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# Warranty claims data analysis considering sales delay

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## Abstract

Sales delay is the time interval from the date of manufacture to the date of sale. In analysing warranty claims data, the existing research relating to the sales delay has mainly focused on estimating the probability distribution of the sales delay. Longer sales delay may lead to more warranty claims as it can have an impact on the post-sale reliability of products. However, research into this problem has received little attention.

This paper estimates the expected number of warranty claims under both renewing and non-renewing warranty policies taking into account the sales delay. We consider the case with three states, the sales delay state, the operating state and the failed state. We extend the three state case into an  $n$  state system case, where  $n \geq 3$ . We then give numerical examples to demonstrate the application of the derived equations. We also present a simulation and a case study where we estimate the reliability of products with three states.

**Keywords:** warranty claims, multistate components, sales delay, failure rate, non-homogeneous Poisson process.

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## Notation

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$t_s$	Time of sale.
$F_0(t)$	Lifetime distribution during operating state.
$F_1(t)$	Lifetime distribution of a product in sales delay state.
$F_2(t t_s)$	Lifetime distribution during operating state for a product sold at time $t_s$ .
$F^{(k)}(t)$	$k^{th}$ convolution of $F$ .
$F^k(t)$	$k^{th}$ power of $F$ .
$\bar{F}_i(t) = 1 - F_i(t)$	Survival function of $F_i(t)$ for $k = 0, 1, 2$ .
$r_i(t)$	Failure rate function of $F_i(t)$ for $k = 0, 1, 2$ .
$G(x)$	Sales delay distribution.
$w$	Warranty period.
$N$	Total production amount.
$s_t$	Number of products sold in month $t$ .
$n_t$	Number of warranty claims in month $t$ .
$\mu_t$	Expected number of warranty claims in month $t$ .

---

## 19 1 Introduction

20 Warranty is a duty attached to a product and requires manufacturers to offer a pre-specified  
21 compensation to buyers when the product fails to perform its designated functions under  
22 normal usage within the warranty period. It is intended to assure buyers that faulty products  
23 will either be repaired or replaced by the manufacturer at no or partial cost. Such an  
24 obligation can be considered a burden imposed on the manufacturer. In the modern economy,  
25 however, warranty is increasingly seen as an opportunity and an effective marketing tool that  
26 can provide a competitive edge. This is due to the prevalent perception that the duration  
27 of the warranty is an indicator of the product's quality. Thus, selecting a suitable warranty  
28 policy is an optimisation process in which both costs and profits should be considered from  
29 the manufacturer's perspective.

30 Warranty requires the manufacturer to reserve a certain amount of its resources to cover  
31 its warranty obligation. Achieving the optimal tradeoff between warranty related reserves  
32 and the actual warranty claims is necessary in order to maintain higher levels of customer  
33 satisfaction and profits. As such, estimation of the expected number of warranty claims  
34 is crucial for formulating an optimal warranty policy and requires warranty claims data  
35 analysis. The warranty claims data is, however, often comes in an incomplete or aggregated  
36 form, where some of the data necessary for statistical analysis is missing.

37 *Sales delay* is the time from the date of manufacture to the date of sale. The sales delay  
38 has been considered in the existing literature, where the main focus has been on estimating  
39 its probability distribution. Suzuki et al.<sup>1,2</sup> estimate the sales delay distribution from a  
40 sample of warranty data, where the dates of sale are known for reported warranty claims.  
41 Wang et al.<sup>3</sup> estimate monthly sales amounts using warranty data where the claims data

42 represent the number of claims for products sold in a given month. They assume that  
43 the total amount of sales for the period of time under study can be obtained from other  
44 additional data sources. Lim<sup>4</sup> uses a non-homogeneous Poisson process (NHPP) model to  
45 estimate the sales delay, where it is referred to as holding times. Karim<sup>5</sup> also estimates the  
46 sales delay distribution using an NHPP model. Karim<sup>6</sup> models the sales delay distribution  
47 using a lognormal distribution, where this distribution is estimated from a sample of sales  
48 delay data. The literature on modelling the sales delay makes extensive use of the ideas and  
49 concepts discussed in relation to the modelling of the so-called reporting delays (Kalbfleisch  
50 and Lawless<sup>7</sup>, Kalbfleisch et al.<sup>8</sup>, and Lawless<sup>9</sup>). The reporting delays themselves can often  
51 be assumed to be negligible.

52 Unlike reporting delays that can have impact only on warranty claims data analysis  
53 without bearing any relevance to product consumers, the sales delay can have an impact  
54 on both manufacturers and product consumers. The impact of the sales delay on product  
55 reliability has been noted before, for example, Robinson and McDonald<sup>10</sup> note that longer  
56 sales delays result in larger number of warranty claims. Nevertheless, the existing literature  
57 has been mainly focused on estimation of the sales delay distribution. So far, little research  
58 has been done on estimating the impact of the sales delay on product reliability and/or  
59 warranty policy optimisation.

60 There are two main types of warranty policies, renewing and non-renewing. Under re-  
61 newing warranty, if a product fails during the warranty period, the warranty of the repaired  
62 or newly replaced product is renewed from anew at no or a pre-specified cost. Under non-  
63 renewing warranty, the warranty of the original product is carried over to the repaired or  
64 newly replaced product. In this paper we consider both of these warranty policies.

65 In its lifecycle, a product can be in a number of states. For example, a product can  
66 be in the following three states, sales delay state, operating state and failed state. This  
67 is an example of a three-state system. Warranty cost analysis for such systems has been  
68 studied by Wu and Li<sup>11</sup> and Wu and Xie<sup>12</sup>, where similar systems with dormant, operating,  
69 and failed states have been considered. These studies discuss three-state building services  
70 systems.

71 In this study we derive the number of expected warranty claims for products under  
72 renewing and non-renewing warranty policies. We also consider estimation of the reliability  
73 of products during the sales delay and operating state from a set of real life data. We show  
74 that models based on considering the reliability of a product during the sales delay period and  
75 during the operating period yield better results than those with a single reliability function.  
76 The novelty of this paper can be summarised in the following points.

- 77 • Sales delay has been studied in warranty claims analysis but has not been considered  
78 from customers perspective. This paper is the first to address this issue.

- The existing literature focuses on estimating the distribution of the sales delay, but does not consider the impact of the sales delay on products reliability. This paper presents the first attempt to explore this issue.

The paper is structured as follows. Section 2 develops models for estimating the expected number of warranty claims for three-state systems under renewing warranty and non-renewing warranty policies. Section 3 presents a numerical example and demonstrates how the models discussed in the previous section can be put into practical use. Section 4 presents a simulation and a case study and discusses the results of implementing models with different failure rates to field data from the electronics industry. The last section discusses future work plans and summarises the main conclusions of this paper.

## 2 Model development

In this paper we make the following assumptions.

