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# Cooper pairing with finite angular momentum: BCS versus Bose limits

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**Abstract**
We revisit the old problem of exotic superconductivity as Cooper pairing with finite angular momentum emerging from a central potential. Based on some general considerations, we suggest that the phenomenon is associated with interactions that keep electrons at some particular, finite distance $r_0$, and occurs at a range of intermediate densities $n \sim 1/r_0^3$. We discuss the ground state and the critical temperature in the framework of a standard functional-integral theory of the BCS to Bose crossover. We find that, due to the lower energy of two-body bound states with $l = 0$, the rotational symmetry of the ground state is always restored on approaching the Bose limit. Moreover in that limit the critical temperature is always higher for pairs with $l = 0$. The breaking of the rotational symmetry of the continuum by the superfluid state thus seems to be a property of weakly-attractive, non-monotonic interaction potentials, at intermediate densities.

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## 1. Introduction

Since the original formulation of BCS theory [1, 2] there has been great interest in possible new phenomena arising from its generalization. The earliest example concerns the possibility of Cooper pairs having angular momentum quantum number $l > 0$, thus breaking the rotational symmetry of the continuum [3, 4]. Such speculations were based on the assumption of a different *shape* of the potential describing the effective attraction between fermions. Another generalization concerned *stronger* values of the fermion–fermion attraction. It was realized that there is a crossover, as this strength is increased, from a BCS superfluid to a Bose–Einstein

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condensate of non-overlapping pairs [5], with a dramatic effect on the critical temperature [6]. It turned out that the ground state can be described by a straightforward generalization of BCS theory [7], while the superconducting instability requires taking into account 'preformed pair' (PP) fluctuations in the normal state [8].

Experimentally, Cooper pairing with $l > 0$ was first observed in the superfluid state of $^3$He. Moreover since the discovery of superconductivity in cuprate perovskites [9], whose pairs have $d_{x^2-y^2}$ symmetry [10], several families of 'anomalous superconductors' have been found. Many of these materials present exotic pairing in the form of a superconducting state that breaks the rotational symmetry of the crystal, and they often deviate from BCS theory in several other ways [11]. This led to a resurgence of interest in the BCS to Bose crossover [12–26], particularly in models with exotic pairing [27–33, 48]. On the other hand, little attention has been paid to the physics of the crossover in the context of the original discussions of exotic pairing [3, 4], namely when a central attraction, in a continuum, leads to Cooper pairing with $l > 0$. In fact this is an especially interesting case since the two-body ground state is guaranteed, under quite general conditions [34], to have $l = 0$, making exotic pairing necessarily a many-body effect. Some obvious questions arise: what type of isotropic interactions lead to exotic pairing, and under what conditions? Is exotic pairing possible in the Bose–Einstein (BE) limit, when the BCS ground state is a condensate of non-overlapping pairs [5–8]? Moreover, the recent achievement of Fermi degeneracy in magnetically trapped, ultra-low temperature gases has stimulated speculations that superfluidity [35–37] and indeed the BCS–Bose crossover [26] may be observable in these systems. Understanding the above questions may guide us as to whether exotic pairing is also a possibility.

Recently, we and our co-authors have studied the above questions in a simple model that features exotic pairing via a central attraction [48]. In this contribution we revisit them with a more general point of view, using some of the results obtained for the 'delta shell model' (DSM) of [48] as an illustration. We begin section 2 by reminding the reader how, as a consequence of the weak-coupling theory of superconductivity, a central potential can, *in principle*, lead to exotic pairing [3, 4]. Then we take this old argument one little step further by asking: what is the essential feature that makes a particular interaction potential, *in practice*, lead to this effect? Having established the existence of such potentials, and knowing what they look like, we move on to outline, in section 3, the main features of a simple, but fairly general theory of the BCS to Bose crossover. Our formalism follows a recipe that has, by now, become standard [38] (though not the only one [13, 25]): to introduce bosonic pairing fields via a Hubbard–Stratonovich transformation (HST) [39, 40] and then expand the action to quadratic (Gaussian) order in those fields. Although limited [16, 18], such a scheme suffices for the discussion of the BCS and BE limits. The novelty of our formulation is that pairs may be created and annihilated with different values of their angular momentum quantum numbers. In contrast, many previous applications of the Gaussian theory assumed these internal degrees of freedom of the Cooper pairs to be 'frozen' to the desired s [14, 20] or $d_{x^2-y^2}$ [28] state. As we shall see these internal degrees of freedom turn out to be important, determining some of the key features of the problem. In sections 4 and 5 we apply the formalism to discuss the ground state and the superconducting instability, respectively, in the BE limit, where the original argument for exotic pairing cannot be applied. Finally, in section 6, we offer our conclusions.

