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# Linear and nonlinear preventive maintenance models

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## Abstract

Preventive maintenance (PM) is a maintenance program with activities initiated at predetermined intervals, or according to prescribed criteria, and intended to reduce the probability of failure, or the degradation of the functioning of an item. In the literature, a number of PM models have been introduced to depict the effectiveness of PM. Based on these models, approaches to scheduling PM policies have been considerably studied. This paper attempts to review existing PM models, and investigate their inter-relationships. We then categorize these models into three classes: linear, nonlinear, and a hybrid of both. These three PM model classes depict the relationships of the hazard functions before, and after a PM. Possible extensions to these three PM models are discussed. The statistical properties for models are derived, and approaches to optimizing the PM policy are given.

## Index Terms

Preventive maintenance, corrective maintenance, linear preventive maintenance, nonlinear preventive maintenance, hazard function, maintenance effectiveness.

## ACRONYM<sup>1</sup>

CM	Corrective Maintenance
PM	Preventive Maintenance
IFR	Increasing Failure Rate
DFR	Decreasing Failure Rate
IFRA	Increasing Failure Rate on Average
DFRA	Decreasing Failure Rate on Average
NBU	New Better than Used
NWU	New Worse than Used

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<sup>1</sup>The singular and plural of an acronym are always spelled the same .

## NOTATION

$t$	A non-negative number, it is reset to zero at each PM.
$h_0(t)$	Hazard function of the system when no PM is conducted on it, or naked hazard function.
$h_k(t)$	Hazard function of the system at time $t$ after the $k$ th PM, where $k = 1, 2, 3, \dots$
$t_0$	Scheduled interval before the first PM.
$t_k$	Scheduled interval between the $k$ th PM, and the $(k + 1)$ th PM, where $k = 1, 2, 3, \dots$
$a, b, \alpha, \beta$	Non-negative parameters.

## I. INTRODUCTION

Maintenance activities (e.g., CM, and PM ) can change a maintained system in one of two ways. Accordingly, the maintenance can be described as better, or worse. A better maintenance decreases the hazard rate and/or virtual age of a system, whereas a worse one increases them, or even brings the system to fail or break down.

The effectiveness of a better maintenance can be further classified into one of the three situations: *perfect*, *minimal*, and *imperfect*. A perfect maintenance is assumed to restore a system to be as good as new (AGAN ). A minimal maintenance restores a system to a state the same as just before the maintenance, or as bad as old (ABAO). An imperfect maintenance may bring a system to any condition between AGAN and ABAO. In reality, however, both CM, and PM are usually imperfect. Pham & Wang [1], and more recently Doyen & Gaudoin [2] have given useful surveys on imperfect maintenance models.

A worse maintenance deteriorates the health of a system compared to what it was prior to maintenance. In the case of technology changes, a system can be brought to a state better than its AGAN state, but these two situations are outside the scope of this article.

Modelling the effectiveness of PM is an active research topic; see [3]–[21] for examples. It is an essential requirement in various scenarios, for example, when people plan maintenance strategies, select maintenance contractors, or estimate the residual lifetime for some important industrial systems (for example, nuclear power plants, planes, trains) put up for re-sale at the end of their planned life.

There are many papers modelling the effectiveness of PM. Most of them model the hazard rates of maintained systems after PM interventions [3]– [4]. According to a taxonomy given by [5], existing PM models are categorized into three groups: age reduction models, hazard rate reduction models, and a hybrid of both.

- *Age reduction models* These models are developed by considering age reduction in the hazard function ( [3], [6]– [7]). Using the concept of age reduction, we might say that a certain PM has reduced the virtual age of a maintained system to a younger age, for example, from an age of  $t$  years old to “as good as  $t - \tau$  years old” where  $\tau > 0$ . That is, the hazard function changes from  $h(t)$  before a PM to  $h(t - \tau)$  after a PM.
- *Hazard rate reduction models* These models are developed considering the reduction of the hazard rate of a system [5]– [8]. This group of models assumes that the hazard rate of a system changes, for example, from  $h(t)$  before a PM to  $ah(t)$  after a PM.

- *Hybrid models* These models are combinations of the above two groups [4], [9]; the hazard rate changes from  $h(t)$  to  $ah(t - t_0)$ , for example.

However, if one reviews existing PM models, the following weaknesses can be found in those models.

- Although many PM models have been developed, little record on comparing these models has been found.
- Parameters in lifetime distributions (such as the Weibull distribution) usually have their physical meanings (for example, the location parameter, and the scale parameter, in the Weibull distribution). However, parameters in the existing PM models are not given any physical explanation, which can limit the applications of these models because of a lack of proper interpretation of the parameters.
- Assumptions in some existing PM models may not be appropriate, which are elaborated upon in Section III.

The main theme of this paper is to briefly review existing PM models, explore their interrelationships, and extend them to three new ones: linear, nonlinear and their hybrid. A PM model is *linear* if the maintained system has hazard rates  $h_k(t)$  ( $k = 1, 2, \dots$ ) immediately after the  $k$ th PM with  $h_k(t) = ah_{k-1}(t) + b$ , *nonlinear* if  $h_k(t) = h_{k-1}(\alpha t + \beta)$ , or *hybrid* if  $h_k(t) = ah_{k-1}(\alpha t + \beta) + b$ , where  $a, b, \alpha$ , and  $\beta$  are non-negative parameters, and  $t > 0$ . Physical interpretation of the parameters in these models will be given. More general extensions of these models will also be provided.