**A1** Products under consideration are non-repairable. A failed product is replaced with an identical product, where replacement times are negligible.

**A2** Products can be in the following three states: sales delay state, operating state and failed state. Products in the sales delay state have a different lifetime distribution from those in the operating state.

**A3** Products that fail during the sales delay state are replaced with identical ones only at the time of sale.

**A4** The sales delay times are assumed to be independent and identically distributed and so are failure times.

The reliability of products during the sales can deteriorate due to different reasons depending on the type of products. For example, electronic equipment can be effected by damp storage conditions.

In a three state system, the first failure of a product can occur in the following three cases.

**Case A.** A product fails to operate at the time of sale,  $t_s$ , where it is replaced with a new identical product with failure rate  $r_0(t)$ , where  $t$  starts from 0.

**Case B.** A product enters an operating state at time  $t_s$ , where its failure rate changes from  $r_1(t)$  to  $r_2(t|t_s)$ . The first failure of the product occurs within  $(t_s, t_s + w)$ , where it is replaced with a new identical product with failure rate  $r_0(t)$ .

**Case C.** The first failure of a product occurs after the warranty term expires. This case does not incur any replacement cost to the manufacturer, so it is not discussed any further.

## 112 2.1 Non-renewing warranty policy

113 From renewal theory (Ross<sup>13</sup>), the number of replacements of a new product,  $N(t)$ , within  
 114 time interval  $(0, t)$  is given by a renewal process with the time between adjacent renewals  
 115 distributed according to  $F_0(t)$ . The probability of  $k$  renewals in  $[0, t)$  is given by  $\Pr\{N(t) =$   
 116  $k\} = F_0^{(k)}(t) - F_0^{(k+1)}(t)$ , where  $F^{(k)}(t)$  is the  $k^{\text{th}}$  convolution of  $F$ . The expected number  
 117 of renewals/replacements,  $M(t)$ , is given by  $M(t) = \sum_{k=1}^{\infty} F_0^{(k)}(t)$ , where  $M(t)$  satisfies the  
 118 renewal integral equation  $M(t) = F_0(t) + \int_0^t M(t-x)dF_0(x)$ .

119 For Case A, the probability of an event that a product fails to operate at time  $t_s$  is  
 120  $F_1(t_s)$ . So, the expected number of replacements for products that fail during the sales  
 121 state,  $M_{1A}(w)$ , is given by:

$$M_{1A}(w) = N(1 + M(w)) \int_0^{\infty} F_1(x)dG(x), \quad (1)$$

122 where  $N$  is the total number of products.

123 For Case B, if the first failure occurs within  $(t_s, t_s + y]$  with  $y < w$ , the expected number  
 124 of replacements within time interval  $(t_s + y, w)$  is  $M(w - y)$ . From time  $t_s$  to the occurrence  
 125 of the first failure, the failure rate of a product is  $r_2(t|t_s)$  and the lifetime distribution is  
 126  $F_2(t|t_s)$ . Thus, the expected number of replacements within  $(t_s, t_s + w]$  for products that fail  
 127 after being sold, Case B, is given by:

$$M_{1B}(w) = N \int_0^{\infty} \left[ \bar{F}_1(x) \int_0^w (1 + M(w - y))dF_2(y|x) \right] dG(x). \quad (2)$$

128 As the expected number of failures for both cases is  $M_{1A}(w) + M_{1B}(w)$ , we can obtain the  
 129 following result:

130 *If products are sold under non-renewing warranty, the expected total number of warranty*  
 131 *claims for  $N$  products within warranty period  $w$  is given by*

$$S_1(w) = N \left\{ (1 + M(w)) \int_0^{\infty} F_1(x)dG(x) + \int_0^{\infty} \left[ \bar{F}_1(x) \int_0^w (1 + M(w - y))dF_2(y|x) \right] dG(x) \right\} \quad (3)$$

## 132 2.2 Renewing warranty policy

133 Under the renewing warranty policy, when a failed product is replaced, the warranty term  
 134 is renewed. Therefore, for Case A, the expected number of replacements is given by

$$\begin{aligned} M_{2A}(w) &= (1 + \sum_{j=1}^{\infty} F_0^j(w))N \int_0^{\infty} F_1(x)dG(x) \\ &= \frac{N}{\bar{F}_0(w)} \int_0^{\infty} F_1(x)dG(x), \end{aligned} \quad (4)$$

135 where  $F^j$  is the  $j^{th}$  power of  $F$ .

136 For Case B, if the first failure occurs within  $(t_s, t_s + w]$ , then the expected number of  
 137 replacements is  $F_2(w|t_s)$ . After the first replacement, the expected number of replacements  
 138 is  $\left(\sum_{j=1}^{\infty} F_0^j(w)\right) N \int_0^{\infty} F_2(w|x) dG(x)$ .

139 Therefore, the expected total number of replacements in Case B is given by

$$\begin{aligned} M_{2B}(w) &= \left(1 + \sum_{j=1}^{\infty} F_0^j(w)\right) N \int_0^{\infty} \bar{F}_1(x) F_2(w|x) dG(x) \\ &= \frac{N}{\bar{F}_0(w)} \int_0^{\infty} \bar{F}_1(x) F_2(w|x) dG(x). \end{aligned} \quad (5)$$

140 Since, the total for both cases is  $M_{2A}(w) + M_{2B}(w)$ , we have the following result:

141 *If products are sold under renewing warranty, the expected total number of warranty*  
 142 *claims for  $N$  products within warranty period  $w$  is given by*

$$S_2(w) = \frac{N}{\bar{F}_0(w)} \int_0^{\infty} \{F_1(x)(1 - F_2(w|x)) + F_2(w|x)\} dG(x) \quad (6)$$

### 143 2.3 Extension to multistate systems

144 From a theoretical and practical point of view it is possible to extend the three state model  
 145 described in the previous section into a multistate model. As discussed earlier a three state  
 146 model has three states, namely, sales delay, operating and failed states. The sales delay state  
 147 and operating state can be considered as two different operating modes. As the intensity  
 148 of usage affects the operating intensity, different operating modes can have different failure  
 149 rate functions. In order to distinguish *mode* from *state*, we define *state* as an operating *mode*  
 150 in which the component has a different failure rate function from others. We refer to such  
 151 components as multistate components.

