Joint importance of multistate systems

Shaomin Wu[1]

School of Construction Management and Engineering,

The University of Reading, Whiteknights, Reading, RG6 6AW, UK

[1]E-mail: s.m.wu@kent.ac.uk.

Suggested citation: Wu, S. (2005). Joint importance of multistate systems. Computers & Industrial Engineering, 49(1), 63-75.

Joint importance of multistate systems

Abstract. Importance measures in reliability engineering are used to identify weak areas of a system and signify the roles of components in either causing or contributing to proper functioning of the system. Traditional importance measures for multistate systems mainly concern reliability importance of an individual component and seldom consider the utility performance of the system. This paper extends the joint importance concepts of two components from the binary system case to the multistate system case. Considering the performance utility of the system, the joint structural importance and the joint reliability importance are defined. The joint structural importance measures the relationship of two components when the reliabilities of components are not available. The joint reliability importance is then inferred when the reliabilities of the components are given. The properties of the importance measures are also investigated. A case study for an offshore electrical power generation system is finally given.

Key words: Birnbaum importance, structural importance, joint structural importance, joint reliability importance, multistate system

1. Introduction

Component importance measures are used to measure the effect of the reliability of individual components on the system reliability. From the design view-point, it is crucial to identify the weaknesses of the system and how the failure of each individual component affects proper functioning of the system; so that efforts can be spent properly to improve the system reliability (Elsayed, 1996).

A wide range of importance measures have been introduced (Andrews & Moss, 2002; Armstrong, 1995; Hong & Lie, 1993; Meng, 1995; Meng, 1996) for binary systems since Birnbaum's work (Birnbaum, 1969). For example, the structural importance is used to measure the topological importance of nodes (or components in the nodes) in the systems (Meng, 1995; Meng, 1996); the Birnbaum reliability importance measures the effect of an improvement in component reliability on system reliability; and the joint importance was introduced to measure how two components in a system interact in contribution to the system reliability (Armstrong, 1995; Armstrong, 1997; Hong & Lie, 1993). The various importance measures can be categorized into either reliability importance or structural importance. The reader is referred to Andrews & Moss (2002) for more discussions on different importance measures.

A binary system is formed from 2-value logic, e.g., functioning/not-functioning. Although such a system has many practical applications, a model based on dichotomizing the system states is often over-simplified and insufficient for describing many commonly encountered situations in real life. As a result, multi-state systems are frequently required. In multi-state systems, the components and/or the system performance have more than 2 states. There are numerous examples of multi-state systems, with more than 2 ordered or unordered states at the system level, the sub-system level, or the component level. For example, a power generator that has 3 states that correspond to supplying electricity of 0, 25MW, 50MW is an example of a multi-state system that has ordered multiple states (Natvig, et. al., 1986).

Research on the importance measure for multistate systems has been conducted by authors (Aven, 1993; Block, 1982; Bueno, 1989; El-Neweihi, Proschan & Sethuraman, 1978; Gandidni, 1990; Griffith, 1980; Meng, 1993; Natvig, 1982; Wu & Chan, 2003), who mainly focus on how to extend the reliability importance of individual components from the binary system case to the multistate system case. Little research has been done on investigating the joint importance of components (or states) for multistate systems based on the utility performance of the system. However, the joint importance provides additional information, which the traditional marginal reliability importance cannot provide, to system designers (Hong, Koo & Lie, 1993). And the performance utility is vitally important from a cost-saving view-point. Investigating the joint importance in multistate systems is therefore important and helpful in practice.

In this paper, the joint importance measures, including the joint structural importance and the joint reliability importance, for multistate systems are introduced. They are defined based on the performance utility function of the system. The joint reliability importance measures how two components (or states) in the system interact in contribution to the system performance utility considering component reliabilities, and the joint structural importance measures how two components interact each other when the reliabilities of the components are not available.

The paper is structured as follows. Assumptions are given in the next section. Section 3 introduces the joint the structural importance for multistate systems. Section 4 introduces the joint reliability importance and its properties are also discussed. Section 5 discusses some special cases of the joint importance measures. Section 6 presents a case study to illustrate the definitions introduced. Section 7 concludes this paper.

2. Assumptions and definitions

Assume the n-component multistate system under study is monotone, i.e., the improvement of any component does not degrade the system's state, and the components are mutually state independent. Assume that a component can only degrade one state each time (that is, from state i to i-1 rather than i-k, where i-k>1), whereas the system may degrade more than one state. For simplicity, assume that components and the system have M+1 states: 0, 1, …,M.

The following two definitions are also used in this paper.

