On the automorphism groups of $q$-enveloping algebras of nilpotent Lie algebras.

Stéphane Launois*

Abstract

We investigate the automorphism group of the quantised enveloping algebra $U_q^+(g)$ of the positive nilpotent part of certain simple complex Lie algebras $g$ in the case where the deformation parameter $q \in \mathbb{C}^*$ is not a root of unity. Studying its action on the set of minimal primitive ideals of $U_q^+(g)$ we compute this group in the cases where $g = \mathfrak{sl}_3$ and $g = \mathfrak{so}_5$ confirming a Conjecture of Andruskiewitsch and Dumas regarding the automorphism group of $U_q^+(g)$. In the case where $g = \mathfrak{sl}_3$, we retrieve the description of the automorphism group of the quantum Heisenberg algebra that was obtained independently by Alev and Dumas, and Caldero. In the case where $g = \mathfrak{so}_5$, the automorphism group of $U_q^+(g)$ was computed in [16] by using previous results of Andruskiewitsch and Dumas. In this paper, we give a new (simpler) proof of the Conjecture of Andruskiewitsch and Dumas in the case where $g = \mathfrak{so}_5$ based both on the original proof and on graded arguments developed in [17] and [18].

Introduction

In the classical situation, there are few results about the automorphism group of the enveloping algebra $U(L)$ of a Lie algebra $L$ over $\mathbb{C}$; except when $\dim L \leq 2$, these groups are known to possess “wild” automorphisms and are far from being understood. For instance, this is the case when $L$ is the three-dimensional abelian Lie algebra $\mathfrak{sl}_2$ [22], when $L = \mathfrak{sl}_2$ [14] and when $L$ is the three-dimensional Heisenberg Lie algebra [1].

In this paper we study the quantum situation. More precisely, we study the automorphism group of the quantised enveloping algebra $U_q^+(g)$ of the positive nilpotent part of a finite dimensional simple complex Lie algebra $g$ in the case where the deformation parameter $q \in \mathbb{C}^*$ is not a root of unity. Although it is a common belief that quantum algebras are ”rigid” and so should possess few symmetries, little is known about the automorphism group of $U_q^+(g)$. Indeed, until recently, this group was known only in the case where $g = \mathfrak{sl}_3$.

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whereas the structure of the automorphism group of the augmented form $\hat{U}_q(b^+)$, where $b^+$ is the positive Borel subalgebra of $\mathfrak{g}$, has been described in [9] in the general case.

The automorphism group of $U_q^+(\mathfrak{sl}_3)$ was computed independently by Alev-Dumas, [2], and Caldero, [8], who showed that

$$\text{Aut}(U_q^+(\mathfrak{sl}_3)) \simeq (\mathbb{C}^*)^2 \rtimes S_2.$$                      

Recently, Andruskiewitsch and Dumas, [4] have obtained partial results on the automorphism group of $U_q^+(\mathfrak{so}_5)$. In view of their results and the description of $\text{Aut}(U_q^+(\mathfrak{sl}_3))$, they have proposed the following conjecture.

**Conjecture (Andruskiewitsch-Dumas, [4, Problem 1]):**

$$\text{Aut}(U_q^+(\mathfrak{g})) \simeq (\mathbb{C}^*)^{rk(\mathfrak{g})} \rtimes \text{autdiagr}(\mathfrak{g}),$$

where $\text{autdiagr}(\mathfrak{g})$ denotes the group of automorphisms of the Dynkin diagram of $\mathfrak{g}$.

Recently we proved this conjecture in the case where $\mathfrak{g} = \mathfrak{so}_5$, [16], and, in collaboration with Samuel Lopes, in the case where $\mathfrak{g} = \mathfrak{sl}_4$, [18]. The techniques in these two cases are very different. Our aim in this paper is to show how one can prove the Andruskiewitsch-Dumas Conjecture in the cases where $\mathfrak{g} = \mathfrak{sl}_3$ and $\mathfrak{g} = \mathfrak{so}_5$ by first studying the action of $\text{Aut}(U_q^+(\mathfrak{g}))$ on the set of minimal primitive ideals of $U_q^+(\mathfrak{g})$ - this was the main idea in [16] -, and then using graded arguments as developed in [17] and [18]. This strategy leads us to a new (simpler) proof of the Andruskiewitsch-Dumas Conjecture in the case where $\mathfrak{g} = \mathfrak{so}_5$.

Throughout this paper, $\mathbb{N}$ denotes the set of nonnegative integers, $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$ and $q$ is a nonzero complex number that is not a root of unity.

\section{Preliminaries.}

In this section, we present the $\mathcal{H}$-stratification theory of Goodearl and Letzter for the positive part $U_q^+(\mathfrak{g})$ of the quantised enveloping algebra of a simple finite-dimensional complex Lie algebra $\mathfrak{g}$. In particular, we present a criterion (due to Goodearl and Letzter) that characterises the primitive ideals of $U_q^+(\mathfrak{g})$ among its prime ideals. In the next section, we will use this criterion in order to describe the primitive spectrum of $U_q^+(\mathfrak{g})$ in the cases where $\mathfrak{g} = \mathfrak{sl}_3$ and $\mathfrak{g} = \mathfrak{so}_5$.

\subsection{Quantised enveloping algebras and their positive parts.}

Let $\mathfrak{g}$ be a simple Lie $\mathbb{C}$-algebra of rank $n$. We denote by $\pi = \{\alpha_1, \ldots, \alpha_n\}$ the set of simple roots associated to a triangular decomposition $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$. Recall that $\pi$ is a basis of an euclidean vector space $E$ over $\mathbb{R}$, whose inner product is denoted by $(\ , \ )$ ($E$ is usually
denoted by $\mathfrak{h}_a$ in Bourbaki). We denote by $W$ the Weyl group of $\mathfrak{g}$, that is, the subgroup of the orthogonal group of $E$ generated by the reflections $s_i := s_{\alpha_i}$, for $i \in \{1, \ldots, n\}$, with reflecting hyperplanes $H_i := \{\beta \in E \mid (\beta, \alpha_i) = 0\}$, $i \in \{1, \ldots, n\}$. The length of $w \in W$ is denoted by $l(w)$. Further, we denote by $w_0$ the longest element of $W$. We denote by $R^+$ the set of positive roots and by $R$ the set of roots. Set $Q^+ := \mathbb{N}\alpha_1 \oplus \cdots \oplus \mathbb{N}\alpha_n$ and $Q := \mathbb{Z}\alpha_1 \oplus \cdots \oplus \mathbb{Z}\alpha_n$. Finally, we denote by $A = (a_{ij}) \in M_n(\mathbb{Z})$ the Cartan matrix associated to these data. As $\mathfrak{g}$ is simple, $a_{ij} \in \{0, -1, -2, -3\}$ for all $i \neq j$.

Recall that the scalar product of two roots $(\alpha, \beta)$ is always an integer. As in [5], we assume that the short roots have length $\sqrt{2}$.

