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# Uniqueness of $A_\infty$ -structures and Hochschild cohomology

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Working over a commutative ground ring, we establish a Hochschild cohomology criterion for uniqueness of derived  $A_\infty$ -algebra structures in the sense of Sagave. We deduce a Hochschild cohomology criterion for intrinsic formality of a differential graded algebra. This generalizes a classical result of Kadeishvili for the case of a graded algebra over a field.

[16E45](#); [16E40](#), [55S30](#)

## Introduction

$A_\infty$ -structures were introduced by Stasheff [[Sta63](#)] in the early 1960s in the study of topological spaces with products. They are now known to arise widely in algebra, geometry and mathematical physics, as well as topology.

We are interested in questions of formality and intrinsic formality for differential graded algebras. Thus we would like to establish conditions under which two differential graded algebras with the same homology are quasi-isomorphic. This has been studied by Keller and others in the case where the ground ring  $k$  is a field. It is related to the existence of different  $A_\infty$ -structures on a minimal model of the differential graded algebra.

An important structural result of Kadeishvili [[Kad80](#)] proves the existence of minimal models of differential graded algebras over a field while another classical theorem by Kadeishvili [[Kad88](#)] gives a criterion for uniqueness of certain minimal models using Hochschild cohomology.

For the applications we have in mind, which are related to rigidity of the model category structures arising in stable homotopy theory, we will be interested in working over local rings rather than fields. When working with a commutative ground ring rather than a field, one has to work with derived  $A_\infty$ -algebras as in the world of “classical”  $A_\infty$ -algebras, a differential graded algebra might not have a minimal model

if its homology is not projective. The theory of derived  $A_\infty$ -algebras was developed by Sagave in [Sag10]. He describes the notion of a minimal model for a differential graded algebra  $A$  over a commutative ground ring by giving a projective resolution of the homology of  $A$  that is compatible with the existing  $A_\infty$ -structure on  $A$ .

Our main result is Theorem 3.7 which extends Kadeishvili's uniqueness theorem to derived  $A_\infty$ -algebras. For this we develop a new notion of Hochschild cohomology. After some further work we again obtain a Hochschild cohomology criterion for intrinsic formality of a differential graded algebra over a commutative ring rather than a field, Theorem 4.4.

In the subsequent sections we return to classical  $A_\infty$ -algebras and derive some further generalizations of Kadeishvili's uniqueness criterion. The first of these is Theorem 5.3 which studies uniqueness of an  $A_\infty$ -structure on a fixed differential graded algebra. The other, Theorem 6.3, discusses differential graded algebras with fixed Massey products on their homology.

An alternative approach is developed by Dugger and Shipley. In [DS07, Section 3] they consider the classification of quasi-isomorphism types of differential graded algebras with given homology. They do this by building differential graded algebras up degreewise via a theory of Postnikov sections and  $k$ -invariants. To do so requires working with bounded below differential graded algebras, a restriction which does not apply to our methods. The  $k$ -invariants live in derived Hochschild cohomology groups of the Postnikov sections with coefficients in the next homology group of the differential graded algebra being built. Their work does not consider  $A_\infty$ -structures and although also formulated in terms of Hochschild cohomology, there does not seem to be a very direct relationship between their methods and ours. However, we are going to put some of their examples in context throughout our paper.

This paper is organized as follows. In Section 1 we recall basic definitions relating to  $A_\infty$ -algebras and Hochschild cohomology. In Section 2 we recall Sagave's construction of derived  $A_\infty$ -algebras and his results about minimal models. This section also introduces the Lie algebra structure which leads to the definition of Hochschild cohomology of a certain class of derived  $A_\infty$ -algebras in Section 3. At the end of Section 3 we show that the vanishing of certain Hochschild cohomology groups gives a sufficient condition for the existence of a unique derived  $A_\infty$ -structure on a fixed underlying object. In Section 4 we deduce the criterion for intrinsic formality of differential graded algebras over a commutative ground ring. Finally, in Sections 5 and 6 we discuss the

previously mentioned analogues of these results for classical  $A_\infty$ -structures. A short appendix is devoted to sign issues.

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## 1 A quick review of $A_\infty$ -algebras

We assume that the reader is familiar with the basic definitions regarding  $A_\infty$ -algebras and Hochschild cohomology, but we are going to recall some of them in this section to establish notation and assumptions. We are going to be very brief with this; the explicit formulas and definitions regarding derived  $A_\infty$ -algebras given in the later Sections 2 and 3 specialize to the case of “classical”  $A_\infty$ -algebras. For greater detail we refer to Keller’s introductory paper [Kel01].

The notion of an  $A_\infty$ -algebra arose with the study of loop spaces in topology and has since become an increasingly important and powerful subject in algebraic topology and homological algebra. Roughly speaking,  $A_\infty$ -algebras are not necessarily associative algebras with given maps for “multiplying”  $n$  elements for each  $n$ , unlike in the case of associative algebras where one knows how to multiply  $n$  elements from knowing how to multiply two elements.

### 1.1 Basic definitions

In Sections 1 and 6 of this paper,  $k$  will denote a field of characteristic not equal to 2. In Sections 2 to 5 we will allow  $k$  to be a commutative ring rather than a field. Note that in fact Sections 1 and 6 do not require a ground field as long as all  $k$ -modules in question are projective.

All unadorned tensor products are over  $k$ . All graded objects will be  $\mathbb{Z}$ -graded unless stated otherwise. Our convention for the degree of a map  $f$  is as follows: a map of graded  $k$ -vector spaces  $f : A \rightarrow B$  of degree  $i$  consists of a sequence of maps  $f^n : A^n \rightarrow B^{n+i}$ . (Later this will be called the internal degree and there will also be a notion of cohomological or external degree.) We often abbreviate ‘differential graded algebra’ to dga.

**Definition 1.1** Let  $A = \bigoplus_{n \in \mathbb{Z}} A^n$  be a graded  $k$ -vector space. An  $A_\infty$ -structure on  $A$  is a sequence of  $k$ -linear maps

$$m_j : A^{\otimes j} \longrightarrow A \quad \text{for } j \geq 1$$

of degree  $2 - j$  satisfying the equation

$$\sum_{n=r+s+t} (-1)^{rs+t} m_{1+r+t}(1^{\otimes r} \otimes m_s \otimes 1^{\otimes t}) = 0$$

for each  $n \geq 1$ . An  $A_\infty$ -algebra is a graded  $k$ -vector space  $A$  together with an  $A_\infty$ -structure on  $A$ .

Further all  $A_\infty$ -algebras are assumed to be strictly unital; c.f. Definition 2.1. We are using the sign convention of Sagave [Sag10, (2.6)] and of Lefèvre-Hasegawa [LH03, 1.2.1.2] rather than of Keller [Kel01].

Note that we are applying the Koszul sign rule when applying such formulas to elements:

$$(f \otimes g)(x \otimes y) = (-1)^{|g||x|} f(x) \otimes g(y).$$

In particular, this definition gives us

$$m_1 m_1 = 0,$$

i.e.  $m_1$  is a differential on  $A$ . It also yields the following special cases: if  $m_k = 0$  for all  $k \neq 2$ , then  $A$  is simply a graded associative algebra. If  $m_k = 0$  for  $k \geq 3$ , then  $A$  is a differential graded algebra.

There are also notions of morphism and quasi-isomorphism of  $A_\infty$ -algebras; these are special cases of Definitions 2.3 and 2.4.

**Notation** We sometimes write an  $A_\infty$ -structure as a formal infinite sum, i.e.

$$m = m_1 + m_2 + \cdots .$$

Note that all infinite sums in this paper are finite in every degree.

## 1.2 Hochschild cohomology and Lie structure

Hochschild cohomology is a very powerful tool in many areas around algebra and topology, from relations to the geometry of loop spaces to deformation theory of algebras and realizability questions in topology. The definition of Hochschild cohomology of associative graded algebras can be extended to a definition of Hochschild cohomology of  $A_\infty$ -algebras. A convenient way of doing this is using a Lie algebra structure on the bigraded  $k$ -vector space

$$C^{n,m}(A, A) = \text{Hom}_k^m(A^{\otimes n}, A) = \prod_i \text{Hom}_k((A^{\otimes n})^i, A^{i+m}),$$

where  $n \in \mathbb{N}$ ,  $m \in \mathbb{Z}$  and  $A$  is a graded  $k$ -vector space.

Explicitly, for  $f \in C^{n,k}(A, A)$  and  $g \in C^{m,l}(A, A)$  the Lie bracket is given by

$$\begin{aligned} [f, g] &= \sum_{i=0}^{n-1} (-1)^{(n-1)(m-1)+(n-1)l+i(m-1)} f(1^{\otimes i} \otimes g \otimes 1^{\otimes n-i-1}) \\ &\quad - (-1)^{(n+k-1)(m+l-1)} \sum_{i=0}^{m-1} (-1)^{(m-1)(n-1)+(m-1)k+i(n-1)} g(1^{\otimes i} \otimes f \otimes 1^{\otimes m-i-1}) \end{aligned}$$

which lies in  $C^{n+m-1, l+k}(A, A)$ . This gives  $C^{*,*}(A, A)$  the structure of a graded Lie algebra, where the grading is by total degree shifted by 1; see e.g. [FP02, Section 2], [Get94, Section 1], [Ger63] or [PS95]. Note that the formula given in some of the references has signs arising from the Koszul rule because it is given evaluated on elements rather than as a formula of morphisms. For details on how this formula arises, see Section 2.2 and the Appendix.

**Lemma 1.2** Let  $m \in C^{*,*}(A, A)$  of total degree 2. Then  $m$  is an  $A_\infty$ -structure on  $A$  if and only if  $[m, m] = 0$ . Further, for such  $m$ ,

$$D := [m, -] : C^{*,*}(A, A) \longrightarrow C^{*,*}(A, A)$$

is a differential on  $C^{*,*}(A, A)$ , i.e.  $D$  raises total degree by 1 and satisfies  $D \circ D = 0$ .

**Proof** The first claim follows immediately from the bracket formula and the fact that 2 is invertible. The fact that  $D \circ D = 0$  is an immediate consequence of the graded Jacobi identity, while the total degree of  $D$  can be computed directly.  $\square$

**Definition 1.3** Let  $A$  be an  $A_\infty$ -algebra with  $A_\infty$ -structure  $m$ . Then the *Hochschild cohomology of the  $A_\infty$ -algebra  $A$*  is defined as

$$\text{HH}^*(A, A) = H^* \left( \prod_i C^{i, *-i}(A, A), [m, -] \right).$$

For this, see, for example, [PS95, §5]. If  $A$  is an associative algebra (i.e.  $m = m_2$ ), a direct computation using the above definitions shows this recovers the usual definition of the Hochschild cohomology of associative algebras, i.e. for  $f \in C^{n,k}(A, A)$ ,

$$[m_2, f] = (-1)^k \left( m_2(1 \otimes f) + \sum_{i=0}^{n-1} (-1)^{i+1} f(1^{\otimes i} \otimes m_2 \otimes 1^{\otimes n-1-i}) + (-1)^{n+1} m_2(f \otimes 1) \right).$$

The grading in Definition 1.3 refers to the total degree. In the case of an associative algebra the differential

$$[m_2, -] : C^{*,*}(A, A) \longrightarrow C^{*+1,*}(A, A)$$

preserves internal degree so we can split the total degree of the Hochschild cohomology into the cohomological degree and the internal degree. We denote the bigraded Hochschild cohomology in this special case by  $\mathrm{HH}_{alg}^{*,*}(A, A)$ . For a general  $A_\infty$ -algebra, we do not have a bigrading, but we can introduce a filtration, see Definition 5.2.

For  $A$  a dga, the definition can be interpreted in terms of bicomplexes. The dga  $A$  has differential  $m_1$  and multiplication  $m_2$ . The bigraded module  $C^{*,*}(A, A)$  becomes a bicomplex by taking

$$d^v := [m_1, -] : C^{*,*} \longrightarrow C^{*,*+1}$$

to be the vertical differential and

$$d^h := [m_2, -] : C^{*,*} \longrightarrow C^{*+1,*}$$

to be the horizontal differential. The condition

$$[m_1 + m_2, m_1 + m_2] = 0$$

translates into  $(d^v)^2 = 0$ ,  $(d^h)^2 = 0$  and  $d^v d^h + d^h d^v = 0$ , which are exactly the conditions for  $C^{*,*}(A, A)$  to be a bicomplex [Wei94, 1.2.4].