- Multistate series (or parallel) system (El-Neweihi, Proschan & Sethuraman, 1978): if $\phi(S) = \min\{s_1, s_2,..., s_n\}$ (or $\phi(S) = \max\{s_1, s_2,..., s_n\}$), then the system is a multistate series (or parallel) system, where $\phi(\bullet)$ is a non-decreasing structure function of the system; $S = \{s_1, s_2,..., s_n\}$, and s_i is a state of component i ($0 \le s_i \le M$).
- Performance utility function (Griffith, 1980): $U_S = \sum_{j=1}^{M} a_j \Pr[\phi(\mathbf{X}) = j]$ is called the performance utility function of the system, where $\mathbf{x} = (x_1,..., x_n)$, and a_i is the utility level of the system when it is at state i, and x_i is a random variable representing the state of component i ($i = 0,1,..., M$ and $a_M > a_{M-1} > \cdots > a_1 > a_0 = 0$).

3. Joint structural importance

Hong & Lie (1993) introduced the concept of the joint reliability importance (JRI) of two components in binary systems as follows:

$$\mathrm{JRI}(i,l) = \frac{\partial^2 h(\mathbf{p})}{\partial p_i \partial p_l}$$

where $h(\bullet)$ is the reliability function of the system with $h(\mathbf{p}) = E[\phi(\mathbf{x})]$, p_i and p_j are reliabilities of component i and component l, respectively. Armstrong (1995) gave the following definition when components in the system are s-independent:

$$JRI(i,l) = h(1_i, 1_l, \mathbf{p}_{il}) + h(0_i, 0_l, \mathbf{p}_{il}) - h(1_i, 0_l, \mathbf{p}_{il}) - h(0_i, 1_l, \mathbf{p}_{il}) \quad (1)$$

where $\mathbf{p}_{il} = (p_1, ..., p_{i-1}, \bullet, p_{i+1}, ..., p_{l-1}, \bullet, p_{l+1}, ..., p_n)$. JRI indicates that a component is more or less important or has the same importance when the other is functioning. As the Birnbaum reliability importance of component i is defined as $h(1_i, \mathbf{p}_i) - h(0_i, \mathbf{p}_i)$, JRI(i,l) can also be interpreted as the change of the Birnbaum reliability importance of component i caused by component l's deteriorating from state 1 to state 0.

The structural importance measures the topological importance of a component when its reliability is not taken into account.

In binary systems, a component is critical in a state vector if its failure will cause system failure. The structural importance of component i in binary systems (shortly, SIB) is obtained from the following index:

$$SIB(i) = \frac{1}{2^{n-1}}(\text{total number of state vectors where component i is critical})$$

$$= \frac{1}{2^{n-1}} \sum_{S_i} \{\phi(1_i, S_i) - \phi(0_i, S_i)\} \quad (2)$$

where $S_i = (s_1, ..., s_{i-1}, \bullet, s_{i+1}, ..., s_n)$.

Similarly, the structural importance of the multistate systems can be defined. In a multistate system, a critical path vector including state m of component i when the system is at state j can be defined as a vector (m_i, S_i) such that $\phi(m_i, S_i) = j$ for $\phi((m-1)_i, S_i) \leq j-1$. As the structural importance of a state of a component depends on the number of critical path vectors including the state of the component, the structural importance of state m of component i when the system is at state j can be defined as follows.

$$\frac{1}{(M+1)^{n-1}} \sum_{S_i} \chi((\phi(m_i, S_i) = j) \cap (\phi((m-1)_i, S_i) \leq j-1))$$

$$= \frac{1}{(M+1)^{n-1}} \sum_{q=1}^{j} \sum_{S_i} \chi\{(\phi(m_i, S_i) = j) \cap (\phi((m-1)_i, S_i) = j-q)\}$$

where $(M+1)^{n-1}$ is the total number of component state vectors with component i in state m, and $\chi(\bullet)$ is an indicator function satisfying $\chi(\text{true})=1$ and $\chi(\text{false})=0$.

Griffith (1980) gave the following performance utility function of a multistate system:

$$U_S = \sum_{j=1}^{M} a_j \Pr[\phi(\mathbf{x}) = j] \quad (3)$$

Further discussion about Griffith's definition can be found in Wu & Chan (2003).

Since the multistate system is monotone, any change of state of a component may change the performance utility of the system. If the system steps from one state down to another caused by component i's stepping from one state down to another, the impacts on the performance utility of the system may be different. The reduction of the system performance level will be $a_j - a_{j-q}$ if the system steps from state j down to state $j-q$. Thus, we can have the following definition on the structural importance of component i in a multistate system (shortly, SIM).

