Twisted Poincaré duality for some quadratic Poisson algebras

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Abstract

We exhibit a Poisson module restoring a twisted Poincaré duality between Poisson homology and cohomology for the polynomial algebra $R = \mathbb{C}[X_1, \ldots, X_n]$ endowed with Poisson bracket arising from a uniparametrised quantum affine space. This Poisson module is obtained as the semiclassical limit of the dualising bimodule for Hochschild homology of the corresponding quantum affine space. As a corollary we compute the Poisson cohomology of R, and so retrieve a result obtained by direct methods (so completely different from ours) by Monnier.

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1 Introduction

Given a Poisson algebra, its Poisson cohomology provides important information about the Poisson structure (the Casimir elements are reflected by the degree zero cohomology, Poisson derivations modulo Hamiltonian derivations by the degree one,...). Computing this cohomology is in general difficult. It has been achieved in some particular cases, see for instance [19] and references therein. One way to study a Poisson algebra is to consider a deformation. If the deformation is "nice", its properties should reflect the corresponding properties of the original Poisson algebra. Results in this spirit have been obtained for instance in [10], or in the framework of symplectic varieties in [3] and recently in [2], where roughly speaking the Poisson homology is shown to match the Hochschild homology of the deformation at least in small degrees.

The aim of this paper is to illustrate this idea that quantisation can provide some intuition on the study of Poisson (co)homology. More precisely, we consider the polynomial algebra $R = \mathbb{C}[X_1, \ldots, X_n]$ endowed with the bracket $\{X_i, X_j\} = a_{ij}X_iX_j$, where $(a_{ij}) \in M_n(\mathbb{Z})$ is a skew-symmetric matrix. The Poisson cohomology of this algebra has been computed in [17] by direct computation. Another approach in order to compute this Poisson cohomology consists in establishing a duality between Poisson homology and cohomology. In general such a duality does not occur. For instance for $n = 2$ and $a_{12} = 1$ the cohomology spaces in degree 0,1 and 2 have respectively dimension 1,2,2, whereas the homology space of degree 2 is null, and infinite-dimensional in degree 0 and 1. Note that this lack of nontrivial Poisson homology in the higher degrees explains the "dimension drop" appearing in the Hochschild homology of the quantum plane (see [15], [26], [23], [11], [6], [8] for details on dimension drop in this and other situations).

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The starting point of this work was the following question: knowing that there is no Poincaré duality between Poisson homology and cohomology in this case (see the paper [22] for $n = 2$), can it be replaced by a "twisted duality"? Such a twisted duality for Poisson (co)homology appears in the literature (see for instance [13], [9], [27]), in a somewhat abstract way. In the present paper we make this twisted duality explicit on the complexes that are used to compute the Poisson (co)homology of the Poisson algebra under consideration.

Our method here relies on the fact that the Poisson algebra R is the semiclassical limit of a uniparametrised quantum affine space U admitting a dualising bimodule for Hochschild (co)homology. This bimodule is easily defined as the algebra U itself, with product twisted on the left by an automorphism. This allows us to define a Poisson module M over R as the semiclassical limit of the dualising bimodule of U. Then we show that the Poisson cohomology of R is dual to the Poisson homology with values in M; in other words, M is a "Poisson dualising module" in the sense that it restores a duality between Poisson homology and cohomology. Finally we use this twisted Poisson duality to compute the Poisson cohomology of R, retrieving so a result of Monnier.

Although we work here with the semiclassical limit of quantum affine space, it is likely that this method will apply to other algebras admitting a "twisted" Poincaré duality for the Hochschild homology at the quantised level, at least when the automorphism used for this twisting can be expressed simply as multiplying generators by a power of the parameter q. This is the case for instance for quantised coordinate rings of semisimple complex algebraic groups, as proved in [6]. Starting from the Van den Bergh duality, [25], at the quantum level, it would be very interesting to retrieve this twisted Poisson duality for semiclassical limits thanks to spectral sequences à la Brylinski, [7], see also [16]. We plan to go back to these questions in a subsequent paper.

The plan of this paper is as follows. First, we show that the automorphism provided by Sitarz to solve the "dimension drop" for the Hochschild homology of the quantum space provides a twisted duality. Next, we produce a Poisson module providing duality between Poisson homology and cohomology of the semiclassical limit R. Finally, we compute the Poisson homology with values in this module.

Throughout this paper we will use the usual following notation for a monomial: if $x_1, \dots, x_n$ are variables and $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ then $x^\alpha = x_1^{\alpha_1} \dots x_n^{\alpha_n}$. We will also denote by ϵ_i the ith vector of the canonical basis of $\mathbb{Z}^n$.

2 Twisted Poincaré duality in Hochschild (co)homology.

In the paper [23], Sitarz provides an automorphism of quantum affine space restoring what one should expect as the "good" Hochschild dimension of these algebras. Namely, Wambst proved in [26] that in the generic case all Hochschild homology groups are null in degree greater than 1. Sitarz proved in his paper that the "twisted" Hochschild homology, for a particular choice of twisting automorphism, provides a vector space of dimension 1 for the homology group of degree n of the quantum affine n-space.

Re-interpreting certain results of [21] in terms of "twisted" Poincaré duality (see [25], [6]), we show here that the automorphism determined by Sitarz actually provides such a duality. This could be as well seen as a corollary of Van den Bergh's results, [25, Proposition 2], but we make it explicit here at the level of the complexes themselves. The method used, in terms of "up to a sign" commuting diagrams, is the same as the one we are going to use for Poisson homology. The following computations are done for quantum affine space, but one can easily check that they remain valid for the "mixed crossed algebras" defined in [21].

2.1 Resolution

We say that a matrix $Q = (q_{ij}) \in M_n(\mathbb{C})$ is multiplicatively skew-symmetric if $q_{ij}q_{ji} = q_{ii} = 1$ for all i, j. Let us recall here the following two quantum algebras.

Definition 2.1.1. *Let V be a $\mathbb{C}$-vector space of dimension n with basis $(v_1, \ldots, v_n)$, and let $Q = (q_{ij})_{1\leq i,j\leq n} \in M_n(\mathbb{C}^*)$ be a multiplicatively skew-symmetric matrix.*

1. *The* quantum affine space $S_QV = \mathcal{O}_Q(\mathbb{C}^n)$ *is the $\mathbb{C}$-algebra generated by n generators $v_1, \ldots, v_n$ with relations $v_iv_j - q_{ij}v_jv_i$ for all $1 \leq i, j \leq n$.*

   It is well-known that S_QV is a noetherian domain, and admits the monomials $\{v^\alpha\}_{\alpha\in\mathbb{N}^n}$ as a PBW-basis.

2. *The* quantum exterior algebra Λ_QV *is the $\mathbb{C}$-algebra generated by n generators $v_1, \ldots, v_n$ with relations $v_iv_j = -q_{ij}v_jv_i$ for all $1 \leq i, j \leq n$. It admits the monomials $\{v^\beta\}_{\beta\in\{0,1\}^n}$ as a $\mathbb{C}$-linear basis.*

Remark 2.1.2. This definition of the quantum exterior algebra differs from the one given in [5] for instance. However, it is the one provided in [26], giving rise to the quantum Koszul complex described below.