### 1.3 Minimal models and uniqueness

We now recall a definition and theorem about minimal models of  $A_\infty$ -algebras. It relates differential graded algebras to  $A_\infty$ -structures on their homology.

**Definition 1.4** An  $A_\infty$ -algebra is called *minimal* if  $m_1 = 0$ .

Over a field, one can replace any  $A_\infty$ -algebra by a quasi-isomorphic minimal one which gives a very convenient way to describe a quasi-isomorphism class of an  $A_\infty$ -algebra. We are particularly interested in the special case of differential graded algebras.

**Theorem 1.5** (Kadeishvili) Let  $A$  be a differential graded algebra over a field  $k$ , and let  $H^*(A)$  be its homology module. Then  $H^*(A)$  has an  $A_\infty$ -structure such that

- $m_1 = 0$  and the multiplication  $m_2$  is induced by the multiplication on  $A$ ,
- there is a morphism of  $A_\infty$ -algebras  $f : H^*(A) \rightarrow A$  such that  $f_1$  is a quasi-isomorphism.

This  $A_\infty$ -algebra  $H^*(A)$  is called the *minimal model* of  $A$ .

For more details, see [Kad80]. Note that the theorem states in particular that the minimal model  $H^*(A)$  is quasi-isomorphic to  $A$  as an  $A_\infty$ -algebra. This is useful in combination with a uniqueness result in [Kad88].

**Definition 1.6** We say that an  $A_\infty$ -structure  $m$  is *trivial* if  $m_n = 0$  for  $n \geq 3$ .

**Theorem 1.7** (Kadeishvili) Let  $C$  be a graded  $k$ -algebra with multiplication  $\mu$ . If

$$\mathrm{HH}_{\mathrm{alg}}^{n,2-n}(C, C) = 0 \quad \text{for } n \geq 3,$$

then every  $A_\infty$ -structure on  $C$  with  $m_1 = 0$  and  $m_2 = \mu$  is quasi-isomorphic to the trivial one.

We can reformulate this in terms of formality of dgas. We recall the following standard definitions.

**Definition 1.8** (1) A dga  $A$  is *formal* if it is quasi-isomorphic to its homology  $H^*(A)$  regarded as a dga with trivial differential.

(2) A dga  $A$  is *intrinsically formal* if any other dga  $A'$  such that  $H^*(A) \cong H^*(A')$  as associative algebras is quasi-isomorphic to  $A$ .

If a dga is intrinsically formal then it is formal, but the converse need not hold. For example, in [DS07, Example 3.15], it is shown that there are two quasi-isomorphism types of dgas with homology an exterior algebra over  $\mathbb{F}_p$  on an even degree generator. The trivial one is therefore formal but not intrinsically formal.

Using Theorem 1.7 for the case  $C = H^*(A)$  yields the following.

**Corollary 1.9** Let  $A$  be a dga and  $H^*(A)$  its homology algebra. Suppose that

$$\mathrm{HH}_{\mathrm{alg}}^{n,2-n}(H^*(A), H^*(A)) = 0 \quad \text{for } n \geq 3.$$

Then  $A$  is intrinsically formal.

In Section 5, we will recover these results as special cases of our derived versions.



## 2 Derived $A_\infty$ -algebras

To work with Kadeishvili's minimal models and to establish the uniqueness theorems, one has to assume all dgas as well as their homology algebras to be degreewise projective, hence the assumption of a ground field. However, there are important examples arising from homotopy theory where projectivity cannot be guaranteed. In 2008, Sagave introduced the notion of derived  $A_\infty$ -algebras, providing a framework for not necessarily projective modules over an arbitrary commutative ground ring [Sag10].

First of all, we recall some definitions and results about derived  $A_\infty$ -algebras; we refer to Sagave's paper for the finer technical details.

The basic idea is to introduce degreewise projective resolutions for an  $A_\infty$ -algebra that are compatible with the  $A_\infty$ -structure. This will introduce another internal grading.

### 2.1 Definitions, conventions and known results

All definitions and results in this subsection have been developed by Sagave in [Sag10] and we refer to his paper for technical details.

Let  $k$  be a commutative ring and let  $A$  be an  $(\mathbb{N}, \mathbb{Z})$ -bigraded  $k$ -module, i.e.

$$A = \bigoplus_{i \in \mathbb{N}, j \in \mathbb{Z}} A_i^j.$$

A morphism of bigraded  $k$ -modules  $f : A \rightarrow B$  of bidegree  $(s, t)$  is a sequence of maps of  $k$ -modules  $f : A_i^j \rightarrow B_{i-s}^{j+t}$  for all  $i \in \mathbb{N}$  and  $j \in \mathbb{Z}$ . Again, we follow the Koszul sign convention: for  $g$  a morphism of bidegree  $(s, t)$  and  $x$  an element of bidegree  $(i, j)$ , we have

$$(f \otimes g)(x \otimes y) = (-1)^{is+jt} f(x) \otimes g(y).$$

The homological (subscript) bidegree is called the *horizontal bidegree* and the cohomological (superscript) bidegree is called the *vertical bidegree*.

Throughout the rest of the paper we also assume that all bigraded modules have no 2-torsion.

**Definition 2.1** [Sag10, Definition 2.1] A *derived  $A_\infty$ -structure* (or  *$dA_\infty$ -structure* for short) on an  $(\mathbb{N}, \mathbb{Z})$ -bigraded  $k$ -module  $A$  consists of  $k$ -linear maps

$$m_{ij} : A^{\otimes j} \rightarrow A$$

of bidegree  $(i, 2 - (i + j))$  for each  $j \geq 1, i \geq 0$ , satisfying the equation

$$(1) \quad \sum_{\substack{u=i+p, v=j+q-1 \\ j=1+r+t}} (-1)^{rq+t+pj} m_{ij}(1^{\otimes r} \otimes m_{pq} \otimes 1^{\otimes t}) = 0$$

for all  $u \geq 0$  and  $v \geq 1$ . A  $dA_\infty$ -algebra is a bigraded  $k$ -module together with a  $dA_\infty$ -structure.

A  $dA_\infty$ -algebra  $A$  is called *strictly unital* if there is a unit map  $\eta : k \rightarrow A$  such that

- $m_{01}(\eta) = 0$ ,
- $m_{02}(\eta \otimes 1) = 1 = m_{02}(1 \otimes \eta)$ ,
- $m_{ij}(1^{\otimes r-1} \otimes \eta \otimes 1^{\otimes j-r}) = 0$  for  $i + j \geq 3, 1 \leq r \leq j$ .

From now on, all  $dA_\infty$ -algebras are assumed to be strictly unital.

**Remark** A  $dA_\infty$ -algebra concentrated in horizontal degree 0 (and hence with  $m_{ij} = 0$  for all  $i \neq 0$ ) is the same as an  $A_\infty$ -algebra.

A  $dA_\infty$ -algebra with  $m_{ij} = 0$  except  $m_{01}$  and  $m_{11}$  is just a bicomplex (with a different sign convention to that encountered earlier) with horizontal differential  $m_{11}$  and vertical differential  $m_{01}$  as the definition in this case forces  $m_{11}m_{11} = 0, m_{01}m_{01} = 0$  and  $m_{01}m_{11} - m_{11}m_{01} = 0$ .

**Definition 2.2** A *bidga* is a monoid in the category of bicomplexes; equivalently, a bidga is a  $dA_\infty$ -algebra with  $m_{ij} = 0$  for  $i + j \geq 3$ . (See [Sag10, Definition 2.10 and Remark 2.11].)

**Definition 2.3** [Sag10, Definition 2.5] Let  $A$  and  $B$  be  $dA_\infty$ -algebras with  $dA_\infty$ -structures  $m$  and  $\bar{m}$ , respectively. A *morphism of  $dA_\infty$ -algebras*  $f : A \rightarrow B$  consists of a family of  $k$ -module maps

$$f_{st} : A^{\otimes t} \rightarrow B$$

of bidegree  $(s, 1 - (s + t))$  satisfying

$$(2) \quad \sum_{\substack{u=i+p, v=j+q-1 \\ j=1+r+t}} (-1)^{rq+t+pj} f_{ij}(1^{\otimes r} \otimes m_{pq} \otimes 1^{\otimes t}) = \sum_{\substack{u=i+p_1+\dots+p_j \\ v=q_1+\dots+q_j}} (-1)^\epsilon \bar{m}_{ij}(f_{p_1q_1} \otimes \dots \otimes f_{p_jq_j})$$

for all  $u \geq 0$  and  $v \geq 1$ . Here,

$$\epsilon = u + \sum_{w=1}^{j-1} \left( jp_w + w(q_{j-w} - p_w) + q_{j-w} \left( \sum_{s=j-w+1}^j p_s + q_s \right) \right).$$

For strictly unital  $dA_\infty$ -algebras, morphisms are required to satisfy the unit conditions  $f_{01}\eta = \bar{\eta}$  and  $f_{ij}(1^{\otimes r-1} \otimes \eta \otimes 1^{\otimes j-r}) = 0$  for  $i+j \geq 2$  and  $1 \leq r \leq j$ .

Recall that a quasi-isomorphism of  $A_\infty$ -algebras is a morphism of  $A_\infty$ -algebras that induces a quasi-isomorphism of complexes with respect to  $m_1$ . In the case of  $dA_\infty$ -algebras, the role of the quasi-isomorphisms is played by the so-called  $E_2$ -equivalences. These are the morphisms that induce an isomorphism of  $E_2$ -terms of the spectral sequence computing the homology of the total complex of a bicomplex, see [McC01, 2.12].

**Notation** The equations defining a  $dA_\infty$ -structure include  $m_{01}m_{01} = 0$ . For a  $dA_\infty$ -algebra  $A$  let  $H_{ver}^*$  denote its homology with respect to the *vertical differential*  $m_{01}$ . The map  $m_{01}$  is called the vertical differential because it raises the vertical degree.

Since the equations defining a  $dA_\infty$ -structure also include

$$m_{21}m_{01} - m_{11}m_{11} + m_{01}m_{21} = 0,$$

it follows that the map  $m_{11}$  becomes a differential in horizontal direction on the bigraded module  $H_{ver}^*(A)$ , so we can form  $H_{hor}^*(H_{ver}^*(A)) = H^*(H_{ver}^*(A), m_{11})$ .

**Definition 2.4** A morphism  $f : A \rightarrow B$  of  $dA_\infty$ -algebras is called an  $E_2$ -equivalence if

$$H_{hor}^*(H_{ver}^*(f_{01}))$$

is an isomorphism of  $k$ -modules, c.f. [Sag10, Definition 2.19].

We would like to extend some applications of  $A_\infty$ -algebras to differential graded algebras that are not necessarily projective over the ground ring  $k$  or whose homology is not projective. The problem we encounter is that not all differential graded algebras possess a minimal model as an  $A_\infty$ -algebra. However, Sagave showed that dgas have reasonable minimal models in the world of  $dA_\infty$ -algebras. For this, one has to apply a special projective resolution.

**Definition 2.5** [Sag10, Definition 3.1] Let  $A$  be a graded algebra. A *termwise  $k$ -projective resolution* of  $A$  is a termwise  $k$ -projective bidga  $P$  with  $m_{01} = 0$  together with an  $E_2$ -equivalence  $P \rightarrow A$ .

**Definition 2.6** [Sag10, Definition 3.2] Let  $A$  be a dga. A  $k$ -projective  $E_1$ -resolution of  $A$  is a bidga  $B$  together with an  $E_2$ -equivalence  $B \rightarrow A$  such that  $H_{ver}^{st}(B)$  is projective for each bidegree. Further, the map  $k \rightarrow H_{ver}^{00}(B)$  induced by the unit  $k \rightarrow B$  is required to split as a  $k$ -module map.

Thus a  $k$ -projective  $E_1$ -resolution of a dga  $A$  induces a termwise  $k$ -projective resolution of the graded homology algebra of  $A$ .

Sagave then proceeds to show that a  $k$ -projective  $E_1$ -resolution is unique up to  $E_2$ -equivalence.

**Theorem 2.7** [Sag10, Theorem 3.4] Every dga  $A$  over  $k$  admits a  $k$ -projective  $E_1$ -resolution. Two such resolutions can be related by a zig-zag of  $E_2$ -equivalences between  $k$ -projective  $E_1$ -resolutions.