152  
 153 [Insert Figure 1 here]

154  
 155 Suppose a component has  $m$  operating modes, and is operated with an operating mode  
 156 pattern,  $M_1 \rightarrow M_2 \rightarrow \dots \rightarrow M_m \rightarrow M_1 \rightarrow M_2 \rightarrow \dots$ , repeatedly. As time passes, the  
 157 component deteriorates, and the adjacent two identical operating modes can have different  
 158 failure rates. Therefore, the component can have more than  $m$  states (see Figure 1), theo-  
 159 retically, it can even have infinite number of states. This type of multistate, which is called  
 160 *type I multistate* hereafter, is different from the common multistate component defined as  
 161 (El-Newehi et al. <sup>14</sup>):

162 *As time passes a component, starting in state  $M$ , deteriorates and enters state  $M - 1$ ,*  
 163 *deteriorates further entering state  $M - 2$ , and so on.*

164 We call the above multistate component as *type II multistate* in what follows. There are two  
 165 main differences between type I and type II multistate components:

- 166 • a transition from state  $i$  to state  $i - 1$  for a type I multistate component is due to an  
 167 artificial interference (e.g., people force the component to change from one state to a  
 168 another by triggering a certain functionality of the component), whereas a transition  
 169 from state  $i$  to state  $i - 1$  for a type II multistate component is due to deterioration,  
 170 and
- 171 • for a type I multistate component, state  $i$  can transit to state 0 without going through  
 172 states  $i - 1, i - 2, \dots, 1$ ; whereas for a type II multistate component, state  $i$  can only  
 173 transit to state  $i - 1$ , then to  $i - 2$ , and so on.

174 Suppose that a type I multistate component has  $n$  states,  $S_1, \dots, S_n$  from the sales date  
 175 to the end of the warranty, where the usage time of  $S_i$  ( $i = 1, \dots, n$ ) is  $t_i$ , with  $t_i$  satisfying  
 176  $\sum_{i=1}^n t_i = w$ . Assume that the lifetime distribution of a component in the full load is  
 177 an exponential distribution with the scale parameter  $\gamma$ , that is  $F(t) = 1 - e^{-\gamma t}$ , and the  
 178 expected failure rate function of  $S_i$  is  $r_i = \alpha_i \gamma$ , where  $0 < \alpha_i \leq 1$ . Then there are  $\alpha_i \gamma t_i$   
 179 failures within  $t_i$  time units. As discussed previously, we also have the lifetime distribution  
 180 during the sales delay given by  $F_1(t) = 1 - e^{-\lambda \gamma t}$ .

### 181 2.3.1 Multistate non-renewing warranty policy for exponential model

182 The expected total number of replacements, from the time of sale to the end of the warranty  
 183 period is  $\sum_{i=1}^n \alpha_i \gamma t_i$ . Because of the memoryless property of the exponential distribution,  
 184  $F_2(y|t_s) = F_0(y)$ , where (3) can be simplified to:

$$\begin{aligned}
 S_1(w) &= N \left\{ (1 + M(w)) \int_0^\infty F_1(x) dG(x) \right. \\
 &\quad \left. + \int_0^\infty \left[ \bar{F}_1(x) \int_0^w (1 + M(w - y)) dF_2(y|x) \right] dG(x) \right\} \\
 &= N \left\{ \int_0^\infty F_1(x) dG(x) + M(w) \right\}
 \end{aligned} \tag{7}$$

185 Thus, we have the following result:

186 *If products are sold under non-renewing warranty, and have  $n$  states after the time of*  
 187 *sale, the expected total number of warranty claims for  $N$  products within warranty period  $w$*   
 188 *is given by:*

$$S_3(w) = N \left\{ \int_0^\infty F_1(x) dG(x) + \sum_{i=1}^n \alpha_i \gamma t_i \right\} \tag{8}$$

189

190



191 **2.3.2 Multistate renewing warranty policy for exponential model**

192 The probability of a failure of type I multistate component within the warranty time period  
 193  $w$ , denoted by  $\bar{H}_0(w)$ , is given by

$$\begin{aligned} H_0(w) &= H_1(t_1) + \bar{H}_1(t_1)H_2(t_2) + \bar{H}_1(t_1)\bar{H}_2(t_2)H_3(t_3) + \dots + \bar{H}_1(t_1)\dots\bar{H}_{n-1}(t_{n-1})H_n(t_n) \\ &= H_1(t_1) + \sum_{i=2}^n \prod_{k=1}^{i-1} \bar{H}_k(t_k)H_i(t_i) \end{aligned}$$

194 where  $H_k(t_k) = 1 - e^{-\alpha_k \gamma t_k}$  for  $k = 1, \dots, n$ .

195 Therefore, the expected total number of replacements after the time of sale, is given by:

$$\sum_{i=1}^n H_0^i(w) = \frac{H_0(w)}{1 - H_0(w)}$$

196 Thus, from (6), we have

197 *If products are sold under renewing warranty, and have  $n$  states after the time of sale, the*  
 198 *expected total number of warranty claims for  $N$  products within warranty period  $w$  is given*  
 199 *by:*

$$S_4(w) = \frac{N}{1 - H_0(w)} \int_0^\infty \{F_1(x)(1 - H_0(w)) + H_0(w)\} dG(x) \quad (9)$$

200 **3 Numerical examples**

201 In this section we present numerical examples on estimating the expected number of warranty  
 202 claims and show how it is possible to establish a relationship between the failure rates of  
 203 products in the sales delay and operating states. Following Wu and Xie<sup>12</sup>, we assume that  
 204 this relationship takes the following form.

205 The failure rate during the sales delay state is assumed to be related to the failure rate  
 206 during the operating state through the following relationship:

$$r_1(t) = \lambda r_0(\nu t), \quad (10)$$

207 where  $t \in (0, t_s]$ ,  $0 < \nu < 1$  and  $0 < \lambda < 1$ . It follows that:

$$F_1(t) = 1 - (\bar{F}_0(\nu t))^{\frac{\lambda}{\nu}} \quad (11)$$

208 The residual lifetime distribution of products that survive after the time of sale,  $t_s$ , or  
 209 the lifetime distribution within time interval  $(t_s, +\infty)$ , can be obtained based on the lifetime  
 210 distribution during the sales delay state,  $F_1(t)$ . However, within the time interval  $(0, t_s]$ , the  
 211 products are in the sales delay state, whereas within the time interval  $(t_s, +\infty)$ , they are in  
 212 the operating state. Hence, the lifetime distribution within  $(t_s, +\infty)$  is different from the

213 residual lifetime distribution that is derived from the distribution within time interval  $(0, t_s]$ .  
 214 The lifetime distribution of a product that survives after the sales delay state,  $t_s$ , is given  
 215 by:

$$F_2(t|t_s) = \frac{F_0(\nu t_s + t) - F_0(\nu t_s)}{1 - F_0(\nu t_s)}, t \geq 0. \quad (12)$$

216  $F_0(\nu t_s)$  is the distribution of the scaled time  $t$ , with scaling parameter  $\nu$ . The scaling of age  
 217 in such a way is commonly used in reliability and maintenance analysis.