Set $U = S_QV$. It follows from [26, Proposition 10.3] that the following is a free resolution of U as a $U^e = U \otimes U^{op}$-module.

The vector space Λ_QV is $\mathbb{N}$-graded, by defining the degree of a monomial $v_1^{\beta_1} \wedge \ldots \wedge v_n^{\beta_n}$ to be $|\beta|$. For all $* \in \mathbb{N}$, one denotes by Λ_Q^*V the homogeneous subspace of Λ_QV of degree $*$. Then the space $U \otimes \Lambda_QV \otimes U$ is graded by the degree of Λ_QV, and $U \otimes \Lambda_QV \otimes U$ becomes a differential complex, with the differential ∂ defined for all $a, b \in U$ by:

$$\begin{aligned}
&\partial(a \otimes v_{i_1} \wedge \ldots \wedge v_{i_*} \otimes b) = \\
&\quad \sum_{k=1}^{*}(-1)^{k-1}\Big(\Big(\prod_{s<k} q_{i_s,i_k}\Big)av_{i_k} \otimes v_{i_1} \wedge \ldots \wedge \widehat{v}_{i_k} \wedge \ldots \wedge v_{i_*} \otimes b \\
&\qquad -\Big(\prod_{s>k} q_{i_k,i_s}\Big)a \otimes v_{i_1} \wedge \ldots \wedge \widehat{v}_{i_k} \wedge \ldots \wedge v_{i_*} \otimes v_{i_k}b\Big)
\end{aligned} \tag{1}$$

where $v_{i_1} \wedge \ldots \wedge \widehat{v}_{i_k} \wedge \ldots \wedge v_{i_*}$ is the outer product $v_{i_1} \wedge \ldots \wedge v_{i_*}$ without v_{i_k}.

2.2 Hochschild homology, cohomology

We recall some of the homological framework that is used in [26], [23].

Definition 2.2.1. *Let $\mathcal{A}$ be a $\mathbb{C}$-algebra. An automorphism σ of $\mathcal{A}$ is said to be a* scaling automorphism *if there exists a basis $\{a_i\}_{i\in I}$ of $\mathcal{A}$ as a vector space such that $\sigma(a_i) = p_ia_i$ with $p_i \in \mathbb{C}^*$ for all $i \in I$.*

Definition 2.2.2. *Consider a $\mathbb{C}$-algebra $\mathcal{A}$, σ an automorphism of A, and let ${}_\sigma\mathcal{A}$ be the $\mathcal{A}^e$-module that is $\mathcal{A}$ as a vector space with module structure: $b.(a_0, a_1) = \sigma(a_0)ba_1$. Denote by $K(\mathcal{A})$ a projective resolution of $\mathcal{A}$ by $\mathcal{A}^e$-modules.*

1. *The* invariant twisted Hochschild homology *is the homology of the subcomplex of ${}_\sigma\mathcal{A} \otimes_{\mathcal{A}^e} K(\mathcal{A})$ consisting only of σ-invariant elements.*

2. *The* twisted Hochschild homology *is the homology of the quotient of the (usual) Hochschild complex by the image of the map* $1-\sigma$*.*

Lemma 2.2.3 ([23], Lemma 2.2)**.** *For any scaling automorphism* σ *the corresponding invariant twisted Hochschild homology is isomorphic to the twisted Hochschild homology.*

So the twisted Hochschild homology is computed as the σ-invariant part of the Hochschild homology with values in ${}_\sigma\mathcal{A}$. Thus we are interested here in $H(U, {}_\sigma U)$. The following scaling automorphism for the algebra S_QV is defined in [23].

Definition 2.2.4. *The automorphism* σ *of* S_QV *defined by* $\sigma(v_i) = p_i v_i$*, with* $p_i = \prod_{j=1}^n q_{j,i}$ *is a scaling automorphism, called the* canonical *scaling automorphism.*

Let us recall - with adapted notation - the results obtained in [26, Section 6] and [23, Section 3]. The twisted Hochschild homology of S_Q is given by the complex below:

$$K^\sigma(S_QV) = \bigoplus_{\substack{\alpha \in \mathbb{N}^n \\ \gamma \in \{0,1\}^n}} \mathbb{C}.v^\alpha \otimes v^\gamma,$$

$$d(v^\alpha \otimes v^\gamma) = \sum_{i=1}^n \Omega_Q(\alpha, \gamma; i) v^{\alpha+[i]} \otimes v^{\gamma-[i]}, \tag{2}$$

with coefficients:

$$\Omega_Q(\alpha,\gamma;i) = (-1)^{\sum\limits_{k<i}\gamma_k} \left(\prod_{k<i} q_{k,i}^{\gamma_k}\right)\left(\prod_{k>i} q_{k,i}^{\alpha_k}\right) \times \left(1 - p_i\left(\prod_{k=1}^n q_{i,k}^{\alpha_k+\gamma_k}\right)\right) \text{ if } \gamma_i = 1, \tag{3}$$

and

$\Omega_Q(\alpha,\gamma;i) = 0$ if $\gamma_i = 0$.

Proposition 2.2.5 ([23], Proposition 3.5)**.** *The complex* $(K^\sigma(S_QV), d)$ *above computes the twisted Hochschild homology of* S_QV*, and* $\deg(v^\alpha \otimes v^\gamma) = \alpha+\gamma$ *defines a* $\mathbb{N}^n$*-grading on it. Moreover, set* $C^\sigma(Q) = \{\rho \in \mathbb{N}^n \mid \forall i,\ \rho_i = 0 \text{ or } p_i v_i v^\rho = v^\rho v_i\}$*. Then for all* $\rho \in \mathbb{N}^n \setminus C^\sigma(Q)$*, the homogeneous subcomplex of* $(K(S_QV), d)$ *of degree* ρ *is acyclic. Further, the twisted Hochschild homology of* $S_Q(V)$ *in degree* k *is given by*

$$H_k(S_QV, {}_\sigma(S_QV)) = \bigoplus_{\substack{\gamma \in \{0,1\}^n \\ |\gamma| = k}} \bigoplus_{\substack{\alpha \in \mathbb{N}^n \\ \alpha+\gamma \in C^\sigma(Q)}} \mathbb{C}.v^\alpha \otimes v^\gamma.$$

The proof of this result relies on a homotopy h_Q given by the linear map $h_Q : K_*^\sigma \to K_{*+1}^\sigma$ defined by $h_Q(v^\alpha \otimes v^\beta) = \frac{1}{||\alpha+\beta||}\sum_{k=1}^n \omega_Q(\alpha,\beta,i) v^{\alpha-\epsilon_i} \otimes v^{\beta+\epsilon_i}$, with

$$\omega_Q(\alpha,\beta,i) = \begin{cases} 0 & \text{if } \alpha+\beta \in C^\sigma(Q) \\ 0 & \text{if } \beta_i = 1 \\ 0 & \text{if } \alpha_i = 0 \\ \Omega_Q(\alpha-\epsilon_i, \beta+\epsilon_i, i)^{-1} & \text{otherwise} \end{cases} \tag{4}$$

where $||\gamma||$ is the cardinal of $\{k \mid \sigma(v_k)v_\gamma \neq v_\gamma v_k \text{ and } \gamma_k \neq 0\}$, see the proof of [26, Theorem 6.1] and [23, Proposition 3.5].