**Definition 2.8** A  $dA_\infty$ -algebra is called *minimal* if  $m_{01} = 0$ .

**Theorem 2.9** [Sag10, Theorem 1.1] Let  $A$  be a dga over  $k$ . Then there is a degreewise  $k$ -projective  $dA_\infty$ -algebra  $E$  together with an  $E_2$ -equivalence  $E \rightarrow A$  such that

- $E$  is minimal,
- $E$  is well-defined up to  $E_2$ -equivalence,
- together with the differential  $m_{11}$  and the multiplication  $m_{02}$ ,  $E$  is a termwise  $k$ -projective resolution of the graded algebra  $H^*(A)$ .

To prove this, Sagave starts with a  $k$ -projective  $E_1$ -resolution  $E \rightarrow A$ . He then shows that the vertical homology  $H_{ver}^*(E)$  admits a  $dA_\infty$ -structure satisfying the claims of the theorem.

However, not every termwise projective resolution of  $H^*(A)$  admits such a structure [Sag10, Remark 4.14.]. For example, consider the dga over  $\mathbb{Z}$

$$A = \mathbb{Z}[e]/(e^4), \quad \partial(e) = p, \quad |e| = -1,$$

also examined by Dugger and Shipley in [DS07, Example 3.13]. The bidga

$$C = \mathbb{Z}\langle a, b \rangle / (a^2, b^2, ab - ba), \quad |a| = (1, 0), |b| = (0, -2), \quad m_{11}(b) = p$$

is a termwise projective resolution of  $H^*(A) = \Lambda_{\mathbb{Z}/p}([e^2])$ , but there is no  $dA_\infty$ -structure on  $C$  admitting an  $E_2$ -equivalence  $C \rightarrow A$ . (For example, equation (2) for  $(u, v) = (2, 2)$  forces  $m_{22}(b \otimes b) \equiv \pm 1 \pmod{p}$  whereas equation (1) for  $(u, v) = (2, 3)$  forces  $m_{22}(b \otimes b) \equiv 0 \pmod{p}$ .)

**Definition 2.10** Let  $A$  and  $E$  be as in Theorem 2.9. Such an  $E$  is called a *minimal model* of  $A$ .

**Remark** Note that in the context of Theorem 2.9, the underlying  $k$ -module of the minimal model  $E$  together with the differentials  $m_{01}$  and  $m_{11}$  and the multiplication  $m_{02}$  form a bidga.

## 2.2 Lie algebra structure on $C_{*}^{*,*}(A, A)$

We would like to establish a reasonable notion of Hochschild cohomology for  $dA_{\infty}$ -algebras. In order to give a simple description, it is our goal to describe the Hochschild cohomology in terms of a graded Lie algebra structure.

Let  $A$  be a  $(\mathbb{N}, \mathbb{Z})$ -bigraded module without 2-torsion over a commutative ring. Define

$$C_k^{n,i}(A, A) = \prod_{u,v} \text{Hom}((A^{\otimes n})_u^v, A_{u-k}^{v+i}).$$

We are going to define a Lie algebra structure on  $C_{*}^{*,*}(A, A)$  generalizing Section 1.2. First of all, we define a bracket operation that is not a Lie bracket. Then we are going to introduce a shift operation on elements of  $C_{*}^{*,*}(A, A)$  and then define the actual Lie bracket using this shift and the previously defined bracket operation.

For  $f \in C_k^{n,i}(A, A)$  and  $g \in C_l^{m,j}(A, A)$  we now define

$$\begin{aligned} \llbracket f, g \rrbracket &= \sum_{v=0}^{n-1} f(1^{\otimes v} \otimes g \otimes 1^{\otimes n-v-1}) \\ &\quad - (-1)^{ij+kl} \sum_{v=0}^{m-1} g(1^{\otimes v} \otimes f \otimes 1^{\otimes m-v-1}) \quad \in C_{k+l}^{n+m-1, i+j}(A, A) \end{aligned}$$

This is not the actual Lie bracket but the first step in our construction. For degree and sign reasons we have to introduce a shift map.

Let  $S(A)$  be the bigraded module with  $S(A)_u^v = A_u^{v+1}$ , and so the suspension map  $S : A \rightarrow S(A)$  given by the identity map in each bigree has internal bigree  $(0, -1)$ . Given  $f \in C_k^{n,i}(A, A)$ , then

$$\sigma(f) = (-1)^{n+i+k-1} S \circ f \circ (S^{-1})^{\otimes n} \in C_k^{n, i+n-1}(S(A), S(A)).$$

Conversely, for  $F \in C_l^{m,j}(S(A), S(A))$ , we define

$$\sigma^{-1}(F) = (-1)^{j+l+\binom{m}{2}} S^{-1} \circ F \circ S^{\otimes m} \in C_l^{m,j+1-m}(A, A),$$

so  $\sigma^{-1}(\sigma(f)) = f$ .

Particularly, for  $m_{ij} \in C_i^{j,2-(i+j)}(A, A)$ , we have  $\sigma(m_{ij}) \in C_i^{j,1-i}(S(A), S(A))$ . Note that the notation  $\sigma(f)$  does not mean applying a shift functor to  $f$ .

We now define

$$\begin{aligned} [f, g] &:= \sigma^{-1} \llbracket \sigma(f), \sigma(g) \rrbracket \\ &= \sum_{v=0}^{n-1} (-1)^{(n-1)(m-1)+v(m-1)+j(n-1)} f(1^{\otimes v} \otimes g \otimes 1^{\otimes n-v-1}) \\ &\quad - (-1)^{\langle f, g \rangle} \sum_{v=0}^{m-1} (-1)^{(m-1)(n-1)+v(n-1)+i(m-1)} g(1^{\otimes v} \otimes f \otimes 1^{\otimes m-v-1}) \\ &\in C_{k+l}^{n+m-1, i+j}(A, A) \end{aligned}$$

for  $f \in C_k^{n,i}(A, A)$  and  $g \in C_l^{m,j}(A, A)$ . Here,  $\langle f, g \rangle := (n+i-1)(m+j-1) + kl$ . (See the Appendix for this computation.) It is easy to see that in the case of bigraded modules concentrated in horizontal degree 0 this specializes to the Lie algebra structure given in Section 1.2.

As earlier, we use formal infinite sums of morphisms. These are now bigraded and any such sum is actually finite in any given bidegree.

**Remark** It is also possible to work with a different definition of the shift  $\sigma$  on morphisms. Instead of our convention

$$\sigma(f) = (-1)^{k+n+i-1} S \circ f \circ (S^{-1})^{\otimes n},$$

it is also possible to work with

$$\bar{\sigma}(f) = (-1)^{k+n+i-1} S \circ f \circ (S^{\otimes n})^{-1}$$

as in [Kel01, 3.6] which differs from the above  $\sigma$  by the sign  $(-1)^{\binom{n}{2}}$ . Working with  $\bar{\sigma}$  would recover Keller's sign convention in the definition of  $A_\infty$ -algebras and their morphisms, whereas our choice of  $\sigma$  recovers the signs of Lefèvre-Hasegawa and Sagave.

It is convenient to describe the above bracket in terms of a composition product as in [Ger63].

**Definition 2.11** For  $f \in C_k^{n,i}(A, A)$  and  $g \in C_l^{m,j}(A, A)$  we define the composition product  $\circ$  by

$$\begin{aligned} f \circ g &= \sum_{v=0}^{n-1} \sigma^{-1} \left( (\sigma(f)(1^{\otimes v} \otimes \sigma(g) \otimes 1^{\otimes n-v-1}) \right) \\ &= \sum_{v=0}^{n-1} (-1)^{(m-1)(n-1)+v(m-1)+j(n-1)} f(1^{\otimes v} \otimes g \otimes 1^{\otimes n-v-1}) \in C_{k+l}^{n+m-1, i+j}(A, A). \end{aligned}$$

Hence, we have that

$$[f, g] = f \circ g - (-1)^{\langle f, g \rangle} g \circ f.$$

We will show that with this bracket  $C_*^{*,*}(A, A)$  can be regarded as a bigraded Lie algebra in the sense of the following definition.

**Definition 2.12** A bigraded  $k$ -module  $X = \bigoplus X_i^j$  is a bigraded Lie algebra if there is a bracket operation  $[-, -] : X \otimes X \rightarrow X$  satisfying

- $[g, f] = -(-1)^{ab+kl}[f, g]$ ,
- $(-1)^{ac+km}[[f, g], h] + (-1)^{ab+kl}[[g, h], f] + (-1)^{bc+lm}[[h, f], g] = 0$ ,

for  $f \in X_k^a$ ,  $g \in X_l^b$ ,  $h \in X_m^c$ .

**Proposition 2.13** The above bracket gives  $C_*^{*,*}(A, A)$  the structure of a bigraded Lie algebra for the bigrading where  $f \in C_k^{n,i}$  is given bidegree  $(k, n + i - 1)$ ; i.e. for all  $f, g, h \in C_*^{*,*}(A, A)$ ,

- $[g, f] = -(-1)^{\langle f, g \rangle}[f, g]$ ,
- $(-1)^{\langle f, h \rangle}[[f, g], h] + (-1)^{\langle g, f \rangle}[[g, h], f] + (-1)^{\langle h, g \rangle}[[h, f], g] = 0$ .

**Proof** The first point is immediate. For the graded Jacobi identity we will show that the composition product  $\circ$  makes  $C_*^{*,*}(A, A)$  a bigraded pre-Lie ring in the sense that for  $f \in C_k^{n,i}(A, A)$ ,  $g \in C_l^{m,j}(A, A)$  and  $h \in C_w^{u,v}(A, A)$ , we have

$$(3) \quad (h \circ f) \circ g - (-1)^{\langle f, g \rangle} (h \circ g) \circ f = h \circ (f \circ g) - (-1)^{\langle f, g \rangle} h \circ (g \circ f).$$

We can then apply a direct computation analogous to the proof of Theorem 1 of [Ger63] which proves the claim. (For this, we note that  $\langle f \circ g, h \rangle = \langle f, h \rangle + \langle g, h \rangle$ .)

To prove the equation (3), we note that

$$f \circ g = \sigma^{-1}(\sigma(f) \odot \sigma(g))$$

with

$$F \odot G := \sum_{r=1}^{n-1} F(1^{\otimes r} \otimes G \otimes 1^{\otimes n-r-1}).$$

This is going to simplify the signs in (3) considerably since this equation is equivalent to

$$(4) \quad (H \odot F) \odot G - (-1)^{\langle f, g \rangle} (H \odot G) \odot F = H \odot (F \odot G) - (-1)^{\langle f, g \rangle} H \odot (G \odot F)$$

for  $F = \sigma(f)$ ,  $G = \sigma(g)$  and  $H = \sigma(h)$ . We have

$$\begin{aligned} (H \odot F) \odot G &= \left( \sum_{r=0}^{u-1} H(1^{\otimes r} \otimes F \otimes 1^{\otimes u-r-1}) \right) \odot G \\ &= (-1)^{\langle f, g \rangle} \sum_{r=0}^{u-1} \sum_{a+b=r-1} H(1^{\otimes a} \otimes G \otimes 1^{\otimes b} \otimes F \otimes 1^{\otimes u-r-1}) \\ &\quad + \sum_{r=0}^{u-1} \sum_{s=0}^{n-1} H(1^{\otimes r} \otimes F(1^{\otimes s} \otimes G \otimes 1^{\otimes n-s-1}) \otimes 1^{\otimes u-r-1}) \\ &\quad + \sum_{r=0}^{u-1} \sum_{a+b=u-r-2} H(1^{\otimes r} \otimes F \otimes 1^{\otimes a} \otimes G \otimes 1^{\otimes b}). \end{aligned}$$

Note that the sign  $(-1)^{\langle f, g \rangle}$  in the first summand arises from the Koszul sign rule for interchanging  $F$  and  $G$ . Using this, we can read off the equation (4), from which (3) follows.  $\square$

Now we would like to describe derived  $A_\infty$ -structures in terms of this Lie algebra structure, but first we have to introduce another operation which alters signs.

**Definition 2.14** For  $f \in C_k^{n,i}(A, A)$  define  $f^\# = (-1)^k f \in C_k^{n,i}(A, A)$ .

This operation satisfies

- $(f^\#)^\# = f$ ,
- $(f \circ g)^\# = f^\# \circ g^\#$ ,
- $[f, g]^\# = [f^\#, g^\#]$ .

