On the automorphism groups of q-enveloping algebras of nilpotent Lie algebras.

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Abstract

We investigate the automorphism group of the quantised enveloping algebra $U_q^+(\mathfrak{g})$ of the positive nilpotent part of certain simple complex Lie algebras $\mathfrak{g}$ in the case where the deformation parameter $q \in \mathbb{C}^*$ is not a root of unity. Studying its action on the set of minimal primitive ideals of $U_q^+(\mathfrak{g})$ we compute this group in the cases where $\mathfrak{g} = \mathfrak{sl}_3$ and $\mathfrak{g} = \mathfrak{so}_5$ confirming a Conjecture of Andruskiewitsch and Dumas regarding the automorphism group of $U_q^+(\mathfrak{g})$. In the case where $\mathfrak{g} = \mathfrak{sl}_3$, we retrieve the description of the automorphism group of the quantum Heisenberg algebra that was obtained independently by Alev and Dumas, and Caldero. In the case where $\mathfrak{g} = \mathfrak{so}_5$, the automorphism group of $U_q^+(\mathfrak{g})$ was computed in [16] by using previous results of Andruskiewitsch and Dumas. In this paper, we give a new (simpler) proof of the Conjecture of Andruskiewitsch and Dumas in the case where $\mathfrak{g} = \mathfrak{so}_5$ based both on the original proof and on graded arguments developed in [17] and [18].

Introduction

In the classical situation, there are few results about the automorphism group of the enveloping algebra $U(\mathcal{L})$ of a Lie algebra $\mathcal{L}$ over $\mathbb{C}$; except when $\dim \mathcal{L} \leq 2$, these groups are known to possess "wild" automorphisms and are far from being understood. For instance, this is the case when $\mathcal{L}$ is the three-dimensional abelian Lie algebra [22], when $\mathcal{L} = \mathfrak{sl}_2$ [14] and when $\mathcal{L}$ is the three-dimensional Heisenberg Lie algebra [1].

In this paper we study the quantum situation. More precisely, we study the automorphism group of the quantised enveloping algebra $U_q^+(\mathfrak{g})$ of the positive nilpotent part of a finite dimensional simple complex Lie algebra $\mathfrak{g}$ in the case where the deformation parameter $q \in \mathbb{C}^*$ is not a root of unity. Although it is a common belief that quantum algebras are "rigid" and so should possess few symmetries, little is known about the automorphism group of $U_q^+(\mathfrak{g})$. Indeed, until recently, this group was known only in the case where $\mathfrak{g} = \mathfrak{sl}_3$

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whereas the structure of the automorphism group of the augmented form $\check{U}_q(\mathfrak{b}^+)$, where $\mathfrak{b}^+$ is the positive Borel subalgebra of $\mathfrak{g}$, has been described in [9] in the general case.

The automorphism group of $U_q^+(\mathfrak{s}l_3)$ was computed independently by Alev-Dumas, [2], and Caldero, [8], who showed that

$$\mathrm{Aut}(U_q^+(\mathfrak{s}l_3)) \simeq (\mathbb{C}^*)^2 \rtimes S_2.$$

Recently, Andruskiewitsch and Dumas, [4] have obtained partial results on the automorphism group of $U_q^+(\mathfrak{so}_5)$. In view of their results and the description of $\mathrm{Aut}(U_q^+(\mathfrak{s}l_3))$, they have proposed the following conjecture.

Conjecture (Andruskiewitsch-Dumas, [4, Problem 1]):

$$\mathrm{Aut}(U_q^+(\mathfrak{g})) \simeq (\mathbb{C}^*)^{\mathrm{rk}(\mathfrak{g})} \rtimes \mathrm{autdiagr}(\mathfrak{g}),$$

where $\mathrm{autdiagr}(\mathfrak{g})$ denotes the group of automorphisms of the Dynkin diagram of $\mathfrak{g}$.

Recently we proved this conjecture in the case where $\mathfrak{g} = \mathfrak{so}_5$, [16], and, in collaboration with Samuel Lopes, in the case where $\mathfrak{g} = \mathfrak{sl}_4$, [18]. The techniques in these two cases are very different. Our aim in this paper is to show how one can prove the Andruskiewitsch-Dumas Conjecture in the cases where $\mathfrak{g} = \mathfrak{sl}_3$ and $\mathfrak{g} = \mathfrak{so}_5$ by first studying the action of $\mathrm{Aut}(U_q^+(\mathfrak{g}))$ on the set of minimal primitive ideals of $U_q^+(\mathfrak{g})$ - this was the main idea in [16] -, and then using graded arguments as developed in [17] and [18]. This strategy leads us to a new (simpler) proof of the Andruskiewitsch-Dumas Conjecture in the case where $\mathfrak{g} = \mathfrak{so}_5$.

Throughout this paper, $\mathbb{N}$ denotes the set of nonnegative integers, $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$ and q is a nonzero complex number that is not a root of unity.

1 Preliminaries.

In this section, we present the $\mathcal{H}$-stratification theory of Goodearl and Letzter for the positive part $U_q^+(\mathfrak{g})$ of the quantised enveloping algebra of a simple finite-dimensional complex Lie algebra $\mathfrak{g}$. In particular, we present a criterion (due to Goodearl and Letzter) that characterises the primitive ideals of $U_q^+(\mathfrak{g})$ among its prime ideals. In the next section, we will use this criterion in order to describe the primitive spectrum of $U_q^+(\mathfrak{g})$ in the cases where $\mathfrak{g} = \mathfrak{sl}_3$ and $\mathfrak{g} = \mathfrak{so}_5$.

1.1 Quantised enveloping algebras and their positive parts.

Let $\mathfrak{g}$ be a simple Lie $\mathbb{C}$-algebra of rank n. We denote by $\pi = \{\alpha_1, \dots, \alpha_n\}$ the set of simple roots associated to a triangular decomposition $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$. Recall that π is a basis of an euclidean vector space E over $\mathbb{R}$, whose inner product is denoted by $(\ ,\ )$ (E is usually denoted by $\mathfrak{h}^*_{\mathbb{R}}$ in Bourbaki). We denote by W the Weyl group of $\mathfrak{g}$, that is, the subgroup of the orthogonal group of E generated by the reflections $s_i := s_{\alpha_i}$, for $i \in \{1,\dots,n\}$, with reflecting hyperplanes $H_i := \{\beta \in E \mid (\beta,\alpha_i) = 0\}$, $i \in \{1,\dots,n\}$. The length of $w \in W$ is denoted by $l(w)$. Further, we denote by w_0 the longest element of W. We denote by R^+ the set of positive roots and by R the set of roots. Set $Q^+ := \mathbb{N}\alpha_1 \oplus \cdots \oplus \mathbb{N}\alpha_n$ and $Q := \mathbb{Z}\alpha_1 \oplus \cdots \oplus \mathbb{Z}\alpha_n$. Finally, we denote by $A = (a_{ij}) \in M_n(\mathbb{Z})$ the Cartan matrix associated to these data. As $\mathfrak{g}$ is simple, $a_{ij} \in \{0,-1,-2,-3\}$ for all $i \neq j$.

Recall that the scalar product of two roots (α,β) is always an integer. As in [5], we assume that the short roots have length $\sqrt{2}$.

