Optimal Maintenance Policies under Different Operational Schedules

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Abstract

In the reliability literature, maintenance time is usually ignored during the optimization of maintenance policies. In some scenarios, costs due to system failures may vary with time, and the ignorance of maintenance time will lead to unrealistic results. This paper develops maintenance policies for such situations where the system under study operates iteratively at two successive states: up or down. The costs due to system failure at the up state consist of both business losses & maintenance costs, whereas those at the down state only include maintenance costs. We consider three models: Model A, B, and C:

- Model A makes only corrective maintenance (CM).
- Model B performs imperfect preventive maintenance (PM) sequentially, and CM.
- Model C executes PM periodically, and CM; this PM can restore the system as good as the state just after the latest CM.

The CM in this paper is imperfect repair. Finally, the impact of these maintenance policies is illustrated through numerical examples.

Acronyms[1]

CM Corrective Maintenance, imperfect.

PM Preventative Maintenance, imperfect .

cdf Cumulative Distribution Function.

Nomenclature

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[1]The singular and plural of an acronym are always spelled the same.

T_u duration of the up state

T_d duration of the down state, $T_d \leq T_u$

T_L designed total operating time

$X_{a,n}$ survival time after the $(n-1)$th CM in Model A, n=1,2,.....

$X_{b,n}$ survival time after the $(n-1)$th PM in Model B, n=1,2,.....

$X_{n,m}$ survival time after the $(m-1)$th CM between the $(n-1)$th PM & nth PM in Model B, n, m=1,2,...

$X_{c,n}$ survival time after the $(n-1)$th CM in Model C, n=1,2,.....

Y_r maintenance time of a CM activity, $Y_r \leq T_d$

$F_{a,n}(t)$ cdf of $X_{a,n}$

$F_{b,n}(t)$ cdf of $X_{b,n}$

$F_{n,m}(t)$ cdf of $X_{n,m}$

$F_{c,n}(t)$ cdf of $X_{c,n}$

$G_r(t)$ cdf of Y_r

c_b business losses per time unit due to the failure of the system during the up time

c_r cost of CM (or repair) per time unit

c_p, μ_p cost of PM per time unit, and expected time per PM respectively

τ_n time interval between the $(n-1)$th & nth PM in Model B

τ time interval between two adjacent PM in Model C

W_a, W_b, W_c the whole life cycle time in Model A, Model B, and Model C, respectively.

$C_{a,L}$ cost incurred in Model A in the whole life cycle time

$C_{b,L}(\tau_n)$ cost incurred in Model B in the whole life time

$C_{c,L}(\tau)$ cost incurred in Model C in the whole life time

C_a expected average cost incurred in Model A in the design life time

$C_b(\tau_n)$ expected average cost incurred in Model B in the design life time

$C_c(\tau)$ expected average cost incurred in Model C in the design life time

1 INTRODUCTION

Critical systems include data processing facilities, call centers, control centrers, and so forth. Most buildings today incorporate some types of critical systems. What they all have in common is their dependence on a reliability infrastructure. For such systems, it is of importance to ensure that the system functions properly. The development of optimal maintenance policies is therefore very important.

Two important points should be highlighted for the critical systems when the optimization of maintenance policies are being studied.

1. The first is the huge difference between maintenance costs, and business losses due to system failures. For example, based on research in office building services systems, Evans et. al. [1] have identified the relationships among three costs incurred within the systems' whole lifecycle: initial capital costs, maintenance & building operating costs, and business operating costs. They found that the operation & maintenance of the building will cost five times the capital costs, and the business operating costs will be two hundred times the capital costs over the life of the building. In other words, business losses are forty times as large as maintenance costs. Another example comes from transportation centers of underground train systems; the business losses will be huge if the system fails to function properly, whereas maintenance activities cost less.

2. The second is that such systems are often operated at two successive states: up, and down. Namely, they are operated in sequence as: up state $\rightarrow$ down state $\rightarrow$ up state $\rightarrow$ down state .... In the up state, the system is operating but may fail, and CM is conducted upon failures. In the down state, the system is not operating, but available for any maintenance. A failure only occurs at the up state, but not at the down state. For example, the up time of some systems can be only eight or sixteen hours a day, and the systems are then put into the down state for rest of the day. If a maintenance activity is executed during the up time, the costs due to system failures consist of both business losses, and maintenance costs. If a maintenance activity is carried out during system down time, the costs due to system failures only include maintenance costs. When we optimize the maintenance policies for such systems, the costs for maintenance should be composed of two elements: business losses, and/or maintenance costs for different time periods. In these scenarios, maintenance time may have two parts: one within the up state, and one within the down state. Ignoring the maintenance time will lead to unrealistic results.

With different maintenance levels, maintenance models can be divided into three classes: models for perfect maintenance, models for normal maintenance, and models for minimal maintenance. A *perfect maintenance* can restore the system as good as new, a *normal maintenance*

can bring the system to any condition, and a *minimal maintenance* can restore the system to the state in which it resided just before failure. Normal maintenance, and minimal maintenance are considered to be imperfect. Maintenance policies are usually broken down into two categories: corrective maintenance (CM), and preventive maintenance (PM). CM is any maintenance one carries out when the system fails. Some authors refer to CM as 'repair', and we will use the terms interchangeably throughout this paper. PM is a planned activity aimed at improving the overall reliability & availability of a system. Sequential PM may be one of the most practical maintenance policies to implement in some industries. We define sequential PM as the situation where a system is preventively maintained at unequal time intervals, and undergoes only CM upon failures between these PM. Usually, the time intervals between PM become shorter as time passes, considering that most units of interest to us degrade, and so need more frequent maintenance with age.

Optimal maintenance policies aim to provide optimum system reliability/availability & safety performance at the lowest possible maintenance costs. In the past several decades, a huge number of maintenance policies have been produced. The reader is referred to Pham & Wang [2], Wang [3], and Scarf [4] for detailed & comprehensive discussions on the theoretic discussion, and the application of maintenance policies, respectively.

This paper considers the optimization of maintenance policies for systems with different costs within the up time & the down time. Section 2 introduces model assumptions that are used in this paper. Section 3 formulates the expected costs of the three models. Section 4 considers the situation when system failure processes can be modeled by geometric processes. Numerical examples are given in Section 5 with a discussion about the three models. Concluding remarks are given in the Section 6.

2 MODEL ASSUMPTIONS

We make the following assumptions:

A. At the beginning, the repairable system under study is new. It operates iteratively in two states: up, and down.

B. A CM is executed immediately upon failure, and a PM is only performed within a system's down state. Both CM & PM can be completed within time interval $(0, T_d)$. The system is only started at time point $0, T_u + T_d, 2(T_u + T_d), 3(T_u + T_d), ...$ In other words, although a CM can be finished before the next down state, the system will not be started until the next up state.