**Proposition 2.15** Let  $A$  be a bigraded  $k$ -module without 2-torsion with given map  $\eta : k \rightarrow A$ . Let

$$m = \sum_{i \geq 0, j \geq 1} m_{ij}$$



with  $m_{ij} \in C_i^{j, 2-(i+j)}(A, A)$  satisfying the unit conditions of Definition 2.1.

Then the following are equivalent

- $m$  is a derived  $A_\infty$ -structure on  $A$ ,
- $m \circ m^\# = 0$ ,
- $[m, m^\#] = 0$ .

**Proof** The equivalence of the first two points follows immediately from the definitions. For the equivalence of the last two points let us consider the part  $[m, m^\#]_u$  of  $[m, m^\#]$  that lies in horizontal degree  $u$ . We have

$$\begin{aligned} [m, m^\#]_u &= \sum_{u=i+p} (m_{ij} \circ m_{pq}^\# - (-1)^{ip+(i-1)(p-1)} m_{ij}^\# \circ m_{pq}) \\ &= \sum_{u=i+p} ((-1)^p m_{ij} \circ m_{pq} - (-1)^{u+1+i} m_{ij} \circ m_{pq}). \end{aligned}$$

We are going to distinguish between the cases  $u$  even and  $u$  odd. For even  $u = i + p$ , the sum splits into the cases where either both  $i$  and  $p$  are even or both  $i$  and  $p$  are odd. In either case, we can read off that

$$[m, m^\#]_u = 2(m \circ m^\#)_u.$$

The case of  $u$  odd follows similarly. □

### 3 Hochschild cohomology and uniqueness of derived $A_\infty$ -algebras

#### 3.1 Hochschild cohomology of $dA_\infty$ -algebras

We would like to define a notion of Hochschild cohomology for  $dA_\infty$ -algebras that extends the classical, non-derived case. However, this is not as straightforward as before. In the classical case of an  $A_\infty$ -algebra  $A$  with  $A_\infty$ -structure  $m$ , we could define a differential on  $C^{*,*}(A, A)$  via  $D = [m, -]$ . This satisfies  $D \circ D = [m, [m, -]] = 0$  since  $[m, m] = 0$ . But in the derived case the signs are slightly more complicated which means we can only guarantee  $[m, m^\#] = 0$ . We can still define Hochschild cohomology for a certain class of  $dA_\infty$ -algebra which includes the cases we are interested in.

**Definition 3.1** Let  $m = \sum_{i \geq 0, j \geq 1} m_{ij}$  be a  $dA_\infty$ -structure. Then we denote the horizontal even degree part by  $m_{\text{even}}$  and the horizontal odd degree part by  $m_{\text{odd}}$ , i.e.,

$$m_{\text{even}} = \sum_{i \text{ even}} m_{ij} \quad \text{and} \quad m_{\text{odd}} = \sum_{i \text{ odd}} m_{ij}.$$

**Remark** Since  $m$  is a  $dA_\infty$ -structure, by Lemma 2.15 we have  $(m_{\text{even}} + m_{\text{odd}}) \circ (m_{\text{even}} - m_{\text{odd}}) = 0$ , which splits as

$$m_{\text{even}} \circ m_{\text{even}} = m_{\text{odd}} \circ m_{\text{odd}} \quad \text{and} \quad m_{\text{even}} \circ m_{\text{odd}} = m_{\text{odd}} \circ m_{\text{even}}.$$

**Definition 3.2** We call a derived  $A_\infty$ -structure  $m$  *orthogonal* if

$$m_{\text{even}} \circ m_{\text{even}} = 0 \quad \text{or, equivalently,} \quad m_{\text{odd}} \circ m_{\text{odd}} = 0.$$

**Example** Bidgas are orthogonal since they have  $m_{\text{odd}} = m_{11}$  and  $m_{11} \circ m_{11} = 0$ .

**Lemma 3.3** Let  $A$  be a bigraded  $k$ -module without 2-torsion and let  $m = \sum_{i,j} m_{ij}$  be an orthogonal derived  $A_\infty$ -structure on  $A$ . Define

$$D : C_*^{*,*}(A, A) \longrightarrow C_*^{*,*}(A, A)$$

via

$$D(f) = [m_{\text{even}}, f^\#] + [m_{\text{odd}}, f] = (-1)^k [m_{\text{even}}, f] + [m_{\text{odd}}, f] \quad \text{for } f \in C_k^{n,i}(A, A).$$

Then  $D$  satisfies  $D \circ D = 0$ . Also,  $D$  raises the total degree by 1, so  $D$  is a differential on  $C_*^{*,*}(A, A)$ .

**Proof** The map  $D$  raises degree by 1 since  $m$  has total degree 2. Let us look at  $D(D(f))$ . Assume that  $f$  has horizontal internal degree  $k$ . Then for even  $p$  the horizontal degree of  $[m_{pq}, f]$  has the same parity as  $k$  whereas for odd  $p$  the horizontal degree of  $[m_{pq}, f]$  has the parity of  $k + 1$ . This means that

$$[m_{\text{even}}, f]^\# = (-1)^k [m_{\text{even}}, f] \quad \text{and} \quad [m_{\text{odd}}, f]^\# = (-1)^{k+1} [m_{\text{odd}}, f].$$

Thus, we obtain

$$D((-1)^k [m_{\text{even}}, f]) = (-1)^k ((-1)^k [m_{\text{even}}, [m_{\text{even}}, f]] + [m_{\text{odd}}, [m_{\text{even}}, f]])$$

and

$$D([m_{\text{odd}}, f]) = (-1)^{k+1} [m_{\text{even}}, [m_{\text{odd}}, f]] + [m_{\text{odd}}, [m_{\text{odd}}, f]]$$

which together give us

$$(5) \quad D(D(f)) = [m_{\text{even}}, [m_{\text{even}}, f]] + (-1)^k [m_{\text{odd}}, [m_{\text{even}}, f]] \\ + (-1)^{k+1} [m_{\text{even}}, [m_{\text{odd}}, f]] + [m_{\text{odd}}, [m_{\text{odd}}, f]].$$

Since  $m$  is assumed to be orthogonal, we can directly compute that

$$[m_{\text{even}}, [m_{\text{even}}, f]] = 0 = [m_{\text{odd}}, [m_{\text{odd}}, f]].$$

From the graded Jacobi identity established in Proposition 2.13 we conclude that

$$[m_{\text{odd}}, [m_{\text{even}}, f]] = [m_{\text{even}}, [m_{\text{odd}}, f]].$$

Putting this together, we can read off the desired equation  $D \circ D = 0$ .  $\square$

**Definition 3.4** Let  $A$  be an orthogonal  $dA_\infty$ -algebra with orthogonal  $dA_\infty$ -structure  $m$ . Then the Hochschild cohomology of  $A$  as a  $dA_\infty$ -algebra is defined as

$$\text{HH}^*(A, A) := H^* \left( \prod_{ij} C_j^{i, *-i-j}(A, A), D \right).$$

The grading in the above definition of Hochschild cohomology denotes the total degree.

**Remark** If  $A$  has  $dA_\infty$ -structure  $m = m_{11} + m_{02}$  (i.e.  $A$  is a bidga with trivial vertical differential), then this definition specializes to Sagave's definition [Sag10, Section 5] of Hochschild cohomology of bidgas with trivial vertical differential.

In this very special case of a bidga with trivial vertical differential, one grading is preserved by both  $m_{11}$  and  $m_{02}$  so that we have bigraded Hochschild cohomology groups:

$$\text{HH}^t(A, A) = \prod_{s \geq 0} \text{HH}^{s, t-s}(A, A),$$

where  $\text{HH}^{s, r}(A, A) = H^s(\prod_n C_{*-n}^{n, r}(A, A), D)$ . We denote the Hochschild cohomology in this special case by  $\text{HH}_{\text{bidga}}^{*,*}(A, A)$ .

## 3.2 Uniqueness of derived $A_\infty$ -algebras

The overall goal of this section is to establish a uniqueness result analogous to Kadeishvili's (Theorem 1.7) for the possibility of extending an existing  $dA_\infty$ -structure

on a minimal model. A minimal model of a differential graded algebra has an underlying bidga with zero vertical differential. Let  $\mu = m_{02}$  denote the multiplication of this bidga and  $\partial = m_{11}$  the horizontal differential.

The first step is to look into how to perturb an existing  $dA_\infty$ -structure by certain elements  $b$  of total degree 1.

**Definition 3.5** Let  $A$  be a bidga with multiplication  $m_{02} = \mu$ , horizontal differential  $m_{11} = \partial$  and vertical differential  $m_{01} = 0$ . Then

$$a = \sum_{i \geq 0, j \geq 1} a_{ij}, \quad a_{ij} \in C_i^{j, 2-(i+j)}(A, A), \quad i + j \geq 3,$$

is a *twisting cochain* if  $\partial + \mu + a$  is a  $dA_\infty$ -structure.

**Remark** Note that by Proposition 2.15  $a$  is a twisting cochain if and only if we have

$$[\partial + \mu + a, \partial^\# + \mu^\# + a^\#] = 0.$$

Letting  $D$  be the differential corresponding to the orthogonal  $dA_\infty$ -structure  $m = \partial + \mu$ , this is equivalent to the *derived Maurer-Cartan formula*

$$(6) \quad 2D(a) = -[a, a^\#] + 4[\partial, a_{odd}],$$

as can be verified quickly by splitting  $a$  into even and odd horizontal degree parts and using that  $[\partial + \mu, \partial + \mu] = 0$ . Hence, an element

$$a = \sum_{i,j} a_{ij}, \quad a_{ij} \in C_i^{j, 2-(i+j)}(A, A), \quad i + j \geq 3$$

is a twisting cochain if and only if  $a$  satisfies the above derived Maurer-Cartan formula.

**Lemma 3.6** Let  $A$  be a bidga with multiplication  $m_{02} = \mu$ , horizontal differential  $m_{11} = \partial$  and vertical differential  $m_{01} = 0$ . Let

$$a = \sum_{i,j} a_{ij}, \quad a_{ij} \in C_i^{j, 2-(i+j)}(A, A), \quad i + j \geq 3,$$

be a twisting cochain. Let either

$$(A) \quad b \in C_k^{n-1, 2-(n+k)}(A, A), \text{ for } k + n \geq 3, \text{ with } [\partial, b] = 0$$

or

$$(B) \quad b \in C_{k-1}^{n, 2-(n+k)}(A, A), \text{ for } k + n \geq 3, \text{ with } [\mu, b] = 0.$$

Then there is a twisting cochain  $\bar{a}$  satisfying

- the  $dA_\infty$ -structures  $\partial + \mu + a$  and  $\bar{m} = \partial + \mu + \bar{a}$  are  $E_2$ -equivalent,
- $\bar{a}_{uv} = a_{uv}$  for  $u < k$  or  $v < n - 1$  or  $(u, v) = (k, n - 1)$  in case **(A)** and for  $u < k - 1$  or  $v < n$  or  $(u, v) = (k - 1, n)$  in case **(B)**,
- $\bar{a}_{kn} = a_{kn} - [\mu, b]$  in case **(A)**,
- $\bar{a}_{kn} = a_{kn} - [\partial, b]$  in case **(B)**.

**Proof** This is a lengthy but direct computation using the definition of a morphism of  $dA_\infty$ -algebras. The twisting cochain  $\bar{a}$  is going to be determined by  $\partial + \mu + a$  being  $E_2$ -equivalent to  $\partial + \mu + \bar{a}$  via the equivalence  $id + b$ . We will only do case **(A)** explicitly since the other case can be read off the proof of this one.