For all $i \in \{1, \ldots, n\}$, set $q_i := q^{(\alpha_i, \alpha_i)/2}$ and

$$\begin{bmatrix} m \\ k \end{bmatrix}_i = \frac{(q_i - q_i^{-1}) \cdots (q_i^{m-1} - q_i^1)(q_i^m - q_i^{-m})}{(q_i - q_i^{-1}) \cdots (q_i^k - q_i^{-k})(q_i - q_i^{-k})(q_i^m - q_i^{-m})}$$

for all integers $0 \leq k \leq m$. By convention,

$$\begin{bmatrix} m \\ 0 \end{bmatrix}_i := 1.$$  

The quantised enveloping algebra $U_q(\mathfrak{g})$ of $\mathfrak{g}$ over $\mathbb{C}$ associated to the previous data is the $\mathbb{C}$-algebra generated by the indeterminates $E_1, \ldots, E_n, F_1, \ldots, F_n, K_i^{\pm 1}, \ldots, K_n^{\pm 1}$ subject to the following relations:

$$K_i K_j = K_j K_i$$

$$K_i E_j K_i^{-1} = q_i^{a_{ij}} E_j$$

$$K_i F_j K_i^{-1} = q_i^{-a_{ij}} F_j$$

$$E_i F_j - F_j E_i = \delta_{ij} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}}$$

and the quantum Serre relations:

$$\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1 - a_{ij} \\ k \end{bmatrix}_i E_i^{1-a_{ij}-k} E_j E_i^k = 0 \quad (i \neq j)$$

(1)

and

$$\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1 - a_{ij} \\ k \end{bmatrix}_i F_i^{1-a_{ij}-k} F_j F_i^k = 0 \quad (i \neq j).$$

We refer the reader to [5, 13, 15] for more details on this (Hopf) algebra. Further, as usual, we denote by $U_q^+(\mathfrak{g})$ (resp. $U_q^-(\mathfrak{g})$) the subalgebra of $U_q(\mathfrak{g})$ generated by $E_1, \ldots, E_n$ (resp. $F_1, \ldots, F_n$) and by $U_q^0$ the subalgebra of $U_q(\mathfrak{g})$ generated by $K_1^{\pm 1}, \ldots, K_n^{\pm 1}$. Moreover, for all $\alpha = a_1 \alpha_1 + \cdots + a_n \alpha_n \in Q$, we set

$$K_{\alpha} := K_1^{a_1} \cdots K_n^{a_n}.$$
As in the classical case, there is a triangular decomposition as vector spaces:

\[ U_q^-(\mathfrak{g}) \otimes U^0 \otimes U_q^+(\mathfrak{g}) \simeq U_q(\mathfrak{g}). \]

In this paper we are concerned with the algebra \( U_q^+(\mathfrak{g}) \) that admits the following presentation, see [13, Theorem 4.21]. The algebra \( U_q^+(\mathfrak{g}) \) is (isomorphic to) the \( \mathbb{C} \)-algebra generated by \( n \) indeterminates \( E_1, \ldots, E_n \) subject to the quantum Serre relations (1).

### 1.2 PBW-basis of \( U_q^+(\mathfrak{g}) \).

To each reduced decomposition of the longest element \( w_0 \) of the Weyl group \( W \) of \( \mathfrak{g} \), Lusztig has associated a PBW basis of \( U_q^+(\mathfrak{g}) \), see for instance [19, Chapter 37], [13, Chapter 8] or [5, I.6.7]. The construction relates to a braid group action by automorphisms on \( U_q^+(\mathfrak{g}) \). Let us first recall this action. For all \( s \in \mathbb{N} \) and \( i \in \{1, \ldots, n\} \), we set

\[
[s]_i := \frac{q_i^s - q_i^{-s}}{q_i - q_i^{-1}} \quad \text{and} \quad [s]_i! := [1]_i \cdots [s - 1]_i [s]_i,
\]

As in [5, I.6.7], we denote by \( T_i \), for \( 1 \leq i \leq n \), the automorphism of \( U_q^+(\mathfrak{g}) \) defined by:

\[
T_i(E_i) = -F_i K_i,
\]

\[
T_i(E_j) = \sum_{s=0}^{-a_{ij}} (-1)^{s-a_{ij}} q_i^{-s} E_i^{(-a_{ij}-s)} E_j E_i^{(s)}, \quad i \neq j
\]

\[
T_i(F_i) = -K_i^{-1} E_i,
\]

\[
T_i(F_j) = \sum_{s=0}^{-a_{ij}} (-1)^{s-a_{ij}} q_i^s E_i^{(s)} E_j F_i^{(-a_{ij}-s)}, \quad i \neq j
\]

\[
T_i(K_\alpha) = K_{s_i(\alpha)}, \quad \alpha \in Q,
\]

where \( E_i^{(s)} := \frac{E_i^s}{[s]_i!} \) and \( F_i^{(s)} := \frac{F_i^s}{[s]_i!} \) for all \( s \in \mathbb{N} \). It was proved by Lusztig that the automorphisms \( T_i \) satisfy the braid relations, that is, if \( s_i s_j \) has order \( m \) in \( W \), then

\[
T_i T_j T_i \cdots = T_j T_i T_j \cdots,
\]

where there are exactly \( m \) factors on each side of this equality.

The automorphisms \( T_i \) can be used in order to describe PBW bases of \( U_q^+(\mathfrak{g}) \) as follows.

It is well-known that the length of \( w_0 \) is equal to the number \( N \) of positive roots of \( \mathfrak{g} \). Let \( s_{i_1} \cdots s_{i_N} \) be a reduced decomposition of \( w_0 \). For \( k \in \{1, \ldots, N\} \), we set \( \beta_k := s_{i_1} \cdots s_{i_{k-1}}(\alpha_{i_k}) \). Then \( \{\beta_1, \ldots, \beta_N\} \) is exactly the set of positive roots of \( \mathfrak{g} \). Similarly, we define elements \( E_{\beta_k} \) of \( U_q(\mathfrak{g}) \) by

\[
E_{\beta_k} := T_{i_1} \cdots T_{i_{k-1}}(E_{i_k}).
\]

Note that the elements \( E_{\beta_k} \) depend on the reduced decomposition of \( w_0 \). The following well-known results were proved by Lusztig and Levendorskii-Soibelman.
Theorem 1.1 (Lusztig and Levendorskii-Soibelman)

1. For all $k \in \{1, \ldots, N\}$, the element $E_{\beta_k}$ belongs to $U_q^+(\mathfrak{g})$.
2. If $\beta_k = \alpha_i$, then $E_{\beta_k} = E_i$.
3. The monomials $E_{\beta_1}^{k_1} \cdots E_{\beta_N}^{k_N}$, with $k_1, \ldots, k_N \in \mathbb{N}$, form a linear basis of $U_q^+(\mathfrak{g})$.
4. For all $1 \leq i < j \leq N$, we have
   
   \[ E_{\beta_j} E_{\beta_i} - q^{-(\beta_i, \beta_j)} E_{\beta_i} E_{\beta_j} = \sum a_{k_{i+1}, \ldots, k_j-1} E_{\beta_{i+1}}^{k_{i+1}} \cdots E_{\beta_{j-1}}^{k_{j-1}}, \]

   where each $a_{k_{i+1}, \ldots, k_{j-1}}$ belongs to $\mathbb{C}$.

As a consequence of this result, $U_q^+(\mathfrak{g})$ can be presented as a skew-polynomial algebra:

\[ U_q^+(\mathfrak{g}) = \mathbb{C}[E_{\beta_1},E_{\beta_2},\ldots,E_{\beta_N};\sigma_1,\delta_1] \cdots [E_{\beta_N};\sigma_N,\delta_N], \]

where each $\sigma_i$ is a linear automorphism and each $\delta_i$ is a $\sigma_i$-derivation of the appropriate subalgebra. In particular, $U_q^+(\mathfrak{g})$ is a noetherian domain and its group of invertible elements is reduced to nonzero complex numbers.