C. We consider three models. In the following three models, a CM is assumed to be an imperfect repair.
In Model A, no PM is conducted; only the effect of CM exists.

In Model B, both PM & CM are carried out. The PM is sequentially executed with τ_n time units after the $(n-1)$th PM, where n=1,2,.... Between two adjacent PM, a CM is carried out immediately on failure. The PM in Model B is assumed to be normal maintenance.

In Model C, periodic PM is executed, and a CM is performed on failure. The PM in Model C can restore the system to the state it was in just after its last CM.

D. When a CM is performed during the up state, both business losses & maintenance costs are taken into account. When a CM or a PM is being performed during the down state, only maintenance costs are incurred.

E. $X_{a,n}$, $X_{b,n}$, $X_{n,m}$, and Y_r are s-independent, where $n, m = 1, 2, 3, ...$

3 PROBLEM FORMULATION

3.1 Model A

A possible scenario is shown in Figure 1 where the system fails at the up state, and is repaired at the down state.

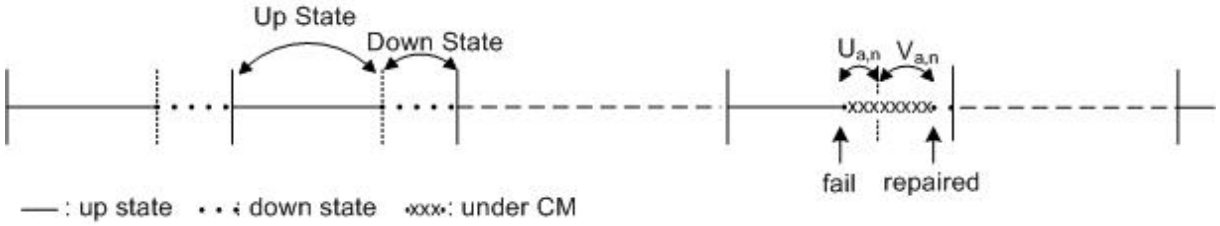

Figure 1: Model A, where only CM is performed.

Costs incurred in Model A include costs on CM, and business losses due to failure at the up state.

Denote the time from the occurrence of the nth failure in an up state to the end of this up state by $U_{a,n}$, the repair time within the coming down state of the nth failure by $V_{a,n}$, the cdf of $U_{a,n}$ by $H_{a,n}(t)$, the cdf of $V_{a,n}$ by $Q_{a,n}(t)$, and the cdf of $\sum_{n=1}^{k} X_{a,n}$ by $F_a^{(k)}(t)$. Let $\upsilon_{a,n} = E[U_{a,n}]$, $\varphi_{a,n} = E[V_{a,n}]$, and $\lambda_{a,n} = E[X_{a,n}]$. Apparently, $U_{a,n}$ is a main factor in the cost function of Model A. With our assumptions, we can obtain

$$U_{a,n} = \begin{cases} T_u - X_{a,n} & \text{if } X_{a,n} < T_u \\ 2T_u - X_{a,n} & \text{if } T_u \le X_{a,n} < 2T_u \\ ... & \\ jT_u - X_{a,n} & \text{if } (j-1)T_u \le X_{a,n} < jT_u \\ ... & \end{cases} \tag{1}$$

and

$$V_{a,n} = \begin{cases} X_{a,n} + Y_r - T_u & \text{if } X_{a,n} < T_u \text{ and } Y_r + X_{a,n} \geq T_u \\ X_{a,n} + Y_r - 2T_u & \text{if } T_u \leq X_{a,n} < 2T_u \text{ and } Y_r + X_{a,n} \geq 2T_u \\ ... & \\ X_{a,n} + Y_r - jT_u & \text{if } (j-1)T_u \leq X_{a,n} < jT_u \text{ and } Y_r + X_{a,n} \geq jT_u \\ ... & \end{cases} \tag{2}$$

The distributions of repair time at the up state, and at the down state, can be obtained through Equations (1), and (2), respectively. That is

Lemma 1

$$H_{a,n}(t) = \sum_{j=1}^{\infty}(F_{a,n}(jT_u) - F_{a,n}(jT_u - t)) \quad \text{where } 0 \leq t < T_u \tag{3}$$

$$Q_{a,n}(t) = \sum_{j=1}^{\infty}\int_{(j-1)T_u}^{jT_u}(G_r(jT_u - x + t) - G_r(jT_u - x))\, dF_{a,n}(x) \quad \text{where } 0 \leq t < T_d \tag{4}$$

The appendix contains proofs of the lemmas & theorems for these equations.

A CM may be started at the up state, and finished at the adjacent down state. Results from Lemma 1 can be used to calculate the expected time of CM at the two states. From Lemma 1, we can get

Lemma 2

$$\upsilon_{a,n} = T_u \sum_{j=1}^{\infty} j(F_{a,n}(jT_u) - F_{a,n}((j-1)T_u)) - \lambda_{a,n} \tag{5}$$

$$\begin{aligned} \varphi_{a,n} = & \sum_{j=1}^{\infty}\left[\int_{jT_u - T_d}^{jT_u}\left(\int_{jT_u - x}^{T_d} y dG(y) + xG(T_d) - xG(jT_u - x) + jT_u G(jT_u - x)\right) dF_{a,n}(x)\right] \\ & - \sum_{j=1}^{\infty}(jT_u G(T_d)(F_{a,n}(jT_u) - F_{a,n}(jT_u - T_d))) \end{aligned} \tag{6}$$

In Lemma 2, $\upsilon_{a,n}$ is the expected time of the nth CM when the system is in the up state, and $\varphi_{a,n}$ is the expected time of the nth CM when the system is in the down state.

The whole life time includes operating time, CM time at the up state, and CM time at the down state. The costs in the whole life time include costs due to CM in different states. That is

Lemma 3 *The expected whole life time, and the expected cost in the whole life time, are*

$$E[W_a] = T_L + \sum_{k=1}^{\infty}\left((F_a^{(k)}(T_L) - F_a^{(k+1)}(T_L))\sum_{n=1}^{k}(\upsilon_n + \varphi_n)\right), \tag{7}$$

and

$$E[C_{a,L}] = \left[\sum_{k=1}^{\infty}\left((F_a^{(k)}(T_L) - F_a^{(k+1)}(T_L))\sum_{n=1}^{k}((c_b + c_r)\upsilon_{a,n} + c_r\varphi_{a,n})\right)\right] \tag{8}$$

respectively.