Let  $f := id + b$ . We consider what it means for there to be a  $dA_\infty$ -structure  $\bar{m} = \partial + \mu + \bar{a}$  on  $A$  such that  $f : (A, m) \rightarrow (A, \bar{m})$  is a morphism of  $dA_\infty$ -structures, i.e. the equation (2) in Definition 2.3 is satisfied. Using  $f_{01} = id$ ,  $f_{k, n-1} = b$  and  $f_{ij} = 0$  in all other degrees as well as  $m = \mu + a$  and  $\bar{m} = \mu + \bar{a}$ , we write down (2). The left-hand side of (2) is only nonzero for  $(i, j) = (0, 1)$  and  $(i, j) = (k, n - 1)$ . Thus, we obtain

$$(-1)^u m_{uv} + \sum_{r=0}^{n-2} (-1)^{r(v-n)+(n-r)+(u-k)(n-1)} b(1^{\otimes r} \otimes m_{u-k, v+2-n} \otimes 1^{\otimes n-2-r}).$$

The sum can only be nonzero if  $u \geq k$  and  $v \geq n - 1$  and  $(u, v) \neq (k, n - 1)$ . In the special case  $(u, v) = (k, n)$  we get

$$(-1)^k a_{kn} + \sum_{r=0}^{n-2} (-1)^{n-r} b(1^{\otimes r} \otimes \mu \otimes 1^{\otimes n-2-r}).$$

For  $(u, v) = (k + 1, n - 1)$ , the result is

$$(-1)^{k+1} a_{k+1, n-1} - \sum_{r=0}^{n-2} b(1^{\otimes r} \otimes \partial \otimes 1^{\otimes n-2-r}).$$

On the right-hand side of (2) we have

$$(7) \quad (-1)^u \bar{m}_{uv} + \sum_{\substack{u=i+p_1+\dots+p_j \\ v=q_1+\dots+q_j}} (-1)^\epsilon \bar{m}_{ij} (f_{p_1 q_1} \otimes \dots \otimes f_{p_j q_j})$$

where at least one of the  $f_{p_r q_r}$  in the sum has to be  $f_{k, n-1} = b$  and  $\epsilon$  is as in Definition 2.3. The following four special cases are to be considered. First, we note that, since we have  $\bar{m}_{01} = 0$ , the sum is zero for  $(u, v) = (k, n - 1)$ . For  $(u, v) = (k, n)$ , we obtain

$$(-1)^k \bar{a}_{kn} + (-1)^n \mu(1 \otimes b) + \mu(b \otimes 1),$$

for  $(u, v) = (k + 1, n - 1)$  we have

$$(-1)^{k+1} \bar{a}_{k+1, n-1} + (-1)^{k+1} \partial(b)$$

and for  $(u, v) = (2k, 2n - 2)$  the result is

$$\bar{a}_{2k, 2n-2} + (-1)^{nk} \mu(b \otimes b) + \sum_{r=0}^{n-1} (-1)^{\epsilon} \bar{a}_{k, n} (1^r \otimes b \otimes 1^{n-1-r}).$$

In all other cases each summand appearing in the sum in (7) has  $i + j \geq 3$ . Further, the sum in (7) can only be nonzero for  $u \geq i + k$  and  $v \geq (n - 1) + (j - 1)$ .

Now recall that

$$[\partial, b] = \partial(b) - (-1)^k \sum_{r=0}^{n-2} b(1^{\otimes r} \otimes \partial \otimes 1^{\otimes n-2-r})$$

and

$$[\mu, b] = (-1)^{n+k} \left( \mu(1 \otimes b) + (-1)^n \mu(b \otimes 1) + \sum_{r=0}^{n-2} (-1)^{r+1} b(1^{\otimes r} \otimes \mu \otimes 1^{\otimes n-2-r}) \right).$$

Further, note that we have assumed that  $[\partial, b] = 0$ .

Putting all this together, we can read off that for  $(u, v)$  with either  $u < k$  or  $v < n - 1$  and for  $(u, v) = (k, n - 1)$ , we have

$$\bar{a}_{uv} = a_{uv}.$$

For  $(u, v) = (k, n)$ , we get

$$\begin{aligned} \bar{a}_{kn} &= a_{kn} - (-1)^k \left( \mu(b \otimes 1) + (-1)^n \mu(1 \otimes b) \right. \\ &\quad \left. + (-1)^{n-1} \sum_{r=0}^{n-2} (-1)^r b(1^{\otimes r} \otimes \mu \otimes 1^{\otimes n-2-r}) \right) \\ &= a_{kn} - [\mu, b]; \end{aligned}$$

for  $(u, v) = (k + 1, n - 1)$  we have

$$\begin{aligned} \bar{a}_{k+1, n-1} &= a_{k+1, n-1} + (-1)^k \sum_{r=0}^{n-2} b(1^{\otimes r} \otimes \partial \otimes 1^{\otimes n-2-r}) - \partial(b) \\ &= a_{k+1, n-1} - [\partial, b] = a_{k+1, n-1}; \end{aligned}$$

for  $(u, v) = (2k, 2n - 2)$  we have

$$\begin{aligned}\bar{a}_{2k, 2n-2} &= a_{2k, 2n-2} \\ &+ \sum_{r=0}^{n-2} (-1)^{m+n-r+k(n-1)} b(1^{\otimes r} \otimes a_k n \otimes 1^{\otimes n-2-r}) - (-1)^{nk} \mu(b \otimes b) \\ &+ \sum_{r=0}^{n-1} (-1)^{\epsilon} \bar{a}_{kn} (1^{\otimes r} \otimes b \otimes 1^{\otimes n-1-r}).\end{aligned}$$

Finally for  $(u, v) \neq (k, n), (k+1, n-1)$  or  $(2k, 2n-2)$  with  $u \geq k$  and  $v \geq n-1$ , we have

$$\begin{aligned}\bar{a}_{uv} &= a_{uv} + (-1)^u \sum_{r=0}^{n-2} (-1)^{r(v-n)+(n-r)+(u-k)(n-1)} b(1^{\otimes r} \otimes m_{u-k, v+2-n} \otimes 1^{\otimes n-2-r}) \\ &- (-1)^u \sum_{\substack{u=i+p_1+\dots+p_j \\ v=q_1+\dots+q_j \\ \text{at least one } q_j \neq 1}} (-1)^{\epsilon} \bar{a}_{ij} (f_{p_1 q_1} \otimes \dots \otimes f_{p_j q_j}).\end{aligned}$$

Note that the second sum in the last equation can only be nonzero if  $i+j \geq 3$ ,  $u \geq k+i$  and  $v \geq (n-1)+(j-1)$ . Also, for fixed  $(u, v)$ , the right-hand side of the last equation only uses  $\bar{a}_{pq}$  with  $p < u$  and  $q < v$ . The same thing happens in the case  $(u, v) = (2k, 2n-2)$ . This proves that the  $\bar{a}$  in the statement of our lemma can be constructed inductively.

One can then check degreewise that  $\bar{m} = \partial + \mu + \bar{a}$  defines a  $dA_{\infty}$ -structure by showing that  $[\bar{m}, \bar{m}^{\#}] = 0$ . The morphism  $f$  is an  $E_2$ -equivalence since  $f_{01} = id$ .  $\square$

**Remark** Note that in the situation of the above lemma, in both cases we have in particular that  $\bar{a}_{uv} = a_{uv}$  whenever  $u+v < k+n$ .

We can now formulate a derived version of Kadeishvili's uniqueness theorem.

**Theorem 3.7** Let  $A$  be a bidga with multiplication  $m_{02} = \mu$ , horizontal differential  $m_{11} = \partial$  and vertical differential  $m_{01} = 0$ . If  $\mathrm{HH}_{bidga}^{r, 2-r}(A, A) = 0$  for  $r \geq 3$ , then every  $dA_{\infty}$ -structure on  $A$  with  $m_{02} = \mu$ ,  $m_{11} = \partial$  and  $m_{01} = 0$  is  $E_2$ -equivalent to the trivial one, i.e. the one with  $m_{02} = \mu$ ,  $m_{11} = \partial$  and  $m_{ij} = 0$  for  $(i, j) \neq (0, 2)$  or  $(1, 1)$ .

**Proof** Let  $m = \partial + \mu + a$  be a  $dA_{\infty}$ -structure on  $A$  with

$$a = \sum_{k+n \geq 3} a_{kn}, \quad a_{kn} \in \mathbb{C}_k^{n, 2-(k+n)}(A, A).$$

We want to show that  $m$  is  $E_2$ -equivalent to the  $dA_\infty$ -structure  $\partial + \mu$ .

We now fix  $t \geq 3$  and show that  $m$  is equivalent to a  $dA_\infty$ -structure with  $a_{kn} = 0$  for  $k + n = t$ . We show this by induction on  $k$ . Assuming that  $a_{ij} = 0$  for  $i + j = t$  and  $i < k$ , we will show that  $m$  is equivalent to a  $dA_\infty$ -structure with  $\bar{m} = \partial + \mu + \bar{a}$  with  $\bar{a}_{kn} = 0$  and  $\bar{a}_{ij} = a_{ij} = 0$  for  $i + j = t$ ,  $i < k$  and  $i + j < t$ .

Because  $m$  is a  $dA_\infty$ -structure, by Lemma 3.3 we have  $[\partial + \mu + a, \partial^\# + \mu^\# + a^\#] = 0$ . Since  $A$  is also a bidga, we have  $[\partial + \mu, \partial^\# + \mu^\#] = 0$ . Hence,  $a$  is a twisting cochain satisfying the Maurer-Cartan formula

$$2D(a) = -[a, a^\#] + 4[\partial, a_{odd}]$$

as explained in (6). Further, we have

$$D(-) = [\mu, (-)^\#] + [\partial, -]$$

with

$$[\mu, (-)^\#] : C_*^{*,*}(A, A) \longrightarrow C_*^{*+1,*}(A, A) \text{ and } [\partial, -] : C_*^{*,*}(A, A) \longrightarrow C_{*+1}^{*,*}(A, A),$$

so  $[\mu, a_{kn}^\#]$  lives in the tridegree  $(n + 1, k, 2 - (k + n))$ -part of  $D(a)$  and  $[\partial, a_{kn}]$  lives in tridegree  $(n, k + 1, 2 - (k + n))$ . However, on the other side of (6) the tridegree  $(n + 1, k, 2 - (k + n))$ -part as well as the  $(n, k + 1, 2 - (k + n))$ -part of  $[a, a^\#]$  is zero since  $[a, a^\#]$  can only be nonzero in degrees  $(u, v, w)$  with  $u + v \geq 5$  whereas  $n + 1 + k = 4$ . Here we are adopting the convention for tridegrees that an element in  $C_k^{n,i}(A, A)$  has tridegree  $(n, k, i)$ .

Thus according to the Maurer-Cartan formula,  $D(a_{kn})$  lives in  $2[\partial, a_{odd}]$ . This information splits into the equations

$$[\mu, a_{kn}^\#] = \epsilon_1 2[\partial, a_{k-1, n+1}], \quad \epsilon_1 \in \{0, 1\}$$

and

$$[\partial, a_{kn}] = \epsilon_2 2[\partial, a_{kn}], \quad \epsilon_2 \in \{0, 1\}$$

where  $\epsilon_2 = 0$  for  $k$  even by definition since the right hand side is supposed to be a summand of  $2[\partial, a_{odd}]$ . Thus, we can also conclude that  $[\mu, a_{kn}^\#] = 0$  since our induction assumption gives  $a_{k-1, n+1} = 0$ .

For  $k$  odd, we are left with  $[\partial, a_{kn}] = \epsilon_2 2[\partial, a_{kn}]$ ,  $\epsilon_2 \in \{0, 1\}$ , from which we can immediately read off that  $[\partial, a_{kn}] = 0$ .

Hence, in any case  $D(a_{kn}) = 0$  and  $a_{kn}$  is a cocycle in  $C_k^{n, 2-(n+k)}(A, A)$ , so

$$[a_{kn}] \in \text{HH}_{bidga}^{k+n, 2-k-n}(A, A).$$



However,  $\mathrm{HH}_{bidga}^{k+n, 2-k-n}(A, A)$  is zero by assumption, so there must be a  $b$  in total degree 1 with  $D(b) = a_{kn}$ .

So, analogously to the proof of Theorem 5.3, there is a  $b_1 \in C_k^{n-1, 2-(k+n)}(A, A)$  with  $[\partial, b_1] = 0$  and  $[\mu, b_1] = a_{kn}$  and  $b_2 \in C_{k-1}^{n, 2-(k+n)}(A, A)$  with  $[\mu, b_2] = 0$  and  $[\partial, b_2] = a_{kn}$ .