Although we discuss PM models and PM policy development in this paper, our primary focus is on the comparison of commonly studied PM models. The paper does not pretend to give a comprehensive view of the topic of existing PM models, and it emphasizes on modelling rather than on statistical inference. We have tried to make Section II reasonably complete; however, those papers which are not included were either considered not to bear directly on the topic of this paper, or were inadvertently overlooked. Our apologies are extended to both the researchers and readers if any relevant papers are omitted.

The paper is structured as follows. Section II briefly reviews the existing PM models, and explores their interrelationships. Section III introduces two new PM models, accompanied by their statistical properties. Section IV studies the periodic PM policy for the two PM models, and provides conditions on the existence of solutions. Section V offers discussion on possible extensions of the newly proposed PM models, and their corresponding properties. Finally, Section VI arrives at conclusions, and further work that might be important.

## II. EXISTING PM MODELS, AND THEIR PROPERTIES

Assume that PM actions are carried out at times  $t_0, t_0 + t_1, t_0 + t_1 + t_2, \dots$ . CM on failure between adjacent PM actions is assumed to be minimal. Time on either PM or CM is negligible.  $h_k(t)$  is the hazard rate of the system after the  $k$ th PM intervention (the initial hazard function is  $h(t) (= h_0(t))$ ). If the  $k$ th PM is performed, the induced hazard function changes from  $h_{k-1}(t)$  to  $h_k(t)$  after the PM.  $h_k(t)$  is assumed to be increasing in  $t$ . The time  $t$  is reset to zero at each PM. Denote  $F_k(t) = 1 - \exp[-\int_0^t h_k(s) ds]$ , and  $F(t) = 1 - \exp[-\int_0^t h_0(s) ds]$ .

*Definition 1: Better PM:* The  $k$ th PM is a better PM if  $h_{k-1}(t + t_{k-1}) \geq h_k(t)$ .

78 *A. Existing PM models*

79 The effectiveness of a PM may be AGAN or ABAO. The criteria are given as follows.

80 *Definition 2:* The  $k$ th PM is AGAN if

$$h_k(t) = h_0(t). \quad (1)$$

81 The  $k$ th PM is ABAO if

$$h_k(t) = h_{k-1}(t + t_{k-1}). \quad (2)$$

82 However, we should note that, in the case of ABAO, a PM is useless.

83 In the Malik model [3], the improvement of the  $k$ th PM is that the  $t$  years old system is no longer that old,  
84 and its post-maintenance age is reduced from  $t$  to  $\mu_k t$  in terms of its reliability, where  $\mu_k$  varies between zero and  
85 one. If we simply concern the age reduction and hazard function, then the effect of the maintenance can also be  
86 expressed by hazard functions as follows.

87 *Definition 3:* (Malik [3]) We say that the  $k$ th PM is MAL if

$$h_k(t) = h_0\left(t + \sum_{i=0}^{k-1} \mu_i t_i\right), \quad (3)$$

88 where  $0 < \mu_i < 1$ .

89 The values of parameter  $\mu_i$  in the MAL model (3) are restricted to (0,1). However, if the range of  $\mu_i$  can be  
90 extended to include the two end points, then the MAL model is an extension of the ABAO model, or the AGAN  
91 model. That is, the MAL model reduces to the ABAO model if  $\mu_i = 1$ , and to the AGAN model if  $\mu_i = 0$ .

92 Nakagawa [5] proposes two PM models; one is an age reduction model, and the other is a hazard rate reduction  
93 model. In what follows, these two models are referred to as NAK1, and NAK2, respectively.

94 *Definition 4:* (Nakagawa [5]) We say that the  $k$ th PM is NAK1 if

$$h_k(t) = h_0\left(t + \sum_{i=0}^{k-1} \left(t_i \prod_{j=i}^{k-1} \nu_{1j}\right)\right), \quad (4)$$

95 where  $0 < \nu_{10} < \nu_{11} < \dots < \nu_{1k-1} < 1$ .

96 If we compare (3) with (4), and let  $\sum_{i=0}^{k-1} (t_i \prod_{j=i}^{k-1} \nu_{1j}) = \sum_{i=0}^{k-1} \mu_i t_i$  for  $\forall k$ , then model NAK1 is equivalent  
97 to model MAL. Obviously,  $\forall j$ , model NAK1 reduces to model ABAO if  $\nu_{1j} = 1$ , and to model AGAN if  $\nu_{1j} = 0$ .

98 *Definition 5:* (Nakagawa [5]) We say that the  $k$ th PM is NAK2 if

$$h_k(t) = h_0(t) \prod_{i=0}^{k-1} \nu_{2i}, \quad (5)$$

99 where  $1 \leq \nu_{20} \leq \nu_{21} \leq \dots \leq \nu_{2k-1}$ .

100 Obviously, if  $\nu_{2k}$  is set to 1 for all  $k$ , then the NAK2 model promotes to the AGAN model.

101 Model NAK1 assumes that PM activities can bring the maintained system younger, and Model NAK2 regards  
102 that a PM can first bring the hazard rate to zero, and then increase more quickly than it did in the previous PM  
103 interval.