In Equation (7) of Lemma 3, the first item (i.e. T_L) is the designed total operating time, and the second element includes the expected total time of CM at the up state, and at the down state. Equation (8) includes business losses due to failure at the up state, and CM costs incurred at both the up state & the down state.

Theorem 1 *The average cost incurred within the design life time in Model A is*

$$C_a = \frac{1}{T_L} E[C_{a,L}] \tag{9}$$

3.2 Model B

There are two main alternatives for modeling an imperfect PM: the one with the assumption that PM is equivalent to minimal repair with probability p & equivalent to replacement with probability $1-p$ [5], and the one which focuses on studying the change of the hazard rate function after PM. Lie and Chun [6] and Nakagawa [7] introduce the concept of adjustment factors in hazard rate functions & effective ages to model the effects of PM. In Model B, we assume that the failure process after each PM is a generalized renewal process.

A possible scenario is shown in Figure 2, where PM activities are executed during the down state. Costs incurred in Model B include costs on CM, costs on PM, and business losses due to failure.

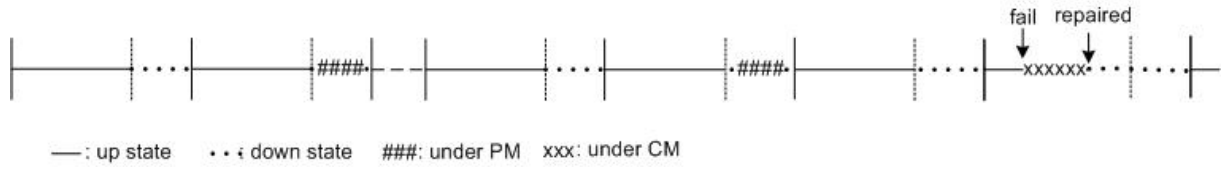

Figure 2: Model B and Model C, where both CM and PM are used.

Denote the operating time interval, the total time, and costs incurred between the $(n-1)$th & the nth PM activity by τ_n, $\tau_{b,n}$, and $C_{b,n}$, respectively. When PM activities are performed, the maintenance situation for failures between two adjacent PM is similar to the case in Model A. The time interval between the $(n-1)$th PM & the nth PM, i.e. τ_n, in Model B can be regarded as the whole life time in Model A. We can obtain Lemma 3 if T_L, $X_{a,n}$, and $F_{a,n}(t)$ are replaced with τ_n, $X_{n,m}$, and $F_{n,m}(t)$ in Section 3.1, respectively. Let M_b satisfy

$$\sum_{n=1}^{M_b} \tau_n \leq T_L < \sum_{n=1}^{M_b+1} \tau_n \tag{10}$$

M_b is the number of PM activities. The expected operating time, and maintenance time between the $(n-1)$th PM & the nth PM can be calculated by mimicking the calculation process in Section 3.1. Let $\tau_{M_b+1} = T_L - \sum_{n=1}^{M_b} \tau_n$. Then, we have

Lemma 4 *The expected values of $\tau_{b,n}$, and $C_{b,n}$ are*

$$E[\tau_{b,n}] = \tau_n + \sum_{k=1}^{\infty}\left[(F_n^{(k)}(\tau_n) - F_n^{(k+1)}(\tau_n))\sum_{m=1}^{k}(\upsilon_{n,m} + \varphi_{n,m})\right] \tag{11}$$

$$E[C_{b,n}] = \frac{1}{E[\tau_{b,n}]}\sum_{k=1}^{\infty}\left((F_n^{(k)}(\tau_n) - F_n^{(k+1)}(\tau_n))\sum_{m=1}^{k}((c_b + c_r)\upsilon_{n,m} + c_r\varphi_{n,m})\right) \tag{12}$$

respectively, where $F_n^{(m)}$ is the cdf of $\sum_{i=1}^{k} X_{n,m}$, $\upsilon_{n,m} = \int_0^\infty t dH_{n,m}(t)$, $\varphi_{n,m} = \int_0^\infty t dQ_{n,m}(t)$,

$$H_{n,m}(t) = \sum_{j=1}^{\infty}(F_{n,m}(j\tau_n) - F_{n,m}(j\tau_n - t)), \tag{13}$$

and

$$Q_{n,m}(t) = \sum_{j=1}^{\infty}\int_{(j-1)\tau_n}^{j\tau_n}(G_r(j\tau_n - x + t) - G_r(j\tau_n - x))\, dF_{n,m}(x) \tag{14}$$

Similarly, $\upsilon_{n,m}$, and $\varphi_{n,m}$ can be obtained through Equations (13), and (14). The expected whole life time includes the total operating time, total time on CM, and the total time on PM. The total cost includes business losses, and costs due to PM & CM.

Lemma 5 *The expected whole life time, and the expected cost in the whole life time are*

$$E[W_b] = \sum_{n=1}^{M_b+1} E[\tau_{b,n}] + M_b\mu_p, \tag{15}$$

and

$$E[C_{b,L}(\tau_n)] = \sum_{n=1}^{M_b+1} E[C_{b,n}] + M_b\mu_p c_p \tag{16}$$

respectively,

where μ_p is the expected time per PM.

Theorem 2 *The average cost incurred within the design life time in Model B is*

$$C_b(\tau_n) = \frac{1}{T_L}\left(\sum_{n=1}^{M_b+1} E[C_{b,n}] + M_b\mu_p c_p\right) \tag{17}$$

Because Equation (17) is a function of τ_n only, we can determine τ_n^* so that

$$\tau_n^* = \min_{\tau_n} C_b(\tau_n) \tag{18}$$

Once the value of τ_n is determined, the optimal period can be obtained. Then $C_{b,L}(\tau_n)$ achieves the minimum.

If τ_n is independent of n, the sequential PM policy becomes a periodic PM policy.

When a sequential PM policy is considered, τ_n is a function of n which can be obtained by minimizing $C_b(\tau_n)$ in Equation (17).

Based on the above discussion, we derive the following algorithm for obtaining the optimal PM schedule.

- Step 1. Calculate $\upsilon_{n,m}$, and $\varphi_{n,m}$ from Equations (13), and (14), respectively.
- Step 2. Solve Equation (10) with respect to M_b.
- Step 3. Substitute $E[C_{b,n}]$ in Equation (12) into (17).
- Step 4. Choose τ_n to minimize the value $C_b(\tau_n)$ in Equation (17).

3.3 Model C

In Model C, a PM is assumed to restore the system to the state it was after last CM. Denote the operating time interval between two adjacent PM by τ. The expected survival time after the $(n-1)$th CM is

$$X_{c,n} = r_n\tau + 1_A(X_{a,n})X_{a,n} \tag{19}$$

where r_n is a random variable with probability distribution $Pr(r_n = k) = (1-F_{a,n}(\tau))^k F_{a,n}(\tau)$; $1_A(x) = 1$ if $x \in A$, and 0 otherwise, while $A = (0, \tau)$.