Applying Lemma 3.6 to  $b_1$ , there is a  $dA_\infty$ -structure  $\bar{m} = \partial + \mu + \bar{a}_{ij}$  with  $\bar{a}_{ij} \in C_i^{j, 2-(i+j)}(A, A)$ ,  $i + j \geq 3$  such that  $\bar{m}$  is  $E_2$ -equivalent to  $m$ ,  $\bar{a}_{kn} = a_{kn} - [\mu, b_1] = 0$  and  $\bar{a}_{ij} = a_{ij}$  for  $i + j < t$  and  $i + j = t$ ,  $i < k$ , which proves our claim.  $\square$

**Example** In [DS09, Proposition 4.2], Dugger and Shipley consider the dga

$$A = \mathbb{Z}\langle e, x, y \rangle / (e^2 = 0, ex + xe = x^2, xy = yx = 1),$$

$$\partial(e) = p, \partial(x) = 0, \partial(y) = 0, \quad |e| = |x| = 1, |y| = -1.$$

This is a dga over  $\mathbb{Z}$  which has homology  $H_n(A) = \mathbb{Z}/p$  in every degree  $n$ . (Note that Dugger and Shipley use homological grading.) They then prove in Theorem 4.5 that  $A$  is not formal.

In [Sag09] Sagave gives a projective  $E_1$ -resolution  $B$  of  $A$ . He then constructs the first degrees of a minimal model structure on the induced termwise projective resolution  $P = H_{ver}^*(B)$  and shows that this gives a nontrivial class in  $\mathrm{HH}_{bidga}^{3, -1}(P, P)$ .

Theorem 3.7 will be used in the next section to give a sufficient criterion for the existence of a unique dga realising a fixed homology algebra over a ground ring rather than a ground field. To prove this derived analogue of Corollary 1.9, we first have to investigate the behaviour of Hochschild cohomology of degreewise projective resolutions under  $E_2$ -equivalence.

## 4 Invariance under $E_2$ -equivalence and intrinsic formality

In order to establish our uniqueness criterion we need an invariance result for Hochschild cohomology under  $E_2$ -equivalence. To prove this we will need to define Hochschild cohomology with coefficients. We will carry this out here only for the special case we need. In future work we hope to study the general case, but this would take us too far afield here.

Thus we will concentrate on the case of relevance to us, namely bidgas with  $m_{01} = 0$ . Invariance under  $E_2$ -equivalence in this situation is also discussed in [Sag10, Section 5]. We begin by spelling out concretely what a bidga with  $m_{01} = 0$  is.

A bidga with  $m_{01} = 0$  is a bigraded module  $A_i^j$  equipped with maps

$$m_{11} : A_i^j \rightarrow A_{i-1}^j$$

and

$$m_{02} : (A \otimes A)_i^j \rightarrow A_i^j$$

with relations which specify that  $m_{02}$  is associative,  $m_{11}$  is a differential and  $m_{11}$  is a derivation with respect to  $m_{02}$ . These relations come from the cases  $(u = 0, v = 3)$ ,  $(u = 2, v = 1)$  and  $(u = 1, v = 2)$  respectively of the defining relations; all other relations are trivial. Notice that this is just a dga with an extra grading.

It is straightforward to see what a module over such a thing should be; it is just a dg module with an extra grading.

**Definition 4.1** Let  $A$  be a bidga with  $m_{01} = 0$ . A *left  $A$ -module*  $M$  is a bigraded module  $\{M_i^j\}$  over the ground ring equipped with a horizontal differential

$$\bar{m}_{11} : M_i^j \rightarrow M_{i-1}^j$$

and an associative action  $\bar{m}_{02}^l : (A \otimes M)_i^j \rightarrow M_i^j$  such that the diagram

$$\begin{array}{ccc} A \otimes M & \xrightarrow{\bar{m}_{02}^l} & M \\ \downarrow m_{11} \otimes 1 + 1 \otimes \bar{m}_{11} & & \downarrow \bar{m}_{11} \\ A \otimes M & \xrightarrow{\bar{m}_{02}^l} & M \end{array}$$

commutes.

A right  $A$ -module is defined in the obvious way, with a right action map

$$\bar{m}_{02}^r : M \otimes A \rightarrow M.$$

And an  $A$ -bimodule is simultaneously a left and right  $A$ -module with the obvious compatibility condition on the left and right actions.

Notice that a morphism of bidgas  $A \rightarrow A'$  between bidgas with  $m_{01} = 0$  makes  $A'$  into an  $A$ -bimodule.

Let us also spell out what an  $E_2$ -equivalence  $f : A \rightarrow A'$  between bidgas with  $m_{01} = 0$  is. This is just a morphism  $f : A \rightarrow A'$  inducing an isomorphism on horizontal

homology. (So we can think of such an  $f$  as a quasi-isomorphism if we think of  $A$  and  $A'$  as complexes with respect to horizontal differentials.)

Now let  $A$  be a bidga with  $m_{01} = 0$  and let  $M$  be an  $A$ -bimodule. Let

$$C_k^{n,i}(A, M) = \prod_{u,v} \text{Hom}((A^{\otimes n})_u^v, M_{u-k}^{v+i})$$

and for  $f \in C_k^{n,i}(A, M)$  define

$$\begin{aligned} Df = & (-1)^{k+n+i-1} \bar{m}_{02}^r(f \otimes 1) + (-1)^{k+i} \bar{m}_{02}^l(1 \otimes f) \\ & + (-1)^{k+n+i} f \circ m_{02} + \bar{m}_{11} \circ f + (-1)^{k+1} f \circ m_{11}. \end{aligned}$$

Then  $D$  is a differential, allowing us to make the following definition.

**Definition 4.2** For  $A$  a bidga with  $m_{01} = 0$  and  $M$  an  $A$ -bimodule the *Hochschild cohomology of  $A$  with coefficients in  $M$*  is defined by

$$\text{HH}_{bidga}^{s,r}(A, M) = H^s \left( \prod_n C_{*-n}^{n,r}(A, M), D \right).$$

This is a covariant functor of  $M$  and a contravariant functor of  $A$ . In the case where  $M = A$ , regarded as a bimodule over itself, this agrees with the earlier definition of  $\text{HH}_{bidga}^{*,*}(A, A)$ . Indeed the formula above for the differential  $D$  just becomes

$$Df = (-1)^k [m_{02}, f] + [m_{11}, f].$$

**Proposition 4.3** Let  $(A, m)$  and  $(A', \bar{m})$  be bidgas with  $m_{01} = \bar{m}_{01} = 0$  and which are degreewise projective over  $k$ . Let  $f : A \rightarrow A'$  be an  $E_2$ -equivalence. Then  $f$  induces an isomorphism of Hochschild cohomology groups

$$\text{HH}_{bidga}^{*,*}(A, A) \cong \text{HH}_{bidga}^{*,*}(A', A').$$

**Proof** For each  $i$ , we can interpret the Hochschild cohomology  $\text{HH}_{bidga}^{*,i}(A, M)$  as the cohomology of a (right half-plane) bicomplex. This works very similarly to the case of Hochschild cohomology of a dga discussed earlier. One differential, say  $D_1$ , is given by the  $m_{11}$  part of the formula for  $D$  and the other, say  $D_2$ , by the  $m_{02}$  part.

Now consider  $A$  and  $M$  as complexes with respect to their differentials  $m_{11}$  and  $\bar{m}_{11}$  (with an extra grading). The differential  $D_1$  on  $\text{Hom}_*^i(A^{\otimes p}, M)$  is the induced differential via the tensor product and Hom functors of complexes. For bounded below and degreewise projective complexes the ordinary Hom and tensor product functors agree with the derived versions and are therefore quasi-isomorphism invariant.

Thus the morphism  $f : A \rightarrow A'$  induces column-wise quasi-isomorphisms of bicomplexes  $C_*^{*,i}(A, A) \rightarrow C_*^{*,i}(A, A')$  and  $C_*^{*,i}(A', A') \rightarrow C_*^{*,i}(A, A')$ . It follows that the induced maps of total complexes are quasi-isomorphisms and

$$\mathrm{HH}_{bidga}^{*,*}(A, A) \cong \mathrm{HH}_{bidga}^{*,*}(A', A').$$

□

Now we are in a position to give our criterion for intrinsic formality.

**Theorem 4.4** Let  $A$  be a dga and  $E$  its minimal model with  $dA_\infty$ -structure  $m$ . By  $\tilde{E}$ , we denote the underlying bidga of  $E$ , i.e.  $\tilde{E} = E$  as  $k$ -modules together with  $dA_\infty$ -structure  $\tilde{m} = m_{11} + m_{02}$ . If

$$\mathrm{HH}_{bidga}^{m, 2-m}(\tilde{E}, \tilde{E}) = 0 \quad \text{for} \quad m \geq 3,$$

then  $A$  is intrinsically formal.

**Proof** Applying Theorem 3.7 to  $\tilde{E}$ , we obtain that every  $dA_\infty$  structure on  $\tilde{E}$  is  $E_2$ -equivalent to the trivial one. By definition of minimal model,  $A$  is  $E_2$ -equivalent to  $\tilde{E}$ . Thus  $A$  is  $E_2$ -equivalent to  $(\tilde{E}, \mathrm{triv})$ . Again by definition of minimal model,  $(\tilde{E}, \mathrm{triv})$  is  $E_2$ -equivalent to  $(H^*(A), \mathrm{triv})$ . Thus we have an  $E_2$ -equivalence between  $A$  and  $(H^*(A), \mathrm{triv})$  and since these are both dgas an  $E_2$ -equivalence is a quasi-isomorphism. So  $A$  is formal.

Now let  $A'$  be a dga with  $H^*(A) \cong H^*(A')$  as associative algebras, let  $E'$  be a minimal model of  $A'$  and let  $\tilde{E}'$  be its underlying bidga. We have  $E_2$ -equivalences

$$\tilde{E}' \simeq (H^*(A'), \mathrm{triv}) \simeq (H^*(A), \mathrm{triv}) \simeq \tilde{E}.$$

Thus  $\tilde{E}'$  and  $\tilde{E}$  are  $E_2$ -equivalent bidgas. By definition of minimal model they are degreewise projective and have  $m_{01} = 0$ . Applying Proposition 4.3 gives

$$\mathrm{HH}_{bidga}^{m, 2-m}(\tilde{E}', \tilde{E}') \cong \mathrm{HH}_{bidga}^{m, 2-m}(\tilde{E}, \tilde{E}).$$

So the Hochschild cohomology of  $\tilde{E}'$  is zero in the relevant range and the argument of the preceding paragraph shows that  $A'$  is also formal.

Since  $A$  and  $A'$  are both formal, the hypothesis  $H^*(A) \cong H^*(A')$  means they are quasi-isomorphic. □

## 5 Uniqueness of classical $A_\infty$ -structures

In this section  $k$  is still a commutative ground ring without 2-torsion unless stated otherwise. We use Hochschild cohomology of differential graded algebras to give a uniqueness criterion for extending the differential and multiplication of a fixed dga to an  $A_\infty$ -structure. In the case of a trivial differential this recovers Kadeishvili's classical Theorem 1.7. We then apply this to an example in homotopy theory.

Fix a differential graded algebra  $A$  with differential  $m_1 = \partial$  and multiplication  $m_2 = \mu$ . We would like to consider the set of all  $A_\infty$ -structures on  $A$  (up to quasi-isomorphism) that extend the differential graded algebra structure, i.e.  $A_\infty$ -structures of the form  $m = \partial + \mu + m_3 + m_4 + \dots$ . Let us write  $a = m_3 + m_4 + \dots$ .

Recall that  $m = \partial + \mu + a$  is an  $A_\infty$ -structure if and only if  $a$  satisfies the Maurer-Cartan formula and that such  $a$  are called *twisting cochains*. In this classical case the Maurer-Cartan formula reads

$$-D(a) = \frac{1}{2}[a, a]$$

if 2 is invertible in  $k$  or, equivalently,  $-D(a) = a \circ a$  where  $\circ$  denotes the composition product, see e.g. [FP02, Section 2] and (6).

**Lemma 5.1** Let  $A$  be a dga with differential  $\partial$  and multiplication  $\mu$ , and let  $a$  be a twisting cochain. Further, for  $n \geq 3$ , let either  $p \in C^{n,1-n}(A, A)$  with

$$d^h(p) = [\mu, p] = 0$$

or  $p \in C^{n-1,2-n}(A, A)$  with

$$d^v(p) = [\partial, p] = 0.$$

Then there is a twisting cochain  $\bar{a}$  such that

- the  $A_\infty$ -structures  $\partial + \mu + \bar{a}$  and  $\partial + \mu + a$  are quasi-isomorphic,
- $\bar{a}_i = a_i$  for  $i \leq n - 1$ ,
- $\bar{a}_n = a_n - D(p)$ . □

We omit the proof since it is very similar to that of Lemma 3.6. For the case where  $A$  is a graded algebra rather than a dga, the analogous result is mentioned without proof in [Kad88, Section 4].