From the free resolution $U \otimes \Lambda_Q^* V \otimes U$ of U we derive a complex $(R^*, {}^t\partial)$ which computes the Hochschild cohomology of U with values in a bimodule M. As a $\mathbb{C}$-vector space,

$$R^* = \mathrm{Hom}_{U^e}(U \otimes \Lambda_Q^* V \otimes U, M),$$

and the differential is the transposition of the differential

$$\partial : U \otimes \Lambda_Q^{*+1} V \otimes U \to U \otimes \Lambda_Q^* V \otimes U,$$

that is:

$$\begin{aligned} {}^t\partial: \quad & R^* \to R^{*+1} \\ & \varphi \mapsto \varphi \circ \partial. \end{aligned}$$

Then $HH^*(U) = H^*(R, {}^t\partial)$.

2.3 Duality

We present in this section the links between the Hochschild homology and cohomology of the algebra $S_Q V$. This section is mainly inspired from Section 6 of [21], although the notation we use is slightly different. We recall the following easy result, which will be our main tool in the following.

Lemma 2.3.1 ([21], Lemma 6.2.1)**.** *Let (C_*, d) be a $\mathbb{C}$-differential complex, and let M_* be a graded $\mathbb{C}$-vector space, such that there exists an isomorphism Φ of graded vector spaces of degree 0, with source C_* and target M_*. Then the map $\widetilde{d} = \Phi \circ d \circ \Phi^{-1}$ is such that $\widetilde{d}^2 = 0$, and $(M_*, \widetilde{d})$ is a differential complex. The map Φ is then an isomorphism of complexes, and one has:*

$$H_*(C, d) = H_*(M, \widetilde{d}).$$

Let us apply this result to the complexes $(R^*, {}^t\partial)$ and (K_*^σ, d) described above. We just give formulae here, the proofs can be found in Section 6 of [21]. Note that the quantum affine space we consider here is just a particular case of the "mixed crossed algebras" studied in [21], and that all the following could apply *verbatim* to these algebras.

There is an isomorphism $\Phi_{1,*}$ from $\mathrm{Hom}_{U^e}(U \otimes \Lambda^* V \otimes U, U)$ onto $\mathrm{Hom}_k(\Lambda_Q^* V, U)$ defined for all $\varphi \in \mathrm{Hom}_{U^e}(U \otimes \Lambda^* V \otimes U, U)$ by:

$$\Phi_{1,*}(\varphi)(v_{i_1} \wedge \ldots \wedge v_{i_*}) = \varphi(1 \otimes v_{i_1} \wedge \ldots \wedge v_{i_*} \otimes 1).$$

Then one computes the conjugate $D = \Phi_{1,*+1} \circ {}^t\partial \circ \Phi_{1,*}^{-1}$ of ${}^t\partial$ by Φ_1. Set

$$L^* = \mathrm{Hom}_k(\Lambda_Q^* V, U).$$

Then one has

$$\begin{aligned} D: \quad L^* & \to L^{*+1} \\ \varphi & \mapsto D(\varphi), \end{aligned}$$

where $D(\varphi)$ is defined by:

$$\begin{aligned} D(\varphi)&(v_{i_1} \wedge \ldots \wedge v_{i_{*+1}}) = \\ & \sum_{k=1}^{*+1} (-1)^{k-1} \Big((\prod_{s<k} q_{i_s, i_k}) v_{i_k} \varphi(v_{i_1} \wedge \ldots \widehat{v}_{i_k} \ldots \wedge v_{i_{*+1}}) \\ & \qquad - (\prod_{s>k} q_{i_k, i_s}) \varphi(v_{i_1} \wedge \ldots \widehat{v}_{i_k} \ldots \wedge v_{i_{*+1}}) v_{i_k} \Big). \end{aligned} \tag{5}$$

By construction, the complex $(R^*, {}^t\partial)$ and the complex (L^*, D) above are isomorphic, and the diagram below commutes:

$$\begin{CD}
R^* @>{{}^t\partial}>> R^{*+1} \\
@V{\Phi_{1,*}}VV \circlearrowleft @VV{\Phi_{1,*+1}}V \\
\mathrm{Hom}(\Lambda_Q^* V, U) @>{D}>> \mathrm{Hom}(\Lambda_Q^{*+1} V, U)
\end{CD}$$

Next, we prove a similar result for the complex (K_*^σ, d), the homology of which is the Hochschild homology of U with values in ${}_\sigma U$. Recall that $K_*^\sigma = U \otimes \Lambda_Q^* V$ as a vector space and that the differential d is given by formula (2).

Lemma 2.3.2 ([21], Lemma 6.2.3)**.** *The canonical map* $\psi_* : \Lambda_Q^* V \otimes \Lambda_Q^{n-*} V \to k \otimes v_1 \wedge \ldots \wedge v_n$ *defined by* $\psi_*(v_{i_1} \wedge \ldots \wedge v_{i_*} \otimes v_{j_1} \wedge \ldots \wedge v_{j_{n-*}}) = v_{i_1} \wedge \ldots \wedge v_{i_*} \wedge v_{j_1} \wedge \ldots \wedge v_{j_{n-*}}$ *induces an isomorphism* $\overline{\psi}_* : \Lambda_Q^{n-*} V \to (\Lambda_Q^* V)'$ *defined by*

$$\overline{\psi}_*(v_{j_1} \wedge \ldots \wedge v_{j_{n-*}}) = \psi_*(\cdot \otimes v_{j_1} \wedge \ldots \wedge v_{j_{n-*}}).$$

Remark 2.3.3. In fact $\overline{\psi}_*$ is just the linear map sending the element $v_{j_1} \wedge \ldots \wedge v_{j_{n-*}}$ to $\Theta_*(i_1, \ldots, i_*)(v_{i_1} \wedge \ldots \wedge v_{i_*})'$ where $\{i_1, \ldots, i_*\}$ is the complementary $*$-uple of $\{j_1, \ldots, j_{n-*}\}$ and $\Theta_*(i_1, \ldots, i_*) \in \mathbb{C}^*$ is defined by:

$$\Theta_*(i_1, \ldots, i_*) = \prod_{k < i_*,\ k \notin \{i_s\}} (-q_{i_*,k}) \prod_{k < i_{*-1},\ k \notin \{i_s\}} (-q_{i_{*-1},k}) \ldots \prod_{k < i_1,\ k \notin \{i_s\}} (-q_{i_1,k}). \tag{6}$$

The isomorphism $\overline{\psi}_*$ induces an isomorphism $\Phi_{2,*} = \mathrm{id} \otimes \overline{\psi}_*$ from ${}_\sigma U \otimes \Lambda_Q^{n-*} V$ to the space ${}_\sigma U \otimes (\Lambda_Q^* V)'$. But ${}_\sigma U \otimes \Lambda_Q^{n-*} V = K_{n-*}^\sigma$, and one thus defines a differential Δ on the complex ${}_\sigma U \otimes (\Lambda_Q V)'$, such that the following diagram commutes:

$$\begin{CD}
{}_\sigma U \otimes (\Lambda_Q^* V)' @>{\Delta}>> {}_\sigma U \otimes (\Lambda_Q^{*+1})' \\
@A{\Phi_{2,*}}AA \circlearrowleft @AA{\Phi_{2,*+1}}A \\
K_{n-*}^\sigma @>{d}>> K_{n-*-1}^\sigma
\end{CD}$$