Definition 1. The structural importance of component i in a multistate system is

$$\mathrm{SIM}(i) = \frac{\sum_{m=1}^{M}\sum_{j=1}^{M}\sum_{q=1}^{j}\sum_{S_i}(a_j - a_{j-q})\chi\{(\phi(m_i, S_i) = j)\cap(\phi((m-1)_i, S_i) = j-q)\}}{(M+1)^{n-1}} \quad (4)$$

Definition 1 defines how topologically important a component is. The larger the SIM(i) is, the more important component i is.

We can infer the definition of the joint structural importance for the multistate system. The following index can be used to define the joint structural importance of component i and component l in the multistate system (shortly, JSIM).

$$\mathrm{JSIM}(i,l) = \sum_{m=1}^{M}\sum_{k=1}^{M}\{\mathrm{SIM}(i,l;m,k) - \mathrm{SIM}(i,l;m,k-1)\} \quad (5)$$

where

$$\mathrm{SIM}(i,l;m,k) = \frac{\sum_{j=1}^{M}\sum_{q=1}^{j}\sum_{S_{il}}(a_j - a_{j-q})\chi\{(\phi(m_i, k_l, S_{il}) = j)\cap(\phi((m-1)_i, k_l, S_{il}) = j-q)\}}{(M+1)^{n-2}} \quad (6)$$

Obviously, $\mathrm{SIM}(i,l;m,k) - \mathrm{SIM}(i,l;m,k-1)$ gives information on how two states of two components interact topologically. Because

$$\sum_{m=1}^{M}\sum_{k=1}^{M}\{\mathrm{SIM}(i,l;m,k) - \mathrm{SIM}(i,l;m,k-1)\} = \sum_{m=1}^{M}\{\mathrm{SIM}(i,l;m,M) - \mathrm{SIM}(i,l;m,0)\}$$

then, the following definition can be obtained.

Definition 2. The joint structural importance of component i and component l in a multistate system (JSIM) is

$$\mathrm{JSIM}(i,l) = \sum_{m=1}^{M}\{\mathrm{SIM}(i,l;m,M) - \mathrm{SIM}(i,l;m,0)\} \quad (7)$$

where $\mathrm{JSIM}(i,l)$ is the sum of the change of the system performance utility caused by component l's stepping from M to 0 while component i changes from state 1 to state M.

$\mathrm{JSIM}(i,l)$ indicates how the topological importance of a component changes when the other is functioning. Here, by the topological importance of a component, we mean the extent of the change of the performance utility if a node is occupied by the component. Namely, when component l is at state M,

(1) $JSIM(i,l) > 0$ (or $JSIM(i,l) < 0$) indicates that component i becomes more (or less) topologically important;

(2) $JSIM(i,l) = 0$ indicates that the topological importance of component i remains unchanged.

4. Joint reliability importance

Section 3 discusses the joint structural importance when the reliabilities of components in the system are not taken into account. In practice, however, it will be more comprehensive and helpful if both the topological structure and the reliabilities of components are taken into account. This section will consider the joint reliability importance based on these two aspects.

Because

$$\begin{aligned}&\Pr[\phi(\mathbf{x}) \ge j]\\ &= \Pr[\phi(0_i,\mathbf{x}_i) \ge j] + \sum_{m=1}^{M} (\Pr[\phi(m_i,\mathbf{x}_i) \ge j] - \Pr[\phi((m-1)_i,\mathbf{x}_i) \ge j]) \Pr[x_i \ge m]\end{aligned}$$

where $\mathbf{x}_i = (x_1,\dots,x_{i-1},\bullet,x_{i+1},\dots,x_n)$. Denote $R_i(m) = \Pr[x_i \ge m]$, we obtain