Denote the time from the occurrence of the nth failure to the end of this up state by $U_{c,n}$, the repair time within the down state of the nth failure by $V_{c,n}$, $\upsilon_{c,n} = E[U_{c,n}]$, and $\varphi_{c,n} = E[V_{c,n}]$. Let M_c satisfy

$$M_c\tau \le T_L < (M_c+1)\tau \tag{20}$$

M_c is the number of PM activities. The cdf of $X_{c,n}$ is given as follows.

$$\begin{aligned} F_{c,n}(t) &= \Pr\{X_{c,n} < t\} \\ &= \Pr\{r_n\tau + 1_A(X_{a,n})X_{a,n} < t\} \\ &= F_{a,n}(t-k\tau)(1-F_{a,n}(\tau))^k + \sum_{j=0}^{k-1}(1-F_{a,n}(\tau))^j F_{a,n}(\tau) \\ &= F_{a,n}(t-k\tau)(1-F_{a,n}(\tau))^k + 1 - (1-F_{a,n}(\tau))^k \end{aligned} \tag{21}$$

where $k\tau \le t < (k+1)\tau$, and $k \ge 1$. Then we have

$$\begin{aligned} \lambda_{c,n} &= \int_0^\infty t dF_{c,n}(t) \\ &= \sum_{k=0}^\infty \int_{k\tau}^{(k+1)\tau} t(1-F_{a,n}(\tau))^k dF_{a,n}(t-k\tau) \\ &= \sum_{k=0}^\infty \int_0^\tau (y+k\tau)(1-F_{a,n}(\tau))^k dF_{a,n}(y) \\ &= \frac{\int_0^\tau t dF_{a,n}(t)}{F_{a,n}(\tau)} + \frac{\tau(1-F_{a,n}(\tau))}{F_{a,n}(\tau)} \end{aligned} \tag{22}$$

$\upsilon_{c,n}$, and $\varphi_{c,n}$ can be obtained by substituting Equations (21), and (22) into Equations (5), and (6).

Let the cdf of $\sum_{n=1}^k X_{c,n}$ be $F_c^{(k)}(t)$. We have

Lemma 6 *The expected whole life time, and the expected cost in the whole life time are*

$$E[W_c] = T_L + M_c\mu_p + \sum_{k=1}^{\infty}\left[\left(F_c^{(k)}(T_L) - F_c^{(k+1)}(T_L)\right)\left(\sum_{n=1}^{k}(\upsilon_{c,n} + \varphi_{c,n})\right)\right], \tag{23}$$

and

$$E[C_{c,L}(\tau)] = M_c\mu_p c_p + \sum_{k=1}^{\infty}\left(F_c^{(k)}(T_L) - F_c^{(k+1)}(T_L)\right)\left(\sum_{n=1}^{k}(\upsilon_{c,n} + \varphi_{c,n})(c_b + c_r)\right) \tag{24}$$

respectively.

Therefore, the following results can be obtained.

Theorem 3 *The average cost incurred within the design life time in Model C is*

$$C_c(\tau) = \frac{1}{T_L}E[C_{c,L}(\tau)] \tag{25}$$

Because Equation (25) is a function of τ only, we can determine τ^* so that $C_{L,c}$ is minimized.

4 GEOMETRIC PROCESS CASE

The geometric process introduced by Lam [8, 9] is a kind of generalized renewal processes that has been commonly used for the optimization of maintenance policies, and reliability analysis [10], [11], [12], [13], [14]. Below we present the definition of the geometric process.

Definition 1 [15] (a) *A random variable ξ is said to be stochastically not less (not greater) than another random variable ζ, denoted by $\xi \geq_{st} \zeta$ ($\xi \leq_{st} \zeta$), if* $\Pr(\xi > \mathrm{a}) \geq \Pr(\zeta > \mathrm{a})$ $(\Pr(\xi > \mathrm{a}) \leq \Pr(\zeta > \mathrm{a}))$ *for all real a.*
(b) *A stochastic process $\{\xi_n\}_{n=1,2,...}$ is said to be stochastically increasing (decreasing) if $\xi_n <_{st} \xi_{n+1}$ ($\xi_n >_{st} \xi_{n+1}$) for all $n = 1, 2, ...$*

(decreasing) processes are defined.

Definition 2 [12] *A sequence of non-negative independent random variables $\{\xi_n\}_{n=1,2,...}$ is called a geometric process if for some $a > 0$, the distribution of ξ_n is $S(a^{n-1}t)$. The constant a is called the parameter of the geometric process.*

Remark 1 : From Definition 2, it follows that

(1) If $a > 1$, then $\{\xi_n\}_{n=1,2,...}$ is stochastically decreasing: $\xi_n \geq \xi_{n+1}$

(2) If $0 < a < 1$, then $\{\xi_n\}_{n=1,2,...}$ is stochastically increasing: $\xi_n \leq \xi_{n+1}$

(3) If $a = 1$, then $\{\xi_n\}_{n=1,2,...}$ is a renewal process.

Some authors [10], [11], [12], [13], [14] apply the geometric process to model life times between failures of a deteriorating system with parameter $a > 1$. Zhang [10] optimized periodic preventive maintenance policies under the assumptions that the failure process after CM activities forms a geometric process, and PM activities can rectify a failed system to a state as good as the state after the last repair. Zhang [10] did not present the relationship between the two probability density functions of the survival lives after the nth repair in Model A and Model C. From Equation (21), the relationship is obtained easily.

The depth of the improvement of PM is an interesting research topic. Malik [16], and Lie & Chun [6] introduce the concept of the improvement factor to model the age restoration of imperfect PM. They regarded that an imperfect PM can reduce a system's age from t to t/k, and result in restoring the system's reliability to $R(t/k)$ from $R(t)$. If we let $k = a^{-n}$, where n is the number of PM, then this failure process is identical to the geometric progress. The difference between these two approaches is that Malik's, and Lie & Chun's work originally focused on modeling the survival life time after PM activities, in which the time between two adjacent PM is fixed. The geometric process was introduced to model the survival life time after a CM, in which the time between two adjacent CM activities is random.

Let the cdf of ξ_1 be $S(t)$, having the hazard rate function $h_1(t)$; then the hazard rate function of $S(a^{n-1}t)$ is given by

$$h_n(t) = a^{n-1}h_1(t) \tag{26}$$

Example 1 : Assume that $S(t) = 1 - e^{(-t/33.32)^{5.28}}$, and $a = 1.2$; the hazard rate functions of $S(t)$, $S(at)$, $S(a^2t)$, $S(a^3t)$, and $S(a^4t)$ are shown in Figure 3. From this figure, we can see that the hazard rate increases more quickly with the increase of n.