The differential Δ is exactly $\Phi_{2,*+1} \circ d \circ \Phi_{2,*}^{-1}$. For all $i_1 < \ldots < i_*$, let $\{j_1, \ldots, j_{n-*}\} = \overline{\{i_1, \ldots, i_*\}}$ be the complementary set. Then

$$\begin{aligned} \Delta(a \otimes (v_{i_1} \wedge \ldots \wedge v_{i_*})') = {} & \Theta_*^{-1}(i_1, \ldots, i_*) \times \\ & \sum_{k=1}^{n-*} (-1)^{k-1} \Big((\prod_{s<k} q_{j_s,j_k}) a v_{j_k} - (\prod_{s>k} q_{j_k,j_s}) p_{j_k} v_{j_k} a \Big) \\ & \qquad \otimes \Theta_{*+1}(i_1, \ldots, j_k, \ldots, i_*)(v_{i_1} \wedge \ldots v_{j_k} \ldots \wedge v_{i_*})'. \end{aligned} \tag{7}$$

Once again, by construction the homology of the above complex is:

$$H^*(U \otimes (\Lambda_Q V)') = HH_{n-*}(U, {}_\sigma U).$$

We have transfered the differential complex structures of (K^σ_*, d) and $(R^*, {}^t\partial)$ onto the graded vector spaces $\mathrm{Hom}(\Lambda^*_Q V, U)$ and $U \otimes (\Lambda^*_Q V)'$. There is a natural linear isomorphism between these two spaces; we use it to compare the two differential complex structures.

Let $\Phi_{3,*}$ be the isomorphism from $U \otimes (\Lambda^*_Q V)'$ to $\mathrm{Hom}(\Lambda^*_Q V, U)$ defined by:

$$\Phi_{3,*}(a \otimes \varphi)(v_{i_1} \wedge \ldots \wedge v_{i_*}) = \varphi(v_{i_1} \wedge \ldots \wedge v_{i_*})a. \tag{8}$$

Then consider the diagram below:

$$\begin{array}{ccc}
\mathrm{Hom}(\Lambda^*_Q V, U) & \xrightarrow{\quad D \quad} & \mathrm{Hom}(\Lambda^{*+1}_Q V, U) \\
\uparrow \Phi_{3,*} & & \uparrow \Phi_{3,*+1} \\
{}_\sigma U \otimes (\Lambda^*_Q V)' & \xrightarrow{\quad \Delta \quad} & {}_\sigma U \otimes (\Lambda^{*+1}_Q V)'
\end{array}$$

This diagram does not commute *a priori*, but we have the following result. The following Proposition relies directly on the computations leading to Lemma 6.3.2 of [21]. But we first must note that a careful check of these computations show that the formulation of this Lemma is wrong in the following sense. In expressing ω_2', the $\tilde{\lambda}_{j_k,t}$ must be replaced by $\tilde{\lambda}_{t,j_k}$. This does not affect the rest of [21], since the only case considered there is the one where the products among t of these elements is always equal to 1. Once this is observed, one gets the following.

Proposition 2.3.4. *With the notation above, we have:* $\Phi_{3,*+1} \circ \Delta = (-1)^{*+1} D \circ \Phi_{3,*}$.

Proof. Lemmas 6.3.1 and 6.3.2 from [21] can be rewritten in our context in the following form (taking care that the Hochschild homology is twisted): $\Phi_{3,*+1} \circ \Delta(a \otimes (v_{i_1} \wedge \ldots \wedge v_{i_*})')$ is the linear map which sends the basis element $v_{\alpha_1} \wedge \ldots \wedge v_{\alpha_{*+1}} \in \Lambda^{*+1}_Q V$ to:

$$\sum_{k=1}^{n-*} (-1)^{k-1} (\omega_1(\alpha_1, \ldots, \alpha_{*+1}; k) a v_{j_k} - p_{j_k} \omega_2(\alpha_1, \ldots, \alpha_{*+1}; k) v_{j_k} a),$$

and $D \circ \Phi_{3,*}(a \otimes (v_{i_1} \wedge \ldots \wedge v_{i_*})')$ is the linear map which sends the basis element $v_{\alpha_1} \wedge \ldots \wedge v_{\alpha_{*+1}} \in \Lambda^{*+1}_Q V$ to:

$$\sum_{k=1}^{n+r-*} (-1)^{k-1} (\omega_1'(\alpha_1, \ldots, \alpha_{*+1}; k) a v_{j_k} - \omega_2'(\alpha_1, \ldots, \alpha_{*+1}; k) v_{j_k} a), \tag{9}$$

with $\omega_1' = (-1)^{*+1}\omega_1$, and $\omega_2' = (-1)^{*+1} p_{j_k} \omega_2$; so we are done. □

The results of this section can now be gathered together in the following

Proposition 2.3.5. *Let $U = S_Q V$ be a quantum affine space. Then there is a duality between the Hochschild cohomology of U and its Hochschild homology with values in the bimodule ${}_\sigma U$:*

$$H_*(U, {}_\sigma U) \equiv H^{n-*}(U, U).$$

This result can be seen as a corollary of Van den Bergh's Theorem [25]. We present the proof above in order to show how the duality occurs at the level of the complexes themselves.

Remark 2.3.6. Let Λ be an $m \times m$ multiplicatively skew-symmetric matrix. Define $Q = \begin{pmatrix} \Lambda & {}^t\Lambda \\ {}^t\Lambda & \Lambda \end{pmatrix}$ by block, an $n \times n$ multiplicatively skew-symmetric matrix with $n = 2m$. Then obviously the canonical automorphism σ associated to Q is the identity. The duality result above implies in particular that for such a matrix Q the Hochschild homology in degree n is nonzero, so that there is no dimension drop of the Hochschild homology with value in the algebra itself. This may explain a result by Connes and Dubois-Violette in the setting of smooth functions over a quantum real affine space $\mathbb{R}^{2m}$ parametrised by such a matrix, see [8, Theorem 8].

3 Twisted Poincaré duality for the semiclassical limit of a quantum affine space

Now, we consider the uniparametrised case, by which we mean that the entries q_{ij} of the matrix Q are all powers of a generic q. Denote $q_{ij} = q^{a_{ij}}$, with $a_{ij} \in \mathbb{Z}$. As q is generic, we have $a_{ii} = a_{ij} + a_{ji} = 0$ for all i, j. The semiclassical limit, [5, Section III.5.4], of the quantum affine space $S_Q V$ is the commutative algebra $R = \mathbb{C}[X_1, \ldots, X_n]$ endowed with the Poisson bracket defined by $\{X_i, X_j\} = a_{ij} X_i X_j$.

The canonical Poisson homology and cohomology are defined respectively thanks to the Kähler differentials and the multiderivations of R. We will denote the homology and cohomology spaces by $HP_*(R)$ and $HP^*(R)$. The complexes computing these homology groups are explicitly written down in the case where $n = 2$ for instance in [22] (see also [20]), where the Poisson cohomology is computed for the affine plane for any Poisson structure defined by a homogenous polynomial.