1.3 Prime and primitive spectra of $U_q^+(\mathfrak{g})$.

We denote by $\text{Spec}(U_q^+(\mathfrak{g}))$ the set of prime ideals of $U_q^+(\mathfrak{g})$. First, as $q$ is not a root of unity, it was proved by Ringel \[21\] (see also \[10\] Theorem 2.3) that, as in the classical situation, every prime ideal of $U_q^+(\mathfrak{g})$ is completely prime.

In order to study the prime and primitive spectra of $U_q^+(\mathfrak{g})$, we will use the stratification theory developed by Goodearl and Letzter. This theory allows the construction of a partition of these two sets by using the action of a suitable torus on $U_q^+(\mathfrak{g})$. More precisely, the torus $\mathcal{H} := (\mathbb{C}^*)^n$ acts naturally by automorphisms on $U_q^+(\mathfrak{g})$ via:

\[ (h_1, \ldots, h_n).E_i = h_i E_i \text{ for all } i \in \{1, \ldots, n\}. \]

(\text{It is easy to check that the quantum Serre relations are preserved by the group $\mathcal{H}$.})

Recall (see \[\text{[14]}\] 3.4.1) that this action is rational. (We refer the reader to \[\text{[3]}\] II.2.] for the definition of a rational action.) A non-zero element $x$ of $U_q^+(\mathfrak{g})$ is an $\mathcal{H}$-eigenvector of $U_q^+(\mathfrak{g})$ if $h.x \in \mathbb{C}^* x$ for all $h \in \mathcal{H}$. An ideal $I$ of $U_q^+(\mathfrak{g})$ is $\mathcal{H}$-invariant if $h.I = I$ for all $h \in \mathcal{H}$. We denote by $\mathcal{H}\text{-Spec}(U_q^+(\mathfrak{g}))$ the set of all $\mathcal{H}$-invariant prime ideals of $U_q^+(\mathfrak{g})$. It turns out that this is a finite set by a theorem of Goodearl and Letzter about iterated Ore extensions, see \[\text{[11]}\] Proposition 4.2. In fact, one can be even more precise in our situation. Indeed, in \[\text{[12]}\], Gorelik has also constructed a stratification of the prime spectrum of $U_q^+(\mathfrak{g})$ using tools coming from representation theory. It turns out that her stratification coincides with the $\mathcal{H}$-stratification, so that we deduce from \[\text{[12]}\] Corollary 7.1.2 that

Proposition 1.2 (Gorelik) $U_q^+(\mathfrak{g})$ has exactly $|W|$ $\mathcal{H}$-invariant prime ideals.
The action of $\mathcal{H}$ on $U_q^+(g)$ allows via the $\mathcal{H}$-stratification theory of Goodearl and Letzter (see [5, II.2]) the construction of a partition of $\text{Spec}(U_q^+(g))$ as follows. If $J$ is an $\mathcal{H}$-invariant prime ideal of $U_q^+(g)$, we denote by $\text{Spec}_J(U_q^+(g))$ the $\mathcal{H}$-stratum of $\text{Spec}(U_q^+(g))$ associated to $J$. Recall that $\text{Spec}_J(U_q^+(g)) := \{ P \in \text{Spec}(U_q^+(g)) \mid \cap_{h \in \mathcal{H}} h.P = J \}$. Then the $\mathcal{H}$-strata $\text{Spec}_J(U_q^+(g))$ $(J \in \mathcal{H}-\text{Spec}(U_q^+(g)))$ form a partition of $\text{Spec}(U_q^+(g))$ (see [5, II.2]):

$$\text{Spec}(U_q^+(g)) = \bigsqcup_{J \in \mathcal{H}-\text{Spec}(U_q^+(g))} \text{Spec}_J(U_q^+(g)).$$

Naturally, this partition induces a partition of the set $\text{Prim}(U_q^+(g))$ of all (left) primitive ideals of $U_q^+(g)$ as follows. For all $J \in \mathcal{H}-\text{Spec}(U_q^+(g))$, we set $\text{Prim}_J(U_q^+(g)) := \text{Spec}_J(U_q^+(g)) \cap \text{Prim}(U_q^+(g))$. Then it is obvious that the $\mathcal{H}$-strata $\text{Prim}_J(U_q^+(g))$ $(J \in \mathcal{H}-\text{Spec}(U_q^+(g)))$ form a partition of $\text{Prim}(U_q^+(g))$:

$$\text{Prim}(U_q^+(g)) = \bigsqcup_{J \in \mathcal{H}-\text{Spec}(U_q^+(g))} \text{Prim}_J(U_q^+(g)).$$

More interestingly, because of the finiteness of the set of $\mathcal{H}$-invariant prime ideals of $U_q^+(g)$, the $\mathcal{H}$-stratification theory provides a useful tool to recognise primitive ideals without having to find all its irreducible representations! Indeed, following previous works of Hodges-Levasseur, Joseph, and Brown-Goodearl, Goodearl and Letzter have characterised the primitive ideals of $U_q^+(g)$ as follows, see [11, Corollary 2.7] or [5, Theorem II.8.4].

**Theorem 1.3 (Goodearl-Letzter)** $\text{Prim}_J(U_q^+(g))$ $(J \in \mathcal{H}-\text{Spec}(U_q^+(g)))$ coincides with those primes in $\text{Spec}_J(U_q^+(g))$ that are maximal in $\text{Spec}_J(U_q^+(g))$.

## 2 Automorphism group of $U_q^+(g)$.

In this section, we investigate the automorphism group of $U_q^+(g)$ viewed as the algebra generated by $n$ indeterminates $E_1, \ldots, E_n$ subject to the quantum Serre relations. This algebra has some well-identified automorphisms. First, there are the so-called torus automorphisms; let $\mathcal{H} = (\mathbb{C}^*)^n$, where $n$ still denotes the rank of $g$. As $U_q^+(g)$ is the $\mathbb{C}$-algebra generated by $n$ indeterminates subject to the quantum Serre relations, it is easy to check that each $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathcal{H}$ determines an algebra automorphism $\phi_\lambda$ of $U_q^+(g)$ with $\phi_\lambda(E_i) = \lambda_i E_i$ for $i \in \{1, \ldots, n\}$, with inverse $\phi_\lambda^{-1} = \phi_{\lambda^{-1}}$. Next, there are the so-called diagram automorphisms coming from the symmetries of the Dynkin diagram of $g$. Namely, let $w$ be an automorphism of the Dynkin diagram of $g$, that is, $w$ is an element of the symmetric group $S_n$ such that $(\alpha_i, \alpha_j) = (\alpha_{w(i)}, \alpha_{w(j)})$ for all $i, j \in \{1, \ldots, n\}$. Then one defines an automorphism, also denoted $w$, of $U_q^+(g)$ by: $w(E_i) = E_{w(i)}$. Observe that

$$\phi_\lambda \circ w = w \circ \phi(\lambda_{w(1)}, \ldots, \lambda_{w(n)}).$$
We denote by $G$ the subgroup of $\text{Aut}(U_+^q(\mathfrak{g}))$ generated by the torus automorphisms and the diagram automorphisms. Observe that

$$G \simeq \mathcal{H} \rtimes \text{autdiagr}(\mathfrak{g}),$$

where $\text{autdiagr}(\mathfrak{g})$ denotes the set of diagram automorphisms of $\mathfrak{g}$.

The group $\text{Aut}(U_+^q(\mathfrak{sl}_3))$ was computed independently by Alev and Dumas, see [2, Proposition 2.3], and Caldero, see [8, Proposition 4.4]; their results show that, in the case where $\mathfrak{g} = \mathfrak{sl}_3$, we have

$$\text{Aut}(U_+^q(\mathfrak{sl}_3)) = G.$$ 

About ten years later, Andruskiewitsch and Dumas investigated the case where $\mathfrak{g} = \mathfrak{so}_5$, see [4]. In this case, they obtained partial results that lead them to the following conjecture.