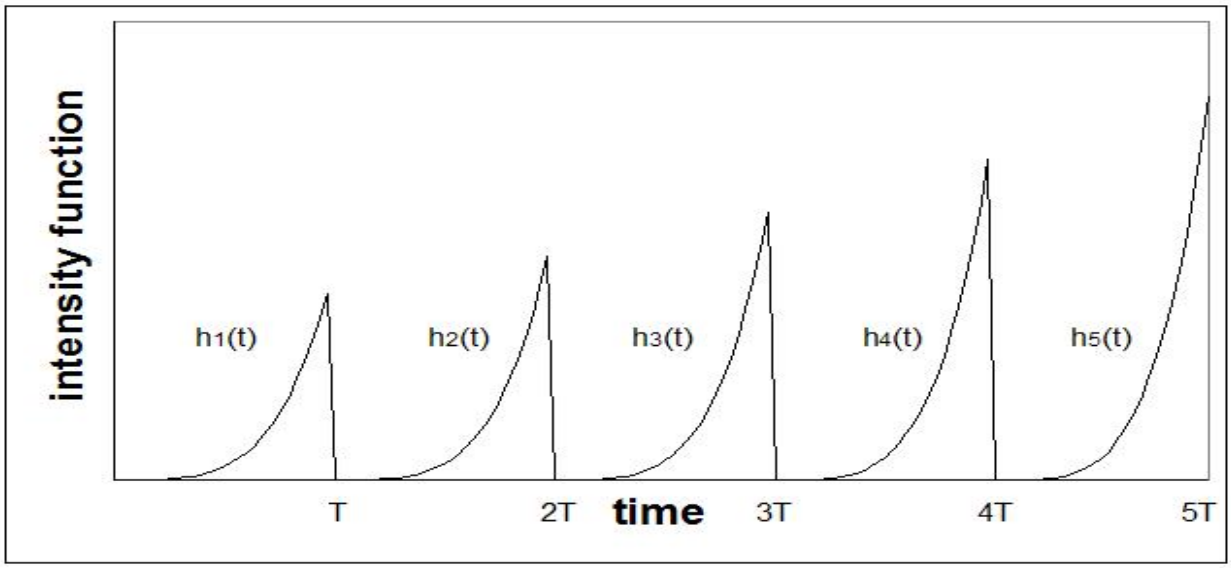

Figure 3: Intensity Function in the First Five Cycles.

5 NUMERICAL EXAMPLES

Numerical examples are discussed in what follows.

Assume the designed life time of a system is 5 years when the default duration of the up state is 16 hours per day, i.e., $T_L = 29,200$(hours). Let $F_{a,1}(t) = F_{b,1}(t) = F_{c,1}(t) = 1-e^{(-t/5,000)^2}$. The repair time distribution of CM activities is a uniform distribution $G_r(t) = \begin{cases} t/T_d & 0 \le t \le T_d \\ 0 & \text{otherwise} \end{cases}$. In all three models, the expected duration of PM is assumed to be $T_d/2$.

5.1 Model A case

Let $F_{a,n}(t) = F_{a,1}(\alpha_a^{n-1}t)$, with $\alpha_a = 1.08$, and $n = 1, 2, ...$ in Model A; and assume $T_d = 24 - T_u$. If the duration of the up state T_u changes from 9 to 18, the change of $C_{a,L}$ is shown in Figure 4. From this figure, the expected cost $C_{a,L}$ increases with the increase of T_u. Therefore, a suggestion from Model A is that the duration of the up state should be as short as possible.

Table 1 presents the numerical results to investigate the pattern changes of T_u & T_d.

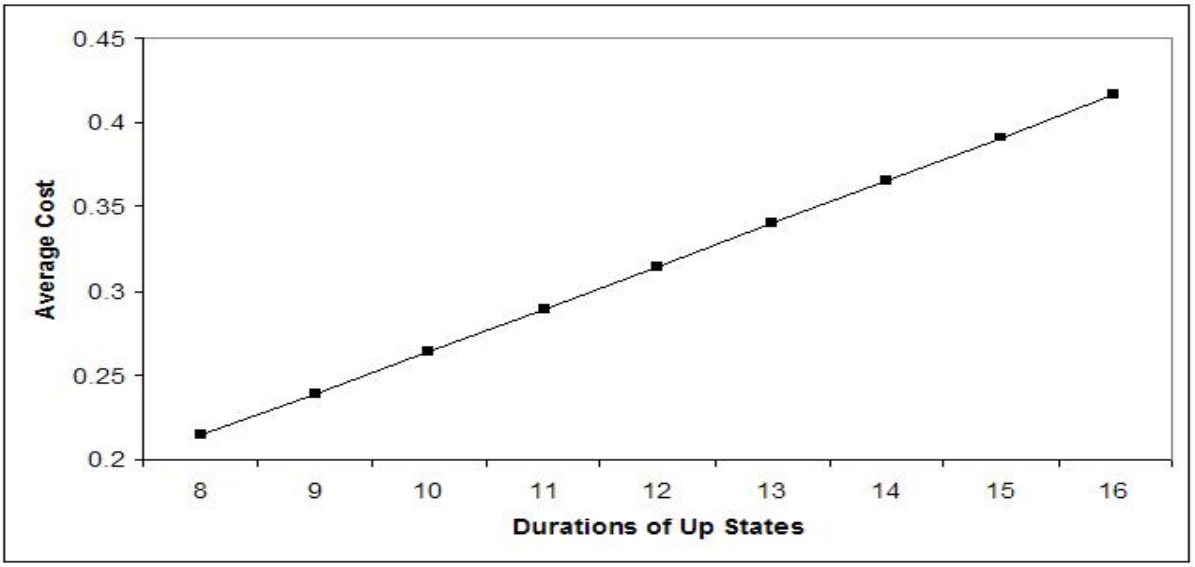

Figure 4: Expected cost of Model A.

Table 1: Average cost of Model A with $T_L = 29200$, $c_b = 200$, $c_c = 5$.

T_u	8	9	10	11	12	13	14	15	16
T_d	16	15	14	13	12	11	10	9	8
C_a	0.2148	0.2393	0.2642	0.2893	0.3146	0.3399	0.3655	0.3911	0.4168

5.2 Model B case

Let $F_{b,n}(t) = F_{b,1}(\beta^{n-1}t)$, and $F_{n,m}(t) = F_{1,1}(\alpha_b^{m-1}\beta^{n-1}t)$, where $\alpha_b = 1.08$, $\beta = 1.05$. Assume that $T_u = 16$, $T_d = 8$, $c_b = 200$, $c_c = 5$, and $\tau_n = \tau_1/\beta^{n-1}$.