In this section we proceed as follows. First, we consider the semiclassical limit M of the twisted bimodule structure of the quantum space ${}_\sigma(S_Q V)$. It turns out that M is a Poisson dualising R-module, in the sense that there is a twisted Poincaré duality between the Poisson cohomology of R and its Poisson homology with values in M:

$$HP_k(R, M) \simeq HP^{n-k}(R). \tag{10}$$

Next, we compute the Poisson homology of R with values in M. As a consequence, we compute the Poisson cohomology of R thanks to the above isomorphism (10), and so retrieve a result of Monnier, [17].

3.1 Poisson algebra and Poisson module.

A commutative algebra R endowed with a Lie bracket $\{.,.\}$ such that, for all $r \in R$, the map $\{r, .\} : R \to R$ is a $\mathbb{C}$-linear derivation of R is called a Poisson algebra. From [16, 18], a Poisson module over the Poisson algebra R is a $\mathbb{C}$-vector space M endowed with two bilinear maps . and $\{.,.\}_M$ such that

1. $(M, .)$ is a (right) module over the commutative algebra R,
2. $(M, \{.,.\}_M)$ is a (right) module over the Lie algebra $(R, \{.,.\})$,
3. $x.\{a, b\} = \{x, a\}_M.b - \{x.b, a\}_M$ for all $a, b \in R$ and $x \in M$.
4. $\{x, ab\}_M = \{x, a\}_M.b + \{x, b\}_M.a$ for all $a, b \in R$ and $x \in M$.

Starting from $S_Q V$, we will now exhibit the Poisson structure on the polynomial algebra $R = \mathbb{C}[X_1, \dots, X_n]$ that arises from the semiclassical limit process, see [5, Section III.5.4]. Let $A = (a_{ij}) \in \mathcal{M}_n(\mathbb{Z})$ be antisymmetric and set $q_{ij} := q^{a_{ij}}$, where q is generic. The semiclassical limit, [5, Section III.5.4] of the quantum affine space $S_Q V$ is the commutative algebra $R = \mathbb{C}[X_1, \dots, X_n]$ endowed with the Poisson bracket defined on the generators of R as follows:

$$\{X_i, X_j\} := ([v_i, v_j]/(q-1))_{|q=1} = ((1 - q^{a_{ji}})/(q-1))_{|q=1} X_i X_j = a_{ij} X_i X_j.$$

Our aim in this section is to construct a Poisson module over the Poisson algebra R (with the Poisson structure as above) that will restore a duality between Poisson homology and Poisson cohomology. As the dualising bimodule is ${}_\sigma(S_Q V)$ in the quantum setting, we will now consider the semiclassical limit of this bimodule. In Section 3.4, we will show that the Poisson module resulting from this procedure is actually a "dualising Poisson module".

We denote by M the semiclassical limit of the twisted bimodule structure of the quantum space ${}_\sigma(S_Q V)$. As a vector space, $M = \mathbb{C}[X_1, \dots, X_n] = R$, and M is endowed with the following two actions of R:

- the external product "." is just the usual product of R.
- the external bracket $\{.,.\}_M$ is defined by

$$\{m, X_i\}_M := \left.\frac{m v_i - \sigma(v_i) m}{q-1}\right|_{q=1}$$

for all $m \in M$ and $i \in \{1, \dots, n\}$. In particular, when $m = X_1^{\alpha_1} \dots X_n^{\alpha_n}$ is a monomial

$$\{X_1^{\alpha_1} \dots X_n^{\alpha_n}, X_i\}_M = \left.\frac{v^\alpha v_i - \sigma(v_i) v^\alpha}{q-1}\right|_{q=1} = \left.\frac{(\prod_{j>i} q_{ji}^{\alpha_j} - \prod_j q_{ji} \prod_{j<i} q_{ij}^{\alpha_j}) v^{\alpha+\epsilon_i}}{q-1}\right|_{q=1}.$$

Recall that $q_{ij} = q^{a_{ij}}$; so

$$(\prod_{j>i} q_{ji}^{\alpha_j} - \prod_j q_{ji} \prod_{j<i} q_{ij}^{\alpha_j}) = \prod_{j>i} q_{ji}^{\alpha_j} (1 - q^{\sum_j a_{ij}(\alpha_j - 1)}),$$

and finally $\{X_1^{\alpha_1} \dots X_n^{\alpha_n}, X_i\}_M = -\sum_j a_{ij}(\alpha_j - 1) X^{\alpha+\epsilon_i}$.

One can easily check that M is a Poisson module over R. Further, observe that

$$\{m, X_i\}_M = -\{X_i, m\} + \left(\sum_{l=1}^n a_{il}\right) X_i m, \tag{11}$$

for all $m \in M$.

3.2 Poisson homology

Let M be a Poisson module over a Poisson algebra R. Then one defines a chain complex on the R-module $C_*^{Poiss}(R, M) = \oplus_{k \in \mathbb{N}} C_k^{Poiss}(R, M)$, where $C_k^{Poiss}(R, M) := M \otimes_R \Omega^k(R)$ and $\Omega^k(R)$ denotes the so-called Kähler differential k-forms, as follows, [16]. The boundary operator $\partial_k : C_k^{Poiss}(R, M) \to C_{k-1}^{Poiss}(R, M)$ is defined by

$$\begin{aligned}\partial_k(m \otimes da_1 \wedge \dots \wedge da_k) = &\sum_{i=1}^k (-1)^{i+1} \{m, a_i\}_M \otimes da_1 \wedge \dots \wedge \widehat{da_i} \wedge \dots \wedge da_k \\ &+ \sum_{1 \le i < j \le k} (-1)^{i+j} m \otimes d\{a_i, a_j\} \wedge da_1 \wedge \dots \wedge \widehat{da_i} \wedge \dots \wedge \widehat{da_j} \wedge \dots \wedge da_k,\end{aligned}$$

where we have removed the expressions under the hats in the previous sums and d denotes the exterior differential.

One can easily check that ∂_k is well-defined and that $\partial_{k-1} \circ \partial_k = 0$. The homology of this complex is denoted by $HP_*(R, M)$. This homology is called the canonical homology in [16] in reference to the canonical homology defined by Brylinski, [7]. It is also called the Poisson homology of the Poisson algebra R with values in the Poisson module M.

In the particular case that we will study, R will be a (commutative) polynomial algebra over $\mathbb{C}$, namely $R = \mathbb{C}[X_1, \dots, X_n]$. In this case, it is clear that $\Omega^*(R)$ is the R-module generated by the wedge products of the 1-differential forms $dX_1, \dots, dX_n$.