For all $i \in \{1,\dots,n\}$, set $q_i := q^{\frac{(\alpha_i,\alpha_i)}{2}}$ and

$$\left[\begin{array}{c} m \\ k \end{array}\right]_i := \frac{(q_i - q_i^{-1})\dots(q_i^{m-1} - q_i^{1-m})(q_i^m - q_i^{-m})}{(q_i - q_i^{-1})\dots(q_i^k - q_i^{-k})(q_i - q_i^{-1})\dots(q_i^{m-k} - q_i^{k-m})}$$

for all integers $0 \leq k \leq m$. By convention,

$$\left[\begin{array}{c} m \\ 0 \end{array}\right]_i := 1.$$

The quantised enveloping algebra $U_q(\mathfrak{g})$ of $\mathfrak{g}$ over $\mathbb{C}$ associated to the previous data is the $\mathbb{C}$-algebra generated by the indeterminates $E_1,\dots,E_n,F_1,\dots,F_n,K_1^{\pm1},\dots,K_n^{\pm1}$ subject to the following relations:

$$K_iK_j = K_jK_i$$

$$K_iE_jK_i^{-1} = q_i^{a_{ij}}E_j \text{ and } K_iF_jK_i^{-1} = q_i^{-a_{ij}}F_j$$

$$E_iF_j - F_jE_i = \delta_{ij}\frac{K_i - K_i^{-1}}{q_i - q_i^{-1}}$$

and the quantum Serre relations:

$$\sum_{k=0}^{1-a_{ij}} (-1)^k \left[\begin{array}{c} 1-a_{ij} \\ k \end{array}\right]_i E_i^{1-a_{ij}-k}E_jE_i^k = 0 \ (i \neq j) \tag{1}$$

and

$$\sum_{k=0}^{1-a_{ij}} (-1)^k \left[\begin{array}{c} 1-a_{ij} \\ k \end{array}\right]_i F_i^{1-a_{ij}-k}F_jF_i^k = 0 \ (i \neq j).$$

We refer the reader to [5, 13, 15] for more details on this (Hopf) algebra. Further, as usual, we denote by $U_q^+(\mathfrak{g})$ (resp. $U_q^-(\mathfrak{g})$) the subalgebra of $U_q(\mathfrak{g})$ generated by $E_1,\dots,E_n$ (resp. $F_1,\dots,F_n$) and by U^0 the subalgebra of $U_q(\mathfrak{g})$ generated by $K_1^{\pm1},\dots,K_n^{\pm1}$. Moreover, for all $\alpha = a_1\alpha_1 + \cdots + a_n\alpha_n \in Q$, we set

$$K_\alpha := K_1^{a_1}\cdots K_n^{a_n}.$$

As in the classical case, there is a triangular decomposition as vector spaces:

$$U_q^-(\mathfrak{g}) \otimes U^0 \otimes U_q^+(\mathfrak{g}) \simeq U_q(\mathfrak{g}).$$

In this paper we are concerned with the algebra $U_q^+(\mathfrak{g})$ that admits the following presentation, see [13, Theorem 4.21]. The algebra $U_q^+(\mathfrak{g})$ is (isomorphic to) the $\mathbb{C}$-algebra generated by n indeterminates $E_1, \ldots, E_n$ subject to the quantum Serre relations (1).

1.2 PBW-basis of $U_q^+(\mathfrak{g})$.

To each reduced decomposition of the longest element w_0 of the Weyl group W of $\mathfrak{g}$, Lusztig has associated a PBW basis of $U_q^+(\mathfrak{g})$, see for instance [19, Chapter 37], [13, Chapter 8] or [5, I.6.7]. The construction relates to a braid group action by automorphisms on $U_q^+(\mathfrak{g})$. Let us first recall this action. For all $s \in \mathbb{N}$ and $i \in \{1, \ldots, n\}$, we set

$$[s]_i := \frac{q_i^s - q_i^{-s}}{q_i - q_i^{-1}} \quad \text{and} \quad [s]_i! := [1]_i \ldots [s-1]_i [s]_i.$$

As in [5, I.6.7], we denote by $\mathrm{T_i}$, for $1 \leq i \leq n$, the automorphism of $U_q^+(\mathfrak{g})$ defined by:

$$T_i(E_i) = -F_i K_i,$$
$$T_i(E_j) = \sum_{s=0}^{-a_{ij}} (-1)^{s-a_{ij}} q_i^{-s} E_i^{(-a_{ij}-s)} E_j E_i^{(s)}, \quad i \neq j$$
$$T_i(F_i) = -K_i^{-1} E_i,$$
$$T_i(F_j) = \sum_{s=0}^{-a_{ij}} (-1)^{s-a_{ij}} q_i^{s} F_i^{(s)} F_j F_i^{(-a_{ij}-s)}, \quad i \neq j$$
$$T_i(K_\alpha) = K_{s_i(\alpha)}, \quad \alpha \in Q,$$

where $E_i^{(s)} := \frac{E_i^s}{[s]_i!}$ and $F_i^{(s)} := \frac{F_i^s}{[s]_i!}$ for all $s \in \mathbb{N}$. It was proved by Lusztig that the automorphisms T_i satisfy the braid relations, that is, if $s_i s_j$ has order m in W, then

$$T_i T_j T_i \cdots = T_j T_i T_j \ldots,$$

where there are exactly m factors on each side of this equality.

The automorphisms T_i can be used in order to describe PBW bases of $U_q^+(\mathfrak{g})$ as follows. It is well-known that the length of w_0 is equal to the number N of positive roots of $\mathfrak{g}$. Let $s_{i_1} \cdots s_{i_N}$ be a reduced decomposition of w_0. For $k \in \{1, \ldots, N\}$, we set $\beta_k := s_{i_1} \cdots s_{i_{k-1}}(\alpha_{i_k})$. Then $\{\beta_1, \ldots, \beta_N\}$ is exactly the set of positive roots of $\mathfrak{g}$. Similarly, we define elements E_{β_k} of $U_q(\mathfrak{g})$ by

$$E_{\beta_k} := T_{i_1} \cdots T_{i_{k-1}}(E_{i_k}).$$

Note that the elements E_{β_k} depend on the reduced decomposition of w_0. The following well-known results were proved by Lusztig and Levendorskii-Soibelman.

Theorem 1.1 (Lusztig and Levendorskii-Soibelman)

1. *For all* $k \in \{1, \ldots, N\}$, *the element* E_{β_k} *belongs to* $U_q^+(\mathfrak{g})$.
2. *If* $\beta_k = \alpha_i$, *then* $E_{\beta_k} = E_i$.
3. *The monomials* $E_{\beta_1}^{k_1} \cdots E_{\beta_N}^{k_N}$, *with* $k_1, \ldots, k_N \in \mathbb{N}$, *form a linear basis of* $U_q^+(\mathfrak{g})$.
4. *For all* $1 \leq i < j \leq N$, *we have*

$$E_{\beta_j} E_{\beta_i} - q^{-(\beta_i, \beta_j)} E_{\beta_i} E_{\beta_j} = \sum a_{k_{i+1}, \ldots, k_{j-1}} E_{\beta_{i+1}}^{k_{i+1}} \cdots E_{\beta_{j-1}}^{k_{j-1}},$$

   where each $a_{k_{i+1}, \ldots, k_{j-1}}$ *belongs to* $\mathbb{C}$.

As a consequence of this result, $U_q^+(\mathfrak{g})$ can be presented as a skew-polynomial algebra:

$$U_q^+(\mathfrak{g}) = \mathbb{C}[E_{\beta_1}][E_{\beta_2}; \sigma_2, \delta_2] \cdots [E_{\beta_N}; \sigma_N, \delta_N],$$

where each σ_i is a linear automorphism and each δ_i is a σ_i-derivation of the appropriate subalgebra. In particular, $U_q^+(\mathfrak{g})$ is a noetherian domain and its group of invertible elements is reduced to nonzero complex numbers.

1.3 Prime and primitive spectra of $U_q^+(\mathfrak{g})$.

We denote by $\mathrm{Spec}(U_q^+(\mathfrak{g}))$ the set of prime ideals of $U_q^+(\mathfrak{g})$. First, as q is not a root of unity, it was proved by Ringel [21] (see also [10, Theorem 2.3]) that, as in the classical situation, every prime ideal of $U_q^+(\mathfrak{g})$ is completely prime.