Table 2 presents the numerical results to investigate the pattern changes of τ_1, and c_p. From this table, when τ_1 changes from 2,700 to 5,500, the expected cost increases monotonously when $C_p = 10$, and changes "wavelike" in other cases.

Four cases are selected from Table 2 to be shown in Figure 5. The X-axis indicates different values of τ_1, and the Y-axis corresponds to the average cost values based on Equation (17).

For example, from Figure 5, when $c_p = 10$, the minimized expected cost can be found if $\tau_1 = 2700$. When $c_p = 50$, the minimized expected cost can be found if $\tau_1 = 4500$.

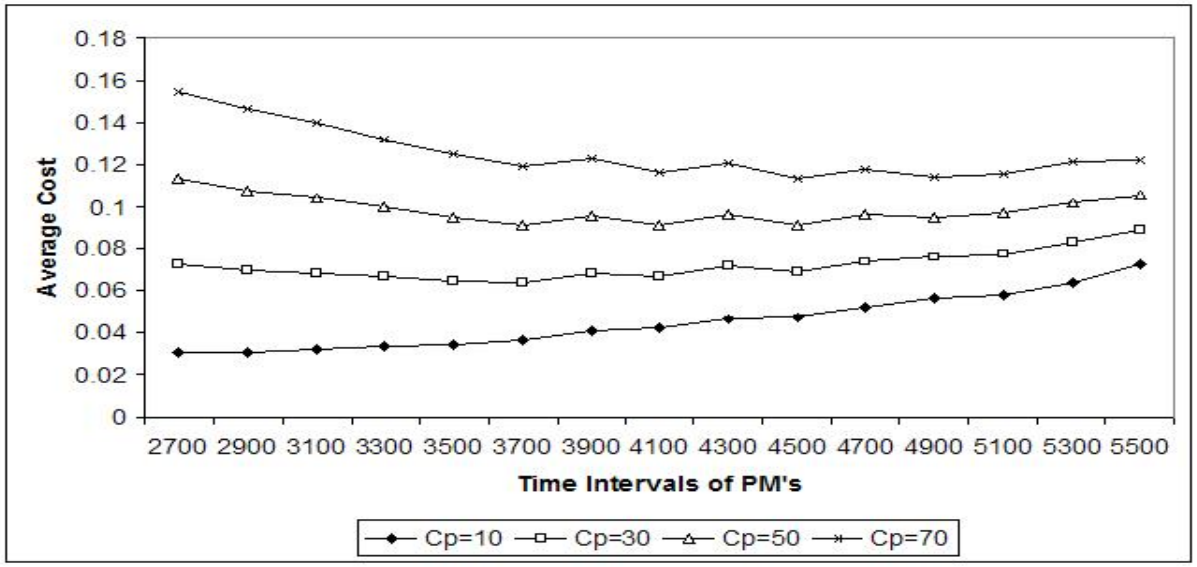

Figure 5: Average cost of Model B, with four cases selected from Table 1.

5.3 Model C case

Let $F_{c,n}(t) = F_{c,1}(\alpha_c^{n-1}t)$, with $\alpha_c = 1.08$, and $n = 1, 2, ...$ in Model C. Assume that $T_u = 16$, $T_d = 8$, $c_b = 200$, and $c_c = 5$.

Table 3 gives the average costs of Model C when τ, and C_p change. For example, when $c_p = 30$, the average cost first decreases from 0.16 to 0.1186 with τ changing from 700 to 1500, then it increases from 0.1186 to 0.3183 when τ changes from 1500 to 4500.

Table 3 indicates that the average cost may decrease with the suitable increase of the time interval of two adjacent PM for a c_p, and it will eventually increase if the time interval is larger than a certain value.

The X-axis indicates different values of τ, and the Y-axis corresponds to the expected cost values based on Equation 25. Four cases are selected from Table 3, to be shown in

Table 2: Average cost of Model B with $T_L = 29200$, $T_u = 16$, $T_d = 8$, $c_b = 200$, $c_c = 5$.

	C_p=10	C_p=20	C_p=30	C_p=40	C_p=50	C_p=60	C_p=70
τ_1=2700	0.031	0.0515	0.0721	0.0926	0.1132	0.1337	0.1543
τ_1=2900	0.0311	0.0503	0.0695	0.0886	0.1078	0.127	0.1462
τ_1=3100	0.0326	0.0504	0.0683	0.0861	0.1039	0.1217	0.1395
τ_1=3300	0.0337	0.0501	0.0666	0.083	0.0994	0.1159	0.1323
τ_1=3500	0.0347	0.0498	0.0649	0.0799	0.095	0.1101	0.1251
τ_1=3700	0.0365	0.0502	0.0639	0.0776	0.0913	0.105	0.1187
τ_1=3900	0.0407	0.0544	0.0681	0.0818	0.0955	0.1092	0.1229
τ_1=4100	0.042	0.0543	0.0666	0.079	0.0913	0.1036	0.116
τ_1=4300	0.0469	0.0592	0.0715	0.0838	0.0962	0.1085	0.1208
τ_1=4500	0.0473	0.0583	0.0692	0.0802	0.0912	0.1021	0.1131
τ_1=4700	0.0521	0.0631	0.074	0.085	0.096	0.1069	0.1179
τ_1=4900	0.0565	0.0661	0.0757	0.0852	0.0948	0.1044	0.114
τ_1=5100	0.0582	0.0678	0.0774	0.087	0.0966	0.1062	0.1158
τ_1=5300	0.0638	0.0734	0.083	0.0926	0.1021	0.1117	0.1213
τ_1=5500	0.0724	0.0806	0.0889	0.0971	0.1053	0.1135	0.1217

Figure 6. For example, when $c_p = 30$, the minimized expected cost can be found if $\tau = 1500$ in Figure 6.