3.3 Poisson cohomology

We denote by $\chi^k(R)$ the R-module of all skew-symmetric k-linear derivations of R, that is, the set of all skew-symmetric $\mathbb{C}$-linear maps $R^k \to R$ that are derivations in each of their arguments. Then we set $\chi^*(R) := \oplus_{k\in\mathbb{N}}\chi^k(R)$, the R-module of so-called skew-symmetric multi-derivations of R. One can define a cochain complex structure on this R-module as follows. The Poisson coboundary operator $\delta_k : \chi^k(R) \to \chi^{k+1}(R)$ is defined by

$$\begin{aligned}\delta_k(P)(f_0, \dots, f_k) \quad := \quad & \sum_{i=0}^{k}(-1)^i \left\{f_i, P(f_0, \dots, \widehat{f_i}, \dots, f_k)\right\} \\ & + \sum_{0\leq i<j\leq k} (-1)^{i+j} P\left(\{f_i, f_j\}, f_0, \dots, \widehat{f_i}, \dots, \widehat{f_j}, \dots, f_k\right)\end{aligned}$$

for all $P \in \chi^k(R)$. It is easy to check that $\delta_k(P)$ belongs indeed to $\chi^{k+1}(R)$ and that $\delta_{k+1} \circ \delta_k = 0$. The cohomology of this complex is called the Poisson cohomology of R; it is denoted by $HP^*(R)$.

3.4 A Poincaré duality result.

For the rest of this paper, we assume that $R = \mathbb{C}[X_1, \dots, X_n]$ is endowed with the Poisson bracket defined by

$$\{X_i, X_j\} = a_{ij} X_i X_j,$$

where $A = (a_{ij}) \in \mathcal{M}_n(\mathbb{Z})$ is skew-symmetric.

Recall that, thanks to the canonical volume form $dX_1 \wedge \ldots \wedge dX_n$, the set $\chi^k(R)$ of all skew-symmetric k-linear derivations of R is isomorphic as a vector space to $\Omega^{n-k}(R)$ via an isomorphism $\dagger$ defined as follows. We denote by S_n the set of all n-permutations. Further, for all $\sigma \in S_n$, we denote by $\varepsilon(\sigma)$ its sign and we set $\sigma_i := \sigma(i)$. For all $P \in \chi^k(R)$ let $\dagger(P)$ be the unique element of $\Omega^{n-k}(R)$ defined by

$$\dagger(P) = \sum_{\sigma\in S_{k,n-k}} \epsilon(\sigma) P(X_{\sigma_1}, \dots, X_{\sigma_k}) dX_{\sigma_{k+1}} \wedge \cdots \wedge dX_{\sigma_n},$$

where $S_{k,n-k}$ denotes the set of those permutations $\sigma \in S_n$ such that $\sigma_1 < \cdots < \sigma_k$ and $\sigma_{k+1} < \cdots < \sigma_n$.

From now on, M denotes the Poisson R-module defined in Section 3.1. Recall that $M = \mathbb{C}[X_1, \dots, X_n] = R$ as a vector space. Hence, we deduce from the above result that $\chi^k(R)$ is isomorphic as a vector space to $M \otimes_R \Omega^{n-k}(R)$ via an isomorphism still denoted by $\dagger$ and defined by:

$$\dagger(P) = \sum_{\sigma \in S_{k,n-k}} \epsilon(\sigma) P(X_{\sigma_1}, \ldots, X_{\sigma_k}) dX_{\sigma_{k+1}} \wedge \cdots \wedge dX_{\sigma_n}$$

for all $P \in \chi^k(R)$. (Observe that we have omitted $\otimes$ in order to simplify the notation.)

So we have a diagram as follows.

$$\begin{array}{ccc} \chi^k(R) & \xrightarrow{\quad\dagger\quad} & M \otimes_R \Omega^{n-k}(R) \\ \Big\downarrow{\scriptstyle \delta_k} & & \Big\downarrow{\scriptstyle \partial_{n-k}} \\ \chi^{k+1}(R) & \xrightarrow{\quad\dagger\quad} & M \otimes_R \Omega^{n-k-1}(R) \end{array}$$

In order to prove that there is a (twisted) Poincaré duality between the Poisson homology of R with values in M and the cohomology of R, we will prove that this diagram is almost commutative.

Proposition 3.4.1. *For all $P \in \chi^k(R)$, the following equality holds:*

$$(\dagger \circ \delta)(P) = (-1)^{k+1}(\partial \circ \dagger)(P).$$

Proof. First, it follows from the definition of δ and $\dagger$ that $(\dagger \circ \delta)(P) = U + V$, where

$$U := \sum_{\substack{\sigma \in S_{k+1,n-k-1} \\ 1 \le i \le k+1}} \varepsilon(\sigma)(-1)^{i+1} \left\{ X_{\sigma_i}, P[X_{\sigma_1}, \ldots, \widehat{X_{\sigma_i}}, \ldots, X_{\sigma_{k+1}}] \right\} dX_{\sigma_{k+2}} \wedge \cdots \wedge dX_{\sigma_n}$$

and

$$V = \sum_{\substack{\sigma \in S_{k+1,n-k-1} \\ 1 \le i < j \le k+1}} \varepsilon(\sigma)(-1)^{i+j} P[\{X_{\sigma_i}, X_{\sigma_j}\}, X_{\sigma_1}, \ldots, \widehat{X_{\sigma_i}}, \ldots, \widehat{X_{\sigma_j}}, \ldots, X_{\sigma_{k+1}}] dX_{\sigma_{k+2}} \wedge \cdots \wedge dX_{\sigma_n}.$$

We proceed in three steps.

• Step 1: we rewrite U.

For r distinct integers $i_1, \ldots, i_r$ let $(i_1 \ldots i_r)$ be the cyclic permutation sending i_1 to i_2, $\ldots$, i_{r-1} to i_r and i_r to i_1. Let $S_{k,1,n-k-1}$ be the set of those permutation $\tau \in S_n$ such that $\tau_1 < \cdots < \tau_k$ and $\tau_{k+2} < \cdots < \tau_n$. Then, the map $(\sigma, i) \in S_{k+1,n-k-1} \times \{1, \ldots, k+1\} \mapsto \tau \in S_{k,1,n-k-1}$ given by

$$\tau = \sigma \circ (i \; i+1 \ldots k+1)$$

is well-defined and turns out to be a bijection. Indeed, it is easy to see that these two sets have each cardinality $\binom{n}{k} \times (n-k)$ and that this map is injective. Observe further that $\varepsilon(\tau) = (-1)^{k+1-i}\varepsilon(\sigma)$. Thus, by means of the change of variables induced by this bijection, we get:

$$U = (-1)^k \sum_{\tau \in S_{k,1,n-k-1}} \varepsilon(\tau) \left\{ X_{\tau_{k+1}}, P[X_{\tau_1}, \ldots, X_{\tau_k}] \right\} dX_{\tau_{k+2}} \wedge \cdots \wedge dX_{\tau_n}. \tag{12}$$

• **Step 2: we rewrite V.**

As $\{X_{\sigma_i}, X_{\sigma_j}\} = a_{\sigma_i,\sigma_j} X_{\sigma_i} X_{\sigma_j}$ and P is a skew-symmetric multiderivation, one can rewrite V as follows.

