1} sgn\left(x_1\right)\right) - \gamma\left(\dot{x}_1+\alpha_1\left|x_1\right|^{a_1} sgn\left(x_1\right)\right) \tag{5.20}$$

Next, introduce an auxiliary variable $z \in \mathbb{R}^m$

$$z = \hat{\theta} - \theta + \beta(\dot{x}_1) \tag{5.21}$$

It can be observed that its dynamics satisfy

$$\dot{z} = \dot{\hat{\theta}} + \frac{d\beta(\dot{x}_1)}{d\dot{x}_1}\ddot{x}_1 = \dot{\hat{\theta}} + \frac{d\beta(\dot{x}_1)}{d\dot{x}_1}[-z^T\varphi(\dot{x}_1) - \left(\alpha_1 a_1\left|x_1\right|^{a_1-1} sgn\left(x_1\right)\right)\dot{x}_1$$
$$-\gamma x_2 - \alpha_2\left|x_2\right|^{a_2} sgn(x_2)];\ x_2 = \dot{x}_1 + \alpha_1\left|x_1\right|^{a_1} sgn\left(x_1\right) \tag{5.22}$$

Let

$$\dot{\hat{\theta}} = \frac{d\beta(\dot{x}_1)}{d\dot{x}_1}[\left(\alpha_1 a_1\left|x_1\right|^{a_1-1} sgn\left(x_1\right)\right)\dot{x}_1 + \alpha_2\left|x_2\right|^{a_2} sgn(x_2) + \gamma x_2] \tag{5.23}$$

and hence

$$\dot{z} = -\left[\frac{d\beta(\dot{x}_1)}{d\dot{x}_1}\varphi^T(\dot{x}_1)\right]z \tag{5.24}$$

Select

$$\beta = \beta(x) \text{ so that } \frac{d\beta(\dot{x}_1)}{d\dot{x}_1}\varphi^T(\dot{x}_1) = Q(\dot{x}_1) \geq 0 \tag{5.25}$$

where $Q(\dot{x}_1) \in \mathbb{R}^{m\times m}$ is a positive semi-definite matrix.. For instance,

$$\frac{d\beta(\dot{x}_1)}{d\dot{x}_1} = \frac{1}{4\gamma}\cdot\frac{\varphi(\dot{x}_1)}{\left|\dot{x}_1\right|} \text{ and } Q = \frac{1}{4\gamma}\cdot\psi(\dot{x}_1)\cdot\psi^T(\dot{x}_1)\ .$$

Finally, the dynamics of the adaptive system (5.15), (5.19), (5.20), (5.23), (5.25) can be presented as

$$\begin{cases} \ddot{x}_1 = -z^T\varphi(\dot{x}_1) - \left(\alpha_1 a_1\left|x_1\right|^{a_1-1} sign\left(x_1\right)\right)\dot{x}_1 \\ \quad -\gamma x_2 - \alpha_2\left|x_2\right|^{a_2} sign(x_2);\quad x_2 = \dot{x}_1 + \alpha_1\left|x_1\right|^{a_1} sign\left(x_1\right) \\ \dot{z} = -Q(\dot{x}_1)z \end{cases} \tag{5.26}$$

The adaptive system dynamics in (5.26) are equivalent to the dynamics of the adaptive system (5.12), (5.13), (5.14) presented in a different basis. Hence, the following theorem is formulated by analogy to Theorem 4.

THEOREM 5: For the system given in (5.26) $x_1(t), \dot{x}_1(t) \to 0$ in finite time while $z(t) \to 0$ as time increases.

Proof. This can be shown by analogy to the proofs of Theorems 1 and 4.

Remark 3. Note that the adaptive control law (5.20) can be interpreted as adaptive continuous second order sliding mode control since $x_1(t), \dot{x}_1(t) \to 0$ in finite time in the presence unknown bounded disturbance term $\theta^T\varphi(x_2)$ with unknown bound.

6. Simulation examples

The first and second order systems using adaptive control are simulated using a numerical example. The parameter θ remains unknown. However a value of $\theta = 10$ is taken for simulation purposes for every case.

6.1 Finite convergence time adaptive SOSM control for the first order system

The first order control system (2.3) with $n = 1$ becomes

$$\dot{x} = \theta^T \varphi(x,t) + u \tag{6.1}$$

The system is simulated using (3.12), (3.16) with $n = 1$ and the function $\varphi(x,t) = x^2$. The function β is selected as the solution of the differential equation $\frac{d\beta}{dx} = \frac{1}{4\gamma_1} x \operatorname{sgn} x$, which is $\beta = \frac{1}{8\gamma_1} x^2$,while the other parameters are selected as $\alpha_1 = 1/2, \gamma_1 = 1,\ k_1 = 1, x(0) = 3, \hat{\theta}(0) = 0$.