**Conjecture (Andruskiewitsch-Dumas, [4, Problem 1]):**

$$\text{Aut}(U_+^q(\mathfrak{g})) = G.$$ 

This conjecture was recently confirmed in two new cases: $\mathfrak{g} = \mathfrak{so}_5$, [16], and $\mathfrak{g} = \mathfrak{sl}_4$, [18]. Our aim in this section is to show how one can use the action of the automorphism group of $U_+^q(\mathfrak{g})$ on the primitive spectrum of this algebra in order to prove the Andruskiewitsch-Dumas Conjecture in the cases where $\mathfrak{g} = \mathfrak{sl}_3$ and $\mathfrak{g} = \mathfrak{so}_5$.

### 2.1 Normal elements of $U_+^q(\mathfrak{g})$.

Recall that an element $a$ of $U_+^q(\mathfrak{g})$ is normal provided the left and right ideals generated by $a$ in $U_+^q(\mathfrak{g})$ coincide, that is, if

$$aU_+^q(\mathfrak{g}) = U_+^q(\mathfrak{g})a.$$

In the sequel, we will use several times the following well-known result concerning normal elements of $U_+^q(\mathfrak{g})$.

**Lemma 2.1** Let $u$ and $v$ be two nonzero normal elements of $U_+^q(\mathfrak{g})$ such that $\langle u \rangle = \langle v \rangle$. Then there exist $\lambda, \mu \in \mathbb{C}^*$ such that $u = \lambda v$ and $v = \mu u$.

**Proof.** It is obvious that units $\lambda, \mu$ exist with these properties. However, the set of units of $U_+^q(\mathfrak{g})$ is precisely $\mathbb{C}^*$. \qed
2.2 N-grading on $U^+_q(\mathfrak{g})$ and automorphisms.

As the quantum Serre relations are homogeneous in the given generators, there is an $\mathbb{N}$-grading on $U^+_q(\mathfrak{g})$ obtained by assigning to $E_i$ degree 1. Let

$$U^+_q(\mathfrak{g}) = \bigoplus_{i \in \mathbb{N}} U^+_q(\mathfrak{g})_i$$

be the corresponding decomposition, with $U^+_q(\mathfrak{g})_i$ the subspace of homogeneous elements of degree $i$. In particular, $U^+_q(\mathfrak{g})_0 = \mathbb{C}$ and $U^+_q(\mathfrak{g})_1$ is the $n$-dimensional space spanned by the generators $E_1, \ldots, E_n$. For $t \in \mathbb{N}$ set $U^+_q(\mathfrak{g})_{\geq t} = \bigoplus_{i \geq t} U^+_q(\mathfrak{g})_i$ and define $U^+_q(\mathfrak{g})_{\leq t}$ similarly.

We say that the nonzero element $u \in U^+_q(\mathfrak{g})$ has degree $t$, and write $\deg(u) = t$, if $u \in U^+_q(\mathfrak{g})_{\leq t} \setminus U^+_q(\mathfrak{g})_{\leq t-1}$ (using the convention that $U^+_q(\mathfrak{g})_{\leq -1} = \{0\}$). As $U^+_q(\mathfrak{g})$ is a domain, $\deg(uv) = \deg(u) + \deg(v)$ for $u, v \neq 0$.

**Definition 2.2** Let $A = \bigoplus_{i \in \mathbb{N}} A_i$ be an $\mathbb{N}$-graded $\mathbb{C}$-algebra with $A_0 = \mathbb{C}$ which is generated as an algebra by $A_1 = \mathbb{C}x_1 \oplus \cdots \oplus \mathbb{C}x_n$. If for each $i \in \{1, \ldots, n\}$ there exist $0 \neq a \in A$ and a scalar $q_{i,a} \neq 1$ such that $x_i = q_{i,a} ax_i$, then we say that $A$ is an $\mathbb{N}$-graded algebra with enough $q$-commutation relations.

The algebra $U^+_q(\mathfrak{g})$, endowed with the grading just defined, is a connected $\mathbb{N}$-graded algebra with enough $q$-commutation relations. Indeed, if $i \in \{1, \ldots, n\}$, then there exists $u \in U^+_q(\mathfrak{g})$ such that $E_i u = q^\bullet u E_i$ where $\bullet$ is a nonzero integer. This can be proved as follows. As $\mathfrak{g}$ is simple, there exists an index $j \in \{1, \ldots, n\}$ such that $j \neq i$ and $a_{ij} \neq 0$, that is, $a_{ij} \in \{-1, -2, -3\}$. Then $s_is_j$ is a reduced expression in $W$, so that one can find a reduced expression of $w_0$ starting with $s_is_j$, that is, one can write

$$w_0 = s_is_js_i3 \cdots s_{iN}.$$

With respect to this reduced expression of $w_0$, we have with the notation of Section 1.2

$$\beta_1 = \alpha_i \quad \text{and} \quad \beta_2 = s_i(\alpha_j) = \alpha_j - a_{ij}\alpha_i$$

Then it follows from Theorem 1.1 that $E_{\beta_1} = E_i$, $E_{\beta_2} = E_{\alpha_j - a_{ij}\alpha_i}$ and

$$E_i E_{\beta_2} = q^{(\alpha_i, \alpha_j - a_{ij}\alpha_i)} E_{\beta_2} E_i,$$

that is,

$$E_i E_{\beta_2} = q^{-(\alpha_i, \alpha_j)} E_{\beta_2} E_i.$$

As $a_{ij} \neq 0$, we have $(\alpha_i, \alpha_j) \neq 0$ and so $q^{-(\alpha_i, \alpha_j)} \neq 1$ since $q$ is not a root of unity. So we have just proved:

**Proposition 2.3** $U^+_q(\mathfrak{g})$ is a connected $\mathbb{N}$-graded algebra with enough $q$-commutation relations.
One of the advantages of \(N\)-graded algebras with enough \(q\)-commutation relations is that any automorphism of such an algebra must conserve the valuation associated to the \(N\)-graduation. More precisely, as \(U_q^+(g)\) is a connected \(N\)-graded algebra with enough \(q\)-commutation relations, we deduce from [18] (see also [17, Proposition 3.2]) the following result.

**Corollary 2.4** Let \(\sigma \in \text{Aut}(U_q^+(g))\) and \(x \in U_q^+(g)_d \setminus \{0\}\). Then \(\sigma(x) = y_d + y_{>d}\) for some \(y_d \in U_q^+(g)_d \setminus \{0\}\) and \(y_{>d} \in U_q^+(g)_{>d+1}\).

### 2.3 The case where \(g = \mathfrak{sl}_3\).

In this section, we investigate the automorphism group of \(U_q^+(g)\) in the case where \(g = \mathfrak{sl}_3\). In this case the Cartan matrix is

\[
A = \begin{pmatrix}
2 & -1 \\
-1 & 2
\end{pmatrix},
\]

so that \(U_q^+(\mathfrak{sl}_3)\) is the \(C\)-algebra generated by two indeterminates \(E_1\) and \(E_2\) subject to the following relations:

\[
E_1^2E_2 - (q + q^{-1})E_1E_2E_1 + E_2E_1^2 = 0 \quad (3)
\]

\[
E_2^2E_1 - (q + q^{-1})E_2E_1E_2 + E_1E_2^2 = 0 \quad (4)
\]

We often refer to this algebra as the quantum Heisenberg algebra, and sometimes we denote it by \(\mathbb{H}\), as in the classical situation the enveloping algebra of \(\mathfrak{sl}_3^+\) is the so-called Heisenberg algebra.