### 218 3.1 Example of a three-state model

219 Without loss of generality we assume that the sales delay distribution is given by the following  
 220 lognormal distribution with mean  $\mu$  and standard deviation  $\sigma$ :

$$G(x) = \int_0^x \frac{1}{x\sigma\sqrt{2\pi}} e^{-\frac{(\ln t - \mu)^2}{2\sigma^2}} dt \quad (13)$$

221 For this example we assume that  $F_0(t)$  is a CDF of Weibull distribution, which is one of the  
 222 most common distributions used in reliability:

$$F_0(t) = 1 - e^{-\left(\frac{t}{\eta}\right)^\alpha}. \quad (14)$$

223 This leads to the following  $F_1(t)$  and  $F_2(t)$ :

$$F_1(t) = 1 - e^{-\frac{\lambda}{\nu} \left(\frac{\nu t}{\eta}\right)^\alpha}, \quad \text{and} \quad (15)$$

$$224 \quad F_2(t|t_s) = 1 - e^{-\left(\frac{\nu t_s + t}{\eta}\right)^\alpha}. \quad (16)$$

225 For this example we approximate the renewal function for Weibull distribution,  $M(t)$ ,  
 226 using methods suggested by Jiang<sup>15</sup>. Setting the relevant parameters, as shown in Table 1,  
 227 we can obtain the following results.

228  
 229 [Insert Table 1 here]

230  
 231 For three-state components, if the values of  $\lambda$  change over interval  $(0, 1)$ , the resulting  
 232 values of  $S_1(w)$  and  $S_2(w)$  are shown in Figure 2. When  $\lambda$  becomes larger, the number of  
 233 warranty claims for  $N$  products under non-renewing warranty and renewing warranty be-  
 234 comes larger. It is also clear that the difference between the two policies remains the same  
 235 for different values of  $\lambda$ .

236  
 237 [Insert Figure 2 here]

238

239 Figure 3 depicts the relationship between the expected number of warranty claims and  
240 values of  $\nu$ . It can be seen from the graph that the difference between the renewing and  
241 non-renewing policy becomes larger as the values of  $\nu$  increase.

242

243 [Insert Figure 3 here]

244

245 Figure 4 depicts the relationship between the expected number of warranty claims and  
246 the mean of the sales delay distribution. It is clear from the figure that as the sales delay  
247 time increases the number of expected claims also increases. The difference between the  
248 renewing and non-renewing policies also increases as the sales delays become longer.

249

250 [Insert Figure 4 here]

251

252 Figure 5 demonstrates the difference between the failure rate during the sales delay and  
253 during the operating state. The magnitude of this difference is dependent on the values of  
254  $\lambda$  and  $\nu$ .

255

256 [Insert Figure 5 here]

257

### 258 **3.2 Example of a multistate model**

259 Using the results for multistate models from the previous section we can set the parameters  
260 as shown in Table 2. The resulting expected number of warranty claims under non-renewing  
261 and renewing warranty policies,  $S_3$  and  $S_4$ , respectively, are given in Table 3. It can be  
262 seen from the table that as  $\lambda$  increases the expected number of warranty claims for both  
263 policies increase. It is also clear that there is little difference between the two policies for  
264 this example. The reason for this is that the parameters  $\alpha_k$  and  $\gamma$  are small. This results in  
265 small  $H_0(w)$ .

266

267 [Insert Table 2 here]

268

269 [Insert Table 3 here]

270

## 271 4 Simulation and case study

272 This section presents the results of simulation and a case study. In both cases we consider  
273 data recorded on monthly basis matched to the months of manufacture. That is, the data  
274 represents failure times since the date of manufacture,  $Y$ , which is the sum of two time  
275 periods, namely, the sales delay,  $S$ , and the failure time  $T$ ,  $Y = S + T$ . In this study, we  
276 assume that the distribution of  $S$  is known. The total number of months considered here is  
277 60, 24 months for fitting the models and 12, 24 and 36 months for estimating the prediction  
278 accuracy. It is not uncommon in the electronics industry to use the first 24 months of the  
279 data to predict the expected number of failures for coming months.

280 For simplicity, we consider only failure times during the sales delay and failure times  
281 during the operating state for products that survived the sales delay period. That is, we  
282 focus only on times to first failure. This is justified as we are looking at how the sales delay  
283 can affect the reliability of the products.

284 In this paper we consider the following three models.

- 285 • **Model 1.** This model assumes that there are no failures during the sales delay time.  
286 For this model, the objective is to estimate the distribution of  $T$ .
- 287 • **Model 2.** This model assumes that a constant proportion,  $p$ , of sold products is found  
288 to be in the failed state at the point of sale. Thus, the objective is to estimate  $p$  and  
289 the distribution of  $T$ .
- 290 • **Model 3.** This model assumes that during the sales delay period the products lifetime  
291 distribution is given by  $F_1(t)$ , and during the operating period the products lifetime  
292 distribution is given by  $F_2(t|t_s)$ , where  $t_s$  is the sales delay time.

293 Based on the above assumptions, the expected number of failures in month  $t$  for each  
294 model is as follows.

295  
296 **Model 1:**

$$\mu_t = \sum_{i=1}^t s_i (F_0(t) - F_0(t - i)). \quad (17)$$

297 where  $s_i$  is the number of products sold in month  $i$ . The expected number of failures,  $\mu_t$ ,  
298 consists of the expected number of failures for products sold in month  $t$  and previous months  
299 with appropriate ages. Here, we assume that products that have been recorded to have failed  
300 in each month have been operated for the whole of the month. That is, the products are  
301 assumed to be sold in the beginning of each month.

302  
303 **Model 2:**

$$\mu_t = ps_t + \sum_{i=1}^{t-1} s_i (F_0(t) - F_0(t - 1)). \quad (18)$$

304 The first term represent the expected number of products that are found in the failed state  
 305 at the time of sale. The second term represents the expected number of failures in month  
 306  $t$  for products sold in previous months with appropriate ages. This model assumes that  
 307 products are sold at the end of each month.

308

309 **Model 3:**

$$\mu_t = s_t F_1(t) + \sum_{i=1}^{t-1} s_i (F_2(t-i|i) - F_2(t-i-1|i)). \quad (19)$$

310 The first term represents the expected number of products found failed at the time of sale.  
 311 The second term represents the sum of the expected number of products that failed in month  
 312  $t$  from sales in previous months excluding the current month. As the case with the previous  
 313 model, this model assumes that the products are sold at the end of each month.

314 Previous studies such as Majeske<sup>16</sup>, Wang et al.<sup>3</sup>, Karim et al.<sup>17</sup> have estimated warranty  
 315 claims using a non-homogeneous Poisson process (NHPP) model. Here, we adopt a similar  
 316 strategy and model our data using the NHPP model where the probability of failures in  
 317 month  $t$  is given by a Poisson distribution with mean  $\mu_t$ . Thus, the log-likelihood function  
 318 for each of the above models is given by:

$$\ln L = \sum_{i=1}^m (n_i \ln \mu_i - \mu_i - \ln n_i!) \quad (20)$$

319 where  $m$  is the number of months used for fitting the models, in this case 24 months. The  
 320 models are fitted by maximising their respective log-likelihood functions. The maximisation  
 321 is done using numerical methods with multiple starting points.