$$\begin{aligned}
U_s &= \sum_{j=1}^{M} a_j \Pr[\phi(\mathbf{x}) = j]\\
&= \sum_{j=1}^{M} (a_j - a_{j-1}) \Pr[\phi(\mathbf{x}) \ge j]\\
&= \sum_{j=1}^{M} \left\{ b_j \Pr[\phi(0_i,\mathbf{x}_i) \ge j] + b_j \sum_{m=1}^{M} R_i(m) \left(\Pr[\phi(m_i,\mathbf{x}_i) \ge j] - \Pr[\phi((m-1)_i,\mathbf{x}_i) \ge j] \right) \right\}\\
&= \sum_{j=1}^{M} \left\{ b_j \left(\Pr[\phi(0_i,0_l,\mathbf{x}_{il}) \ge j] + \sum_{k=1}^{M} R_l(k)(\Pr[\phi(0_i,k_l,\mathbf{x}_{il}) \ge j] - \Pr[\phi(0_i,(k-1)_l,\mathbf{x}_{il}) \ge j]) \right) \right\}\\
&\quad + \sum_{j=1}^{M} \{ b_j \sum_{m=1}^{M} \Big(R_i(m) \Pr[\phi(m_i,0_l,\mathbf{x}_{il}) \ge j] - \Pr[\phi((m-1)_i,0_l,\mathbf{x}_{il}) \ge j]\\
&\quad + R_i(m) \sum_{k=1}^{M} R_l(k)(\Pr[\phi(m_i,k_l,\mathbf{x}_{il}) \ge j] - \Pr[\phi(m_i,(k-1)_l,\mathbf{x}_{il}) \ge j])\\
&\quad - R_i(m) \sum_{k=1}^{M} R_l(k)(\Pr[\phi((m-1)_i,k_l,\mathbf{x}_{il}) \ge j] + \Pr[\phi((m-1)_i,(k-1)_l,\mathbf{x}_{il}) \ge j]) \Big) \Big\}\\
&= \sum_{j=1}^{M} b_j \Pr[\phi(0_i,0_l,\mathbf{x}_{il}) \ge j]\\
&\quad + \sum_{j=1}^{M} \sum_{m=1}^{M} b_j R_i(m) \left\{ \Pr[\phi(m_i,0_l,\mathbf{x}_{il}) \ge j] - \Pr[\phi((m-1)_i,0_l,\mathbf{x}_{il}) \ge j] \right\}\\
&\quad + \sum_{j=1}^{M} \sum_{k=1}^{M} b_j R_l(k) \{ \Pr[\phi(0_i,k_l,\mathbf{x}_{il}) \ge j] - \Pr[\phi(0_i,(k-1)_l,\mathbf{x}_{il}) \ge j]) \}\\
&\quad + \sum_{j=1}^{M} \sum_{m=1}^{M} \sum_{k=1}^{M} b_j R_i(m) R_l(k) \left\{ \Pr[\phi(m_i,k_l,\mathbf{x}_{il}) \ge j] - \Pr[\phi(m_i,(k-1)_l,\mathbf{x}_{il}) \ge j] \right.
\end{aligned}$$

$$-\Pr[\phi((m-1)_i, k_l, \mathbf{x}_{il}) \geq j] + \Pr[\phi((m-1)_i, (k-1)_l, \mathbf{x}_{il}) \geq j]\} \quad (8)$$

where $b_j = a_j - a_{j-1}$ and $R_l(k) = \Pr[x_l \geq k]$. Then

$$\frac{\partial^2 U_s}{\partial R_i(m) \partial R_l(k)} = \sum_{j=1}^{M} (a_j - a_{j-1}) \{\Pr[\phi(m_i, k_l, \mathbf{x}_{il}) \geq j] - \Pr[\phi(m_i, (k-1)_l, \mathbf{x}_{il}) \geq j]$$

$$-\Pr[\phi((m-1)_i, k_l, \mathbf{x}_{il}) \geq j] + \Pr[\phi((m-1)_i, (k-1)_l, \mathbf{x}_{il}) \geq j]\} \quad (9)$$

Just as the definition of the joint reliability importance for a binary system was introduced by Hong & Lie (1993), so one can propose the following definition to measure the joint reliability importance of two states of two components in a multistate system.

Definition 3. The joint reliability importance of state m of component i and state k of component l of a multistate system (shortly, JPIM) is

$$\mathrm{JPIM}(i, l; m, k) = \frac{\partial^2 U_s}{\partial R_i(m) \partial R_l(k)} \quad (10)$$

Griffith (1980) discussed the concept of reliability importance of component i in a multistate system. He proposed the following importance vector for component i (called as Griffith importance hereafter):

$$I_i = (I_i^G(1), \ldots, I_i^G(M))$$

where $I_i^G(m) = \sum_{j=1}^{M} (a_j - a_{j-1})(\Pr[\phi(m_i, \mathbf{x}_i) \geq j] - \Pr[\phi((m-1)_i, \mathbf{x}_i) \geq j])$

$I_i^G(m)$ can be interpreted as the change of the performance utility of the system caused by component i's deteriorating from state m to state $m-1$, and it can be called the importance of state m of component i. Similarly, $\mathrm{JPIM}(i, l; m, k)$ in Eq. (10) can be regarded as the change of the importance of state m of component i caused by component l's deteriorating from state k to state $k-1$, namely,

$$\mathrm{JPIM}(i, l; m, k) = I_i^G(m \mid k_l) - I_i^G(m \mid (k-1)_l) \quad (11)$$

where $I_i^G(m \mid k_l) = \sum_{j=1}^{M} (a_j - a_{j-1})(\Pr[\phi(m_i, k_l, \mathbf{x}_{il}) \geq j] - \Pr[\phi((m-1)_i, k_l, \mathbf{x}_{il}) \geq j])$.