We now make explicit a PBW basis of \(\mathbb{H}\). The Weyl group of \(\mathfrak{sl}_3\) is isomorphic to the symmetric group \(S_3\), where \(s_1\) is identified with the transposition \((1\ 2)\) and \(s_2\) is identified with \((2\ 3)\). Its longest element is then \(w_0 = (13)\); it has two reduced decompositions: \(w_0 = s_1s_2s_1 = s_2s_1s_2\). Let us choose the reduced decomposition \(s_1s_2s_1\) of \(w_0\) in order to construct a PBW basis of \(U_q^+(\mathfrak{sl}_3)\). According to Section \[1.2\] this reduced decomposition leads to the following root vectors:

\[
E_{\alpha_1} = E_1, \quad E_{\alpha_1+\alpha_2} = T_1(E_2) = -E_1E_2 + q^{-1}E_2E_1 \quad \text{and} \quad E_{\alpha_2} = T_1T_2(E_1) = E_2.
\]

In order to simplify the notation, we set \(E_3 := -E_1E_2 + q^{-1}E_2E_1\). Then, it follows from Theorem \[1.3\] that

- The monomials \(E_1^{k_1}E_3^{k_3}E_2^{k_2}\), with \(k_1, k_2, k_3\) nonnegative integers, form a PBW-basis of \(U_q^+(\mathfrak{sl}_3)\).

- \(\mathbb{H}\) is the iterated Ore extension over \(C\) generated by the indeterminates \(E_1, E_3, E_2\) subject to the following relations:

\[
E_3E_1 = q^{-1}E_1E_3, \quad E_2E_3 = q^{-1}E_3E_2, \quad E_2E_1 = qE_1E_2 + qE_3.
\]

In particular, \(\mathbb{H}\) is a Noetherian domain, and its group of invertible elements is reduced to \(C^*\).
It follows from the previous commutation relations between the root vectors that $E_3$ is a normal element in $H$, that is, $E_3 H = HE_3$.

In order to describe the prime and primitive spectra of $H$, we need to introduce two other elements. The first one is the root vector $E_3' := T_2(E_1) = -E_2 E_1 + q^{-1} E_1 E_2$. This root vector would have appeared if we have chosen the reduced decomposition $s_2 s_1 s_2$ of $w_0$ in order to construct a PBW basis of $H$. It follows from Theorem 1.1 that $E_3'$ $q$-commutes with $E_1$ and $E_2$, so that $E_3'$ is also a normal element of $H$. Moreover, one can describe the centre of $H$ using the two normal elements $E_3$ and $E_3'$. Indeed, in [3, Corollaire 2.16], Alev and Dumas have described the centre of $U_q(sln)$; independently Caldero has described the centre of $U_q^+(g)$ for arbitrary $g$, see [7]. In our particular situation, their results show that the centre $Z(H)$ of $H$ is a polynomial ring in one variable $Z(H) = \mathbb{C}[\Omega]$, where $\Omega = E_3 E_3'$.

We are now in position to describe the prime and primitive spectra of $H = U_q^+(sl(3))$; this was first achieved by Malliavin who obtained the following picture for the poset of prime ideals of $H$, see [20, Théorème 2.4]:

\[
\begin{align*}
\langle\langle E_1, E_2 - \beta \rangle\rangle & \quad \langle\langle E_1, E_2 \rangle\rangle & \quad \langle\langle E_1 - \alpha, E_2 \rangle\rangle \\
\langle E_1 \rangle & \quad \langle E_2 \rangle & \\
\langle\langle E_3 \rangle \rangle & \quad \langle\langle \Omega - \gamma \rangle \rangle & \quad \langle\langle E_3' \rangle \rangle \\
\langle 0 \rangle & \quad & \end{align*}
\]

where $\alpha, \beta, \gamma \in \mathbb{C}^*$.

Recall from Section 1.3 that the torus $H = (\mathbb{C}^*)^2$ acts on $U_q^+(sl(3))$ by automorphisms and that the $H$-stratification theory of Goodearl and Letzter constructs a partition of the prime spectrum of $U_q^+(sl(3))$ into so-called $H$-strata, this partition being indexed by the $H$-invariant prime ideals of $U_q^+(sl(3))$. Using this description of $\text{Spec}(U_q^+(sl(3)))$, it is easy to identify the $6 = |W|$ $H$-invariant prime ideals of $H$ and their corresponding $H$-strata. As $E_1, E_2, E_3$ and $E_3'$ are $H$-eigenvectors, the $6$ $H$-invariant primes are:

$\langle 0 \rangle, \langle E_3 \rangle, \langle E_3' \rangle, \langle E_1 \rangle, \langle E_2 \rangle$ and $\langle E_1, E_2 \rangle$.

Moreover the corresponding $H$-strata are:

$\text{Spec}_{\langle 0 \rangle}(H) = \{\langle 0 \rangle\} \cup \{\langle \Omega - \gamma \rangle \mid \gamma \in \mathbb{C}^*\}$,
\[ \text{Spec}_\mathcal{G}(\mathbb{H}) = \{ \langle E_3 \rangle \}, \]
\[ \text{Spec}_\mathcal{G}(\mathbb{H}) = \{ \langle E'_3 \rangle \}, \]
\[ \text{Spec}_\mathcal{G}(\mathbb{H}) = \{ \langle E_1 \rangle \} \cup \{ \langle E_1, E_2 - \beta \rangle \mid \beta \in \mathbb{C}^* \}, \]
\[ \text{Spec}_\mathcal{G}(\mathbb{H}) = \{ \langle E_2 \rangle \} \cup \{ \langle E_1 - \alpha, E_2 \rangle \mid \alpha \in \mathbb{C}^* \} \]
and \[ \text{Spec}_\mathcal{G}(\mathcal{G}_1, \mathcal{G}_2)(\mathbb{H}) = \{ \langle E_1, E_2 \rangle \}. \]

We deduce from this description of the \( \mathcal{H} \)-strata and the fact that primitive ideals are exactly those primes that are maximal within their \( \mathcal{H} \)-strata, see Theorem 1.3, that the primitive ideals of \( U_q^+(\mathfrak{sl}_3) \) are exactly those primes that appear in double brackets in the previous picture.

We now investigate the group of automorphisms of \( \mathbb{H} = U_q^+(\mathfrak{sl}_3) \). In that case, the torus acting naturally on \( U_q^+(\mathfrak{sl}_3) \) is \( \mathcal{H} = (\mathbb{C}^*)^2 \), there is only one non-trivial diagram automorphism \( w \) that exchanges \( E_1 \) and \( E_2 \), and so the subgroup \( G \) of \( \text{Aut}(U_q^+(\mathfrak{sl}_3)) \) generated by the torus and diagram automorphisms is isomorphic to the semi-direct product \( (\mathbb{C}^*)^2 \rtimes S_2 \). We want to prove that \( \text{Aut}(U_q^+(\mathfrak{sl}_3)) = G \).

In order to do this, we study the action of \( \text{Aut}(U_q^+(\mathfrak{sl}_3)) \) on the set of primitive ideals that are not maximal. As there are only two of them, \( \langle E_3 \rangle \) and \( \langle E'_3 \rangle \), an automorphism of \( \mathbb{H} \) will either fix them or permute them.