## 322 4.1 Simulation

323 The sales delay times are simulated from a lognormal distribution with the mean and variance  
 324 of the associated normal distribution given by  $\mu = 2.05$ , and  $\sigma^2 = 0.06$ . This means that the  
 325 expected sales delay is about 8 months and that 95% of products are sold within the first  
 326 12 months.

327 Both  $F_1(t)$  and  $F_2(t)$  are derived from a Weibull distribution with scale parameter  $\eta = 75$   
 328 and shape parameter  $\alpha = 1.2$ , given by equations (14) - (16). The scaling parameters for the  
 329 hazard function are  $\lambda = 0.03$  and  $\nu = 0.15$ . This means that during the sales delay time the  
 330 average failure time is 1175 months and during the operating time with average sales delay  
 331 of 8 months, the expected failure time is around 69 months.

332 The simulated data is the average of 1000 runs of 10000 products. If the products  
 333 fail during the sales delay, the failure time is taken to be the sales delay time, as failures  
 334 discovered only at the point of sale. The failure time for products that survive beyond the  
 335 sales delay time are taken to be the sales delay time plus the failure time generated from

336  $F_2(t)$ . As mentioned previously, in this paper we focus on only on times to first failure.  
337 Thus, the renewals of newly replaced products are not generated. This will be done in our  
338 future works.

339 Thus, we have artificially generated data from a process where products fail during the  
340 sales delay time. However, it is not possible do discern from the data the number of failures  
341 that occurred during the sales delay period. The only data available is the records of monthly  
342 failures. This data was generated in a way that matches the data available for the case study,  
343 which is discussed in the next subsection.

344 The results of fitting the models to the simulation data are presented in Table 4, where  
345 columns headed with  $K$  represent prediction horizons of 12, 24, and 36 months with their  
346 respective means squared errors. Predictions were done based on  $\hat{\mu}_t$  for  $t = m + 1, m +$   
347  $2, \dots, m + K$ . Table 5 presents the estimated parameters for each model. It is clear from  
348 Table 4 that Model 3 has the highest likelihood, however, its AIC is bigger than the AIC of  
349 Model 2. Nevertheless, Model 3 seems to be more preferable as it has much better prediction  
350 accuracy. In practise, the estimation of the reliability of the products is often done for pur-  
351 poses of improving product quality and predicting the expected number of failures. Thus,  
352 the prediction accuracy is an important issue (Wu and Akbarov<sup>18</sup>, Fredette and Lawless<sup>19</sup>,  
353 and Wasserman<sup>20</sup>).

354

355 [Insert Table 4 here]

356

357 [Insert Table 5 here]

358

## 359 4.2 Case study

360 The products under consideration are electronic products, specifically, Internet networking  
361 equipment that have lifetime warranty. Such products can fail during the sales delay time  
362 due to reasons such as damp storage conditions and damage due to poor handling during  
363 transportation and storage. In relation to such products, Yang et al.<sup>21</sup> note that up to 66%  
364 of the products found failed at the time of sale can be attributed to damage during the sales  
365 delay period.

366 The available data also includes products that are in the failed state due to manufactur-  
367 ing process. Thus, the failure data also includes information about manufacturing quality.  
368 However, in practise, it is not always possible to distinguish between products that fail during  
369 the sales delay and product that are poorly manufactured.

370 The available data represents a collection of records over a period of time categorised  
371 into calendar months. It consists of the following three pieces of information: the month  
372 of manufacture,  $j$ , the shipment amount in the month of manufacture,  $N_j$ , and the num-

373 ber of products returned in month  $i$  that were manufactured in month  $j$ ,  $n_{ji}$ . The shipment  
374 amount in month  $j$  consists of products manufactured in month  $j$  and products manufactured  
375 in previous months. However, for the purposes of this study, we assume that the shipment  
376 amounts,  $N_j$ , adequately represent the number of products manufactured in month  $j$ . The  
377 data used for fitting the models discussed earlier and prediction consists of the aggregate  
378 data for 10 production batches. That is, we have the total number of failures for each month,  
379  $n_t = \sum_{j=1}^{10} n_{jt}$ , for the period of 60 months along with the total number of shipments for  
380 10 production batches,  $N$ . The data used in this study is given in Table 6 and plotted in  
381 Figure 6. The sales amounts for each month,  $s_t$ , were estimated from subjective data, which  
382 represents monthly sales as a percentage of the monthly shipments.

383

384 [Inset Table 6 here]

385

386 [Insert Figure 6 here]

387

388 Tables 7 and 8 show the results of applying the models to the case study data. As the  
389 case with the simulated data, it can be seen that Model 3 has bigger AIC than Model 2.  
390 However, Model 3 has better prediction accuracy.

## 391 5 Conclusions

392 Many manufacturers offer warranty on their products from the date of sale to a pre-specified  
393 point in time. For the cases where products spend prolonged periods of time before being  
394 sold it is necessary to take these periods into account as they can have a significant impact  
395 on the expected number of warranty claims.

396 In this study we have achieved the following. The expected number of warranty claims for  
397 products with several states under both non-renewing warranty and renewing warranty have  
398 been formulated and derived. A numerical example to examine the methods proposed has  
399 been demonstrated. This paper also considers three different models applied to simulated  
400 data and data from electronics industry. The results show that the models that take into  
401 account failures during the sales delay period result in better predictions. We also show that  
402 longer sales delays result in larger numbers of warranty claims reinforcing the remark made  
403 by Robinson and McDonald<sup>10</sup>.

404 In the future work we can consider the following issues:

- 405 • Consider the expected number of warranty claims in a more general framework that  
406 takes into account different costs associated with inventory holding, replacements and  
407 so on, in the same line as some of the recent studies on warranty analysis.

408 • Consider the role of human factors in the sales delay such as the ones consider by  
409 Wu<sup>22</sup>.

410 • Consider the case of extended warranties and how the sales delay can impact the  
411 formulation of extended warranty policies .