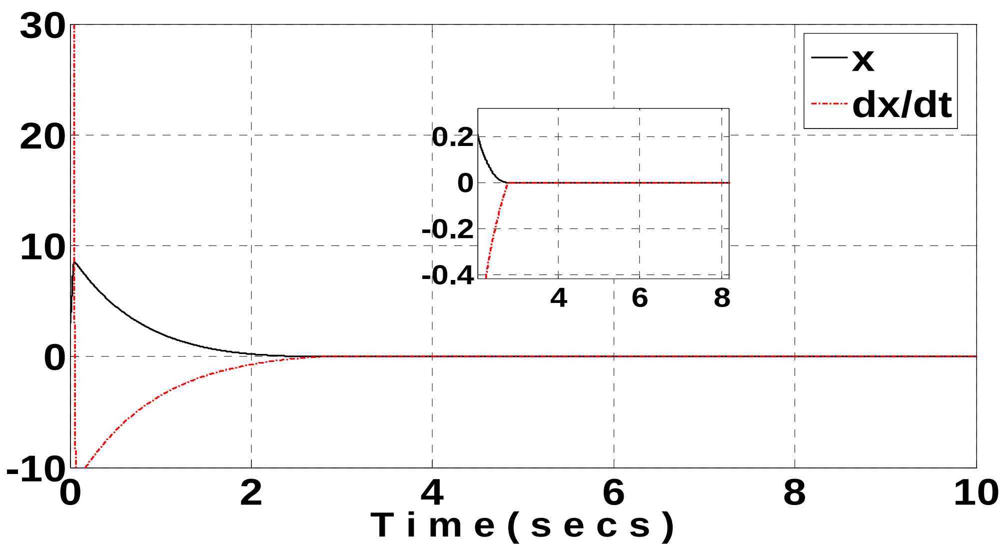

FIG.1. Time history of the state variable and its derivative in the first order system