In order to study the prime and primitive spectra of $U_q^+(\mathfrak{g})$, we will use the stratification theory developed by Goodearl and Letzter. This theory allows the construction of a partition of these two sets by using the action of a suitable torus on $U_q^+(\mathfrak{g})$. More precisely, the torus $\mathcal{H} := (\mathbb{C}^*)^n$ acts naturally by automorphisms on $U_q^+(\mathfrak{g})$ via:

$$(h_1, \ldots, h_n).E_i = h_i E_i \text{ for all } i \in \{1, \ldots, n\}.$$

(It is easy to check that the quantum Serre relations are preserved by the group $\mathcal{H}$.) Recall (see [4, 3.4.1]) that this action is rational. (We refer the reader to [5, II.2.] for the defintion of a rational action.) A non-zero element x of $U_q^+(\mathfrak{g})$ is an $\mathcal{H}$-*eigenvector of* $U_q^+(\mathfrak{g})$ if $h.x \in \mathbb{C}^* x$ for all $h \in \mathcal{H}$. An ideal I of $U_q^+(\mathfrak{g})$ is $\mathcal{H}$-*invariant* if $h.I = I$ for all $h \in \mathcal{H}$. We denote by $\mathcal{H}$-$\mathrm{Spec}(U_q^+(\mathfrak{g}))$ the set of all $\mathcal{H}$-invariant prime ideals of $U_q^+(\mathfrak{g})$. It turns out that this is a finite set by a theorem of Goodearl and Letzter about iterated Ore extensions, see [11, Proposition 4.2]. In fact, one can be even more precise in our situation. Indeed, in [12], Gorelik has also constructed a stratification of the prime spectrum of $U_q^+(\mathfrak{g})$ using tools coming from representation theory. It turns out that her stratification coincides with the $\mathcal{H}$-stratification, so that we deduce from [12, Corollary 7.1.2] that

Proposition 1.2 (Gorelik) $U_q^+(\mathfrak{g})$ *has exactly* $|W|$ $\mathcal{H}$-*invariant prime ideals.*

The action of $\mathcal{H}$ on $U_q^+(\mathfrak{g})$ allows via the $\mathcal{H}$-stratification theory of Goodearl and Letzter (see [5, II.2]) the construction of a partition of $\mathrm{Spec}(U_q^+(\mathfrak{g}))$ as follows. If J is an $\mathcal{H}$-invariant prime ideal of $U_q^+(\mathfrak{g})$, we denote by $\mathrm{Spec}_J(U_q^+(\mathfrak{g}))$ the *$\mathcal{H}$-stratum of* $\mathrm{Spec}(U_q^+(\mathfrak{g}))$ *associated to* J. Recall that $\mathrm{Spec}_J(U_q^+(\mathfrak{g})) := \{P \in \mathrm{Spec}(U_q^+(\mathfrak{g})) \mid \bigcap_{h\in\mathcal{H}} h.P = J\}$. Then the $\mathcal{H}$-strata $\mathrm{Spec}_J(U_q^+(\mathfrak{g}))$ ($J \in \mathcal{H}\text{-}\mathrm{Spec}(U_q^+(\mathfrak{g}))$) form a partition of $\mathrm{Spec}(U_q^+(\mathfrak{g}))$ (see [5, II.2]):

$$\mathrm{Spec}(U_q^+(\mathfrak{g})) = \bigsqcup_{J\in\mathcal{H}\text{-}\mathrm{Spec}(U_q^+(\mathfrak{g}))} \mathrm{Spec}_J(U_q^+(\mathfrak{g})).$$

Naturally, this partition induces a partition of the set $\mathrm{Prim}(U_q^+(\mathfrak{g}))$ of all (left) primitive ideals of $U_q^+(\mathfrak{g})$ as follows. For all $J \in \mathcal{H}\text{-}\mathrm{Spec}(U_q^+(\mathfrak{g}))$, we set $\mathrm{Prim}_J(U_q^+(\mathfrak{g})) := \mathrm{Spec}_J(U_q^+(\mathfrak{g})) \cap \mathrm{Prim}(U_q^+(\mathfrak{g}))$. Then it is obvious that the $\mathcal{H}$-strata $\mathrm{Prim}_J(U_q^+(\mathfrak{g}))$ ($J \in \mathcal{H}\text{-}\mathrm{Spec}(U_q^+(\mathfrak{g}))$) form a partition of $\mathrm{Prim}(U_q^+(\mathfrak{g}))$:

$$\mathrm{Prim}(U_q^+(\mathfrak{g})) = \bigsqcup_{J\in\mathcal{H}\text{-}\mathrm{Spec}(U_q^+(\mathfrak{g}))} \mathrm{Prim}_J(U_q^+(\mathfrak{g})).$$

More interestingly, because of the finiteness of the set of $\mathcal{H}$-invariant prime ideals of $U_q^+(\mathfrak{g})$, the $\mathcal{H}$-stratification theory provides a useful tool to recognise primitive ideals without having to find all its irreductible representations! Indeed, following previous works of Hodges-Levasseur, Joseph, and Brown-Goodearl, Goodearl and Letzter have characterised the primitive ideals of $U_q^+(\mathfrak{g})$ as follows, see [11, Corollary 2.7] or [5, Theorem II.8.4].

Theorem 1.3 (Goodearl-Letzter) $\mathrm{Prim}_J(U_q^+(\mathfrak{g}))$ *(*$J \in \mathcal{H}\text{-}\mathrm{Spec}(U_q^+(\mathfrak{g}))$*) coincides with those primes in* $\mathrm{Spec}_J(U_q^+(\mathfrak{g}))$ *that are maximal in* $\mathrm{Spec}_J(U_q^+(\mathfrak{g}))$.

2 Automorphism group of $U_q^+(\mathfrak{g})$.

In this section, we investigate the automorphism group of $U_q^+(\mathfrak{g})$ viewed as the algebra generated by n indeterminates $E_1, \ldots, E_n$ subject to the quantum Serre relations. This algebra has some well-identified automorphisms. First, there are the so-called torus automorphisms; let $\mathcal{H} = (\mathbb{C}^*)^n$, where n still denotes the rank of $\mathfrak{g}$. As $U_q^+(\mathfrak{g})$ is the $\mathbb{C}$-algebra generated by n indeterminates subject to the quantum Serre relations, it is easy to check that each $\bar{\lambda} = (\lambda_1, \ldots, , \lambda_n) \in \mathcal{H}$ determines an algebra automorphism $\phi_{\bar{\lambda}}$ of $U_q^+(\mathfrak{g})$ with $\phi_{\bar{\lambda}}(E_i) = \lambda_i E_i$ for $i \in \{1, \ldots, n\}$, with inverse $\phi_{\bar{\lambda}}^{-1} = \phi_{\bar{\lambda}^{-1}}$. Next, there are the so-called diagram automorphisms coming from the symmetries of the Dynkin diagram of $\mathfrak{g}$. Namely, let w be an automorphism of the Dynkin diagram of $\mathfrak{g}$, that is, w is an element of the symmetric group S_n such that $(\alpha_i, \alpha_j) = (\alpha_{w(i)}, \alpha_{w(j)})$ for all $i, j \in \{1, \ldots, n\}$. Then one defines an automorphism, also denoted w, of $U_q^+(\mathfrak{g})$ by: $w(E_i) = E_{w(i)}$. Observe that

$$\phi_{\bar{\lambda}} \circ w = w \circ \phi_{(\lambda_{w(1)}, \ldots, , \lambda_{w(n)})}.$$

We denote by G the subgroup of $\mathrm{Aut}(U_q^+(\mathfrak{g}))$ generated by the torus automorphisms and the diagram automorphisms. Observe that

$$G \simeq \mathcal{H} \rtimes \mathrm{autdiagr}(\mathfrak{g}),$$

where $\mathrm{autdiagr}(\mathfrak{g})$ denotes the set of diagram automorphisms of $\mathfrak{g}$.

The group $\mathrm{Aut}(U_q^+(\mathfrak{s}l_3))$ was computed independently by Alev and Dumas, see [2, Proposition 2.3] , and Caldero, see [8, Proposition 4.4]; their results show that, in the case where $\mathfrak{g} = \mathfrak{sl}_3$, we have

$$\mathrm{Aut}(U_q^+(\mathfrak{s}l_3)) = G.$$

About ten years later, Andruskiewitsch and Dumas investigated the case where $\mathfrak{g} = \mathfrak{so}_5$, see [4]. In this case, they obtained partial results that lead them to the following conjecture.