104 Lin et al. [4] combine models NAK1 and NAK2, and propose the following PM model to link the hazard functions  
 105 between two adjacent working periods.

106 *Definition 6:* (Lin et al [4]) We say that the  $k$ th PM is LIN if

$$h_k(t) = \lambda_{1k} h_{k-1}(\lambda_{2k} t_{k-1} + t). \quad (6)$$

107 Obviously, if  $\lambda_{1k} = 1$  (or  $\lambda_{2k} = 0$ ) for all  $k$ , then the LIN model is equivalent to the NAK1 (or NAK2) model.

108 All of the above PM models can be applied to both periodic, and sequential PM modelling. Canfield [6] only  
 109 considers the periodic PM case. He distinguishes between the level of the hazard, and the shape of the hazard  
 110 function as they are related to system degradation with time. The hazard level reflects the extent of the system  
 111 degradation. The shape of the hazard function at a given time reflects the rate at which the hazard is changing. He  
 112 regards the effective age after PM reduces to  $t - \tau$  if the item's effective age was  $t$  just prior to this PM, while the  
 113 hazard level remains unchanged, where  $\tau (\geq 0)$  is the restoration interval at the effective age of the item due to the  
 114  $k$ th PM. The restoration interval  $\tau$  in this model is an index for measuring the quality of PM.

115 *Definition 7:* We say that the  $k$ th PM is CAN if

$$h_k(t) = h_0(t + k(T - \tau)) + \sum_{i=1}^k \{h_0((i-1)(T - \tau) + T) - h_0(i(T - \tau))\}, \quad (7)$$

116 where  $T$  is a fixed constant time length between two adjacent PM actions.

117 When  $\tau = T$ , and suppose  $h_0(0) = 0$ , the CAN model reduces to

$$h_k(t) = h_0(t) + kh_0(T). \quad (8)$$

118 Parameter  $\tau$  in the CAN model is assumed to be a fixed constant. Reference [10] considers  $\tau$  as a random  
 119 variable and develops PM policies.

120 Kijima et al. [11], and Kijima [12] introduce two types of CM models, type I and type II, using the concept  
 121 of virtual age. The idea is to distinguish between the system's age, which is the time elapsed since the system  
 122 was new, usually at time  $t = 0$ ; and the virtual age of the system, which describes its present health condition  
 123 when compared to a new system. The two models are  $V_k = V_{k-1} + \kappa_k X_k$ , and  $V_k = \kappa_k (V_{k-1} + X_k)$ , where  
 124  $V_k$  is the virtual age of the system immediately after the  $k$ th repair, and  $\kappa_k$  is a parameter. Interesting extensions  
 125 on the virtual age concept have been made by other authors. For example, Dagpunar [14] considers the case in  
 126 which the virtual age after the  $k$ th CM can be expressed as  $V_k = \phi(V_{k-1} + X_k)$  (where  $\phi(\cdot)$  is an arbitrary scaling  
 127 function that models the effects of CM); Dorado [15] studied nonparametric statistical inference in a model slightly  
 128 more general than Kijima's models. More references can be found in [16], [17]. Kijima's virtual age concept was  
 129 originally introduced to model the effectiveness of CM activities. It has been applied to the PM case recently by  
 130 some authors (for example, [18], [19]).

131 The type I, and type II models [11], [12] share similar expressions of hazard functions. Therefore, we only  
 132 discuss type I as an example.

133 *Definition 8:* We say that the  $k$ th PM is KIJ if

$$h_k(t) = h_0(\kappa_k t + \sum_{i=1}^{k-1} \kappa_i t_i). \quad (9)$$

134 The KIJ model can be seen as an extension of the NAK1, and MAL models.

135 Seo & Bai [13] introduced a periodic PM model. They define  $h_k(\omega_{k-1}(T)) = h_{k-1}(\Omega(\omega_{k-2}(T), T))$ , where  
 136  $\Omega(\cdot)$  and  $\omega(\cdot)$  are specified functions, and  $T$  is a fixed constant time length between two adjacent PM actions. This  
 137 model can be regarded as an extension of the KIJ model for periodic PM.

138 Another interesting model is the geometric process for CM. Lam [22] defines the geometric process as an  
 139 alternative to the NHPP: a sequence of random variables  $\{X_k, k = 1, 2, \dots\}$  is a geometric process if the distribution  
 140 function of  $X_k$  is given by  $F(\alpha^{k-1}t)$  for  $k = 1, 2, \dots$ , and  $\alpha$  is a positive constant. The hazard rate changes from  
 141  $h(t)$  before a CM activity to  $\alpha h_{k-1}(\alpha t)$  after the CM. The change is similar to the hybrid PM models. Wang  
 142 & Pham [23] later refer a process similar to the geometric process as a quasi-renewal process. Finkelstein [24]  
 143 develops a very similar model where he defines a general deteriorating renewal process such that  $F_{k+1}(t) \leq F_k(t)$ .  
 144 Wu & Clements-Croome [25] extend the geometric process by replacing its parameter  $\alpha^{k-1}$  with  $\alpha_1 \alpha^{k-1} + \beta_1 \beta^{k-1}$ ,  
 145 where  $\alpha > 1$ , and  $0 < \beta < 1$ . The geometric process has been studied by many authors (for example, see [7],  
 146 [26]–[29]). However, we have found very few works in the application of the geometric process to modelling PM.  
 147 We hence will not discuss this model in detail.