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Table 1: Parameters (Three state model)

$\eta$	$\alpha$	$\lambda$	$\nu$	$w$	$N$	$\sigma$	$\mu$
100	1.5	0.2	0.2	24	1000	1.5	0.7

Table 2: Parameters (Multistate model)

$\gamma$	$w$	$\mu$	$\sigma$	$N$	$t_1$	$t_2$	$t_3$	$t_4$	$\alpha_1$	$\alpha_2$	$\alpha_3$	$\alpha_4$
0.04	12	1.5	0.7	100	2	3	2	5	0.008	0.016	0.024	0.032

Table 3: Number of warranty claims versus  $\lambda$  (Multistate model)

$\lambda$	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
$S_3$	3.336	5.504	7.596	9.615	11.567	13.453	15.279	17.045	18.756	20.415
$S_4$	3.342	5.510	7.602	9.621	11.573	13.459	15.284	17.051	18.762	20.412

Table 4: Simulation results.  $K$ -prediction horizon columns show mean squared errors.

	lnL	AIC	$K = 12$	$K = 24$	$K = 36$
Model 1	5300.4	-10596.8	1.89	2.33	2.25
Model 2	5303.5	-10601.0	14.48	20.32	22.62
Model 3	5303.9	-10599.8	0.10	0.12	0.17

Table 5: Simulation results. Estimated parameters.

	$\hat{\eta}$	$\hat{\alpha}$	$\hat{p}$	$\hat{\lambda}$	$\hat{\nu}$
M1	73.98	1.19	-	-	-
M2	79.31	1.12	0.0048	-	-
M3	75.05	1.19	-	0.13	0.11

Table 7: Case study results.  $K$ -prediction horizon columns show mean squared errors.

	lnL	AIC	$K = 12$	$K = 24$	$K = 36$
Model 1	51308	-102612	2033	1688	1619
Model 2	51339	-102672	1131	821	754
Model 3	51333	-102658	371	461	639

Table 8: Case study results. Estimated parameters.

	$\hat{\eta}$	$\hat{\alpha}$	$\hat{p}$	$\hat{\lambda}$	$\hat{\nu}$
M1	467.85	0.80	-	-	-
M2	540.82	0.75	0.0028	-	-
M3	874.34	0.56	-	0.078	1.057

Table 6: Case study data.

$t$	$n_t$	$t$	$n_t$	$t$	$n_t$	$t$	$n_t$
1	51	16	391	31	291	46	273
2	163	17	380	32	288	47	254
3	299	18	384	33	269	48	246
4	484	19	370	34	262	49	244
5	597	20	333	35	265	50	224
6	662	21	312	36	277	51	280
7	671	22	332	37	272	52	253
8	623	23	308	38	257	53	235
9	552	24	333	39	258	54	241
10	530	25	317	40	255	55	215
11	501	26	287	41	282	56	264
12	460	27	345	42	251	57	240
13	462	28	325	43	240	58	211
14	413	29	262	44	260	59	210
15	447	30	276	45	273	60	214

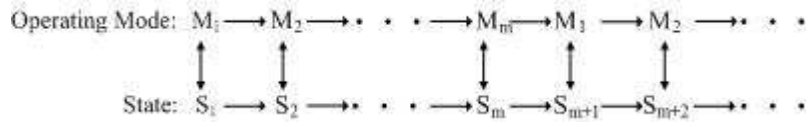


Figure 1: Operating modes and their corresponding states

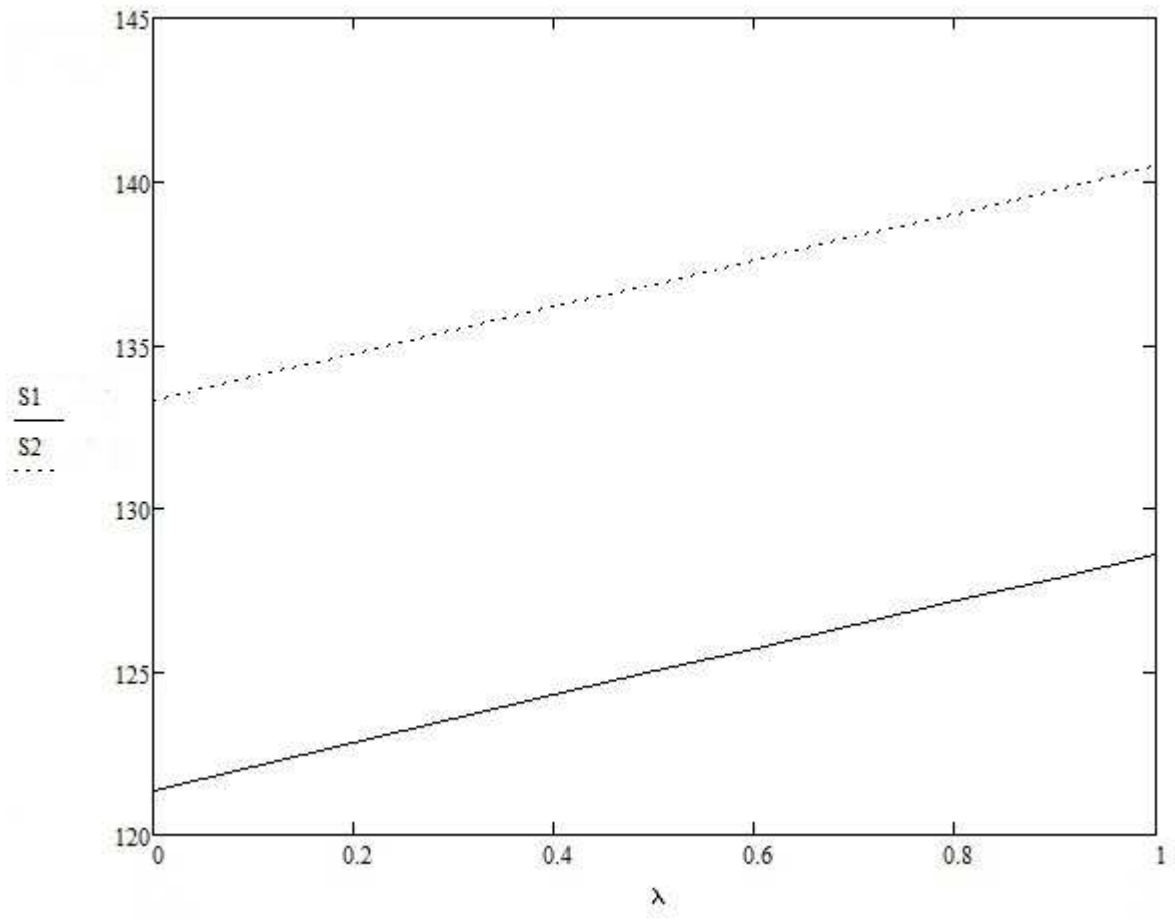


Figure 2: Expected number of warranty claims versus  $\lambda$ .