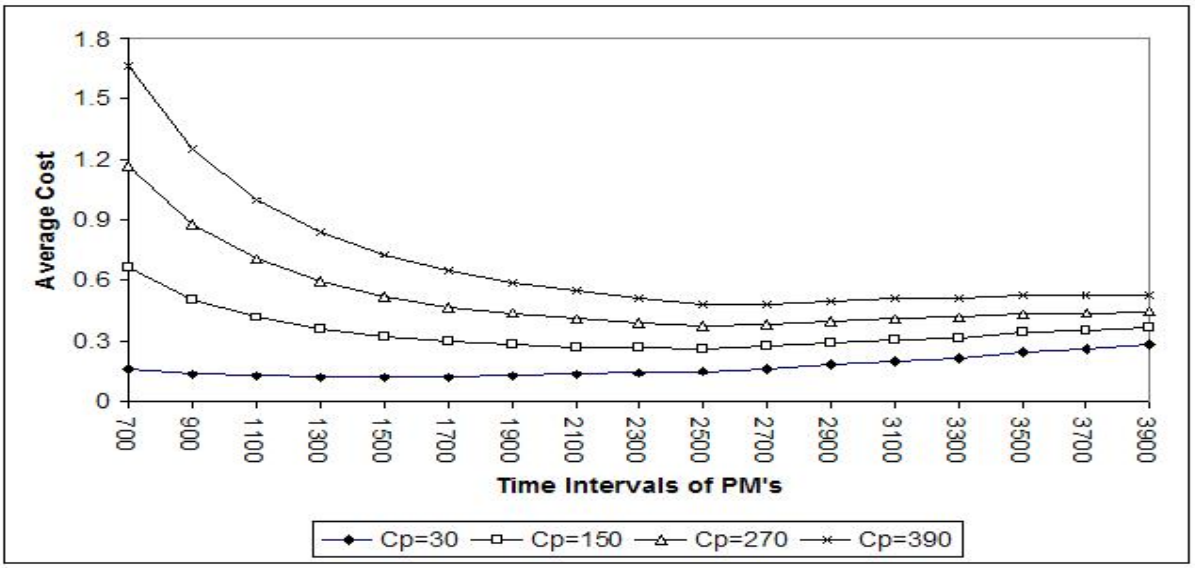

Figure 6: Expected cost of Model C, with four cases selected from Table 2.

The above numerical examples show that optimal PM policies for Model B, and Model C can be obtained by minimizing Equations (17), and (25), respectively.

5.4 Discussions

The above mentioned three models can be applied in accordance with the real situation in practice. If no PM is conducted, the average cost of Model B (or Model C) equals to the

Table 3: Average cost of Model C with $T_L = 29200$, $T_u = 16$, $T_d = 8$, $c_b = 200$, and $c_c = 5$.

	C_p=30	C_p=90	C_p=150	C_p=210	C_p=270	C_p=330	C_p=390	C_p=450
τ= 700	0.16	0.4111	0.6622	0.9134	1.1645	1.4156	1.6667	1.9178
τ= 900	0.1353	0.3206	0.5059	0.6912	0.8765	1.0618	1.247	1.4323
τ= 1100	0.1238	0.2695	0.4152	0.5608	0.7065	0.8522	0.9979	1.1436
τ= 1300	0.1193	0.239	0.3586	0.4782	0.5978	0.7175	0.8371	0.9567
τ= 1500	0.1186	0.2199	0.3213	0.4226	0.524	0.6253	0.7266	0.828
τ= 1700	0.1206	0.2085	0.2963	0.3842	0.4721	0.5599	0.6478	0.7357
τ= 1900	0.1265	0.2039	0.2814	0.3588	0.4362	0.5136	0.5911	0.6685
τ= 2100	0.1327	0.2018	0.2708	0.3399	0.409	0.478	0.5471	0.6162
τ= 2300	0.1403	0.2025	0.2647	0.3269	0.3892	0.4514	0.5136	0.5758
τ= 2500	0.1482	0.2046	0.2609	0.3172	0.3736	0.4299	0.4863	0.5426
τ= 2700	0.164	0.2178	0.2716	0.3254	0.3792	0.433	0.4868	0.5406
τ= 2900	0.1849	0.2375	0.2902	0.3429	0.3955	0.4482	0.5009	0.5536
τ= 3100	0.2002	0.2518	0.3035	0.3552	0.4069	0.4585	0.5102	0.5619
τ= 3300	0.2167	0.266	0.3153	0.3646	0.4139	0.4632	0.5125	0.5618
τ= 3500	0.2459	0.2927	0.3395	0.3864	0.4332	0.48	0.5269	0.5737
τ= 3700	0.2637	0.3075	0.3513	0.3951	0.4388	0.4826	0.5264	0.5702
τ= 3900	0.2827	0.3238	0.365	0.4061	0.4473	0.4884	0.5296	0.5707

average cost of Model A. That is, $C_b(\tau_n) = C_a$ if $\tau_n > T_L$ for $n = 1, 2, ...$ in Model B, and $C_c(\tau) = C_a$ if $\tau > T_L$ in Model C.

Comparing the average cost of Model A to Model B, we can find that costs of Model B in Table 2 is smaller than those of Model A in Table 1. With the increase of C_p, the average cost of Model B becomes larger. Comparing the average cost of Model A to Model C, we get that the average cost of Model C from Table 3 is smaller that those in any case of Model A, when $C_p = 30$ & $\tau \leq 3100$. When C_p becomes larger, the average cost of both Model B & Model C becomes larger. We therefore suggest that PM should be applied when PM cost is small. From Tables 2 & 3, when $C_p = 30$, and the time to first PM changes from 2700 to 3900, the costs of Model B is smaller than those of Model C.

In practice, a maintenance policy can be selected from the three models based on the real situation, and the depth of PM.

6 CONCLUSIONS AND FURTHER WORK

In this paper, we have developed the optimal PM policies for the critical system that operates periodically, and is maintained with higher cost at the up state than at the down state.

The criterion used to determine the optimality of the PM is the average maintenance cost per unit time during the designed life time. Given the cost structures of maintaining the system, we determine the optimal time intervals of PM. We have considered three models: maintenance with no PM, maintenance with sequential PM, and maintenance with periodic

PM. In addition, we investigate the pattern changes of those objective parameters of our interests, such as the average cost, and optimal PM period, by assuming that the failure processes after CM are geometric processes.

Numerical analysis indicates that (1) the increase in the duration of the up state increases the average cost when no PM is performed, and (2) the increase of the time interval of PM may decrease or increase the average cost when PM is performed.

The following achievements have been made in this paper.

(1) The average costs for the three models, Model A, Model B, and Model C, are obtained. Optimal time interval PM activities can therefore be developed by minimizing the average cost of Model B & Model C.

(2) Numerical examples are presented when the failure process is the geometric process.

(3) The relationship between the geometric process, and improvement factors of the imperfect PM is investigated.

The application of geometric process models is rather restricted in the sense that a geometric process model with a fixed parameter θ (see Definition 2) can only describe a system in which the hazard rate function decreases or increases exponentially with the number of repairs. Further work may focus on the application of non-homogeneous Poisson processes following different laws to optimize maintenance policies under the consideration of maintenance costs within different time periods.