#### 148 B. Interrelationships

149 On the basis of the above discussion, if we use  $Y \implies Z$  to denote that  $Z$  can be derived from  $Y$ , the chain of  
 150 implications in Fig. 1 exists among the existing sequential PM models.

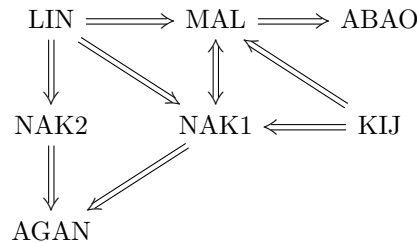


Fig. 1. Interrelationships of existing PM models.

151 From the relationships shown in Fig. 1, we conclude that all of the existing PM models can be categorized to  
 152 be special cases of the LIN model, and the KIJ model. The CAN model, and the model by Seo & Bai [13] are not  
 153 included in Fig. 1 as these two models are periodic ones.

154 *C. A new categorization*

155 Unlike the classification of the PM models used by [5], the following category is created from the perspective of  
 156 the aging property. All of the PM models in the preceding subsection can be categorized into one of the following  
 157 classes, or a combination of the two classes.

158 **Age reduction PM models:** PM modelled by AGAN, MAL, NAK1, or CAN assumes that the PM restores  
 159 the maintained system to a younger age. Apparently, the model introduced by Seo & Bai [13] also falls  
 160 into this category. After a PM, the system will follow the deteriorating speed from its younger age point.  
 161 The parameters in the age reduction PM models indicate how much a PM has reduced the functional  
 162 age of a maintained system. These parameters are  $\mu_i$ ,  $\nu_{1j}$ , and  $\tau$  in the MAL, NAK1, and CAN models,  
 163 respectively. They are called *age reduction parameters* in what follows.

164 **Age defying PM models:** PM modelled by NAK2 defies the age of the system after it has been maintained.  
 165 The ageing of the system after a PM will slow down (or speed up in some cases). The effect of the PM  
 166 mainly influences the future system degradation. The parameters in this category measure the speed of  
 167 deterioration of the maintained system after PM has been conducted. These parameters are  $\nu_{2i}$  in the  
 168 NAK2 model, and they are called *age defying parameters*.

169 PM in the LIN model can function as both age reduction, and age defying. The parameters  $\lambda_{2k}$ , and  $\lambda_{1k}$  in the  
 170 LIN model are the age reduction parameter, and the age defying parameter, respectively.

171 **III. LINEAR, AND NONLINEAR PM**

172 Aside from the LIN model, the PM models reviewed in Section II consider the PM improvement simply from one  
 173 aspect: either age reduction, or age defying. The PM models introduced in this section depict the PM effectiveness  
 174 from both aspects. These models are called linear PM models, and nonlinear PM models, respectively.

175 *A. Linear PM*

176 The NAK2 model assumes that the hazard rate right after PM reduces to 0, and then increases more quickly  
 177 than it did in the previous PM interval. These assumptions might be too rigorous, and even unrealistic in some  
 178 scenarios, which can limit the models applications in practice. For example, in some scenarios, a PM, such as  
 179 cleaning, adjustment, alignment, and lubrication work, may not always reduce the system's age or hazard rate to  
 180 zero [30]. Instead, it may only reduce the degradation rate of the system to a certain level. Therefore, a reasonable  
 181 extension is to relax the assumptions that the hazard rate reduces to a certain level after better maintenance, and  
 182 then increases more quickly than it did in the previous PM interval. This relaxation of the assumption leads to the  
 183 following model.

184 *Definition 9:* The  $k$ th PM is called *linear PM* if

$$h_k(t) = ah_{k-1}(t) + b, \quad (10)$$

185 where  $a$ , and  $b$  are parameters;  $t \in (0, t_k)$  for  $k = 1, 2, \dots$ , and  $a, b > 0$ .



186 The reason we call the  $k$ th PM a *linear PM* is that the relationship between the two adjacent hazard functions  
187 before and after the  $k$ th PM is linear. We also call (10) a *linear PM model*.

188 Parameters in (10) can have their physical meanings. Parameter  $a$  indicates a degree of deterioration after PM.  
189 The system deteriorates faster than before if  $a > 1$ , deteriorates slower than before if  $a < 1$ , or keeps the same  
190 shape but different locations of the hazard rate as before if  $a = 1$  and  $b \neq 0$ .  $a$  is called an *age defying parameter*  
191 of the linear PM model, although the PM does not defy the age of the maintained system in the case of  $a > 1$ .  
192 Parameter  $b$  indicates the starting value of the hazard rate immediately after a PM. The PM is a worse maintenance  
193 if  $b > h_{k-1}(t_{k-1})$ , and it is a better maintenance if  $b < h_{k-1}(t_{k-1})$ . Therefore, we call  $b$  a *location parameter*.

194 Equation (10) reduces to the NAK2 model if  $b = 0$ ; and to the AGAN model if  $b = 0$ , and  $a = 1$ . Parameters  
195  $a$ , and  $b$  reflect the performance of the linear PM.  $a$  can be limited within  $(0, 1)$  for better PM.

196 If all of the first  $k$  PM are linear PM, (10) can also be written as

$$h_k(t) = A_k h_0(t) + B_k, \quad (11)$$

197 where  $A_0 = 1$ ,  $B_0 = 0$ ,  $B_1 = b$ ,  $A_k = a^k$ , and  $B_k = \frac{a^k + a - 2}{a - 1} b$ .