Eq. (11) indicates the relationship between the joint reliability importance of two states and the Griffith importance of a state.

Definition 4. The joint reliability importance of component i and component l in a multistate system is

$$\mathrm{JPIM}(i, l) = \sum_{m=1}^{M} \sum_{k=1}^{M} \mathrm{JPIM}(i, l; m, k) \quad (12)$$

Lemma 1.

$$\mathrm{JPIM}(i,l)=\sum_{j=1}^{M}(a_j-a_{j-1})\{\Pr[\phi(M_i,M_l,\mathbf{x}_{il})\geq j]-\Pr[\phi(0_i,M_l,\mathbf{x}_{il})\geq j]$$

$$-\Pr[\phi(M_i,0_l,\mathbf{x}_{il})\geq j]+\Pr[\phi(0_i,0_l,\mathbf{x}_{il})\geq j]\} \quad (13)$$

Proof. From Eqs. (10) and (12), we have

$$\mathrm{JPIM}(i,l)=\sum_{k=1}^{M}\sum_{m=1}^{M}\mathrm{JPIM}(i,l;m,k)$$

$$=\sum_{j=1}^{M}\sum_{k=1}^{M}\sum_{m=1}^{M}\{(a_j-a_{j-1})\{\Pr[\phi(m_i,k_l,\mathbf{x}_{il})\geq j]-\Pr[\phi(m_i,(k-1)_l,\mathbf{x}_{il})\geq j]$$

$$-\Pr[\phi((m-1)_i,k_l,\mathbf{x}_{il})\geq j]+\Pr[\phi((m-1)_i,(k-1)_l,\mathbf{x}_{il})\geq j]\}\}$$

$$=\sum_{j=1}^{M}(a_j-a_{j-1})\{\Pr[\phi(M_i,M_l,\mathbf{x}_{il})\geq j]-\Pr[\phi(0_i,M_l,\mathbf{x}_{il})\geq j]$$

$$-\Pr[\phi(M_i,0_l,\mathbf{x}_{il})\geq j]+\Pr[\phi(0_i,0_l,\mathbf{x}_{il})\geq j]\} \quad (14)$$

Then we obtain the lemma. □

JPIM(i,l) indicates how the performance utility of the system changes with the change of states of a component when the other is functioning. Namely, when component l is at state M,

(1) JPIM(i,l)>0 (JPIM(i,l)<0) indicates that component i becomes more (less) important;

(2) JPIM(i,l)=0 indicates that both the performance utility of the system and the importance of component i remain unchanged.

Theorem 1. The joint reliability importance of state m of component i and state k of component l in a series system is

$$\mathrm{JPIM}(i,l;m,k)=\begin{cases}0 & \text{for } k\neq m\\ (a_m-a_{m-1})\prod_{\substack{s=1\\ s\neq i,s\neq l}}^{n}R_s(m) & \text{for } k=m\end{cases} \quad (15)$$

where $R_s(m)=\Pr\{x_s>m\}$.

Proof. By the definition of a series system, $\phi(S)=\min\{s_1,s_2,...,s_n\}$. Let $s_i=m-1$ or m, and $s_l=k-1$ or k. When $m=k=j$, $\Pr[\phi(s_i,s_l,\mathbf{x}_{il})\geq j]=0$ if either $s_i=m-1$ or $s_l=m-1$, and

$$\Pr[\phi(m_i,k_l,\mathbf{x}_{il})\geq j]=\Pr[\phi(\mathbf{x}_{il})\geq j]=\prod_{\substack{s=1\\ s\neq i,s\neq l}}^{n}R_s(m). \quad (16)$$

When $m\neq k$, there are 4 cases based on Eq. (9). All of the following four cases imply that $\mathrm{JPIM}(i,l;m,k)=0$.

1. If $m>j$ and $k>j$, then $\Pr[\phi(s_i,s_l,\mathbf{x}_{il})\geq j]=\Pr[\min\{\mathbf{x}_{il}\}\geq j]$.

2. If $m<j$ or $k<j$, then $\Pr[\phi(s_i,s_l,\mathbf{x}_{il})\geq j]=0$.

3. If $m > j$ and $k = j$, then $\Pr[\phi(m_i, k_l, \mathbf{x}_{il}) \geq j] = \Pr[\phi((m-1)_i, k_l, \mathbf{x}_{il}) \geq j]$ and $\Pr[\phi(m_i, (k-1)_l, \mathbf{x}_{il}) \geq j] = \Pr[\phi((m-1)_i, (k-1)_l, \mathbf{x}_{il}) \geq j]$.