Let \( \sigma \) be an automorphism of \( U_q^+(\mathfrak{sl}_3) \). It follows from the previous observation that

\[ \text{either } \sigma(\langle E_3 \rangle) = \langle E_3 \rangle \text{ and } \sigma(\langle E'_3 \rangle) = \langle E'_3 \rangle, \]

or \( \sigma(\langle E_3 \rangle) = \langle E'_3 \rangle \text{ and } \sigma(\langle E'_3 \rangle) = \langle E_3 \rangle \).

As it is clear that the diagram automorphism \( w \) permutes the ideals \( \langle E_3 \rangle \) and \( \langle E'_3 \rangle \), we get that there exists an automorphism \( g \in G \) such that

\[ g \circ \sigma(\langle E_3 \rangle) = \langle E_3 \rangle \text{ and } g \circ \sigma(\langle E'_3 \rangle) = \langle E'_3 \rangle \].

Then, as \( E_3 \) and \( E'_3 \) are normal, we deduce from Lemma 2.1 that there exist \( \lambda, \lambda' \in \mathbb{C}^* \) such that

\[ g \circ \sigma(E_3) = \lambda E_3 \text{ and } g \circ \sigma(E'_3) = \lambda' E'_3. \]

In order to prove that \( g \circ \sigma \) is an element of \( G \), we now use the \( \mathbb{N} \)-gradation of \( U_q^+(\mathfrak{sl}_3) \) introduced in Section 2.2. With respect to this graduation, \( E_1 \) and \( E_2 \) are homogeneous of degree 1, and so \( E_3 \) and \( E'_3 \) are homogeneous of degree 2. Moreover, as \( (q^{-2} - 1)E_1E_2 = E_3 + q^{-1}E'_3 \), we deduce from the above discussion that

\[ g \circ \sigma(E_1E_2) = \frac{1}{q^{-2} - 1} (\lambda E_3 + q^{-1}\lambda' E'_3) \]

has degree two. On the other hand, as \( U_q^+(\mathfrak{sl}_3) \) is a connected \( \mathbb{N} \)-graded algebra with enough \( q \)-commutation relations by Proposition 2.3, it follows from Corollary 2.4 that \( \sigma(E_1) = a_1E_1 + a_2E_2 + u \) and \( \sigma(E_2) = b_1E_1 + b_2E_2 + v \), where \( (a_1, a_2), (b_1, b_2) \in \mathbb{C}^2 \setminus \{(0, 0)\} \), and \( u, v \in U_q^+(\mathfrak{sl}_3) \) are linear combinations of homogeneous elements of degree greater than one. As \( g \circ \sigma(E_1), g \circ \sigma(E_2) \) has degree two, it is clear that \( u = v = 0 \). To conclude that
follows from Theorem 1.1 that $g \circ \sigma \in G$, it just remains to prove that $a_2 = 0 = b_1$. This can be easily shown by using the fact that $g \circ \sigma (-E_1 E_2 + q^{-1} E_2 E_1) = g \circ \sigma (E_3) = \lambda E_3$; replacing $g \circ \sigma (E_2)$ by $a_1 E_1 + a_2 E_2$ and $b_1 E_1 + b_2 E_2$ respectively, and then identifying the coefficients in the PBW basis, leads to $a_2 = 0 = b_1$, as required. Hence we have just proved that $g \circ \sigma \in G$, so that $\sigma$ itself belongs to $G$ the subgroup of $\text{Aut}(U^+_q(\mathfrak{sl}_3))$ generated by the torus and diagram automorphisms. Hence one can state the following result that confirms the Andruskiewitsch-Dumas Conjecture.

**Proposition 2.5** $\text{Aut}(U^+_q(\mathfrak{sl}_3)) \simeq (\mathbb{C}^*)^2 \rtimes \text{autdiagr}(\mathfrak{sl}_3)$

This result was first obtained independently by Alev and Dumas, [2 Proposition 2.3], and Caldero, [8, Proposition 4.4], but using somehow different methods; they studied this automorphism group by looking at its action on the set of normal elements of $U^+_q(\mathfrak{sl}_3)$.

### 2.4 The case where $g = \mathfrak{so}_5$.

In this section we investigate the automorphism group of $U^+_q(\mathfrak{g})$ in the case where $\mathfrak{g} = \mathfrak{so}_5$. In this case there are no diagram automorphisms, so that the Andruskiewitsch-Dumas Conjecture asks whether every automorphism of $U^+_q(\mathfrak{so}_5)$ is a torus automorphism. In [16] we have proved their conjecture when $\mathfrak{g} = \mathfrak{so}_5$. The aim of this section is to present a slightly different proof based both on the original proof and on the recent proof by S.A. Lopes and the author of the Andruskiewitsch-Dumas Conjecture in the case where $\mathfrak{g}$ is of type $A_3$.

In the case where $\mathfrak{g} = \mathfrak{so}_5$, the Cartan matrix is $A = \begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix}$, so that $U^+_q(\mathfrak{so}_5)$ is the $\mathbb{C}$-algebra generated by two indeterminates $E_1$ and $E_2$ subject to the following relations:

\begin{align*}
E_1^3 E_2 - (q^2 + 1 + q^{-2}) E_1^2 E_2 E_1 + (q^2 + 1 + q^{-2}) E_1 E_2^2 + E_2 E_1^3 & = 0 \\
E_2^3 E_1 - (q^2 + 1 + q^{-2}) E_2 E_1 E_2 + E_1 E_2^3 & = 0
\end{align*}

We now make explicit a PBW basis of $U^+_q(\mathfrak{so}_5)$. The Weyl group of $\mathfrak{so}_5$ is isomorphic to the dihedral group $D(4)$. Its longest element is $w_0 = -id$; it has two reduced decompositions: $w_0 = s_1 s_2 s_1 s_2 = s_2 s_1 s_2 s_1$. Let us choose the reduced decomposition $s_1 s_2 s_1 s_2$ of $w_0$ in order to construct a PBW basis of $U^+_q(\mathfrak{so}_5)$. According to Section 1.2, this reduced decomposition leads to the following root vectors:

\begin{align*}
E_{a_1} & = E_1, \quad E_{2a_1+a_2} = T_1(E_2) = \frac{1}{(q + q^{-1})} (E_1^2 E_2 - q^{-1}(q + q^{-1}) E_1 E_2 E_1 + q^{-2} E_2 E_1^2), \\
E_{a_1+a_2} & = T_1 T_2(E_1) = -E_1 E_2 + q^{-2} E_2 E_1 \quad \text{and} \quad E_{a_2} = T_1 T_2 T_1(E_2) = E_2.
\end{align*}

In order to simplify the notation, we set $E_3 := -E_{a_1+a_2}$ and $E_4 := E_{2a_1+a_2}$. Then, it follows from Theorem 1.1 that

- The monomials $E_1^{k_1} E_2^{k_2} E_3^{k_3} E_4^{k_4}$, with $k_1, k_2, k_3, k_4$ nonnegative integers, form a PBW-basis of $U^+_q(\mathfrak{so}_5)$.
• $U_q^+(\mathfrak{so}_5)$ is the iterated Ore extension over $\mathbb{C}$ generated by the indeterminates $E_1$, $E_4$, $E_3$, $E_2$ subject to the following relations:

\begin{align*}
E_4E_1 &= q^{-2}E_1E_4 \\
E_3E_1 &= E_1E_3 - (q + q^{-1})E_4, \quad E_3E_4 = q^{-2}E_4E_3, \\
E_2E_1 &= q^2E_1E_2 - q^2E_3, \quad E_2E_4 = E_4E_2 - \frac{q^2-1}{q+q^{-1}}E_3^2, \quad E_2E_3 = q^{-2}E_3E_2.
\end{align*}