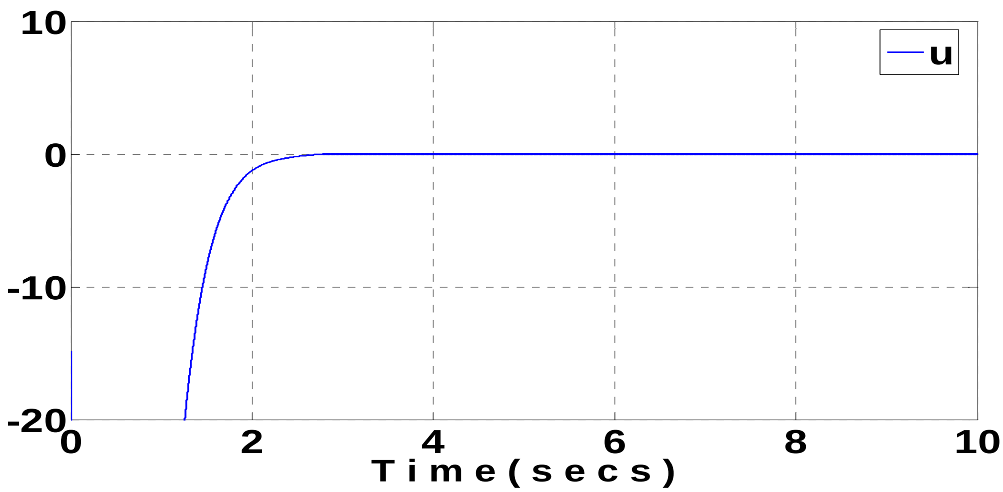

FIG.2. Time history of the ASOSM control in the first order system

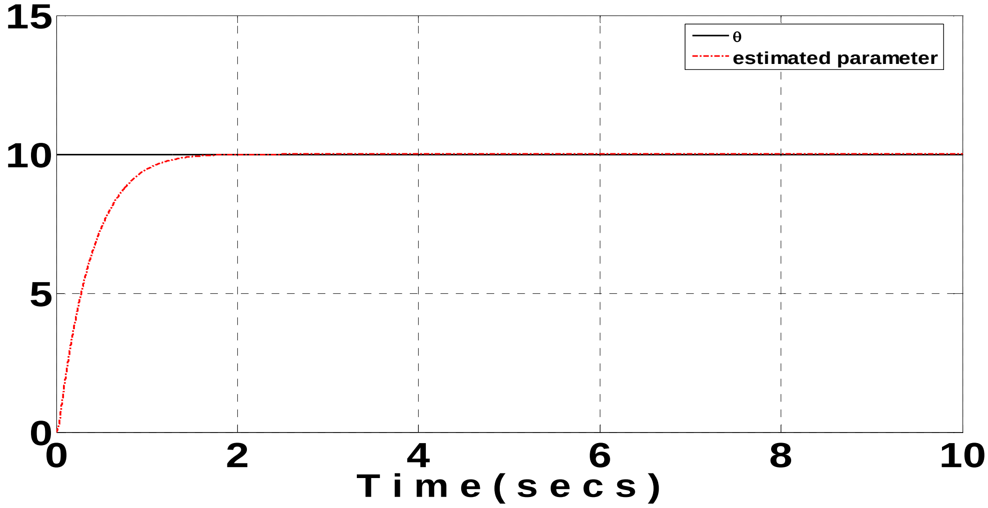

FIG.3. Time history of the parameter adaptation in the first order system

6.2 Finite convergence time adaptive SOSM control for the second order system, Case1

The second order control system (2.3) is studied with $n = 2$ where

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \theta^T \varphi(x_1, x_2, t) + u \end{aligned} \tag{6.2}$$

and the function $\varphi(x_1, x_2, t) = x_2^2$ .The adaptation function is selected as $\beta = \frac{1}{8\gamma_0} {x_2}^2$, while the value of the parameters are $\alpha_1 = 1/3, \alpha_2 = 1/2, \gamma_0 = \gamma_1 = \gamma_2 = 1, k_1 = 10, k_2 = 10, x_1(0) = 5, x_2(0) = -2, \hat{\theta}(0) = 0$.

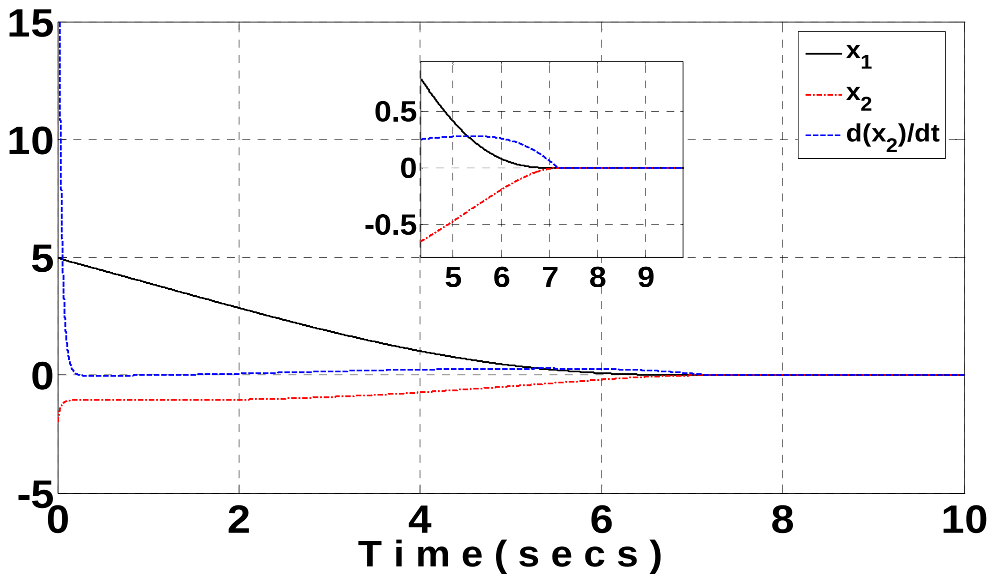

FIG.4. Time history of the state variable, its first and second derivative in the second order system