4. If $m = j$ and $k > j$, then $\Pr[\phi(m_i, k_l, \mathbf{x}_{il}) \geq j] = \Pr[\phi(m_i, (k-1)_l, \mathbf{x}_{il}) \geq j]$ and $\Pr[\phi((m-1)_i, k_l, \mathbf{x}_{il}) \geq j] = \Pr[\phi((m-1)_i, (k-1)_l, \mathbf{x}_{il}) \geq j]$.

This obtains Theorem 9. □

Similarly, the following theorem can be proved.

Theorem 2. The joint reliability importance of state m of component i and state k of component l in a parallel system is

$$\mathrm{JPIM}(i,l;m,k) = \begin{cases} 0 & \text{for } k \neq m \\ -(a_m - a_{m-1}) \prod_{\substack{s=1 \\ s\neq i, s\neq l}}^{n} (1 - R_s(m)) & \text{for } k = m \end{cases} \tag{16}$$

5. Discussion

One of the assumptions in the paper is that all components and the system have the same number of states here. If this assumption does not hold, the joint structural importance measures and the joint reliability importance can be extended after the number of states M+1 has replaced by the corresponding numbers of states.

Another assumption in the paper is that only one-state degradation for components is allowed. If a component can degrade over two states (e.g., from state i to state i-k with k>1), the definitions would become very complicated but can be defined similarly.

When M =1, a multistate system becomes a binary system and the following lemmas hold.

Lemma 2. If M=1, $a_1 = 1$ and $a_0 = 0$, then SIM(i)=SIB(i), and JSIM(i,l)=JRI(i,l).

Proof. It is easy to obtain SIM(i)=SIB(i) from Eqs. (2) and (4); JSIM(i,l)=JRI(i,l) from Eqs. (1) and (7). □

On the other hand, if the state occupancy probabilities of the components are the same, Lemma 2 holds.

Lemma 3. If $\Pr[x_i = j] = \frac{1}{M+1}$, then $\mathrm{JPIM}(i,l;m,k) \leq \mathrm{JSIM}(i,l;m,k)$ for $i,l = 1,\ldots,n$ and $j = 1,\ldots,M$.

Proof. Let A and B be two sets, then $\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$ and $\Pr(B) = 1 - \Pr(\bar{B})$. We have

$$\Pr[\phi(m_i, k_l, \mathbf{x}_{il}) \geq j] - \Pr[\phi((m-1)_i, k_l, \mathbf{x}_{il}) \geq j]$$
$$= \Pr[\phi(m_i, k_l, \mathbf{x}_{il}) \geq j] + \Pr[\phi((m-1)_i, k_l, \mathbf{x}_{il}) < j] - 1$$
$$= \Pr[(\phi(m_i, k_l, \mathbf{x}_{il}) \geq j) \cup (\phi((m-1)_i, k_l, \mathbf{x}_{il}) < j)]$$

$$+\Pr[(\phi(m_i,k_l,\mathbf{x}_{il})\ge j)\cap(\phi((m-1)_i,k_l,\mathbf{x}_{il})<j)]-1$$

$$<\Pr[(\phi(m_i,k_l,\mathbf{x}_{il})\ge j)\cap(\phi((m-1)_i,k_l,\mathbf{x}_{il})<j)]$$

$$=\frac{1}{(M+1)^{n-2}}\sum_{S_{il}}\chi[(\phi(m_i,k_l,S_{il})\ge j)\cap(\phi((m-1)_i,k_l,S_{il})<j)]$$

Therefore, from Eq. (6), we have

$$\sum_{j=1}^{M}\{\Pr[\phi(m_i,k_l,\mathbf{x}_{il})\ge j]-\Pr[\phi((m-1)_i,k_l,\mathbf{x}_{il})\ge j]\}$$

$$\le\frac{1}{(M+1)^{n-2}}\sum_{j=1}^{M}\{\sum_{S_{il}}\chi[(\phi(m_i,k_l,S_{il})\ge j)\cap(\phi((m-1)_i,k_l,S_{il})<j)]\}$$

$$=\mathrm{SIM}(i,l;m,k) \tag{17}$$

Similarly,

$$\sum_{j=1}^{M}\{\Pr[\phi(m_i,(k-1)_l,\mathbf{x}_{il})\ge j]-\Pr[\phi((m-1)_i,(k-1)_l,\mathbf{x}_{il})\ge j]\}$$

$$\le\mathrm{SIM}(i,l;m,k-1) \tag{18}$$

Therefore, from Definitions 2 and 4, we have $\mathrm{JPIM}(i,l)\le\mathrm{JSIM}(i,l)$.□

Lemma 3 shows the relationship between the joint structural importance and the joint reliability importance.