$$\begin{aligned}
V = & \sum_{\substack{\sigma\in S_{k+1,n-k-1}\\ 1\le i<j\le k+1}} \varepsilon(\sigma)(-1)^{i+j} a_{\sigma_i,\sigma_j} (-1)^j X_{\sigma_i} P[X_{\sigma_1},\dots,\widehat{X_{\sigma_i}},\dots,X_{\sigma_{k+1}}] dX_{\sigma_{k+2}}\wedge\cdots\wedge dX_{\sigma_n}\\
& + \sum_{\substack{\sigma\in S_{k+1,n-k-1}\\ 1\le i<j\le k+1}} \varepsilon(\sigma)(-1)^{i+j} a_{\sigma_i,\sigma_j} (-1)^{i+1} X_{\sigma_j} P[X_{\sigma_1},\dots,\widehat{X_{\sigma_j}},\dots,X_{\sigma_{k+1}}] dX_{\sigma_{k+2}}\wedge\cdots\wedge dX_{\sigma_n}\\
= & \sum_{\sigma\in S_{k+1,n-k-1}} \varepsilon(\sigma) \sum_{1\le i<j\le k+1} (-1)^i a_{\sigma_i,\sigma_j} X_{\sigma_i} P[X_{\sigma_1},\dots,\widehat{X_{\sigma_i}},\dots,X_{\sigma_{k+1}}] dX_{\sigma_{k+2}}\wedge\cdots\wedge dX_{\sigma_n}\\
& + \sum_{\sigma\in S_{k+1,n-k-1}} \varepsilon(\sigma) \sum_{1\le i<j\le k+1} (-1)^{j+1} a_{\sigma_i,\sigma_j} X_{\sigma_j} P[X_{\sigma_1},\dots,\widehat{X_{\sigma_j}},\dots,X_{\sigma_{k+1}}] dX_{\sigma_{k+2}}\wedge\cdots\wedge dX_{\sigma_n}\\
= & \sum_{\sigma\in S_{k+1,n-k-1}} \varepsilon(\sigma) \sum_{i=1}^{k} (-1)^i \left(\sum_{j=i+1}^{k+1} a_{\sigma_i,\sigma_j}\right) X_{\sigma_i} P[X_{\sigma_1},\dots,\widehat{X_{\sigma_i}},\dots,X_{\sigma_{k+1}}] dX_{\sigma_{k+2}}\wedge\cdots\wedge dX_{\sigma_n}\\
& + \sum_{\sigma\in S_{k+1,n-k-1}} \varepsilon(\sigma) \sum_{j=2}^{k+1} (-1)^j \left(\sum_{i=1}^{j-1} a_{\sigma_j,\sigma_i}\right) X_{\sigma_j} P[X_{\sigma_1},\dots,\widehat{X_{\sigma_j}},\dots,X_{\sigma_{k+1}}] dX_{\sigma_{k+2}}\wedge\cdots\wedge dX_{\sigma_n}\\
= & \sum_{\sigma\in S_{k+1,n-k-1}} \varepsilon(\sigma) \sum_{i=1}^{k+1} (-1)^i \left(\sum_{j=1}^{k+1} a_{\sigma_i,\sigma_j}\right) X_{\sigma_i} P[X_{\sigma_1},\dots,\widehat{X_{\sigma_i}},\dots,X_{\sigma_{k+1}}] dX_{\sigma_{k+2}}\wedge\cdots\wedge dX_{\sigma_n}
\end{aligned}$$

Hence, using the same change of variables as in the previous step, we get:

$$V = (-1)^{k+1} \sum_{\tau\in S_{k,1,n-k-1}} \varepsilon(\tau) \left(\sum_{l=1}^{k} a_{\tau_{k+1},\tau_l}\right) X_{\tau_{k+1}} P[X_{\tau_1},\dots,X_{\tau_k}] dX_{\tau_{k+2}}\wedge\cdots\wedge dX_{\tau_n} \qquad (13)$$

• **Step 3: we rewrite $(\partial\circ\dagger)(P)$ and conclude.**

First, it follows from the definition of ∂ and $\dagger$ that

$$\begin{aligned}
(\partial\circ\dagger)(P) = & \sum_{\sigma\in S_{k,n-k}} \varepsilon(\sigma) \Bigg[\sum_{i=1}^{n-k} (-1)^{i+1} \big\{P[X_{\sigma_1},\dots,X_{\sigma_k}], X_{\sigma_{i+k}}\big\}_M dX_{\sigma_{k+1}}\wedge\cdots\wedge\widehat{dX_{\sigma_{k+i}}}\wedge\cdots\wedge dX_{\sigma_n}\\
& + \sum_{1\le i<j\le n-k} (-1)^{i+j} P[X_{\sigma_1},\dots,X_{\sigma_k}] d\left\{X_{\sigma_{k+i}}, X_{\sigma_{k+j}}\right\}\wedge dX_{\sigma_{k+1}}\wedge\cdots\wedge\widehat{dX_{\sigma_{k+i}}}\wedge\cdots\wedge\widehat{dX_{\sigma_{k+j}}}\cdots\wedge dX_{\sigma_n}\Bigg].
\end{aligned}$$

Next, using (11) and the fact that $\left\{X_{\sigma_{k+i}}, X_{\sigma_{k+j}}\right\} = a_{\sigma_{k+i}\sigma_{k+j}} X_{\sigma_{k+i}} X_{\sigma_{k+j}}$, we obtain:

$$
\begin{aligned}
(\partial \circ \dagger)(P) \ &= \sum_{\sigma \in S_{k,n-k}} \varepsilon(\sigma) \sum_{i=1}^{n-k} (-1)^i \left\{X_{\sigma_{i+k}}, P[X_{\sigma_1}, \ldots, X_{\sigma_k}]\right\} dX_{\sigma_{k+1}} \wedge \cdots \wedge \widehat{dX_{\sigma_{k+i}}} \wedge \cdots \wedge dX_{\sigma_n} \\
&+ \sum_{\sigma \in S_{k,n-k}} \varepsilon(\sigma) \sum_{i=1}^{n-k} (-1)^{i+1} \left(\sum_{l=1}^{n} a_{\sigma_{i+k} l} \right) X_{\sigma_{i+k}} P[X_{\sigma_1}, \ldots, X_{\sigma_k}] dX_{\sigma_{k+1}} \wedge \cdots \wedge \widehat{dX_{\sigma_{k+i}}} \wedge \cdots \wedge dX_{\sigma_n} \\
&+ \sum_{\sigma \in S_{k,n-k}} \varepsilon(\sigma) \sum_{1 \le i < j \le n-k} (-1)^{i+j} a_{\sigma_{k+i}\sigma_{k+j}} (-1)^{j-2} X_{\sigma_{k+i}} P[X_{\sigma_1}, \ldots, X_{\sigma_k}] dX_{\sigma_{k+1}} \wedge \cdots \wedge \widehat{dX_{\sigma_{k+i}}} \wedge \cdots \wedge dX_{\sigma_n} \\
&+ \sum_{\sigma \in S_{k,n-k}} \varepsilon(\sigma) \sum_{1 \le i < j \le n-k} (-1)^{i+j} a_{\sigma_{k+i}\sigma_{k+j}} (-1)^{i-1} X_{\sigma_{k+j}} P[X_{\sigma_1}, \ldots, X_{\sigma_k}] dX_{\sigma_{k+1}} \wedge \cdots \wedge \widehat{dX_{\sigma_{k+j}}} \wedge \cdots \wedge dX_{\sigma_n}
\end{aligned}
$$