Conjecture (Andruskiewitsch-Dumas, [4, Problem 1]):

$$\mathrm{Aut}(U_q^+(\mathfrak{g})) = G.$$

This conjecture was recently confirmed in two new cases: $\mathfrak{g} = \mathfrak{so}_5$, [16], and $\mathfrak{g} = \mathfrak{sl}_4$, [18]. Our aim in this section is to show how one can use the action of the automorphism group of $U_q^+(\mathfrak{g})$ on the primitive spectrum of this algebra in order to prove the Andruskiewitsch-Dumas Conjecture in the cases where $\mathfrak{g} = \mathfrak{sl}_3$ and $\mathfrak{g} = \mathfrak{so}_5$.

2.1 Normal elements of $U_q^+(\mathfrak{g})$.

Recall that an element a of $U_q^+(\mathfrak{g})$ is normal provided the left and right ideals generated by a in $U_q^+(\mathfrak{g})$ coincide, that is, if

$$aU_q^+(\mathfrak{g}) = U_q^+(\mathfrak{g})a.$$

In the sequel, we will use several times the following well-known result concerning normal elements of $U_q^+(\mathfrak{g})$.

Lemma 2.1 *Let u and v be two nonzero normal elements of $U_q^+(\mathfrak{g})$ such that $\langle u\rangle = \langle v\rangle$. Then there exist $\lambda, \mu \in \mathbb{C}^*$ such that $u = \lambda v$ and $v = \mu u$.*

Proof. It is obvious that units λ, μ exist with these properties. However, the set of units of $U_q^+(\mathfrak{g})$ is precisely $\mathbb{C}^*$. $\square$

2.2 $\mathbb{N}$-grading on $U_q^+(\mathfrak{g})$ and automorphisms.

As the quantum Serre relations are homogeneous in the given generators, there is an $\mathbb{N}$-grading on $U_q^+(\mathfrak{g})$ obtained by assigning to E_i degree 1. Let

$$U_q^+(\mathfrak{g}) = \bigoplus_{i\in\mathbb{N}} U_q^+(\mathfrak{g})_i \tag{2}$$

be the corresponding decomposition, with $U_q^+(\mathfrak{g})_i$ the subspace of homogeneous elements of degree i. In particular, $U_q^+(\mathfrak{g})_0 = \mathbb{C}$ and $U_q^+(\mathfrak{g})_1$ is the n-dimensional space spanned by the generators $E_1, \dots, E_n$. For $t \in \mathbb{N}$ set $U_q^+(\mathfrak{g})_{\geq t} = \bigoplus_{i\geq t} U_q^+(\mathfrak{g})_i$ and define $U_q^+(\mathfrak{g})_{\leq t}$ similarly.

We say that the nonzero element $u \in U_q^+(\mathfrak{g})$ has degree t, and write $\deg(u) = t$, if $u \in U_q^+(\mathfrak{g})_{\leq t} \setminus U_q^+(\mathfrak{g})_{\leq t-1}$ (using the convention that $U_q^+(\mathfrak{g})_{\leq -1} = \{0\}$). As $U_q^+(\mathfrak{g})$ is a domain, $\deg(uv) = \deg(u) + \deg(v)$ for $u, v \neq 0$.

Definition 2.2 *Let $A = \bigoplus_{i\in\mathbb{N}} A_i$ be an $\mathbb{N}$-graded $\mathbb{C}$-algebra with $A_0 = \mathbb{C}$ which is generated as an algebra by $A_1 = \mathbb{C}x_1 \oplus \cdots \oplus \mathbb{C}x_n$. If for each $i \in \{1, \dots, n\}$ there exist $0 \neq a \in A$ and a scalar $q_{i,a} \neq 1$ such that $x_i a = q_{i,a} a x_i$, then we say that A is an $\mathbb{N}$-graded algebra with enough q-commutation relations.*

The algebra $U_q^+(\mathfrak{g})$, endowed with the grading just defined, is a connected $\mathbb{N}$-graded algebra with enough q-commutation relations. Indeed, if $i \in \{1, \dots, n\}$, then there exists $u \in U_q^+(\mathfrak{g})$ such that $E_i u = q^{\bullet} u E_i$ where $\bullet$ is a nonzero integer. This can be proved as follows. As $\mathfrak{g}$ is simple, there exists an index $j \in \{1, \dots, n\}$ such that $j \neq i$ and $a_{ij} \neq 0$, that is, $a_{ij} \in \{-1, -2, -3\}$. Then $s_i s_j$ is a reduced expression in W, so that one can find a reduced expression of w_0 starting with $s_i s_j$, that is, one can write

$$w_0 = s_i s_j s_{i_3} \dots s_{i_N}.$$

With respect to this reduced expression of w_0, we have with the notation of Section 1.2:

$$\beta_1 = \alpha_i \quad \text{and} \quad \beta_2 = s_i(\alpha_j) = \alpha_j - a_{ij}\alpha_i$$

Then it follows from Theorem 1.1 that $E_{\beta_1} = E_i$, $E_{\beta_2} = E_{\alpha_j - a_{ij}\alpha_i}$ and

$$E_i E_{\beta_2} = q^{(\alpha_i, \alpha_j - a_{ij}\alpha_i)} E_{\beta_2} E_i,$$

that is,

$$E_i E_{\beta_2} = q^{-(\alpha_i, \alpha_j)} E_{\beta_2} E_i.$$

As $a_{ij} \neq 0$, we have $(\alpha_i, \alpha_j) \neq 0$ and so $q^{-(\alpha_i, \alpha_j)} \neq 1$ since q is not a root of unity. So we have just proved:

Proposition 2.3 $U_q^+(\mathfrak{g})$ *is a connected $\mathbb{N}$-graded algebra with enough q-commutation relations.*

One of the advantages of $\mathbb{N}$-graded algebras with enough q-commutation relations is that any automorphism of such an algebra must conserve the valuation associated to the $\mathbb{N}$-graduation. More precisely, as $U_q^+(\mathfrak{g})$ is a connected $\mathbb{N}$-graded algebra with enough q-commutation relations, we deduce from [18] (see also [17, Proposition 3.2]) the following result.

Corollary 2.4 *Let $\sigma \in \mathrm{Aut}(U_q^+(\mathfrak{g}))$ and $x \in U_q^+(\mathfrak{g})_d \setminus \{0\}$. Then $\sigma(x) = y_d + y_{>d}$, for some $y_d \in U_q^+(\mathfrak{g})_d \setminus \{0\}$ and $y_{>d} \in U_q^+(\mathfrak{g})_{\geq d+1}$.*

2.3 The case where $\mathfrak{g} = \mathfrak{sl}_3$.

In this section, we investigate the automorphism group of $U_q^+(\mathfrak{g})$ in the case where $\mathfrak{g} = \mathfrak{sl}_3$. In this case the Cartan matrix is $A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$, so that $U_q^+(\mathfrak{s}l_3)$ is the $\mathbb{C}$-algebra generated by two indeterminates E_1 and E_2 subject to the following relations:

$$E_1^2E_2 - (q+q^{-1})E_1E_2E_1 + E_2E_1^2 = 0 \tag{3}$$
$$E_2^2E_1 - (q+q^{-1})E_2E_1E_2 + E_1E_2^2 = 0 \tag{4}$$

We often refer to this algebra as the quantum Heisenberg algebra, and sometimes we denote it by $\mathbb{H}$, as in the classical situation the enveloping algebra of $\mathfrak{sl}_3^+$ is the so-called Heisenberg algebra.