With the help of Lemma 5.1, we can now prove the sufficient condition for a unique  $A_\infty$ -structure on a dga  $A$  extending the existing differential and multiplication. This

is only a minor generalization of Kadeishvili's classical result [Kad88, Theorem 1] in the zero differential case, but we have not been able to find a reference.

To formulate the uniqueness results of this section and Section 6 we have to look deeper into the grading of the Hochschild cohomology of  $A_\infty$ -algebras and the internal grading of representing cocycles. An element of  $\mathrm{HH}^n(A, A)$  can be non-uniquely expressed as

$$[x] = [x_0 + x_1 + x_2 + \cdots] \quad \text{with} \quad x_i \in C^{i, n-i}(A, A).$$

However, while the sum of the  $x_i$  is a cocycle the individual summands are not necessarily cocycles themselves. So generally we do not get a decomposition of  $\mathrm{HH}^n(A, A)$  as  $\prod_i \mathrm{HH}^{i, n-i}(A, A)$ . To keep track of the internal degrees we introduce a decreasing filtration on  $\mathrm{HH}^*(A, A)$ .

**Definition 5.2** For an  $A_\infty$ -algebra  $A$ , let

$$F^k \mathrm{HH}^n(A, A) = \{[x] \in \mathrm{HH}^n(A, A) \mid x \in \prod_{i \geq k} C^{i, n-i}(A, A)\}.$$

This means that  $F^k \mathrm{HH}^n(A, A)$  consists of all those elements of  $\mathrm{HH}^n(A, A)$  whose representing cocycles can be written as a sum of  $x_i \in C^{i, n-i}(A, A)$  with  $i \geq k$ .

Note that in the case of a bidga the filtration  $F^*$  given in Definition 5.2 agrees with the usual filtration arising from the column-wise filtration on the bicomplex, see e.g. [McC01, 2.2 and 2.4].

**Theorem 5.3** Let  $A$  be a dga with differential  $\partial$  and multiplication  $\mu$ . If

$$F^3 \mathrm{HH}^2(A, A) = 0,$$

then any  $A_\infty$ -structure on  $A$  with  $m_1 = \partial$  and  $m_2 = \mu$  is quasi-isomorphic to  $\partial + \mu$ .

**Proof** Let  $a$  be a twisting cochain. Assuming that there is a  $k \geq 3$  such that  $a_i = 0$  for  $i < k$ , we are going to show that there is a twisting cochain  $\bar{a}$  that is equivalent to  $a$  and satisfies  $\bar{a}_i = 0$  for  $i \leq k$ , i.e. we are killing off the bottom summand. By induction, it follows that  $a$  is equivalent to zero.

So let  $a$  now be a twisting cochain such that there is a  $k \geq 3$  with  $a_i = 0$  for  $i < k$ . Considering the Maurer-Cartan equation

$$-D(a) = a \circ a$$

in bidegrees  $(k+1, 2-k)$  and  $(k, 3-k)$ , we see that  $D(a_k) = 0$  for degree reasons, so  $a_k$  is a cocycle and  $[a_k] \in F^k \mathrm{HH}^2(A, A)$ . Since  $F^k \mathrm{HH}^2(A, A) = 0$ ,  $a_k$  also has to

be a coboundary, i.e. there is a cochain  $p$  in total degree 1 with  $D(p) = a_k$ . This  $p$  is the sum of two cochains  $p_1$  and  $p_2$  with  $p_1 \in C^{k,1-k}(A, A)$  and  $p_2 \in C^{k-1,2-k}(A, A)$ . We have  $d^v(p_1) + d^h(p_2) = a_k$  and  $d^h(p_1) = d^v(p_2) = 0$  for degree reasons.

$$\begin{array}{ccccc}
C^{k-1,3-k}(A, A) & & C^{k,3-k}(A, A) & & \dots \\
\uparrow d^v & & \uparrow d^v & & \\
C^{k-1,2-k}(A, A) & \xrightarrow{d^h} & C^{k,2-k}(A, A) & \xrightarrow{d^h} & C^{k+1,2-k}(A, A) \\
& & \uparrow d^v & & \\
\dots & & C^{k,1-k}(A, A) & \xrightarrow{d^h} & C^{k+1,1-k}(A, A)
\end{array}$$

Applying Lemma 5.1 for  $p_1$  and  $p_2$ , we obtain that there is a twisting cochain  $\bar{a}$  quasi-isomorphic to  $a$  with  $\bar{a}_i = 0$  for  $i < k$  and  $\bar{a}_k = a_k - D(p) = 0$ , which completes our proof.  $\square$

**Example** Consider the dga over the  $p$ -local integers

$$A = \mathbb{Z}_{(p)}[x] \otimes \Lambda_{\mathbb{Z}_{(p)}}(e)/(x^m, x^{m-1}e), \quad \partial(x) = pe, \quad |e| = -(2p-3), \quad |x| = -(2p-2)$$

where  $m \geq 2$ . We can compute its Hochschild cohomology as a dga by applying the spectral sequence for the homology of the total complex of a bicomplex [McC01, 2.15]. Its  $E_1$ -term is the Hochschild cohomology of  $A$  as a graded algebra.

To obtain this, we note that for an  $A$ -bimodule  $M$

$$\begin{aligned}
\mathrm{HH}_{alg}^{*,*}(A, M) &\cong \mathrm{HH}_{alg}^{*,*}(\mathbb{Z}_{(p)}[x]/(x^m) \otimes \Lambda_{\mathbb{Z}_{(p)}}(e), M) \\
&\cong \mathrm{HH}_{alg}^{*,*}(\mathbb{Z}_{(p)}[x]/(x^m), M) \otimes \mathrm{HH}_{alg}^{*,*}(\Lambda_{\mathbb{Z}_{(p)}}(e), \mathbb{Z}_{(p)}).
\end{aligned}$$

(Use [CE56, XI.1] for the second isomorphism. The first follows from a change-of-rings spectral sequence, see [McC01].) Computing each factor separately, we obtain

$$\mathrm{HH}_{alg}^{*,*}(A, A) = \mathbb{Z}_{(p)}[f, \tau] \otimes \Lambda_{\mathbb{Z}_{(p)}}(\sigma) \otimes A$$

with  $|f| = (1, -|e|)$ ,  $|\tau| = (2, -m|x|)$  and  $|\sigma| = (1, -|x|)$  for  $A$  viewed as a graded algebra.

Already at this  $E_1$ -stage we can read off that

$$\mathrm{HH}_{alg}^{n,2-n}(A, A) = 0 \text{ for } n \geq 3,$$

so  $F^3 \mathrm{HH}^2(A, A) = 0$  for  $A$  as a dga. Hence  $\mu + \partial$  is the only  $A_\infty$ -structure on  $A$  with  $m_1 = \partial$  and  $m_2 = \mu$ .

Also note that the homology of  $A$  coincides with the stable homotopy groups of the  $K(p)$ -local sphere in a certain range, i.e.

$$H^{-i}(A) = \pi_i(L_1S^0) \text{ for } 0 \leq i \leq (m-1)(2p-2) - 1.$$

Combining Kadeishvili's result on minimal models with Theorem 5.3, we recover the following result which we already stated earlier as Corollary 1.9.

**Corollary 5.4** Let  $A$  be a dga over a ground field and  $H^*(A)$  its homology algebra. Suppose that

$$\mathrm{HH}_{alg}^{n, 2-n}(H^*(A), H^*(A)) = 0 \quad \text{for } n \geq 3.$$

Then  $A$  is intrinsically formal.

**Proof** We apply Theorem 5.3 to  $H^*(A)$  with the trivial differential to see that any  $A_\infty$ -structure on this is quasi-isomorphic to the trivial one. So in particular the minimal model is quasi-isomorphic to the trivial structure. But the minimal model is quasi-isomorphic to  $A$ , so  $A$  is formal.

Now given a dga  $A'$  with  $H^*(A') \cong H^*(A)$ , the same argument shows that  $A'$  is also formal and thus that  $A'$  is quasi-isomorphic to  $A$ .  $\square$

We note that the corollary follows from the special case of Theorem 5.3 where the dga has trivial differential.

## 6 Massey products

Massey products provide some very useful additional structure when studying differential graded algebras and their homology. They are closely related to Toda brackets in triangulated categories which have strong applications in homotopy theory. Here we explain the relationship between Massey products and the  $m_3$  part of  $A_\infty$ -structures; see also [BKS05, Lemma 5.14].

In this section,  $k$  denotes a field of characteristic not 2.

Let  $A$  be a differential graded algebra and  $\alpha_1, \alpha_2, \alpha_3$  elements in the homology  $H^*(A)$  such that  $\alpha_1\alpha_2 = 0$  and  $\alpha_2\alpha_3 = 0$ . That means that for chosen representing cocycles  $a_i$  of  $\alpha_i$  there is an element  $u_i$  such that  $d(u_i) = (-1)^{1+|a_i|}a_i a_{i+1}$ . With those elements, one can now define the Massey product of  $\alpha_1, \alpha_2$  and  $\alpha_3$  as follows.



**Definition 6.1** Let  $\alpha_1, \alpha_2$  and  $\alpha_3$  be as above. Then the *Massey product*

$$\langle \alpha_1, \alpha_2, \alpha_3 \rangle \subset H^{|\alpha_1|+|\alpha_2|+|\alpha_3|-1}(A)$$

is defined as the set of homology classes of the elements

$$(-1)^{1+|\alpha_1|}a_1u_2 + (-1)^{1+|\alpha_1|}u_1a_3$$

ranging over all possible choices of representing cocycles  $a_i$  of the  $\alpha_i$  and  $u_i$  such that  $d(u_i) = (-1)^{1+|\alpha_i|}a_i a_{i+1}$ .

Note that the Massey product  $\langle \alpha_1, \alpha_2, \alpha_3 \rangle$  is a set rather than an element as the choices one makes can be altered by appropriate cocycles. Hence, if one fixes any  $x$  in the Massey product, for any other  $x'$  in the Massey product there is a

$$y \in \alpha_1 H^{|\alpha_3|+|\alpha_2|-1}(A) \oplus H^{|\alpha_2|+|\alpha_1|-1}(A)\alpha_3$$

such that  $x' = x + y$ . The group

$$\alpha_1 H^{|\alpha_3|+|\alpha_2|-1}(A) \oplus H^{|\alpha_2|+|\alpha_1|-1}(A)\alpha_3$$

is called the *indeterminacy* of  $\langle \alpha_1, \alpha_2, \alpha_3 \rangle$ . So a Massey product consists of only one element if and only if its indeterminacy is zero. For more details on Massey products, see e.g. [Rav86, A.1.4].

**Example** Let  $k$  be a field of characteristic different from 2. Consider the following noncommutative differential graded algebra

$$A = k \langle x, y \rangle / (x^3, y^2, xy = -yx), \quad \partial(x) = 0, \partial(y) = x^2, \quad |x| = 2, |y| = 3.$$

Its homology has a copy of  $k$  in degrees 0, 2, 5 and 7 and zero elsewhere. Let  $[x]$  and  $[xy]$  denote the homology classes of  $x$  and  $xy$  respectively. Then

$$2[xy] = \langle [x], [x], [x] \rangle \in H^5(A),$$

the indeterminacy being zero for degree reasons.

**Example** The dga

$$A = \mathbb{Z}_{(p)}[x] \otimes \Lambda_{\mathbb{Z}_{(p)}}(e) / (x^m, x^{m-1}e), \quad \partial(x) = pe, \quad |e| = -(2p-3), |x| = -(2p-2)$$

considered in the previous section has nontrivial Massey products. Take  $a_k$  to be an order  $p$  element in  $H^{-(2p-2)k+1}(A)$ . Then

$$\langle a_i, p, a_j \rangle = a_{i+j}.$$

This is related to the Toda bracket relation  $\langle \alpha_i, p, \alpha_j \rangle = \alpha_{i+j}$  in the homotopy groups of the  $K_{(p)}$ -local sphere  $\pi_* L_1 S^0$ .