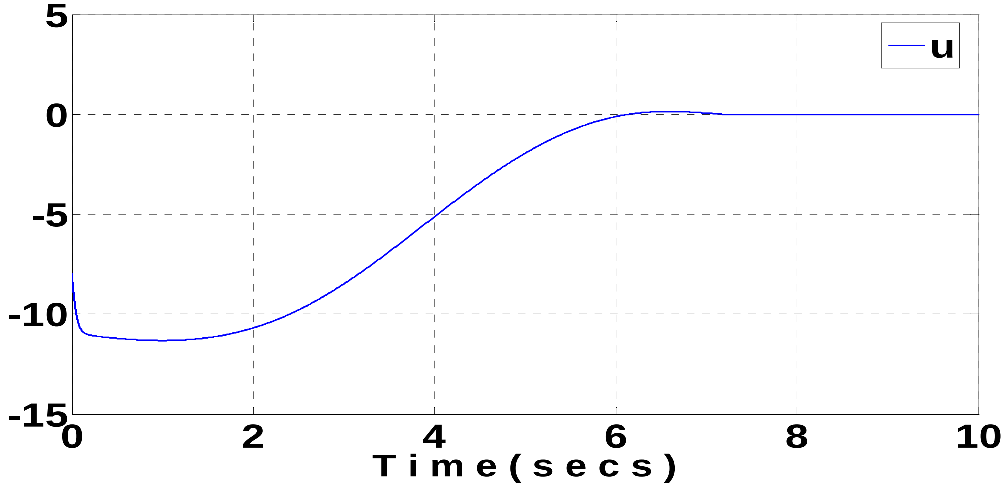

FIG.5. Time history of the ASOSM control in the second order system

It is clear from Figs. 4-5 that $x_1, x_2, \dot{x}_2 \rightarrow 0$ in finite time and stay at the origin thereafter in the presence of the bounded disturbance with the unknown bound, while the finite convergent-time adaptive control law is continuous/smooth.

6.3 Finite convergence time adaptive SOSM control for the second order system, Case2

Figs.6-8. show the response of the second order control system given by (5.4), (5.6), (5.7), (5.10) and simulated with the function $\varphi(x_2) = x_2^2$. The adaptation function $\beta = \beta(x_2)$ is selected as $\beta = \frac{1}{12\gamma} {x_2}^2$, while the parameter values are $a_1 = 2/3, a_2 = 1/3, \gamma = 1, \alpha_1 = 3, \alpha_2 = 3, x_1(0) = 5, x_2(0) = 1, \hat{\theta}(0) = 0$.