We now make explicit a PBW basis of $\mathbb{H}$. The Weyl group of $\mathfrak{sl}_3$ is isomorphic to the symmetric group S_3, where s_1 is identified with the transposition $(1\ 2)$ and s_2 is identified with $(2\ 3)$. Its longest element is then $w_0 = (13)$; it has two reduced decompositions: $w_0 = s_1s_2s_1 = s_2s_1s_2$. Let us choose the reduced decomposition $s_1s_2s_1$ of w_0 in order to construct a PBW basis of $U_q^+(\mathfrak{s}l_3)$. According to Section 1.2, this reduced decomposition leads to the following root vectors:

$$E_{\alpha_1} = E_1,\ E_{\alpha_1+\alpha_2} = T_1(E_2) = -E_1E_2 + q^{-1}E_2E_1 \text{ and } E_{\alpha_2} = T_1T_2(E_1) = E_2.$$

In order to simplify the notation, we set $E_3 := -E_1E_2 + q^{-1}E_2E_1$. Then, it follows from Theorem 1.1 that

- The monomials $E_1^{k_1}E_3^{k_3}E_2^{k_2}$, with k_1, k_2, k_3 nonnegative integers, form a PBW-basis of $U_q^+(\mathfrak{s}l_3)$.
- $\mathbb{H}$ is the iterated Ore extension over $\mathbb{C}$ generated by the indeterminates E_1, E_3, E_2 subject to the following relations:

  $$E_3E_1 = q^{-1}E_1E_3, \quad E_2E_3 = q^{-1}E_3E_2, \quad E_2E_1 = qE_1E_2 + qE_3.$$

  In particular, $\mathbb{H}$ is a Noetherian domain, and its group of invertible elements is reduced to $\mathbb{C}^*$.

- It follows from the previous commutation relations between the root vectors that E_3 is a normal element in $\mathbb{H}$, that is, $E_3\mathbb{H} = \mathbb{H}E_3$.

In order to describe the prime and primitive spectra of $\mathbb{H}$, we need to introduce two other elements. The first one is the root vector $E_3' := T_2(E_1) = -E_2E_1 + q^{-1}E_1E_2$. This root vector would have appeared if we have choosen the reduced decomposition $s_2s_1s_2$ of w_0 in order to construct a PBW basis of $\mathbb{H}$. It follows from Theorem 1.1 that E_3' q-commutes with E_1 and E_2, so that E_3' is also a normal element of $\mathbb{H}$. Moreover, one can describe the centre of $\mathbb{H}$ using the two normal elements E_3 and E_3'. Indeed, in [3, Corollaire 2.16], Alev and Dumas have described the centre of $U_q^+(\mathfrak{sl}_n)$; independently Caldero has described the centre of $U_q^+(\mathfrak{g})$ for arbitrary $\mathfrak{g}$, see [7]. In our particular situation, their results show that the centre $Z(\mathbb{H})$ of $\mathbb{H}$ is a polynomial ring in one variable $Z(\mathbb{H}) = \mathbb{C}[\Omega]$, where $\Omega = E_3E_3'$.

We are now in position to describe the prime and primitive spectra of $\mathbb{H} = U_q^+(sl(3))$; this was first achieved by Malliavin who obtained the following picture for the poset of prime ideals of $\mathbb{H}$, see [20, Théorème 2.4]:

$$\begin{tikzcd}
\langle\langle E_1, E_2-\beta\rangle\rangle \arrow[dr, dash] & & \langle\langle E_1, E_2\rangle\rangle \arrow[dl, dash] \arrow[dr, dash] & & \langle\langle E_1-\alpha, E_2\rangle\rangle \arrow[dl, dash] \\
& \langle E_1\rangle \arrow[dl, dash] \arrow[drrr, dash] & & \langle E_2\rangle \arrow[dlll, dash] \arrow[dr, dash] & \\
\langle\langle E_3\rangle\rangle \arrow[drr, dash] & & \langle\langle \Omega-\gamma\rangle\rangle \arrow[d, dash] & & \langle\langle E_3'\rangle\rangle \arrow[dll, dash] \\
& & \langle 0\rangle & &
\end{tikzcd}$$

where $\alpha, \beta, \gamma \in \mathbb{C}^*$.

Recall from Section 1.3 that the torus $\mathcal{H} = (\mathbb{C}^*)^2$ acts on $U_q^+(\mathfrak{sl}_3)$ by automorphisms and that the $\mathcal{H}$-stratification theory of Goodearl and Letzter constructs a partition of the prime spectrum of $U_q^+(\mathfrak{sl}_3)$ into so-called $\mathcal{H}$-strata, this partition being indexed by the $\mathcal{H}$-invariant prime ideals of $U_q^+(\mathfrak{sl}_3)$. Using this description of $\mathrm{Spec}(U_q^+(\mathfrak{sl}_3))$, it is easy to identify the $6 = |W|$ $\mathcal{H}$-invariant prime ideals of $\mathbb{H}$ and their corresponding $\mathcal{H}$-strata. As E_1, E_2, E_3 and E_3' are $\mathcal{H}$-eigenvectors, the 6 $\mathcal{H}$-invariant primes are:

$$\langle 0\rangle,\ \langle E_3\rangle,\ \langle E_3'\rangle,\ \langle E_1\rangle,\ \langle E_2\rangle \text{ and } \langle E_1, E_2\rangle.$$

Moreover the corresponding $\mathcal{H}$-strata are:
$\mathrm{Spec}_{\langle 0\rangle}(\mathbb{H}) = \{\langle 0\rangle\} \cup \{\langle \Omega-\gamma\rangle \mid \gamma \in \mathbb{C}^*\}$,

$\mathrm{Spec}_{\langle E_3\rangle}(\mathbb{H}) = \{\langle E_3\rangle\}$,
$\mathrm{Spec}_{\langle E_3'\rangle}(\mathbb{H}) = \{\langle E_3'\rangle\}$,
$\mathrm{Spec}_{\langle E_1\rangle}(\mathbb{H}) = \{\langle E_1\rangle\} \cup \{\langle E_1, E_2 - \beta\rangle \mid \beta \in \mathbb{C}^*\}$,
$\mathrm{Spec}_{\langle E_2\rangle}(\mathbb{H}) = \{\langle E_2\rangle\} \cup \{\langle E_1 - \alpha, E_2\rangle \mid \alpha \in \mathbb{C}^*\}$
and $\mathrm{Spec}_{\langle E_1,E_2\rangle}(\mathbb{H}) = \{\langle E_1, E_2\rangle\}$.

We deduce from this description of the $\mathcal{H}$-strata and the the fact that primitive ideals are exactly those primes that are maximal within their $\mathcal{H}$-strata, see Theorem 1.3, that the primitive ideals of $U_q^+(\mathfrak{s}l_3)$ are exactly those primes that appear in double brackets in the previous picture.

We now investigate the group of automorphisms of $\mathbb{H} = U_q^+(\mathfrak{s}l_3)$. In that case, the torus acting naturally on $U_q^+(\mathfrak{s}l_3)$ is $\mathcal{H} = (\mathbb{C}^*)^2$, there is only one non-trivial diagram automorphism w that exchanges E_1 and E_2, and so the subgroup G of $\mathrm{Aut}(U_q^+(\mathfrak{s}l_3))$ generated by the torus and diagram automorphisms is isomorphic to the semi-direct product $(\mathbb{C}^*)^2 \rtimes S_2$. We want to prove that $\mathrm{Aut}(U_q^+(\mathfrak{s}l_3)) = G$.

In order to do this, we study the action of $\mathrm{Aut}(U_q^+(\mathfrak{s}l_3))$ on the set of primitive ideals that are not maximal. As there are only two of them, $\langle E_3\rangle$ and $\langle E_3'\rangle$, an automorphism of $\mathbb{H}$ will either fix them or permute them.