In the context of  $A_\infty$ -algebras, Massey products can be reformulated using minimal models which were introduced in the previous section. We quote the following well-known result (see also [BKS05, Lemma 5.14]).

**Lemma 6.2** Let  $A$  be a dga and  $H^*(A)$  its minimal model with  $A_\infty$ -structure  $m$ . Let  $\alpha_1, \alpha_2, \alpha_3 \in H^*(A)$ . If the Massey product  $\langle \alpha_1, \alpha_2, \alpha_3 \rangle$  is defined in  $H^*(A)$ , then

$$(-1)^{|\alpha_1|+|\alpha_2|+1} m_3(\alpha_1 \otimes \alpha_2 \otimes \alpha_3) \in \langle \alpha_1, \alpha_2, \alpha_3 \rangle.$$

□

Hence, if  $A$  and  $B$  are differential graded algebras with isomorphic homology algebras  $H^*(A)$  and  $H^*(B)$ , then they have the same Massey products if the  $A_\infty$ -structures of the minimal models have identical  $m_3$ . (The converse is not necessarily true, see the discussion at the end of this section.)

**Theorem 6.3** Let  $A$  be a dga whose minimal model  $H^*(A)$  satisfies  $m_i = 0$  for  $i \neq 2, 3$  and let  $\bar{m}$  be an  $A_\infty$ -structure on  $H^*(A)$  with  $\bar{m}_2 = m_2$  and  $\bar{m}_3 = m_3$ . If  $F^4 \text{HH}^2(H^*(A), H^*(A)) = 0$ , then  $\bar{m}$  and  $m$  are quasi-isomorphic.

**Proof** The proof is extremely similar to the proof of Theorem 5.3. The differential in the Hochschild complex for  $H^*(A)$  is  $D = D_2 + D_3$  with

$$D_2 = [m_2, -] : C^{n,k}(H^*(A), H^*(A)) \longrightarrow C^{n+1,k}(H^*(A), H^*(A))$$

and

$$D_3 = [m_3, -] : C^{n,k}(H^*(A), H^*(A)) \longrightarrow C^{n+2,k-1}(H^*(A), H^*(A)).$$

Assume there is an  $A_\infty$ -structure  $\bar{m}$  on  $H^*(A)$  with

$$\bar{m} = m_2 + m_3 + a_4 + a_5 + \cdots.$$

Let

$$a = a_4 + a_5 + \cdots.$$

Because  $m = m_2 + m_3$  is an  $A_\infty$ -structure on the minimal model by assumption, we know that  $a$  is a twisting cochain, i.e.  $a$  satisfies the Maurer-Cartan equation. Again, for degree reasons  $D(a_4) = 0$  and so there is

$$p_2 \in C^{3,-2}(H^*(A), H^*(A)) \text{ and } p_3 \in C^{2,-1}(H^*(A), H^*(A))$$

with

$$D_2(p_2) + D_3(p_3) = a_4 \text{ and } D_3(p_2) = D_2(p_3) = 0.$$

The analogue of Proposition 5.1 also holds in this case: for any

$$p \in C^{n,1-n}(H^*(A), H^*(A)) \text{ with } D_3(p) = 0$$

or

$$p \in C^{n+1,-n}(H^*(A), H^*(A)) \text{ with } D_2(p) = 0,$$

there is a twisting cochain  $\bar{a} = \bar{a}_4 + \bar{a}_5 + \dots$  such that

- $\bar{a}$  is equivalent to  $a$ ,
- $\bar{a}_k = a_k$  for  $k \leq n$ ,
- $\bar{a}_{n+1} = a_{n+1} - D(p)$ .

The rest of the proof follows the same steps as the proof of Theorem 5.3. □

Of course one would like to apply this theorem to a minimal model

$$(H^*(A), m = m_2 + m_3)$$

of a dga  $A$  to obtain a uniqueness result analogous to Corollary 1.9 and conclude that the vanishing of the right Hochschild cohomology groups implies that  $A$  is the only dga up to quasi-isomorphism with the given homology and Massey products.

This does not quite work- to give the same Massey products on minimal models of dgas with the same homology algebras,  $m_3$  only needs to agree on triples  $(a, b, c)$  with  $ab = 0 = bc$ . For example, in [BKS05, Example 5.15 and Proposition 5.16] Benson, Krause and Schwede constructed an example of a dga with trivial Massey products but nontrivial  $m_3$ .

It would also be interesting to study the implication of Massey products regarding uniqueness criteria in the derived case.

## A Signs in the Lie bracket

In this appendix we verify the signs appearing in the Lie bracket of Section 2.2. The special case where  $k = l = 0$  recovers the signs in Section 1.2.

**Lemma A.1** In the context of Section 2.2,

$$\begin{aligned}
[f, g] &:= \sigma^{-1} \llbracket \sigma(f), \sigma(g) \rrbracket \\
&= \sum_{v=0}^{n-1} (-1)^{(n-1)(m-1)+v(m-1)+j(n-1)} f(1^{\otimes v} \otimes g \otimes 1^{\otimes n-v-1}) \\
&\quad - (-1)^{\langle f, g \rangle} \sum_{v=0}^{m-1} (-1)^{(m-1)(n-1)+v(n-1)+i(m-1)} g(1^{\otimes v} \otimes f \otimes 1^{\otimes m-v-1}) \\
&\in C_{k+l}^{n+m-1, i+j}(A, A)
\end{aligned}$$

for  $f \in C_k^{n,i}(A, A)$  and  $g \in C_l^{m,j}(A, A)$ . Here,  $\langle f, g \rangle := (n+i-1)(m+j-1) + kl$ .

**Proof** Throughout this proof, by  $\circ$ , we mean the actual composition of morphisms rather than the previously used composition product.

The signs arise from the Koszul sign rule for interchanging morphisms. For morphisms  $f, g, h$  and  $u$ , we have

$$(f \otimes g) \circ (h \otimes u) = (-1)^{is+jt} (f \circ h) \otimes (g \circ u)$$

with  $g$  having internal bidegree  $(i, j)$  and  $h$  having internal bidegree  $(s, t)$ .

We then obtain

$$\begin{aligned}
\sigma^{-1} \llbracket \sigma(f), \sigma(g) \rrbracket &= \sigma^{-1} \left( \sum_{v=0}^{n-1} \sigma(f)(1^{\otimes v} \otimes \sigma(g) \otimes 1^{\otimes n-v-1}) \right) \\
&\quad - (-1)^{\langle f, g \rangle} \sigma^{-1} \left( \sum_{v=0}^{m-1} \sigma(g)(1^{\otimes v} \otimes \sigma(f) \otimes 1^{\otimes m-v-1}) \right).
\end{aligned}$$

For reasons of symmetry and linearity we are only going to explicitly compute

$$\sigma^{-1} \left( \sigma(f)(1^{\otimes v} \otimes \sigma(g) \otimes 1^{\otimes n-v-1}) \right).$$

Up to sign, this is  $f(1^{\otimes v} \otimes g \otimes 1^{\otimes n-v-1})$  and we now calculate the sign.

The term  $\sigma(f)(1^{\otimes v} \otimes \sigma(g) \otimes 1^{\otimes n-v-1})$  lies in  $C_{k+l}^{n+m-1, i+j+n+m-2}(S(A), S(A))$ , so

$$\begin{aligned}
& \sigma^{-1}\left(\sigma(f)(1^{\otimes v} \otimes \sigma(g) \otimes 1^{\otimes n-v-1})\right) \\
&= (-1)^{i+j+n+m+k+l+\binom{n+m-1}{2}} S^{-1} \circ \left(\sigma(f)(1^{\otimes v} \otimes \sigma(g) \otimes 1^{\otimes n-v-1})\right) \circ S^{\otimes n+m-1} \\
&= (-1)^{\binom{n+m-1}{2}} S^{-1} \circ S \circ f \circ (S^{-1})^{\otimes n} \\
&\quad \circ \left(1^{\otimes v} \otimes (S \circ g \circ (S^{-1})^{\otimes m}) \otimes 1^{\otimes n-v-1}\right) \circ S^{\otimes n+m-1} \\
&= (-1)^{\binom{n+m-1}{2}} f \circ \left((S^{-1})^{\otimes v} \otimes S^{-1} \otimes (S^{-1})^{\otimes n-v-1}\right) \\
&\quad \circ \left(1^{\otimes v} \otimes (S \circ g \circ (S^{-1})^{\otimes m}) \otimes 1^{\otimes n-v-1}\right) \circ S^{\otimes n+m-1}.
\end{aligned}$$

In the next step we are obtaining a new sign  $(-1)^{(n-v-1)(j+m-1)}$  by interchanging  $(S^{-1})^{\otimes n-v-1}$  with  $1^{\otimes v} \otimes (S \circ g \circ (S^{-1})^{\otimes m})$ . Interchanging  $S^{-1}$  and  $1^{\otimes v}$  does not introduce any new signs, so we continue with

$$\begin{aligned}
& (-1)^{\binom{n+m-1}{2}+(n-v-1)(j+m-1)} \\
& f \circ \left((S^{-1})^{\otimes v} \otimes (S^{-1} \circ S \circ g \circ (S^{-1})^{\otimes m}) \otimes (S^{-1})^{\otimes n-v-1}\right) \circ S^{\otimes n+m-1} \\
&= (-1)^{\binom{n+m-1}{2}+(n-v-1)(j+m-1)} \\
& f \circ \left((S^{-1})^{\otimes v} \otimes g \circ (S^{-1})^{\otimes m} \otimes (S^{-1})^{\otimes n-v-1}\right) \circ \left(S^{\otimes v} \otimes S^{\otimes m} \otimes S^{\otimes n-v-1}\right).
\end{aligned}$$

Since  $(S^{-1})^{\otimes a} = (-1)^{\binom{a}{2}}(S^{\otimes a})^{-1}$ , we continue with

$$\begin{aligned}
& (-1)^{\binom{n+m-1}{2}+\binom{n-v-1}{2}+\binom{m}{2}+\binom{v}{2}+(n-v-1)(j+m-1)} \\
& f \circ \left((S^{\otimes v})^{-1} \otimes g \circ (S^{\otimes m})^{-1} \otimes (S^{\otimes n-v-1})^{-1}\right) \circ \left(S^{\otimes v} \otimes S^{\otimes m} \otimes S^{\otimes n-v-1}\right).
\end{aligned}$$

We have that

$$\binom{n+m-1}{2} + \binom{n-v-1}{2} + \binom{m}{2} + \binom{v}{2} \equiv (n-1)(v+m) + v \pmod{2}$$

so we can simplify the sign in the above expression to give

$$\begin{aligned}
& (-1)^{(n-1)(v+m)+v+(n-v-1)(j+m-1)} \\
& f \circ \left((S^{\otimes v})^{-1} \otimes g \circ (S^{\otimes m})^{-1} \otimes (S^{\otimes n-v-1})^{-1}\right) \circ \left(S^{\otimes v} \otimes S^{\otimes m} \otimes S^{\otimes n-v-1}\right).
\end{aligned}$$

We then interchange  $S^{\otimes v}$  with  $g \circ (S^{\otimes m})^{-1} \otimes (S^{\otimes n-v-1})^{-1}$  which in addition gives us the new sign  $(-1)^{(j+m+n-v-1)v}$ , so we have

$$\begin{aligned}
& (-1)^{(n-1)(v+m)+v+(n-v-1)(j+m-1)+(j+m+n-v-1)v} \\
& f \circ \left(1^{\otimes v} \otimes g \circ (S^{\otimes m})^{-1} \otimes (S^{\otimes n-v-1})^{-1}\right) \circ \left(1^{\otimes v} \otimes S^{\otimes m} \otimes S^{\otimes n-v-1}\right).
\end{aligned}$$

Finally, we add to the sign by interchanging  $S^{\otimes m}$  with  $(S^{\otimes n-v-1})^{-1}$ , so we end up with

$$(-1)^{(n-1)(v+m)+v+(n-v-1)(j+m-1)+(j+m+n-v-1)v+m(n-v-1)} f \circ (1^{\otimes v} \otimes g \otimes 1^{\otimes n-v-1}).$$

We can then simplify the above sign to

$$(-1)^{(n-1)(m-1)+v(m-1)+(n-1)j}$$

which proves our claim.  $\square$

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