## 2. Central potentials leading to exotic pairing

Let us begin by recalling some central ideas of the original weak-coupling theory of exotic pairing [3, 4]. Consider a system of electrons, in a three-dimensional continuum, interacting via some local, non-retarded, central potential $V(\mathbf{r}-\mathbf{r}') = V(|\mathbf{r}-\mathbf{r}'|)$. For simplicity, we will assume the interaction to take place between electrons with opposite spins and limit our discussion to the case of singlet pairing, with angular momentum quantum number $l = 0, 2, 4, \ldots$. As is well known the Fourier transform

$$V(\mathbf{k}-\mathbf{k}') = \int \mathrm{d}^3\mathbf{r}\, \mathrm{e}^{\mathrm{i}(\mathbf{k}-\mathbf{k}')\mathbf{r}} V(\mathbf{r}) \tag{1}$$

admits the partial-wave decomposition

$$V(\mathbf{k}-\mathbf{k}') = \sum_{l=0}^{\infty} K_l(|\mathbf{k}|, |\mathbf{k}'|)(2l+1) P_l(\hat{\mathbf{k}}\hat{\mathbf{k}}') \tag{2}$$

in terms of the Legendre polynomials $P_l(\hat{\mathbf{k}}\hat{\mathbf{k}}') = (2l+1)^{-1} 4\pi \sum_m Y_{l,m}(\hat{\mathbf{k}}) Y^*_{l,m}(\hat{\mathbf{k}}')$. It was soon pointed out [3] that in the weak-coupling limit one can use the approximation

$$V(\mathbf{k}-\mathbf{k}') \approx K_{l_{\max}}(2l_{\max}+1) P_{l_{\max}}(\hat{\mathbf{k}}\hat{\mathbf{k}}') \tag{3}$$

where $l_{\max}$ is the value of $l$ for which the coupling constant on the Fermi surface,

$$K_l \equiv K_l(k_F, k_F) \tag{4}$$

is largest ($k_F$ is the Fermi vector). The approximate form (3) of the potential $V(\mathbf{k}-\mathbf{k}')$ is, for $l_{\max} > 0$, anisotropic, and it leads to pairing with finite angular momentum quantum number $l_{\max}$ [3].

Let us now try to understand how the preference for a particular value of $l$ comes about. To do this one has to examine the relationship between the coupling constant $K_l$ and the interaction potential $V(r)$. It can be found by expanding in spherical waves the two exponentials $\mathrm{e}^{\mathrm{i}\mathbf{k}\mathbf{r}}$ and $\mathrm{e}^{-\mathrm{i}\mathbf{k}'\mathbf{r}}$ in equation (1); comparison to (2) yields

$$K_l(|\mathbf{k}|, |\mathbf{k}'|) = (-1)^l \int_0^{\infty} \mathrm{d}r\, 4\pi r^2 j_l(|\mathbf{k}|r) V(r) j_l(|\mathbf{k}'|r) \tag{5}$$

where $j_l(x)$ is a spherical Bessel function. Substituting (5) in (4) we obtain

$$K_l = (-1)^l \int_0^{\infty} \mathrm{d}r\, 4\pi r^2 j_l(k_F r)^2 V(r). \tag{6}$$

Thus $K_l$ is a weighted integral of the interaction potential $V(r)$. The weighting factors $j_l(k_F r)^2$ are shown in figure 1 for $l = 0, 2, 4, \ldots$. Evidently for a purely attractive, monotonic potential, such as that in figure 2($a$), $l_{\max} = 0$. To have $l_{\max} > 0$, $V(r)$ has to be most attractive at distances at which the weighting factors for a finite value of the angular momentum quantum number are largest. That is achieved by non-monotonic potentials that lead to maximum attraction at some finite distance $r_0$, such as that in figure 2($b$). For example, provided that the width of the potential well centred on $r_0$, namely $r_c$ (see the figure), is sufficiently small, choosing $r_0$ so that $k_F r_0 \sim 3$ yields $l_{\max} = 2$.