Then, rewriting the last three sums in the right-hand side leads to

$$
\begin{aligned}
(\partial \circ \dagger)(P) \ &= \sum_{\sigma \in S_{k,n-k}} \varepsilon(\sigma) \sum_{i=1}^{n-k} (-1)^i \left\{X_{\sigma_{i+k}}, P[X_{\sigma_1}, \ldots, X_{\sigma_k}]\right\} dX_{\sigma_{k+1}} \wedge \cdots \wedge \widehat{dX_{\sigma_{k+i}}} \wedge \cdots \wedge dX_{\sigma_n} \\
&+ \sum_{\sigma \in S_{k,n-k}} \varepsilon(\sigma) \sum_{i=1}^{n-k} (-1)^{i+1} \left(\sum_{l=1}^{k} a_{\sigma_{i+k}\sigma_l} \right) X_{\sigma_{i+k}} P[X_{\sigma_1}, \ldots, X_{\sigma_k}] dX_{\sigma_{k+1}} \wedge \cdots \wedge \widehat{dX_{\sigma_{k+i}}} \wedge \cdots \wedge dX_{\sigma_n}
\end{aligned}
$$

Finally, observe that the map $(\sigma, i) \in S_{k,n-k} \times \{1, \ldots, n-k\} \mapsto \tau \in S_{k,1,n-k-1}$ given by

$$\tau = \sigma \circ (k+i \ldots k+1)$$

is well-defined and is a bijection. Observe further that $\varepsilon(\tau) = (-1)^{i-1}\varepsilon(\sigma)$. Hence, by means of the change of variable induced by this bijection, we get

$$
\begin{aligned}
(\partial \circ \dagger)(P) \ &= \ - \sum_{\tau \in S_{k,1,n-k-1}} \varepsilon(\tau) \left\{X_{\tau_{k+1}}, P[X_{\tau_1}, \ldots, X_{\tau_k}]\right\} dX_{\tau_{k+2}} \wedge \cdots \wedge dX_{\tau_n} \\
&+ \sum_{\tau \in S_{k,1,n-k-1}} \varepsilon(\tau) \left(\sum_{l=1}^{k} a_{\tau_{k+1}\tau_l} \right) X_{\tau_{k+1}} P[X_{\tau_1}, \ldots, X_{\tau_k}] dX_{\tau_{k+2}} \wedge \cdots \wedge dX_{\tau_n}
\end{aligned}
$$

So, we deduce from (12) and (13) that

$$(\partial \circ \dagger)(P) = (-1)^{k+1}(U+V) = (-1)^{k+1}(\dagger \circ \delta_\pi)(P),$$

as desired. □

It follows from Proposition 3.4.1 that the diagram

$$
\begin{array}{ccc}
\chi^k(R) & \xrightarrow{\ \dagger\ } & M \otimes_R \Omega^{n-k}(R) \\
\Big\downarrow \delta_k & \circlearrowleft & \Big\downarrow (-1)^{k+1}\partial_{n-k} \\
\chi^{k+1}(R) & \xrightarrow{\ \dagger\ } & M \otimes_R \Omega^{n-k-1}(R)
\end{array}
$$

is commutative. Naturally, this leads to a (twisted) Poincaré duality between the Poisson homology of R with values in M and the Poisson cohomology of R. More precisely, we can now state the main result of this paper.

Theorem 3.4.2. *For all $k \in \mathbb{N}$, we have:*

$$HP_k(R, M) \simeq HP^{n-k}(R).$$

3.5 Application to Poisson cohomology.

In view of Theorem 3.4.2, in order to compute the Poisson cohomology of R, it is enough to compute $HP_k(R, M)$. The final part of this paper is dedicated to this computation.

First, if $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$, then we set $X^\alpha := X_1^{\alpha_1} \dots X_n^{\alpha_n}$. Similary, if $\beta = (\beta_1, \dots, \beta_n) \in \mathbb{N}^n$, then we set $dX^\beta := dX_1^{\beta_1} \wedge \dots \wedge dX_n^{\beta_n}$. Then, by following the proof of [26, Theorem 6.1], we obtain the following result.

Proposition 3.5.1.

$$HP_k(R, M) = \bigoplus_{\substack{|\beta|=k \\ \alpha+\beta \in C}} \mathbb{C} X^\alpha dX^\beta$$

where

$$C := \{\gamma \in \mathbb{N}^n \mid \gamma_i = 0 \text{ or } \sum_{j=1}^n a_{ij}(\gamma_j - 1) = 0\}.$$

Proof. Easy computations show that

$$\partial(X^\alpha dX^\beta) = \sum_{i=1}^n \delta_{1,\beta_i} \Omega(\alpha, \beta, i) X^{\alpha+\varepsilon_i} dX^{\beta-\varepsilon_i},$$

where δ_{1,β_i} is the Kronecker symbol, and

$$\Omega(\alpha, \beta, i) := (-1)^{\sum_{j=1}^{i-1} \beta_j} \sum_{j=1}^n a_{ij}(\alpha_j + \beta_j - 1).$$

Observe that $\Omega(\alpha, \beta, i) = \frac{\Omega_Q(\alpha,\beta,i)}{1-q} \mid_{q=1}$, where $Q := (q^{a_{ij}})$ and $\Omega_Q(\alpha, \beta, i)$ has been defined in (3). We also set

$$\omega(\alpha, \beta, i) := (1-q)\omega_Q(\alpha, \beta, i) \mid_{q=1},$$

where ω_Q has been defined in (4).

It is clear that $X^\alpha dX^\beta$ is in the homology group of the complex when $\alpha + \beta \in C$. Next we have to prove that there exists an homotopy h such that $\partial_{\pi,\sigma} h + h\partial_{\pi,\sigma}(X^\alpha dX^\beta) = X^\alpha dX^\beta$ when $\alpha + \beta \notin C$. We set

$$h(X^\alpha dX^\beta) := \frac{1}{\|\, \alpha + \beta \,\|} \sum_{i=1}^n \omega(\alpha, \beta, i) X^{\alpha-\varepsilon_i} dX^{\beta+\varepsilon_i}.$$

In order to prove that h is a homotopy, it is enough to prove that certain equalities hold between the Ω and the ω. As these equalities hold at the quantum level, i.e. the linear map h_Q defined by (4) is a homotopy, the desired equalities also hold at the "semiclassical level" thanks to a specialisation. □

In the case where $n = 2$ and $a_{12} = 1$, we obtain the following result:

$$HP_0(R, M) = \mathbb{C} \oplus \mathbb{C}X_1X_2, \quad HP_1(R, M) = \mathbb{C}X_1dX_2 \oplus \mathbb{C}X_2dX_1, \quad HP_2(R, M) = \mathbb{C}dX_1 \wedge dX_2.$$

In this way, we retrieve the dimensions computed for the cohomology in [22], see also the Introduction of the present work.

Finally, in view of Theorem 3.4.2 and Proposition 3.5.1, we obtain the following result regarding the Poisson cohomology of R. This result has been previously obtained by a direct (and so completely different) method in [17].

Corollary 3.5.2.

$$HP^k(R) \simeq \bigoplus_{\substack{|\beta|=n-k \\ \alpha+\beta\in C}} \mathbb{C}X^\alpha dX^\beta$$

where

$$C := \{\gamma \in \mathbb{N}^n \mid \gamma_i = 0 \ \text{or} \ \sum_{j=1}^{n} a_{ij}(\gamma_j - 1) = 0\}.$$

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