6. A case study

Natvig, et. al. (1986) considered an offshore electrical power generation system (see Figure 1). The amount of power that can be supplied to platform 1 depends on three components: control unit U, generator A_1, and standby generator A_2. The components have three states 0, 2, 4 in Natvig, et. al. (1986), for simplicity, we denote it by S={0,1,2}. The meaning of the states of the components is as in Table 1. The reliabilities of the components are in Table 2. The structure function $\phi_1:\{0,1,2\}^3\to\{0,1,2\}$ is

$$\phi_1(\mathbf{x})=\chi(U>0)\min\{A_1+A_2\chi(U=2),2\}$$

For our use, assume that $a_0=0$, $a_1=1000$, $a_2=2000$. We have results shown in Tables 3, 4, 5, 6, where $I^{P}_{ilmk}=\mathrm{JPIM}(i,l;m,k)$ in Table 6.

Table 3 shows the structural importance of a component based on Definition 1. It shows that, topologically, the control unit U is more important than the generator A_1 and A_2, whereas the generator A_1 is more important than the generator A_2. This result is true because the control unit U controls both A_1 and A_2, whereas the generator A_2 is a standby component.

Table 4 shows the joint structural importance of two components based on Definition 2.

$\mathrm{JSIM}(U,A_1)>0$ means that the control unit U becomes topologically more important when the generator A_1 is functioning. $\text{JSIM}(A_2, A_1) < 0$ means the generator A_2 becomes topologically less important when the generator A_1 is functioning. As A_2 is a standby generator, it is less important when A_1 is functioning.

Table 5 shows the joint reliability importance of states of components based on Definition 3. $I^P_{UA_111}$ is the largest and positive, which indicates state 1 of the control unit U becomes much more important when generator A_1 steps from state 1 down to state 0. The value of I^P_{2312} is the smallest, which indicates state 1 of the generator A_1 becomes much less important when the generator A_2 steps from state 2 down to state 1.

From Table 6, any component is more important when another is functioning. This result is different from the analysis results from Table 4 because the state occupancy probabilities have been taken into account in Table 6.

Obviously, as Table 5 and Table 6 are obtained based on reliabilities of components, any change of reliabilities of the components may result in the change of the joint reliability importance. If the utility levels a_i changes, the relationship between the components and all results in Tables 3, 4, 5, 6 may change.

The joint importance measures can hence be an objective function subjected to optional utility levels and reliabilities, which can lead to an optimal design considering the trade-off between reliability and performance utility perspective.

7. Conclusions

Importance measures for multistate systems are more complex than binary systems. They can be defined from different perspectives. The performance utility of a system can provide engineers with information that can be associated with costs such as purchase cost, business losses and operating costs; importance measures defined on the basis of the performance utility are therefore vitally important in practice.

At the reliability design phase, the joint importance can improve system designer's understanding of the relationships between components, components and the system, which is very helpful.

This paper introduced two novel importance measures for multistate systems on the basis of the performance utilities. The joint structural importance and the joint reliability importance were extended from the binary system case to the multistate system case. The relationship between the joint structural importance and the joint reliability importance were discussed.

The case study shows the application of the definitions to a simple offshore electrical power generation system. The joint importance proposed in the paper can not only be applied to more complex systems but also offer precise measures on the relationship between components,

components and systems.

Some future research topics in this direction include further investigation of the importance measure for the multistate systems and the importance measures for the continuum reliability systems.

Acknowledgments

The authors would like to thank the referees and the editor for their constructive suggestions for improving the paper.

References

Andrews, J. D. & Moss, T. R. (1996). Reliability and risk assessment. London: Professional Engineering Publishing.

Armstrong, M. (1995). Joint reliability importance of components. IEEE Transactions on Reliability, 44(3), 408-412.

Armstrong, M. (1997). Reliability importance and dual failure-mode components. IEEE Transactions on Reliability, 46(2), 212-221.

Aven, T. (1993). On performance measures for multistate monotone systems. Reliability Engineering & System Safety, 41(3), 259-266.

Birnbaum, Z. W. (1969). On the importance of different components in a multicomponent systems. In: P. R. Krishnaiah, Multivariate analysis II (pp. 581-592). New York: Academic Press.