198 It is easy to derive the following Lemma.

199 *Lemma 1:* If all of the first  $k$  PM are linear PM, we have

$$F_k(t) = 1 - e^{-B_k t} (1 - F(t))^{A_k}. \quad (12)$$

200

201 *Theorem 1:* If all of the first  $k$  PM are linear PM, and  $1 - F(t)$  belongs to IFR, DFR, IFRA, DFRA, NBU, or  
202 NWU, then  $1 - F_k(t)$  is in the same category for  $k = 2, 3, \dots$

203 The proof of Theorem 1 is given in Appendix.

## 204 B. Nonlinear PM

205 Models MAL, NAK1, and CAN consider age-reduction phenomenon after PM. If a PM can defy and also reduce  
206 the age of a maintained system, then the following model is more appropriate to describe such scenarios.

207 *Definition 10:* Assume  $h_0(t)$  is a nonlinear function with respect to  $t$ . The  $k$ th PM is called *nonlinear PM* if

$$h_k(t) = h_{k-1}(\alpha t + \beta), \quad (13)$$

208 where  $\alpha (> 0)$ , and  $\beta (\geq 0)$  are parameters; and  $t \in (0, t_k)$ .

209 In this model,  $\alpha$  plays a role in defying or accelerating degradation of a maintained system due to the effectiveness  
210 of PM. A PM defies the age of a maintained system for  $\alpha \in (0, 1)$ , and it accelerates the deterioration of the system  
211 for  $\alpha \in (1, \infty)$ .  $\beta$  is the value that a PM brings the system's age to in terms of the immediate preceding PM  
212 interval. Hence, we call  $\alpha$  an *age defying parameter*, and  $\beta$  a *location parameter*.

213 The linearity between  $h_k(t)$  and  $h_{k-1}(t)$  in (13) depends on  $h_{k-1}(t)$ : if  $h_{k-1}(t)$  is linear with respect to  $t$ , then  
214 the relationship is linear; otherwise, it is nonlinear. In the case that  $h_{k-1}(t)$  is linear, (13) is equivalent to (10).

215 This equivalence is the reason we assume  $h_0(t)$  is nonlinear in the definition. We also call (13) a *nonlinear PM*  
 216 *model*.

217 The two parameters  $\alpha$ , and  $\beta$  in (13) estimate the effectiveness of the PM: a better maintenance if  $0 < \alpha < 1$ ,  
 218 and  $\beta < t_{k-1}$ .

219 If all of the first  $k$  PM are nonlinear PM, another expression of (13) is given as

$$h_k(t) = h_0(\Phi_k t + \Psi_k), \quad (14)$$

220 where  $\Phi_0 = 1$ ,  $\Psi_0 = 0$ ,  $\Psi_1 = \beta$ ,  $\Phi_k = \alpha^k$ , and  $\Psi_k = \frac{\alpha^k + \alpha - 2}{\alpha - 1} \beta$ .

221 *Lemma 2:* If all of the first  $k$  PM are nonlinear PM, for nonlinear PM, we have

$$F_k(t) = 1 - (1 - F(\Phi_k t + \Psi_k))^{\frac{1}{\Phi_k}} (1 - F(\Psi_k))^{-\frac{1}{\Phi_k}}. \quad (15)$$

222

223 *Theorem 2:* If all of the first  $k$  PM are nonlinear PM, and  $1 - F(t)$  is IFR (or DFR), then  $1 - F_k(t)$  of the  
 224 nonlinear PM model (13) is IFR (or DFR) for  $k = 1, 2, 3, \dots$

225

#### IV. PM POLICY OPTIMIZATION

226 A PM policy specifies how PM activities should be scheduled. In the reliability and maintenance literature,  
 227 two PM policies are commonly discussed: periodic PM, and sequential PM. The periodic policy schedules PM  
 228 activities at fixed time periods, for example,  $T, 2T, 3T, \dots$ , whereas the sequential policy schedules PM activities  
 229 at a sequence of time intervals,  $t_1, t_2, \dots$ , that can be unequal. Obviously, if we let  $t_1 = t_2 = \dots = T$ , then  
 230 the sequential PM is equivalent to the periodic PM. Hence, we focus on the sequential PM in what follows, and  
 231 searching maintenance intervals aiming at optimizing overall cost.

232 We make the following assumptions for the maintenance policy optimization.

- 233 • The planning horizon is infinite.
- 234 • The hazard functions,  $h_0(t)$ , is continuous, and strictly increasing if there are no PM interventions.
- 235 • The times for PM, minimal repair, and replacement are negligible.
- 236 • PM is performed at  $t_1, t_1 + t_2, \dots, \sum_{i=1}^{N-1} t_i$ , and the system is replaced at  $\sum_{i=1}^N t_i$ .
- 237 • Minimal repairs are used for failures between PM.
- 238 • The system is restored to as good as new state at replacement.

239 To derive the expected cost expression, we assume that the planning horizon is infinite, the system is replaced  
 240 after  $N - 1$  PM, and the system is brought to an AGAN state at replacement.