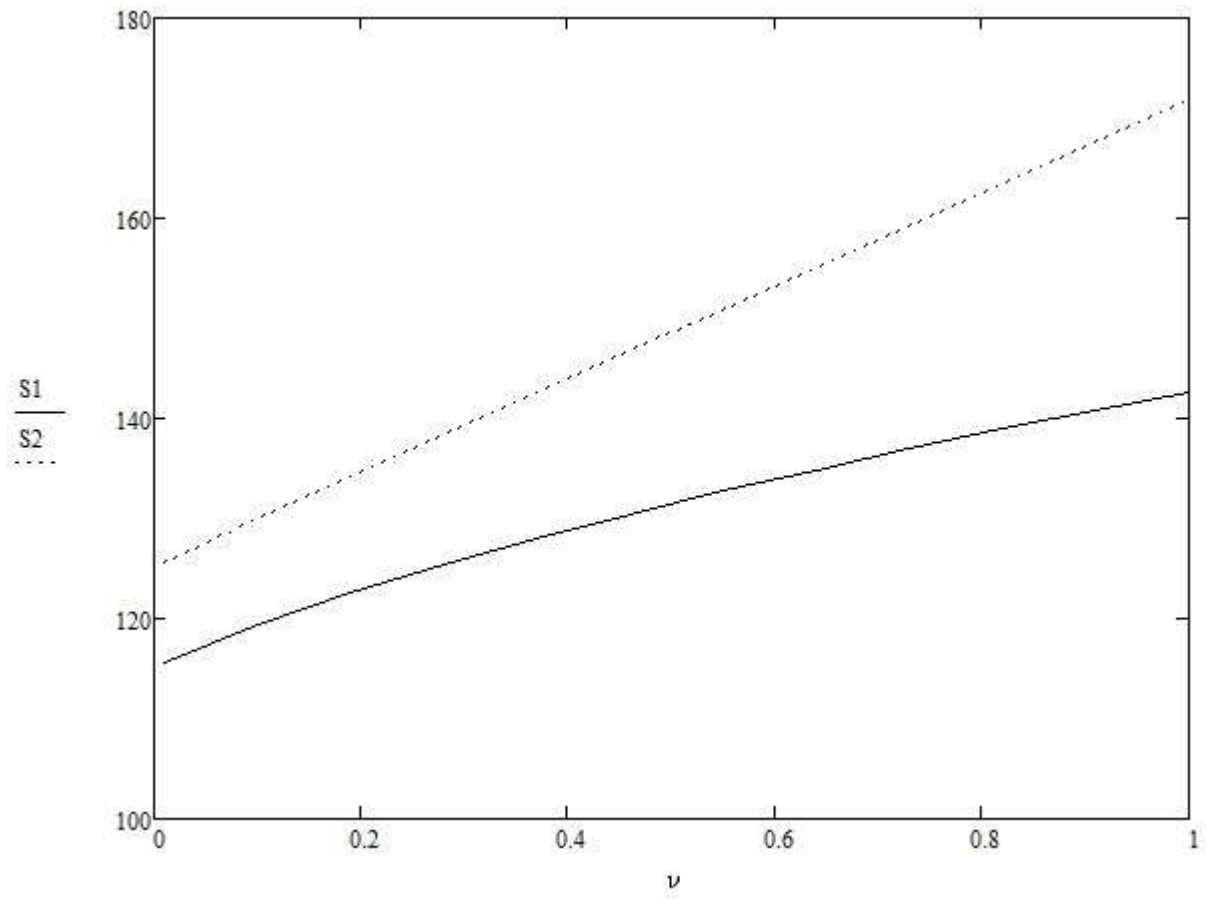


Figure 3: Expected number of warranty claims versus  $\nu$ .

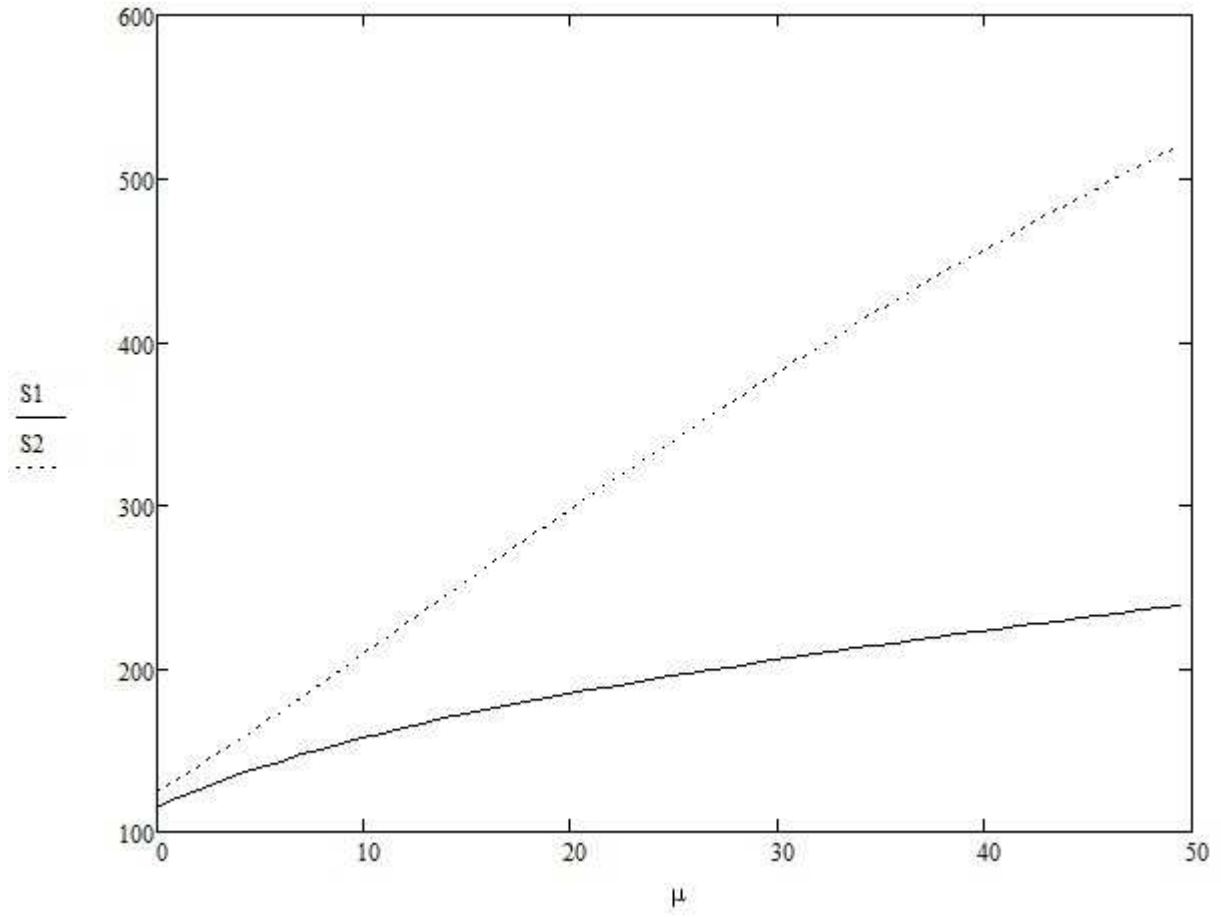


Figure 4: Expected number of warranty claims versus the mean of sales delay distribution,  $\mu$ .