APPENDIX

Proof of Lemma 1

$$
\begin{aligned}
H_{a,n}(t) &= \Pr\{U_{a,n} < t\} \\
&= \sum_{j=1}^{\infty} \Pr\{jT_u - X_{a,n} < t | (j-1)T_u \le X_{a,n} < jT_u) P((j-1)T_u \le X_{a,n} < jT_u\} \\
&= \sum_{j=1}^{\infty} \Pr\{(jT_u - t < X_{a,n}) \bigcap ((j-1)T_u \le X_{a,n} < jT_u)\} \\
&= \sum_{j=1}^{\infty} \Pr\{jT_u - t < X_{a,n} < jT_u\} \\
&= \sum_{j=1}^{\infty} (F_{a,n}(jT_u) - F_{a,n}(jT_u - t)) \qquad (27) \\
Q_{a,n}(t) &= \Pr\{V_{a,n} < t)\} \\
&= \sum_{j=1}^{\infty} \Pr\{jT_u \le X_{a,n} + Y_r < jT_u + t | (j-1)T_u \le X_{a,n} < jT_u) P((j-1)T_u \le X_{a,n} < jT_u\} \\
&= \sum_{j=1}^{\infty} \Pr\{(jT_u - X_{a,n} \le Y_r < jT_u + t - X_{a,n}) \bigcap ((j-1)T_u \le X_{a,n} < jT_u)\}
\end{aligned}
$$

$$= \sum_{j=1}^{\infty} \int_{(j-1)T_u}^{jT_u} \left(G_r(jT_u - x + t) - G_r(jT_u - x)\right) dF_{a,n}(x) \tag{28}$$

The proof is completed.

Proof of Lemma 2

$$\begin{aligned}
\upsilon_{a,n} &= \int_0^{\infty} t dH_{a,n}(t) \\
&= -\sum_{j=1}^{\infty} \left(\int_0^{T_u} t dF_{a,n}(jT_u - t) \right) \\
&= \sum_{j=1}^{\infty} \left(\int_{(j-1)T_u}^{jT_u} (jT_u - x) dF_{a,n}(x) \right) \\
&= \sum_{j=1}^{\infty} \int_{(j-1)T_u}^{jT_u} jT_u dF_{a,n}(x) - \sum_{j=1}^{\infty} \left(\int_{(j-1)T_u}^{jT_u} x dF_{a,n}(x) \right) \\
&= T_u \sum_{j=1}^{\infty} j(F_{a,n}(jT_u) - F_{a,n}((j-1)T_u)) - \lambda_{a,n}
\end{aligned} \tag{29}$$

When $(j-1)T_u < x < jT_u$, then $jT_u - x > 0$, and $T_d > jT_u - x$

$$\begin{aligned}
\varphi_{a,n} &= \int_0^{\infty} t dQ_{a,n}(t) \\
&= \sum_{j=1}^{\infty} \left[\int_{(j-1)T_u}^{jT_u} \left(\int_0^{T_d} t dG(jT_u - x + t) \right) dF_{a,n}(x) \right] \\
&= \sum_{j=1}^{\infty} \left[\int_{(j-1)T_u}^{jT_u} \left(\int_{jT_u - x}^{jT_u - x + T_d} (y + x - jT_u) dG(y) \right) dF_{a,n}(x) \right] \\
&= \sum_{j=1}^{\infty} \left[\int_{jT_u - T_d}^{jT_u} \left(\int_{jT_u - x}^{T_d} (y + x - jT_u) dG(y) \right) dF_{a,n}(x) \right] \\
&= \sum_{j=1}^{\infty} \left[\int_{jT_u - T_d}^{jT_u} \left(\int_{jT_u - x}^{T_d} y dG(y) + (x - jT_u)(G(T_d) - G(jT_u - x)) \right) dF_{a,n}(x) \right] \\
&= \sum_{j=1}^{\infty} \left[\int_{jT_u - T_d}^{jT_u} \left(\int_{jT_u - x}^{T_d} y dG(y) + xG(T_d) - xG(jT_u - x) + jT_u G(jT_u - x) \right) dF_{a,n}(x) \right] \\
&\quad - \sum_{j=1}^{\infty} (jT_u G(T_d)(F_{a,n}(jT_u) - F_{a,n}(jT_u - T_d)))
\end{aligned} \tag{30}$$

This completes the proof.

Proof of Lemma 3

From the definition of W_a, it follows

$$W_a = T_L + \sum_{n=1}^{K(T_L)} (U_{a,n} + V_{a,n}) \tag{31}$$

where $K(T_L)$ is the number of failures within time interval $(0, T_L)$. $K(T_L)$ is a random variable. Because

$$\Pr\{K(T_L) \geq k) = P(\sum_{n=1}^{k} X_{a,n} < T_L\} = F_a^{(k)}(T_L) \tag{32}$$

the expected value of W_a is

$$\begin{aligned}
E[W_a] &= T_L + E\left[\sum_{n=1}^{K(T_L)} (U_{a,n} + V_{a,n})\right] \\
&= T_L + E\left[E\left(\sum_{n=1}^{K(T_L)} (U_{a,n} + V_{a,n})|K\right)\right] \\
&= T_L + \sum_{k=1}^{\infty}\left(\Pr\{K(T_L) = k\} \sum_{n=1}^{k} E[U_{a,n} + V_{a,n})\right) \\
&= T_L + \sum_{k=1}^{\infty}\left((\Pr\{\sum_{n=1}^{k} X_{a,n} < T_L\} - \Pr\{\sum_{n=1}^{k+1} X_{a,n} < T_L)\} \sum_{n=1}^{k} (\upsilon_{a,n} + \varphi_{a,n})\right) \\
&= T_L + \sum_{k=1}^{\infty}\left((F_a^{(k)}(T_L) - F_a^{(k+1)}(T_L)) \sum_{n=1}^{k} (\upsilon_{a,n} + \varphi_{a,n})\right)
\end{aligned} \tag{33}$$

The costs due to failure at the up state are $\sum_{k=1}^{\infty}\left((F_a^{(k)}(T_L) - F_a^{(k+1)}(T_L)) \sum_{n=1}^{k} (c_b + c_r)\upsilon_{a,n}\right)$, and the costs due to failure at the down state are $\sum_{k=1}^{\infty}\left((F_a^{(k)}(T_L) - F_a^{(k+1)}(T_L)) \sum_{n=1}^{k} c_r\varphi_{a,n}\right)$. By adding these costs together, we can obtain Equation (8).

This completes the proof.

Proofs of other lemmas & theorems All of the remaining lemmas & theorems can be obtained by following the methods as in Lemmas 1, 2, & 3.

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