241 In what follows, we consider both the linear, and nonlinear cases.

##### 242 A. Linear PM model case

243 Assume all PM are linear, the total number of minimal repairs is given by  $\sum_{k=0}^{N-1} \int_0^{t_k} (A_k h_0(x) + B_k) dx$ , the  
 244 total number of PM is  $N - 1$ , and there is one replacement. Then the expected cost per unit time between two

---

Inputs:
$h_0(t)$ : hazard function;
$c_m$ : cost of minimal repair;
$c_p$ : unit cost of PM;
$c_r$ : replacement cost;
Outputs:
$N^*$ : optimal number of PM before a replacement;
$t_k^*$ : optimal time interval for the $k$ th PM;
Steps:
1: for $N_L = 1, 2, \dots, N$ do;
2: obtain $t_k$ by solving equation (17);
3: if inequalities (18) and (19) are satisfied, then;
4: calculate $C_L(t_1, t_2, \dots, t_{N-1}, N)$ ;
5: set $T^* \leftarrow T$ ; $N^* \leftarrow N$ ;
6: break;
7: end;
8: end;

---

TABLE I  
SEARCHING THE OPTIMAL  $t_k^*$ , AND  $N^*$  FOR THE LINEAR PM MODEL.

245 adjacent replacements is given by

$$C_L(t_0, t_1, \dots, t_{N-1}, N) = \frac{c_m \sum_{k=0}^{N-1} \int_0^{t_k} (A_k h_0(x) + B_k) dx + c_p(N-1) + c_r}{\sum_{k=0}^{N-1} t_k}, \quad (16)$$

246 where,  $c_m$  is the cost per CM,  $c_p$  is the cost per PM,  $c_r$  is replacement cost, and  $N-1$  is the number of PM  
247 between two adjacent replacements.

248 The optimal  $t_k^*$  should satisfy the following conditions.  $\frac{\partial C_L(t_0, t_1, \dots, t_{N-1}, N)}{\partial t_k} |_{t_k=t_k^*} = 0$ , for  $k = 0, 1, 2, \dots, N-1$ .

249 This implies that the optimum  $t_0^*, t_1^*, \dots, t_{N-1}^*$  should satisfy

$$A_k h_0(t_k^*) + B_k = \frac{C_L(t_0, t_1, \dots, t_{N-1}, N)}{c_m}, \quad (17)$$

250 for  $k = 0, 1, 2, \dots, N-1$ . The optimal  $N^*$  should satisfy the following conditions.

$$C_L(t_0^*, t_1^*, \dots, t_{N^*-2}^*, (N^*-1)) \geq C_L(t_0^*, t_1^*, \dots, t_{N^*-1}^*, N^*), \quad (18)$$

251 and

$$C_L(t_0^*, t_1^*, \dots, t_{N^*}^*, (N^*+1)) \geq C_L(t_0^*, t_1^*, \dots, t_{N^*-1}^*, N^*). \quad (19)$$

252 Table I presents a method to search the optimal values of  $t_k^*$ , and  $N^*$  for the linear PM model case.

### 253 B. Nonlinear PM model case

254 Assume all PM are nonlinear, and the system is aged from  $\Psi_k$  just before the  $k$ th PM to  $\Phi_k t_k + \Psi_k$  just before  
255 the next PM. Then the total cost is given by

$$C_N(t_0, t_1, \dots, t_{N-1}, N) = \frac{c_m \sum_{k=0}^{N-1} \int_0^{t_k} h_0(\Phi_k x + \Psi_k) dx + c_p(N-1) + c_r}{\sum_{k=0}^{N-1} t_k}. \quad (20)$$

256 The optimal  $t_k^*$  should satisfy the conditions  $\frac{\partial C_N(t_0, t_1, \dots, t_{N-1}, N)}{\partial t_k} \Big|_{t_k=t_k^*} = 0$ , for  $k = 0, 1, 2, \dots, N-1$ . This  
 257 implies that the optimum  $t_0^*, t_1^*, \dots, t_{N-1}^*$  should satisfy  $h_0(t_0^*) = h_0(\Phi_1 t_1^* + \Psi_1) = h_0(\Phi_2 t_2^* + \Psi_2) = \dots =$   
 258  $h_0(\Phi_{N-1} t_{N-1}^* + \Psi_{N-1})$ , and  $c_m h_0(\Phi_k t_k^* + \Psi_k) = C_N(t_0^*, t_1^*, t_2^*, \dots, t_{N-1}^*, N)$ .

259 Similarly, the optimum value  $N^*$  should satisfy the conditions  $C_N(t_1^*, t_2^*, \dots, t_{N^*-2}^*, (N^*-1)) \geq C_N(t_1^*, t_2^*, \dots, t_{N^*-1}^*, N^*)$ ,  
 260 and  $C_N(t_1^*, t_2^*, \dots, t_{N^*}^*, (N^*+1)) \geq C_N(t_1^*, t_2^*, \dots, t_{N^*-1}^*, N^*)$ .

## 261 V. DISCUSSION

262 Comparing to the existing PM models reviewed in Section II, we can see that the linear PM model relaxes the  
 263 assumption of the NAK2 model, while the nonlinear PM model relaxes the assumption of models MAL, NAK1,  
 264 and CAN. Here, by relaxation, we mean that the proposed models either relax the assumption of parameters  
 265  $0 < \nu_{10} < \nu_{11} < \dots < \nu_{1k-1} < 1$  in the NAK1 model, and  $1 \leq \nu_{20} \leq \nu_{21} \leq \dots \leq \nu_{2k-1}$  in the NAK2 model;  
 266 or add one more parameter,  $b_k$  in the linear PM model, and  $\beta_k$  in the nonlinear PM model. The following hybrid  
 267 model can be seen as extensions of all of the existing PM models.