Let σ be an automorphism of $U_q^+(\mathfrak{s}l_3)$. It follows from the previous observation that

$$\text{either } \sigma(\langle E_3\rangle) = \langle E_3\rangle \text{ and } \sigma(\langle E_3'\rangle) = \langle E_3'\rangle,$$

$$\text{or } \sigma(\langle E_3\rangle) = \langle E_3'\rangle \text{ and } \sigma(\langle E_3'\rangle) = \langle E_3\rangle.$$

As it is clear that the diagram automorphism w permutes the ideals $\langle E_3\rangle$ and $\langle E_3'\rangle$, we get that there exists an automorphism $g \in G$ such that

$$g \circ \sigma(\langle E_3\rangle) = \langle E_3\rangle \text{ and } g \circ \sigma(\langle E_3'\rangle) = \langle E_3'\rangle.$$

Then, as E_3 and E_3' are normal, we deduce from Lemma 2.1 that there exist $\lambda, \lambda' \in \mathbb{C}^*$ such that

$$g \circ \sigma(E_3) = \lambda E_3 \text{ and } g \circ \sigma(E_3') = \lambda' E_3'.$$

In order to prove that $g \circ \sigma$ is an element of G, we now use the $\mathbb{N}$-graduation of $U_q^+(\mathfrak{s}l_3)$ introduced in Section 2.2. With respect to this graduation, E_1 and E_2 are homogeneous of degree 1, and so E_3 and E_3' are homogeneous of degree 2. Moreover, as $(q^{-2}-1)E_1E_2 = E_3 + q^{-1}E_3'$, we deduce from the above discussion that

$$g \circ \sigma(E_1E_2) = \frac{1}{q^{-2}-1}\left(\lambda E_3 + q^{-1}\lambda' E_3'\right)$$

has degree two. On the other hand, as $U_q^+(\mathfrak{s}l_3)$ is a connected $\mathbb{N}$-graded algebra with enough q-commutation relations by Proposition 2.3, it follows from Corollary 2.4 that $\sigma(E_1) = a_1E_1 + a_2E_2 + u$ and $\sigma(E_2) = b_1E_1 + b_2E_2 + v$, where $(a_1, a_2), (b_1, b_2) \in \mathbb{C}^2 \setminus \{(0,0)\}$, and $u, v \in U_q^+(\mathfrak{s}l_3)$ are linear combinations of homogeneous elements of degree greater than one. As $g \circ \sigma(E_1).g \circ \sigma(E_2)$ has degree two, it is clear that $u = v = 0$. To conclude that

$g \circ \sigma \in G$, it just remains to prove that $a_2 = 0 = b_1$. This can be easily shown by using the fact that $g \circ \sigma(-E_1E_2 + q^{-1}E_2E_1) = g \circ \sigma(E_3) = \lambda E_3$; replacing $g \circ \sigma(E_1)$ and $g \circ \sigma(E_2)$ by $a_1E_1 + a_2E_2$ and $b_1E_1 + b_2E_2$ respectively, and then identifying the coefficients in the PBW basis, leads to $a_2 = 0 = b_1$, as required. Hence we have just proved that $g \circ \sigma \in G$, so that σ itself belongs to G the subgroup of $\mathrm{Aut}(U_q^+(\mathfrak{s}l_3))$ generated by the torus and diagram automorphisms. Hence one can state the following result that confirms the Andruskiewitsch-Dumas Conjecture.

Proposition 2.5 $\mathrm{Aut}(U_q^+(\mathfrak{s}l_3)) \simeq (\mathbb{C}^*)^2 \rtimes \mathrm{autdiagr}(\mathfrak{sl}_3)$

This result was first obtained independently by Alev and Dumas, [2, Proposition 2.3], and Caldero, [8, Proposition 4.4], but using somehow different methods; they studied this automorphism group by looking at its action on the set of normal elements of $U_q^+(\mathfrak{s}l_3)$.

2.4 The case where $\mathfrak{g} = \mathfrak{so}_5$.

In this section we investigate the automorphism group of $U_q^+(\mathfrak{g})$ in the case where $\mathfrak{g} = \mathfrak{so}_5$. In this case there are no diagram automorphisms, so that the Andruskiewitsch-Dumas Conjecture asks whether every automorphism of $U_q^+(\mathfrak{so}_5)$ is a torus automorphism. In [16] we have proved their conjecture when $\mathfrak{g} = \mathfrak{so}_5$. The aim of this section is to present a slightly different proof based both on the original proof and on the recent proof by S.A. Lopes and the author of the Andruskiewitsch-Dumas Conjecture in the case where $\mathfrak{g}$ is of type A_3.

In the case where $\mathfrak{g} = \mathfrak{so}_5$, the Cartan matrix is $A = \begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix}$, so that $U_q^+(\mathfrak{so}_5)$ is the $\mathbb{C}$-algebra generated by two indeterminates E_1 and E_2 subject to the following relations:

$$E_1^3E_2 - (q^2 + 1 + q^{-2})E_1^2E_2E_1 + (q^2 + 1 + q^{-2})E_1E_2E_1^2 + E_2E_1^3 = 0 \tag{5}$$

$$E_2^2E_1 - (q^2 + q^{-2})E_2E_1E_2 + E_1E_2^2 = 0 \tag{6}$$

We now make explicit a PBW basis of $U_q^+(\mathfrak{so}_5)$. The Weyl group of $\mathfrak{so}_5$ is isomorphic to the dihedral group $D(4)$. Its longest element is $w_0 = -id$; it has two reduced decompositions: $w_0 = s_1s_2s_1s_2 = s_2s_1s_2s_1$. Let us choose the reduced decomposition $s_1s_2s_1s_2$ of w_0 in order to construct a PBW basis of $U_q^+(\mathfrak{so}_5)$. According to Section 1.2, this reduced decomposition leads to the following root vectors:

$$E_{\alpha_1} = E_1,\ E_{2\alpha_1+\alpha_2} = T_1(E_2) = \frac{1}{(q+q^{-1})}\left(E_1^2E_2 - q^{-1}(q+q^{-1})E_1E_2E_1 + q^{-2}E_2E_1^2\right),$$

$$E_{\alpha_1+\alpha_2} = T_1T_2(E_1) = -E_1E_2 + q^{-2}E_2E_1 \text{ and } E_{\alpha_2} = T_1T_2T_1(E_2) = E_2.$$

In order to simplify the notation, we set $E_3 := -E_{\alpha_1+\alpha_2}$ and $E_4 := E_{2\alpha_1+\alpha_2}$. Then, it follows from Theorem 1.1 that

- The monomials $E_1^{k_1}E_4^{k_4}E_3^{k_3}E_2^{k_2}$, with k_1, k_2, k_3, k_4 nonnegative integers, form a PBW-basis of $U_q^+(\mathfrak{so}_5)$.

- $U_q^+(\mathfrak{so}_5)$ is the iterated Ore extension over $\mathbb{C}$ generated by the indeterminates E_1, E_4, E_3, E_2 subject to the following relations:

  $$\begin{array}{lll} E_4E_1 = q^{-2}E_1E_4 & & \\ E_3E_1 = E_1E_3 - (q+q^{-1})E_4, & E_3E_4 = q^{-2}E_4E_3, & \\ E_2E_1 = q^2E_1E_2 - q^2E_3, & E_2E_4 = E_4E_2 - \frac{q^2-1}{q+q^{-1}}E_3^2, & E_2E_3 = q^{-2}E_3E_2. \end{array}$$

  In particular, $U_q^+(\mathfrak{so}_5)$ is a Noetherian domain, and its group of invertible elements is reduced to $\mathbb{C}^*$.