In particular, $U_q^+(\mathfrak{so}_5)$ is a Noetherian domain, and its group of invertible elements is reduced to $\mathbb{C}^*$. Before describing the automorphism group of $U_q^+(\mathfrak{so}_5)$, we first describe the centre and the primitive ideals of $U_q^+(\mathfrak{so}_5)$. The centre of $U_q^+(\mathfrak{g})$ has been described in general by Caldero, [7]. In the case where $\mathfrak{g} = \mathfrak{so}_5$, his result shows that $Z(U_q^+(\mathfrak{so}_5))$ is a polynomial algebra in two indeterminates

$$Z(U_q^+(\mathfrak{so}_5)) = \mathbb{C}[z, z'],$$

where

$$z = (1 - q^2)E_1E_3 + q^2(q + q^{-1})E_4$$

and

$$z' = -(q^2 - q^{-2})(q + q^{-1})E_4E_2 + q^2(q^2 - 1)E_3^2.$$

Recall from Section 1.3 that the torus $\mathcal{H} = (\mathbb{C}^*)^2$ acts on $U_q^+(\mathfrak{so}_5)$ by automorphisms and that the $\mathcal{H}$-stratification theory of Goodearl and Letzter constructs a partition of the prime spectrum of $U_q^+(\mathfrak{so}_5)$ into so-called $\mathcal{H}$-strata, this partition being indexed by the $8 = |W| \mathcal{H}$-invariant prime ideals of $U_q^+(\mathfrak{so}_5)$. In [15], we have described these eight $\mathcal{H}$-strata. More precisely, we have obtained the following picture for the poset $\text{Spec}(U_q^+(\mathfrak{so}_5))$, 13
where $\alpha, \beta, \gamma, \delta \in \mathbb{C}^*$, $E_3' := E_1E_2 - q^2E_2E_1$ and

$$I = \{\langle P(z, z') \rangle \mid P \text{ is a unitary irreducible polynomial of } \mathbb{C}[z, z'], \ P \neq z, z'\}.$$ 

As the primitive ideals are those primes that are maximal in their $\mathcal{H}$-strata, see Theorem 1.3, we deduced from this description of the prime spectrum that the primitive ideals of $U_q^+(\mathfrak{so}_5)$ are the following:

- $\langle z - \alpha, z' - \beta \rangle$ with $(\alpha, \beta) \in \mathbb{C}^2 \setminus \{(0, 0)\}$.
- $\langle E_3 \rangle$ and $\langle E_3' \rangle$.
- $\langle E_1 - \alpha, E_2 - \beta \rangle$ with $(\alpha, \beta) \in \mathbb{C}^2$ such that $\alpha\beta = 0$.

(They correspond to the “double brackets” prime ideals in the above picture.)

Among them, two only are not maximal, $\langle E_3 \rangle$ and $\langle E_3' \rangle$. Unfortunately, as $E_3$ and $E_3'$ are not normal in $U_q^+(\mathfrak{so}_5)$, one cannot easily obtain information using the fact that any automorphism of $U_q^+(\mathfrak{so}_5)$ will either preserve or exchange these two prime ideals. Rather than using this observation, we will use the action of $\text{Aut}(U_q^+(\mathfrak{so}_5))$ on the set of maximal ideals of height two. Because of the previous description of the primitive spectrum of $U_q^+(\mathfrak{so}_5)$, the height two maximal ideals in $U_q^+(\mathfrak{so}_5)$ are those $\langle z - \alpha, z' - \beta \rangle$ with $(\alpha, \beta) \in \mathbb{C}^2 \setminus \{(0, 0)\}$. In [16, Proposition 3.6], we have proved that the group of units of the factor algebra $U_q^+(\mathfrak{so}_5)/\langle z - \alpha, z' - \beta \rangle$ is reduced to $\mathbb{C}^*$ if and only if both $\alpha$ and $\beta$ are nonzero. Consequently, if $\sigma$ is an automorphism of $U_q^+(\mathfrak{so}_5)$ and $\alpha \in \mathbb{C}^*$, we get that:

$$\sigma(\langle z - \alpha, z' \rangle) = \langle z - \alpha', z' \rangle \text{ or } \langle z, z' - \beta' \rangle,$$
where \( \alpha', \beta' \in \mathbb{C}^* \). Similarly, if \( \sigma \) is an automorphism of \( U^+_q(\mathfrak{so}_5) \) and \( \beta \in \mathbb{C}^* \), we get that:

\[
\sigma(\langle z, z' - \beta \rangle) = \langle z - \alpha', z' \rangle \text{ or } \langle z, z' - \beta' \rangle,
\]

where \( \alpha', \beta' \in \mathbb{C}^* \).

We now use this information to prove that the action of \( \text{Aut}(U^+_q(\mathfrak{so}_5)) \) on the centre of \( U^+_q(\mathfrak{so}_5) \) is trivial. More precisely, we are now in position to prove the following result.

**Proposition 2.6** Let \( \sigma \in \text{Aut}(U^+_q(\mathfrak{so}_5)) \). There exist \( \lambda, \lambda' \in \mathbb{C}^* \) such that

\[
\sigma(z) = \lambda z \quad \text{and} \quad \sigma(z') = \lambda' z'.
\]

**Proof.** We only prove the result for \( z \). First, using the fact that \( U^+_q(\mathfrak{so}_5) \) is noetherian, it is easy to show that, for any family \( \{\beta_i\}_{i \in \mathbb{N}} \) of pairwise distinct nonzero complex numbers, we have:

\[
\langle z \rangle = \bigcap_{i \in \mathbb{N}} P_{0, \beta_i} \quad \text{and} \quad \langle z' \rangle = \bigcap_{i \in \mathbb{N}} P_{\beta_i, 0},
\]

where \( P_{\alpha, \beta} := \langle z - \alpha, z' - \beta \rangle \). Indeed, if the inclusion

\[
\langle z \rangle \subseteq I := \bigcap_{i \in \mathbb{N}} P_{0, \beta_i}
\]

is not an equality, then any \( P_{0, \beta_i} \) is a minimal prime over \( I \) for height reasons. As the \( P_{0, \beta_i} \) are pairwise distinct, \( I \) is a two-sided ideal of \( U^+_q(\mathfrak{so}_5) \) with infinitely many prime ideals minimal over it. This contradicts the noetherianity of \( U^+_q(\mathfrak{so}_5) \). Hence

\[
\langle z \rangle = \bigcap_{i \in \mathbb{N}} P_{0, \beta_i} \quad \text{and} \quad \langle z' \rangle = \bigcap_{i \in \mathbb{N}} P_{\beta_i, 0},
\]

and so

\[
\sigma(\langle z \rangle) = \bigcap_{i \in \mathbb{N}} \sigma(P_{0, \beta_i}).
\]

It follows from (7) that, for all \( i \in \mathbb{N} \), there exists \( (\gamma_i, \delta_i) \neq (0, 0) \) with \( \gamma_i = 0 \) or \( \delta_i = 0 \) such that

\[
\sigma(P_{0, \beta_i}) = P_{\gamma_i, \delta_i}.
\]

Naturally, we can choose the family \( \{\beta_i\}_{i \in \mathbb{N}} \) such that either \( \gamma_i = 0 \) for all \( i \in \mathbb{N} \), or \( \delta_i = 0 \) for all \( i \in \mathbb{N} \). Moreover, observe that, as the \( \beta_i \) are pairwise distinct, so are the \( \gamma_i \) or the \( \delta_i \).