## 3. Basic theory of the BCS to Bose crossover

The above discussion implies that, in the BCS limit, exotic pairing can be described by an effectively anisotropic interaction, equation (3). On the other hand, we would like to describe the ground state and the superconducting instability also in the BE limit, where such approximation may not (and, as we shall see, does not) apply. For this we need to develop our theory in a more general framework, and in particular it is important to work with a complete

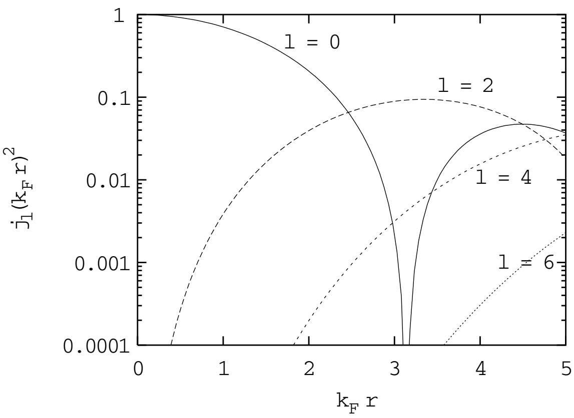


**Figure 1.** The weighting factors in the integral on the right-hand side of equation (6) for pairing with the first four even values of the angular momentum quantum number, $l = 0, 2, 4, 6$. Note the logarithmic scale on the vertical axis.

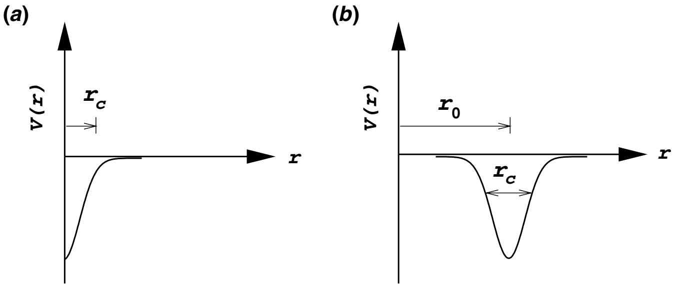


**Figure 2.** Interaction potentials representing on-site attraction (*a*) and attraction at a finite distance (*b*).

description of the interaction potential. For concreteness let us write explicitly the grand-canonical Hamiltonian for the situation at hand. At chemical potential $\mu$, it is [8, equation (1) (for example)]

$$\hat{H} - \mu\hat{N} = \sum_{\mathbf{k},\sigma} \varepsilon_{\mathbf{k}} \hat{c}^{+}_{\mathbf{k},\sigma} \hat{c}_{\mathbf{k},\sigma} + \frac{1}{L^3} \sum_{\mathbf{q},\mathbf{k},\mathbf{k}'} V(\mathbf{k}-\mathbf{k}') \hat{c}^{+}_{\mathbf{q}/2+\mathbf{k},\uparrow} \hat{c}^{+}_{\mathbf{q}/2-\mathbf{k},\downarrow} \hat{c}_{\mathbf{q}/2-\mathbf{k}',\downarrow} \hat{c}_{\mathbf{q}/2+\mathbf{k}',\uparrow}. \tag{7}$$

It is particularly illustrative to re-write it in the following way:

$$\hat{H} - \mu\hat{N} = \sum_{\mathbf{k},\sigma} \varepsilon_{\mathbf{k}} \hat{c}^{+}_{\mathbf{k},\sigma} \hat{c}_{\mathbf{k},\sigma} + \frac{1}{L^3} \sum_{\kappa,l,m} \sum_{\mathbf{q}} V_{\kappa,l} \hat{b}^{+}_{\kappa,l,m,\mathbf{q}} \hat{b}_{\kappa,l,m,\mathbf{q}}. \tag{8}$$