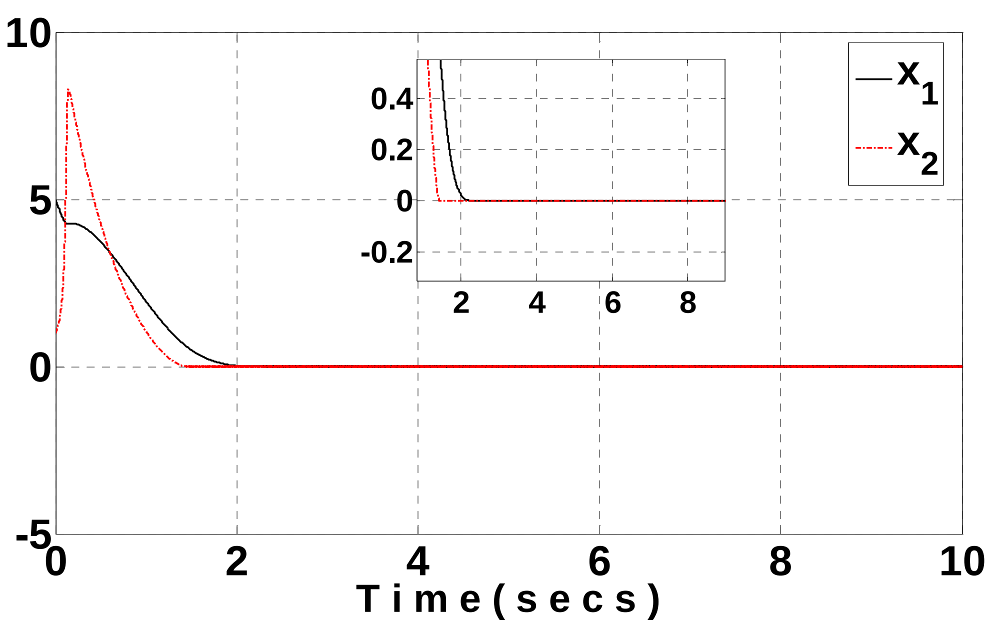

FIG.6. Time history of the state variables in the second order system, Case 2

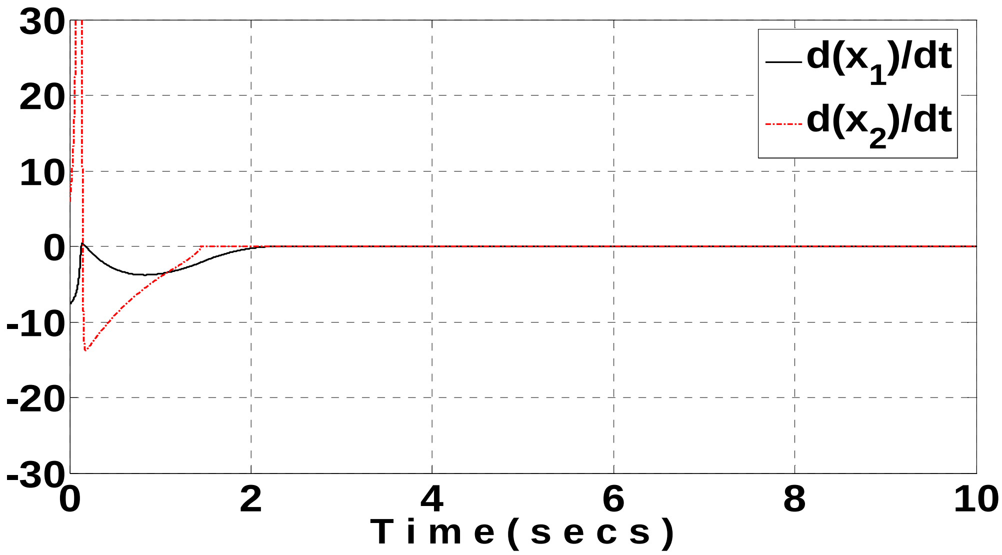

FIG.7. Time history of the derivatives of the state variables in the second order system, Case 2

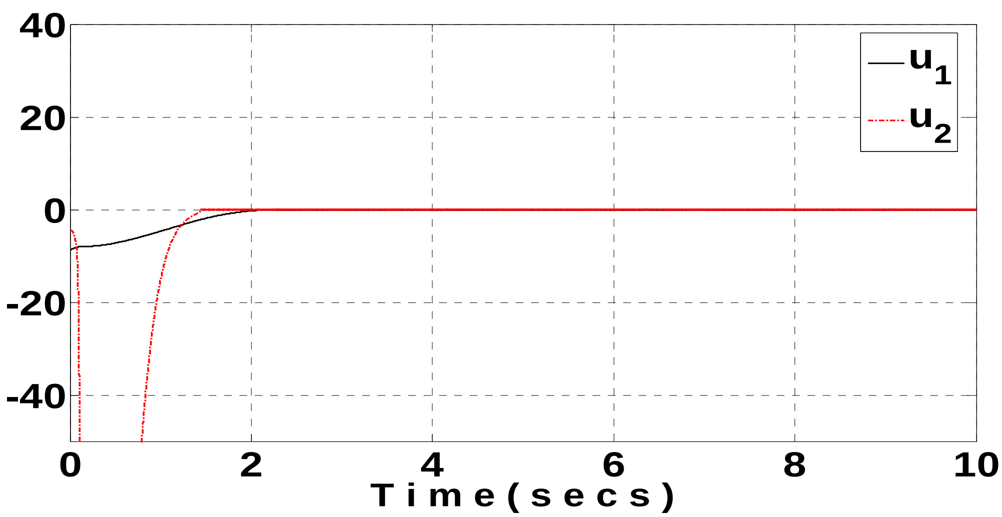

FIG.8. Time history of the adaptive controls in the second order system, Case 2

It is clear from Figs. 6-8 that $x_1, \dot{x}_1, x_2, \dot{x}_2 \to 0$ in finite time and stay at the origin thereafter in the presence of the bounded disturbance with the unknown bound, while the finite convergent-time adaptive control laws are continuous. This means that the proposed adaptive nonlinear control law $u \in \mathbb{R}^2$ is a continuous vector SOSM control.

6.4 Finite time adaptive SOSM control for the second order system, Case 3.

The second order control system given by (5.15), (5.20), (5.23) is taken with the function $\varphi(x_1, \dot{x}_1) = \dot{x}_1^2$. While $\beta = \dfrac{1}{8\gamma}\dot{x}_1^{\,2}$, the other parameters are $a_1 = 2/3, a_2 = 1/3, \gamma = 1, \alpha_1 = 6, \alpha_2 = 8, x_1(0) = 1, x_2(0) = 4$. The simulation of the second order system is shown in Figures 9-10. As can be seen from Fig.9, $x_1, \dot{x}_1, \ddot{x}_1 \to 0$ in finite time and the system stays at the origin thereafter in the presence of the bounded disturbance. The corresponding continuous control law $u \in \mathbb{R}$ (Fig. 10) can be interpreted as continuous SOSM control.