Hence, either

\[
\sigma(\langle z \rangle) = \bigcap_{i \in \mathbb{N}} P_{\gamma_i, 0},
\]

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or
\[ \sigma((z)) = \bigcap_{i \in \mathbb{N}} P_{0,i}, \]
that is,
\[ \text{either } \langle \sigma(z) \rangle = \sigma(\langle z \rangle) = \langle z' \rangle \text{ or } \langle \sigma(z) \rangle = \sigma(\langle z \rangle) = \langle z \rangle. \]

As \( z, \sigma(z) \) and \( z' \) are all central, it follows from Lemma 2.1 that there exists \( \lambda \in \mathbb{C}^\ast \) such that either \( \sigma(z) = \lambda z \) or \( \sigma(z) = \lambda z' \).

To conclude, it just remains to show that the second case cannot happen. In order to do this, we use a graded argument. Observe that, with respect to the \( \mathbb{N} \)-graduation of \( U_q^+(\mathfrak{so}_5) \) defined in Section 2.2, \( z \) and \( z' \) are homogeneous of degree 3 and 4 respectively. Thus, if \( \sigma(z) = \lambda z' \), then we would obtain a contradiction with the fact that every automorphism of \( U_q^+(\mathfrak{so}_5) \) preserves the valuation, see Corollary 2.4. Hence \( \sigma(z) = \lambda z \), as desired. The corresponding result for \( z' \) can be proved in a similar way, so we omit it. \( \square \)

Andruskiewitsch and Dumas, \([4, \text{Proposition 3.3}]\), have proved that the subgroup of those automorphisms of \( U_q^+(\mathfrak{so}_5) \) that stabilize \( \langle z \rangle \) is isomorphic to \((\mathbb{C}^\ast)^2\). Thus, as we have just shown that every automorphism of \( U_q^+(\mathfrak{so}_5) \) fixes \( \langle z \rangle \), we get that \( \text{Aut}(U_q^+(\mathfrak{so}_5)) \) itself is isomorphic to \((\mathbb{C}^\ast)^2\). This is the route that we have followed in \([16]\) in order to prove the Andruskiewitsch-Dumas Conjecture in the case where \( \mathfrak{g} = \mathfrak{so}_5 \). Recently, with Samuel Lopes, we proved this Conjecture in the case where \( \mathfrak{g} = \mathfrak{sl}_4 \) using different methods and in particular graded arguments. We are now using (similar) graded arguments to prove that every automorphism of \( U_q^+(\mathfrak{so}_5) \) is a torus automorphism (without using results of Andruskiewitsch and Dumas).

In the proof, we will need the following relation that is easily obtained by straightforward computations.

**Lemma 2.7** \((q^2 - 1)E_3E'_3 = (q^4 - 1)zE_2 + q^2z'\).

**Proposition 2.8** Let \( \sigma \) be an automorphism of \( U_q^+(\mathfrak{so}_5) \). Then there exist \( a_1, b_2 \in \mathbb{C}^\ast \) such that
\[ \sigma(E_1) = a_1E_1 \quad \text{and} \quad \sigma(E_2) = b_2E_2. \]

**Proof.** For all \( i \in \{1, \ldots, 4\} \), we set \( d_i := \deg(\sigma(E_i)) \). We also set \( d'_3 := \deg(\sigma(E'_3)) \). It follows from Corollary 2.4 that \( d_1, d_2 \geq 1, \; d_3, d'_3 \geq 2 \) and \( d_4 \geq 3 \). First we prove that \( d_1 = d_2 = 1 \).

Assume first that \( d_1 + d_3 \geq 3 \). As \( z = (1 - q^2)E_1E_3 + q^2(q + q^{-1})E_4 \) and \( \sigma(z) = \lambda z \) with \( \lambda \in \mathbb{C}^\ast \) by Proposition 2.6, we get:
\[ \lambda z = (1 - q^2)\sigma(E_1)\sigma(E_3) + q^2(q + q^{-1})\sigma(E_4). \] (8)

Recall that \( \deg(uv) = \deg(u) + \deg(v) \) for \( u, v \neq 0 \), as \( U_q^+(\mathfrak{g}) \) is a domain. Thus, as \( \deg(z) = 3 < \deg(\sigma(E_1)\sigma(E_3)) = d_1 + d_3 \), we deduce from (8) that \( d_1 + d_3 = d_4 \). As \( z' = -(q^2 - q^{-2})(q + q^{-1})E_4E_2 + q^2(q^2 - 1)E_3^2 \) and \( \deg(z') = 4 < d_1 + d_3 + d_2 = d_4 + d_2 = \)
deg(σ(E_4)σ(E_2)), we get in a similar manner that d_2 + d_4 = 2d_3. Thus d_1 + d_2 = d_3. As d_1 + d_3 > 3, this forces d_3 > 2 and so d_3 + d'_3 > 4. Thus we deduce from Lemma 2.7 that d_3 + d'_3 = 3 + d_2. Hence d_1 + d'_3 = 3. As d_1 ≥ 1 and d'_3 ≥ 2, this implies d_1 = 1 and d'_3 = 2.

Thus we have just proved that d_1 = deg(σ(E_1)) = 1 and either d_3 = 2 or d'_3 = 2. To prove that d_2 = 1, we distinguish between these two cases.

If d_3 = 2, then as previously we deduce from the relation z' = (q^2 - q^{-2})(q + q^{-1})E_4E_2 + q^2(q^2 - 1)E_3^2 that d_2 + d_4 = 4, so that d_2 = 1, as desired.

If d'_3 = 2, then one can use the definition of E'_3 and the previous expression of z' in order to prove that z' = q^{-2}(q^2 - 1)E_3^2 + E_2u, where u is a nonzero homogeneous element of U_q^+(sl_5) of degree 3. (u is nonzero since (z') is a completely prime ideal and E_3 \notin (z') for degree reasons.) As d'_3 = 2 and deg(σ(z')) = 4, we get as previously that d_2 = 1.

To summarise, we have just proved that deg(σ(E_1)) = 1 = deg(σ(E_2)), so that σ(E_1) = a_1E_1 + a_2E_2 and σ(E_2) = b_1E_1 + b_2E_2, where (a_1, a_2), (b_1, b_2) ∈ C^2 \{ (0,0) \}. To conclude that a_2 = b_1 = 0, one can for instance use the fact that σ(E_1) and σ(E_2) must satisfy the quantum Serre relations.

We have just confirmed the Andruskiewitsch-Dumas Conjecture in the case where g = sl_5.

**Theorem 2.9** Every automorphism of U_q^+(sl_5) is a torus automorphism, so that

\[
\text{Aut}(U_q^+(sl_5)) \simeq (\mathbb{C}^*)^2.
\]

### 2.5 Beyond these two cases.

To finish this overview paper, let us mention that recently the Andruskiewitsch-Dumas Conjecture was confirmed by Samuel Lopes and the author, in the case where g = sl_4. The crucial step of the proof is to prove that, up to an element of G, every normal element of U_q^+(sl_4) is fixed by every automorphism. This step was dealt with by first computing the Lie algebra of derivations of U_q^+(sl_4), and this already requires a lot of computations!

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**References**


Stéphane Launois:
Institute of Mathematics, Statistics and Actuarial Science,
University of Kent at Canterbury, CT2 7NF, UK.
Email: S.Launois@kent.ac.uk