Before describing the automorphism group of $U_q^+(\mathfrak{so}_5)$, we first describe the centre and the primitive ideals of $U_q^+(\mathfrak{so}_5)$. The centre of $U_q^+(\mathfrak{g})$ has been described in general by Caldero, [7]. In the case where $\mathfrak{g} = \mathfrak{so}_5$, his result shows that $Z(U_q^+(\mathfrak{so}_5))$ is a polynomial algebra in two indeterminates

$$Z(U_q^+(\mathfrak{so}_5)) = \mathbb{C}[z, z'],$$

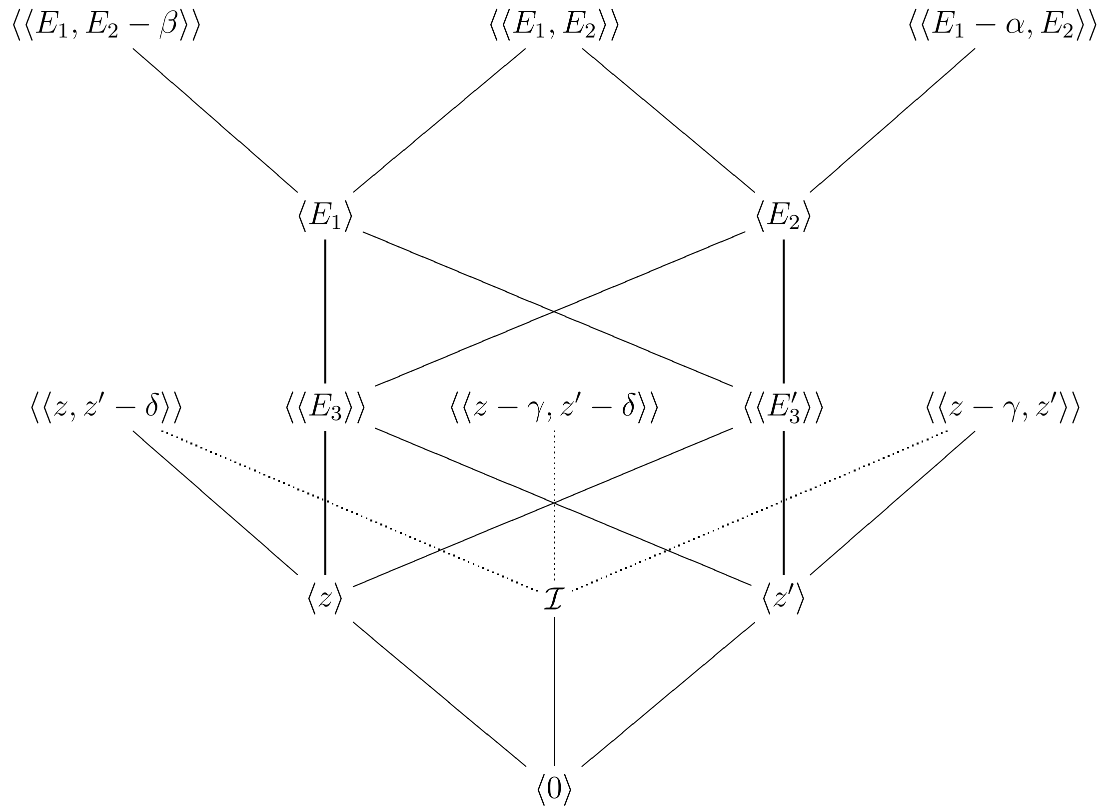
where

$$z = (1-q^2)E_1E_3 + q^2(q+q^{-1})E_4$$

and

$$z' = -(q^2-q^{-2})(q+q^{-1})E_4E_2 + q^2(q^2-1)E_3^2.$$

Recall from Section 1.3 that the torus $\mathcal{H} = (\mathbb{C}^*)^2$ acts on $U_q^+(\mathfrak{so}_5)$ by automorphisms and that the $\mathcal{H}$-stratification theory of Goodearl and Letzter constructs a partition of the prime spectrum of $U_q^+(\mathfrak{so}_5)$ into so-called $\mathcal{H}$-strata, this partition being indexed by the $8 = |W|$ $\mathcal{H}$-invariant prime ideals of $U_q^+(\mathfrak{so}_5)$. In [16], we have described these eight $\mathcal{H}$-strata. More precisely, we have obtained the following picture for the poset $\mathrm{Spec}(U_q^+(\mathfrak{so}_5))$,



where $\alpha, \beta, \gamma, \delta \in \mathbb{C}^*$, $E_3' := E_1E_2 - q^2E_2E_1$ and

$$\mathcal{I} = \{\langle P(z, z')\rangle \mid P \text{ is a unitary irreductible polynomial of } \mathbb{C}[z, z'],\ P \neq z, z'\}.$$

As the primitive ideals are those primes that are maximal in their $\mathcal{H}$-strata, see Theorem 1.3, we deduced from this description of the prime spectrum that the primitive ideals of $U_q^+(\mathfrak{so}_5)$ are the following:

- $\langle z - \alpha, z' - \beta\rangle$ with $(\alpha, \beta) \in \mathbb{C}^2 \setminus \{(0, 0)\}$.
- $\langle E_3\rangle$ and $\langle E_3'\rangle$.
- $\langle E_1 - \alpha, E_2 - \beta\rangle$ with $(\alpha, \beta) \in \mathbb{C}^2$ such that $\alpha\beta = 0$.

(They correspond to the "double brackets" prime ideals in the above picture.)

Among them, two only are not maximal, $\langle E_3\rangle$ and $\langle E_3'\rangle$. Unfortunately, as E_3 and E_3' are not normal in $U_q^+(\mathfrak{so}_5)$, one cannot easily obtain information using the fact that any automorphism of $U_q^+(\mathfrak{so}_5)$ will either preserve or exchange these two prime ideals. Rather than using this observation, we will use the action of $\mathrm{Aut}(U_q^+(\mathfrak{so}_5))$ on the set of maximal ideals of height two. Because of the previous description of the primitive spectrum of $U_q^+(\mathfrak{so}_5)$, the height two maximal ideals in $U_q^+(\mathfrak{so}_5)$ are those $\langle z - \alpha, z' - \beta\rangle$ with $(\alpha, \beta) \in \mathbb{C}^2 \setminus \{(0, 0)\}$. In [16, Proposition 3.6], we have proved that the group of units of the factor algebra $U_q^+(\mathfrak{so}_5)/\langle z - \alpha, z' - \beta\rangle$ is reduced to $\mathbb{C}^*$ if and only if both α and β are nonzero. Consequently, if σ is an automorphism of $U_q^+(\mathfrak{so}_5)$ and $\alpha \in \mathbb{C}^*$, we get that:

$$\sigma(\langle z - \alpha, z'\rangle) = \langle z - \alpha', z'\rangle \text{ or } \langle z, z' - \beta'\rangle,$$

where $\alpha', \beta' \in \mathbb{C}^*$. Similarly, if σ is an automorphism of $U_q^+(\mathfrak{so}_5)$ and $\beta \in \mathbb{C}^*$, we get that:

$$\sigma(\langle z, z' - \beta \rangle) = \langle z - \alpha', z' \rangle \text{ or } \langle z, z' - \beta' \rangle, \tag{7}$$

where $\alpha', \beta' \in \mathbb{C}^*$.

We now use this information to prove that the action of $\mathrm{Aut}(U_q^+(\mathfrak{so}_5))$ on the centre of $U_q^+(\mathfrak{so}_5)$ is trivial. More precisely, we are now in position to prove the following result.

Proposition 2.6 *Let $\sigma \in \mathrm{Aut}(U_q^+(\mathfrak{so}_5))$. There exist $\lambda, \lambda' \in \mathbb{C}^*$ such that*

$$\sigma(z) = \lambda z \quad \textit{and} \quad \sigma(z') = \lambda' z'.$$

Proof. We only prove the result for z. First, using the fact that $U_q^+(\mathfrak{so}_5)$ is noetherian, it is easy to show that, for any family $\{\beta_i\}_{i\in\mathbb{N}}$ of pairwise distinct nonzero complex numbers, we have:

$$\langle z \rangle = \bigcap_{i\in\mathbb{N}} P_{0,\beta_i} \text{ and } \langle z' \rangle = \bigcap_{i\in\mathbb{N}} P_{\beta_i,0},$$

where $P_{\alpha,\beta} := \langle z - \alpha, z' - \beta \rangle$. Indeed, if the inclusion

$$\langle z \rangle \subseteq I := \bigcap_{i\in\mathbb{N}} P_{0,\beta_i}$$

is not an equality, then any P_{0,β_i} is a minimal prime over I for height reasons. As the P_{0,β_i} are pairwise distinct, I is a two-sided ideal of $U_q^+(\mathfrak{so}_5)$ with infinitely many prime ideals minimal over it. This contradicts the noetherianity of $U_q^+(\mathfrak{so}_5)$. Hence

$$\langle z \rangle = \bigcap_{i\in\mathbb{N}} P_{0,\beta_i} \text{ and } \langle z' \rangle = \bigcap_{i\in\mathbb{N}} P_{\beta_i,0},$$

and so

$$\sigma(\langle z \rangle) = \bigcap_{i\in\mathbb{N}} \sigma(P_{0,\beta_i}).$$

It follows from (7) that, for all $i \in \mathbb{N}$, there exists $(\gamma_i, \delta_i) \neq (0,0)$ with $\gamma_i = 0$ or $\delta_i = 0$ such that

$$\sigma(P_{0,\beta_i}) = P_{\gamma_i,\delta_i}.$$

Naturally, we can choose the family $\{\beta_i\}_{i\in\mathbb{N}}$ such that either $\gamma_i = 0$ for all $i \in \mathbb{N}$, or $\delta_i = 0$ for all $i \in \mathbb{N}$. Moreover, observe that, as the β_i are pairwise distinct, so are the γ_i or the δ_i.