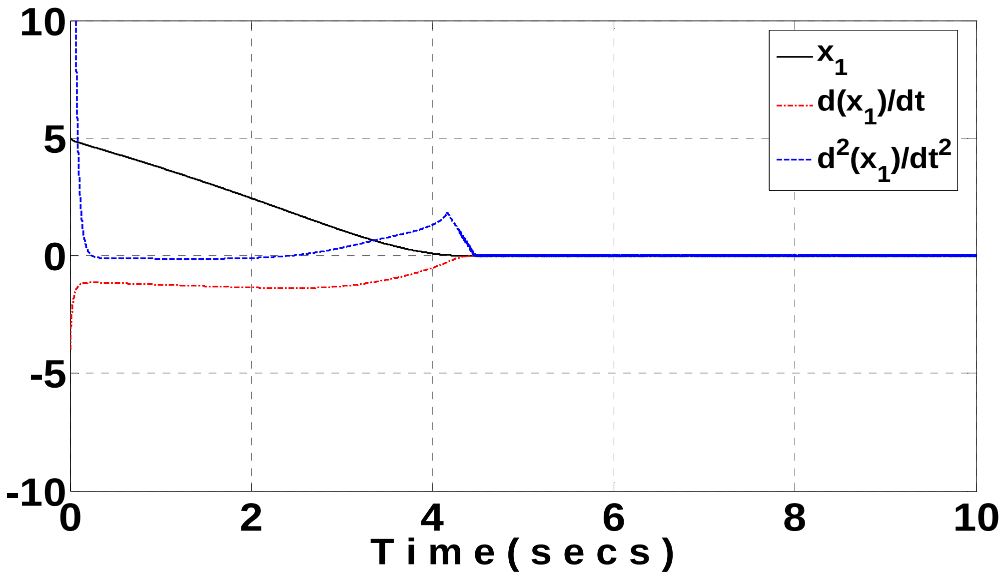

FIG.9. Time history of the state variable and its first and second order derivatives in the second order system, Case 3

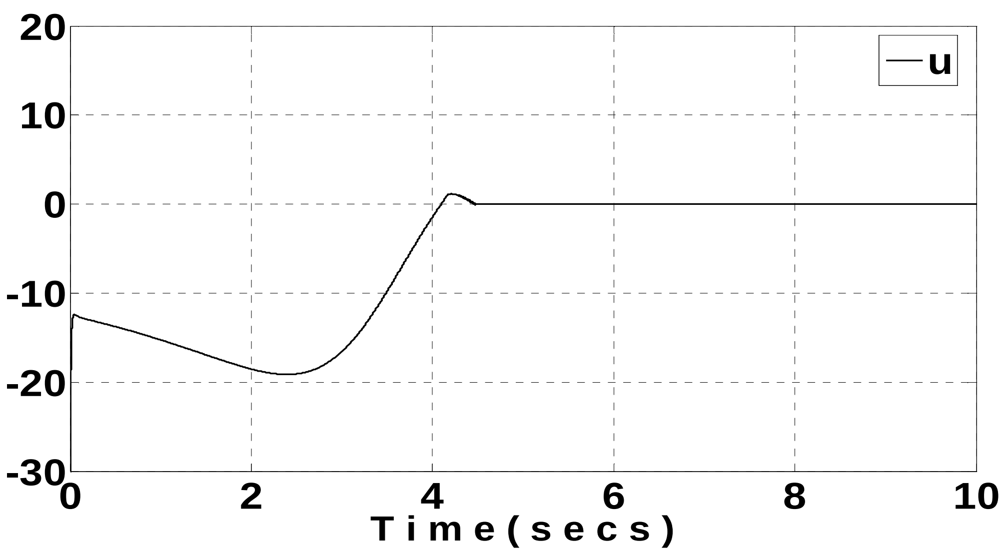

FIG.10. Time history of the adaptive control in the second order system, Case 3

7. Conclusion

Adaptive continuous finite convergent-time second order sliding mode control laws have been proposed for first and second order dynamic systems. These control algorithms completely eliminate chattering, since they do not contain any discontinuous terms, while providing robustness to bounded disturbances with unknown bounds.

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