268 If we combine the linear and nonlinear PM models, a hybrid PM model can be derived. For all PM models,  
 269 parameter estimation is of interest in both practical, and theoretical perspectives. This section addresses both the  
 270 hybrid PM model, and its parameter estimation.

### 271 A. Hybrid PM

272 Combining both the linear and nonlinear PM models, we can extend them to a hybrid PM model as follows.

273 *Definition 11:* The  $k$ th PM is called *hybrid PM* if the hazard functions before, and after the  $k$ th maintenance  
 274 have the relationship

$$h_k(t) = ah_{k-1}(\alpha t + \beta) + b, \quad (21)$$

275 for  $k = 1, 2, \dots$

### 276 B. More generic extensions

277 All PM models discussed in Section II assume different parameters after each PM. Similarly, we have the  
 278 following extensions for the linear, nonlinear, and hybrid PM models.

#### 279 1) Extended linear PM model:

280 *Definition 12:* The  $k$ th PM is called an *extended linear PM* if

$$h_k(t) = a_k h_{k-1}(t) + b_k, \quad (22)$$

281 where  $a_k$ , and  $b_k$  are parameters;  $t \in (0, t_k)$ , and  $a_k, b_k > 0$ . A typical, reasonable choice for the age reduction  
 282 parameter  $b_k$  is to assume that it depends on  $t_{k-1}$ . For example, set  $b_k = \rho_k h_{k-1}(t_{k-1})$  with  $\rho_k \in (0, 1)$  for better  
 283 maintenance.

284 If all of the first  $k$  PM are extended linear PM, then (22) can also be written as

$$h_k(t) = A'_k h_0(t) + B'_k, \quad (23)$$

285 where

$$A'_0 = 1, B'_0 = 0, B'_1 = b_1, A'_k = \prod_{i=1}^k a_i, \text{ and } B'_k = \sum_{i=1}^{k-1} \left( b_i \prod_{j=i+1}^k a_j \right) + b_k.$$

286 It is easy to derive the following Lemma.

287 *Lemma 3:* If all of the first  $k$  PM are extended linear PM, for (23), we have

$$F_k(t) = 1 - e^{-B'_k t} (1 - F(t))^{A'_k}. \quad (24)$$

288 If  $1 - F(t)$  belongs to IFR, DFR, IFRA, DFRA, NBU, or NWU, then  $1 - F_k(t)$  is in the same category for  
289  $k = 2, 3, \dots$

290 2) *Extended nonlinear PM model:*

291 *Definition 13:* Assume  $h_0(t)$  is a nonlinear function with respect to  $t$ . The  $k$ th PM is called an *extended nonlinear*  
292 *PM* if

$$h_k(t) = h_{k-1}(\alpha_k t + \beta_k), \quad (25)$$

293 where  $t \in (0, t_k)$ ,  $\alpha_k$  is an age defying parameter, and  $\beta_k$  is an age reduction parameter.

294 Similar to the linear PM model, the expressions of parameters,  $\alpha_k$ , and  $\beta_k$ , in the extended nonlinear PM model  
295 are important. If we recall the existing PM models, parameters  $\mu_k$  in the MAL model,  $\nu_{1k}$  in the NAK1 model,  
296  $\lambda_{2k}$  in the LIN model, and  $kT - k\tau$  in the CAN model, are related to the time intervals  $t_k$ , and playing a similar  
297 role as the parameter  $\beta_k$  in the extended nonlinear PM model. Therefore,  $\beta_k$  can be set to  $\gamma_k t_{k-1}$ , which will be  
298 used in the PM policy optimization section, where  $\gamma_k \in (0, 1)$ .

299 If all of the first  $k$  PM are extended nonlinear PM, another expression of e (25) is

$$h_k(t) = h_0(\Phi'_k t + \Psi'_k), \quad (26)$$

300 where

$$\Phi'_0 = 1, \Psi'_0 = 0, \Psi'_1 = \beta_1, \Phi'_k = \prod_{i=1}^k \alpha_i, \text{ and } \Psi'_k = \sum_{i=1}^{k-1} \left( \beta_i \prod_{j=i+1}^k \alpha_j \right) + \beta_k.$$

301 The proof for the following Lemma is simple, so we do not show the proof in this paper.

302 *Lemma 4:* If all of the first  $k$  PM are extended nonlinear PM, for (25), we have

$$F_k(t) = 1 - (1 - F(\Phi'_k t + \Psi'_k))^{\frac{1}{\Phi'_k}} (1 - F(\Psi'_k))^{-\frac{1}{\Phi'_k}}. \quad (27)$$

303 If  $1 - F(t)$  is IFR (or DFR), then  $1 - F_k(t)$  is IFR (or DFR) for  $k = 1, 2, 3, \dots$

304 3) *Extended hybrid PM model:*

305 *Definition 14:* The  $k$ th PM is called an *extended hybrid PM* if the hazard functions before, and after the  $k$ th  
306 maintenance have the relationship

$$h_k(t) = a_k h_{k-1}(\alpha_k t + \beta_k) + b_k. \quad (28)$$

307 All PM models in Section II are special cases of the extended hybrid model.

308 If all of the first  $k$  PM are extended hybrid linear PM, then another expression of model (28) is given as

$$h_k(t) = A_k h_0(\Phi_k t + \Psi_k) + B_k, \quad (29)$$

309 and we have the following lemma.