Hence, either

$$\sigma(\langle z \rangle) = \bigcap_{i\in\mathbb{N}} P_{\gamma_i,0},$$

or

$$\sigma\left(\langle z\rangle\right)=\bigcap_{i\in\mathbb{N}}P_{0,\delta_i},$$

that is,

$$\text{either } \langle\sigma(z)\rangle=\sigma\left(\langle z\rangle\right)=\langle z'\rangle \text{ or } \langle\sigma(z)\rangle=\sigma\left(\langle z\rangle\right)=\langle z\rangle.$$

As z, $\sigma(z)$ and z' are all central, it follows from Lemma 2.1 that there exists $\lambda\in\mathbb{C}^*$ such that either $\sigma(z)=\lambda z$ or $\sigma(z)=\lambda z'$.

To conclude, it just remains to show that the second case cannot happen. In order to do this, we use a graded argument. Observe that, with respect to the $\mathbb{N}$-graduation of $U_q^+(\mathfrak{so}_5)$ defined in Section 2.2, z and z' are homogeneous of degree 3 and 4 respectively. Thus, if $\sigma(z)=\lambda z'$, then we would obtain a contradiction with the fact that every automorphism of $U_q^+(\mathfrak{so}_5)$ preserves the valuation, see Corollary 2.4. Hence $\sigma(z)=\lambda z$, as desired. The corresponding result for z' can be proved in a similar way, so we omit it. $\square$

Andruskiewitsch and Dumas, [4, Proposition 3.3], have proved that the subgroup of those automorphisms of $U_q^+(\mathfrak{so}_5)$ that stabilize $\langle z\rangle$ is isomorphic to $(\mathbb{C}^*)^2$. Thus, as we have just shown that every automorphism of $U_q^+(\mathfrak{so}_5)$ fixes $\langle z\rangle$, we get that $\mathrm{Aut}(U_q^+(\mathfrak{so}_5))$ itself is isomorphic to $(\mathbb{C}^*)^2$. This is the route that we have followed in [16] in order to prove the Andruskiewitsch-Dumas Conjecture in the case where $\mathfrak{g}=\mathfrak{so}_5$. Recently, with Samuel Lopes, we proved this Conjecture in the case where $\mathfrak{g}=\mathfrak{sl}_4$ using different methods and in particular graded arguments. We are now using (similar) graded arguments to prove that every automorphism of $U_q^+(\mathfrak{so}_5)$ is a torus automorphism (witout using results of Andruskiewitsch and Dumas).

In the proof, we will need the following relation that is easily obtained by straightforward computations.

Lemma 2.7 $(q^2-1)E_3E_3'=(q^4-1)zE_2+q^2z'$.

Proposition 2.8 *Let σ be an automorphism of $U_q^+(\mathfrak{so}_5)$. Then there exist $a_1,b_2\in\mathbb{C}^*$ such that*

$$\sigma(E_1)=a_1E_1 \quad \textit{and} \quad \sigma(E_2)=b_2E_2.$$

Proof. For all $i\in\{1,\dots,4\}$, we set $d_i:=\deg(\sigma(E_i))$. We also set $d_3':=\deg(\sigma(E_3'))$. It follows from Corollary 2.4 that $d_1,d_2\geq 1$, $d_3,d_3'\geq 2$ and $d_4\geq 3$. First we prove that $d_1=d_2=1$.

Assume first that $d_1+d_3>3$. As $z=(1-q^2)E_1E_3+q^2(q+q^{-1})E_4$ and $\sigma(z)=\lambda z$ with $\lambda\in\mathbb{C}^*$ by Proposition 2.6, we get:

$$\lambda z \;=\; (1-q^2)\sigma(E_1)\sigma(E_3)+q^2(q+q^{-1})\sigma(E_4). \tag{8}$$

Recall that $\deg(uv)=\deg(u)+\deg(v)$ for $u,v\neq 0$, as $U_q^+(\mathfrak{g})$ is a domain. Thus, as $\deg(z)=3<\deg(\sigma(E_1)\sigma(E_3))=d_1+d_3$, we deduce from (8) that $d_1+d_3=d_4$. As $z'=-(q^2-q^{-2})(q+q^{-1})E_4E_2+q^2(q^2-1)E_3^2$ and $\deg(z')=4<d_1+d_3+d_2=d_4+d_2=$

$\deg(\sigma(E_4)\sigma(E_2))$, we get in a similar manner that $d_2 + d_4 = 2d_3$. Thus $d_1 + d_2 = d_3$. As $d_1 + d_3 > 3$, this forces $d_3 > 2$ and so $d_3 + d_3' > 4$. Thus we deduce from Lemma 2.7 that $d_3 + d_3' = 3 + d_2$. Hence $d_1 + d_3' = 3$. As $d_1 \geq 1$ and $d_3' \geq 2$, this implies $d_1 = 1$ and $d_3' = 2$.

Thus we have just proved that $d_1 = \deg(\sigma(E_1)) = 1$ and either $d_3 = 2$ or $d_3' = 2$. To prove that $d_2 = 1$, we distinguish between these two cases.

If $d_3 = 2$, then as previously we deduce from the relation $z' = -(q^2 - q^{-2})(q + q^{-1})E_4E_2 + q^2(q^2 - 1)E_3^2$ that $d_2 + d_4 = 4$, so that $d_2 = 1$, as desired.

If $d_3' = 2$, then one can use the definition of E_3' and the previous expression of z' in order to prove that $z' = q^{-2}(q^2 - 1)E_3'^2 + E_2u$, where u is a nonzero homogeneous element of $U_q^+(\mathfrak{so}_5)$ of degree 3. (u is nonzero since $\langle z' \rangle$ is a completely prime ideal and $E_3' \notin \langle z' \rangle$ for degree reasons.) As $d_3' = 2$ and $\deg(\sigma(z')) = 4$, we get as previously that $d_2 = 1$.

To summarise, we have just proved that $\deg(\sigma(E_1)) = 1 = \deg(\sigma(E_2))$, so that $\sigma(E_1) = a_1E_1 + a_2E_2$ and $\sigma(E_2) = b_1E_1 + b_2E_2$, where $(a_1, a_2), (b_1, b_2) \in \mathbb{C}^2 \setminus \{(0,0)\}$. To conclude that $a_2 = b_1 = 0$, one can for instance use the fact that $\sigma(E_1)$ and $\sigma(E_2)$ must satisfy the quantum Serre relations. $\square$

We have just confirmed the Andruskiewitsch-Dumas Conjecture in the case where $\mathfrak{g} = \mathfrak{so}_5$.

Theorem 2.9 *Every automorphism of $U_q^+(\mathfrak{so}_5)$ is a torus automorphism, so that*

$$\mathrm{Aut}(U_q^+(\mathfrak{so}_5)) \simeq (\mathbb{C}^*)^2.$$

2.5 Beyond these two cases.

To finish this overview paper, let us mention that recently the Andruskiewitsch-Dumas Conjecture was confirmed by Samuel Lopes and the author, [18], in the case where $\mathfrak{g} = \mathfrak{sl}_4$. The crucial step of the proof is to prove that, up to an element of G, every normal element of $U_q^+(\mathfrak{sl}_4)$ is fixed by every automorphism. This step was dealt with by first computing the Lie algebra of derivations of $U_q^+(\mathfrak{sl}_4)$, and this already requires a lot of computations!

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