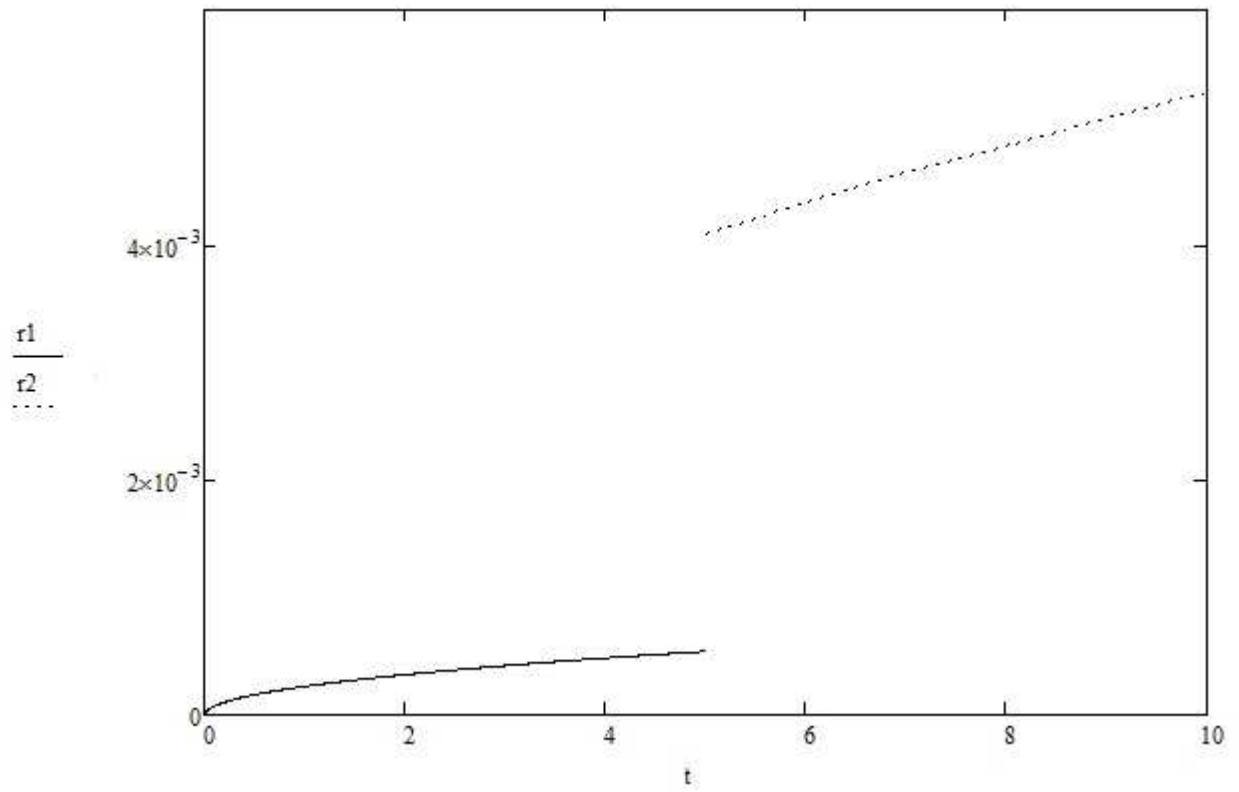


Figure 5: Failure rate during the sales delay and operating states for a product sold at time  $t = 5$ .



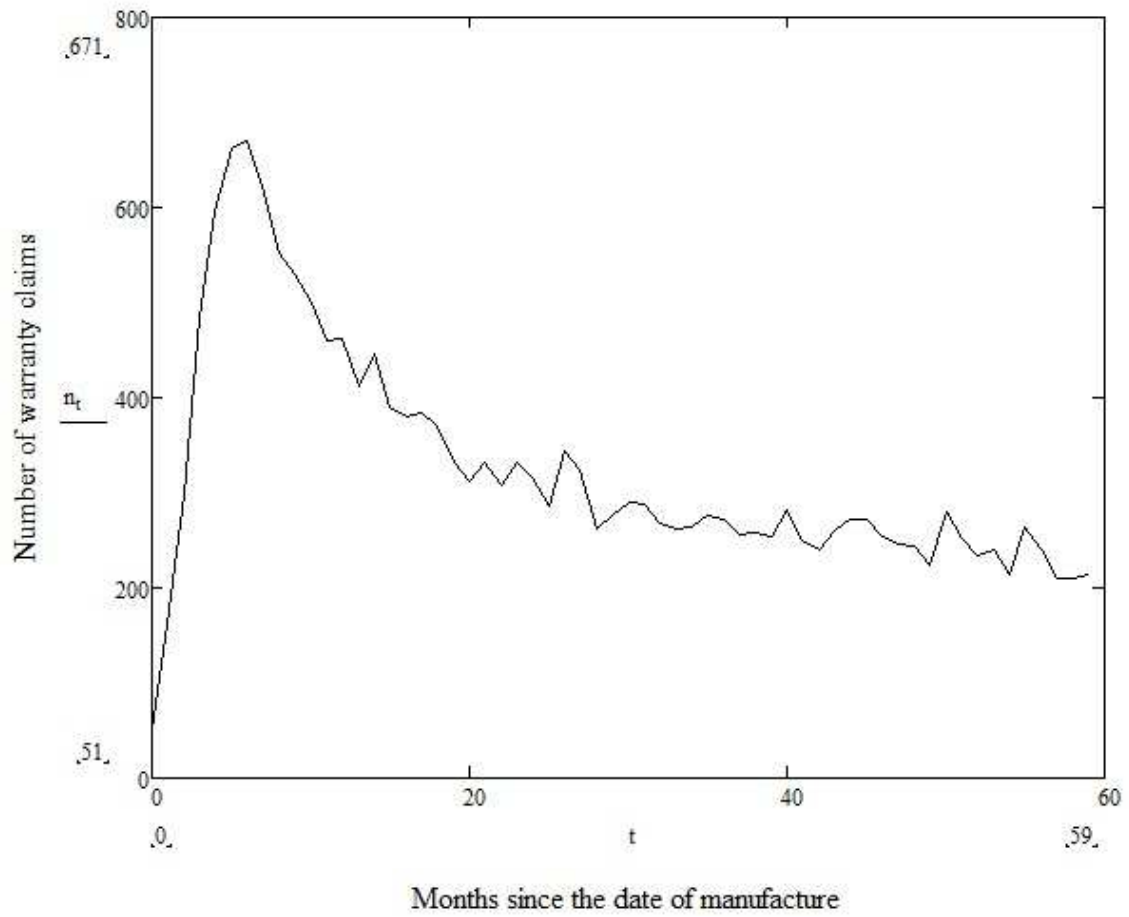


Figure 6: Data used the case study.