Block, H. W. (1982). A decomposition for multistate monotone system. Journal of Applied Probability, 19, 391-402.

Bueno, V. C. (1989). On the importance of components for multistate monotone systems. Statistics & Probability Letters, 7, 51-59.

El-Neweihi, E., Proschan, F. & Sethuraman. J. (1978). Multistate coherent systems, Journal of Applied Probability, 15, 675-688.

Elsayed, E. A. (1996). Reliability engineering. New York: Addison Wesley Longman.

Gandini, A. (1990). Importance and sensitivity analysis in assessing system reliability. IEEE Transactions on Reliability, 39(1), 61-70.

Griffith, W. S. (1980). Multistate reliability models. Journal of Applied Probability, 17, 735-744.

Hong, J. S. & Lie, C. H. (1993). Joint reliability importance of two edges in an undirected network. IEEE Transactions on Reliability, 42(1), 17-23.

Hong, J. S., Koo, H. Y. & Lie, C. H. (2002). Joint reliability importance of k-out-n systems. European Journal of Operational Research, 142, 539-547.

Meng, F. C. (1993). Component relevancy and characterization results in multistate systems. IEEE Transactions on Reliability, 42(3), 478-483.

Meng, F. C. (1995). Some further results on ranking the importance of system components. Reliability Engineering & System Safety, 47, 97-101.

Meng, F. C. (1996). Comparing the importance of system components by some structural characteristics. IEEE Transactions on Reliability, 45(1), 59-65.

Natvig, B. (1982). Two suggestions of how to define a multistate coherent system. Advances in Applied Probability, 14, 434-457.

Natvig, B., Sϕrmo, S., Holen, A. T. & Hϕgasen, G. (1986). Multistate reliability theory — a case study. Advances in Applied Probability, 18, 921-932.

Wu, S. & Chan, L. Y. (2003). Performance utility analysis of multistate systems. IEEE Transactions on Reliability, 52(1), 14-21.

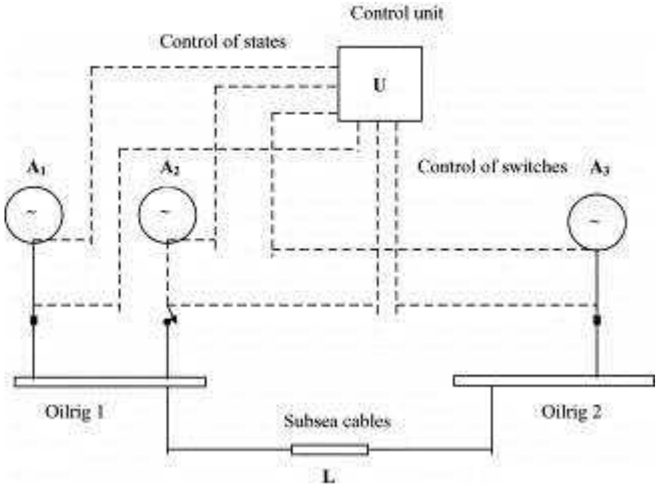

Figure 1. The offshore electrical power generation system

Table 1. Component states

Variable	State		
	0	1	2
U	A_1 & A_2 off	A_2 off	perfect functioning
A_1	supply 0MW	supply maximum 25MW	supply maximum 50MW
A_2	supply 0MW	supply maximum 25MW	supply maximum 50MW

Table 2. Reliabilities of components and the system

Variable	State		
	0	1	2
U	0.182	0.572	0.246
A_1	0.138	0.808	0.054
A_2	0.138	0.808	0.054
System	0.266	0.516	0.218

Table 3. Structural importance of components

SIM(U)	**SIM(A_1)**	**SIM(A_2)**
1555.6	1000	333.06

Table 4. Joint structural importance of components

JSIM(U, A_1)	JSIM(U, A_2)	JSIM(A_2, A_1)
1000	1000	-666.67

Table 5. Joint reliability importance of states

$I^{P}_{UA_1 11}$	$I^{P}_{UA_1 12}$	$I^{P}_{UA_1 22}$	$I^{P}_{UA_2 11}$	$I^{P}_{UA_2 12}$	$I^{P}_{UA_2 22}$	$I^{P}_{A_1 A_2 11}$	$I^{P}_{A_1 A_2 12}$	$I^{P}_{A_1 A_2 22}$
460.56	-399.00	55.46	349.82	-393.68	50.14	388.75	-425.08	153.72

Table 6. Joint reliability importance of components

JPIM(U, A_1)	JPIM(U, A_2)	JPIM(A_1, A_2)
117.02	6.28	117.39