310 *Lemma 5:* For Model (29), we have

$$F_k(t) = 1 - e^{-B_k t} (1 - F_0(\Phi_k t + \Psi_k))^{\frac{A_k}{\Phi_k}} (1 - F_0(\Psi_k))^{-\frac{A_k}{\Phi_k}}. \quad (30)$$

311 If  $h_k(t)$  is IFR (DFR), then  $h_{k+1}(t)$  is IFR (DFR).

### 312 C. More complex situations

313 Obviously, the introduced PM models do not consider more complex situations that can exhibit more complex  
314 non-linear relationship between  $h_k(t)$  and  $h_{k-1}(t)$ . For example,  $h_k(t)$  can be a  $G(h_{k-1}(t))$ , or  $h_k(t) = h_{k-1}(g(t))$   
315 where  $G(\cdot)$  and  $g(\cdot)$  are nonlinear functions. In practice, however, estimating parameters for a nonlinear relationship  
316 can be problematic as there might be limited data available.

### 317 D. Parameter estimation

318 In practice, there are two approaches to estimating the parameters in PM models. The first approach estimates  
319 the parameters on the basis of reliability data sets. For example, one can utilize maximum likelihood estimation  
320 to estimate the parameters of the linear, and non-linear PM models. The second one uses domain experience to  
321 estimate the parameters. This approach is used only if few failure data are collected (see [31]). A combination of  
322 these two approaches can also be used. For example, [7] considers the scenarios where the maintenance effect is a  
323 random variable. It assumes that both parameter  $\tau$  in the CAN model, and parameter  $\nu_{1i}$  in the NAK1 model are  
324 random variables with certain probability distributions. Under such assumptions, they show that more cost-effective  
325 PM policies can be obtained.

326 Note that a PM model with many parameters might not be applicable in practice. This is due to a lack of  
327 sufficient data for parameter estimation. However, (10) of the linear PM model, (13) of the nonlinear PM model,  
328 and (21) of the hybrid PM model include fewer parameters which should be easier to estimate, and more realistic  
329 for applications in practice.

## VI. CONCLUSIONS

PM models are important in both designing maintenance policies, and assessing the residual lifetime of systems being preventively maintained. Many PM models are proposed in the reliability literature. As discussed in Section III, however, existing PM models present weaknesses in the sense that either their parameters might not have physical meanings, or their model assumptions are too restrictive. The linear, nonlinear, and hybrid PM models proposed in this paper overcome such weaknesses.

The main contributions of this paper are as follows.

- We have reviewed the existing PM models, investigated their interrelationships, and proposed a new classification of the PM models.
- Three PM models are introduced, and their relationships are investigated.
- The properties of the PM models are derived.
- The expected costs for the three PM models for sequential PM are formulated, and the necessary conditions of obtaining the optimal PM policies for both the general, and special cases are derived.

Our future research will include

- estimating the parameters within the three models, and comparing the three models with those reviewed in Section II with respect to their performance on the basis of field test data; and
- investigating the application of the proposed PM models to various scenarios, including optimizing warranty policies for products with linear or nonlinear preventive maintenance.

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## APPENDIX

411

412 **The proof of Theorem 1 is as follows.**

412

413 **Proof** If  $h_0(t)$  is increasing (or decreasing), then (11)  $h_k(t)$  is increasing (or decreasing).

413

414  $1 - F(t)$  is IFRA (or DFRA) if  $[1 - F(t)]^{1/t}$  is decreasing (or increasing).

414

415 Because

415

$$(1 - F_k(t))^{1/t} = (e^{-B_k t} (1 - F(t))^{A_k})^{1/t} = e^{-B_k} \left( (1 - F(t))^{1/t} \right)^{A_k} \quad (31)$$

416 from (11), assuming that  $(1 - F(t))^{1/t}$  is increasing (decreasing) with respect to  $t$ , then  $(1 - F_k(t))^{1/t}$  is increasing  
417 (decreasing) in  $t$ .

416

418  $1 - F(t)$  is NBU (or NWU) if  $1 - F(t_1 + t_2) \leq (1 - F(t_1))(1 - F(t_2))$  (or  $1 - F(t_1 + t_2) \geq (1 - F(t_1))(1 - F(t_2))$ ).

418

419 Assume that  $(1 - F(t_1))(1 - F(t_2)) \geq 1 - F(t_1 + t_2)$ . According to (1), then it follows that

419

$$\begin{aligned} 1 - F_k(t_1 + t_2) &= e^{-B_k(t_1+t_2)} (1 - F(t_1 + t_2))^{A_k} \\ &\geq e^{-B_k(t_1+t_2)} (1 - F(t_1))^{A_k} (1 - F(t_2))^{A_k} \\ &= (1 - F_k(t_1))(1 - F_k(t_2)) \end{aligned} \quad (32)$$

420 A similar proof exists for the case  $1 - F(t_1 + t_2) \geq (1 - F(t_1))(1 - F(t_2))$ . This proves the theorem.

420