Here, $\hat{c}^{+}_{\mathbf{k},\sigma}$, $\hat{c}_{\mathbf{k},\sigma}$ create and annihilate, respectively, an electron with momentum $\hbar\mathbf{k}$ and spin $\sigma = \uparrow, \downarrow$. $\varepsilon_{\mathbf{k}} \equiv \hbar^2|\mathbf{k}|^2/2m^* - \mu$ is the single-particle dispersion relation ($m^*$ is the effective mass of an electron) and $L^3$ is the (very large) sample volume. The additional operators

$$\hat{b}^{+}_{\kappa,l,m,\mathbf{q}} \equiv \sum_{\mathbf{k}} \phi_{\kappa,l,m,\mathbf{k}} \hat{c}^{+}_{\mathbf{q}/2+\mathbf{k}\uparrow} \hat{c}^{+}_{\mathbf{q}/2-\mathbf{k}\downarrow} \tag{9}$$

$$\hat{b}_{\kappa,l,m,\mathbf{q}} \equiv \sum_{\mathbf{k}} \phi^{*}_{\kappa,l,m,\mathbf{k}} \hat{c}_{\mathbf{q}/2-\mathbf{k}\downarrow} \hat{c}_{\mathbf{q}/2+\mathbf{k}\uparrow} \tag{10}$$

create and annihilate, respectively, a pair with opposite spins, total momentum $\hbar\mathbf{q}$ and internal wavefunction $\phi_{\kappa,l,m,\mathbf{k}} \equiv R_{\kappa,l}(|\mathbf{k}|)Y_{l,m}(\hat{\mathbf{k}})$, where $\hbar\mathbf{k}$ is the momentum of one of the two components of the pair with respect to its centre of mass. To put the generic Hamiltonian (7) in the form (8) it suffices to define the 'kernel factors' $R_{\kappa,l}(|\mathbf{k}|)$ and 'coupling constants' $V_{\kappa,l}$ so that they yield the following parametrization:

$$K_l(|\mathbf{k}|, |\mathbf{k}'|) = \frac{1}{4\pi}\sum_{\kappa} V_{\kappa,l} R_{\kappa,l}(|\mathbf{k}|) R^*_{\kappa,l}(|\mathbf{k}'|). \tag{11}$$

We may construct the $\phi_{\kappa,l,m,\mathbf{k}}$ and $V_{\kappa,l}$ as a complete set of solutions to the eigenvalue problem

$$\sum_{\mathbf{k}'} V(\mathbf{k}-\mathbf{k}')\phi_{\kappa,l,m,\mathbf{k}'} = V_{\kappa,l}\phi_{\kappa,l,m,\mathbf{k}} \tag{12}$$

which can be regarded as the result of neglecting, for strong attraction, the kinetic energy contribution to the Schrödinger equation for a two-body bound state.

Equation (8) generalizes the usual BCS Hamiltonian by including interaction terms corresponding to pairs with $\hbar\mathbf{q} \neq 0$ and different values of the internal angular momentum, given by $l, m$. This is required for the correct description of the normal state above $T_c$ in the strong-coupling limit [8] and, as we shall see, to capture the essential physics of exotic pairing via a central potential, respectively. The latter can be understood, in essence, by noting that the angular momentum quantum numbers $l, m$ describe the internal rotational degrees of freedom of the Cooper pairs, and so they are a key ingredient to describe their dynamics in the case of a central attraction. The additional index $\kappa$ has been introduced to ensure the generality of (11), and it refers to the relative motion of the electrons in a pair in the radial direction. Evidently, for interaction potentials of the type we are interested in, such radial motion is 'locked' so that the distance between the two electrons remains constant, and is equal to $r_0$. We will therefore disregard these vibrational modes, assuming that there is a single kernel factor $R_l(|\mathbf{k}|)$ for each value of $l$. Obviously, in that case there is a single coupling constant $V_l$ for each value of the angular momentum quantum number. This approximation becomes exact in the limit when $r_c \to 0$ and $V(r_0) \to -\infty$, keeping $V(r_0)r_c \equiv -g$ constant. Then we obtain the DSM [41], featuring the central 'delta shell' potential [42]

$$V(\mathbf{r}-\mathbf{r}') = -g\delta(|\mathbf{r}-\mathbf{r}'| - r_0) \tag{13}$$

for which [48]

$$K_l(|\mathbf{k}|, |\mathbf{k}'|) = -g4\pi r_0^2(-1)^l j_l(|\mathbf{k}|r_0) j_l(|\mathbf{k}'|r_0). \tag{14}$$

More generally, it must be regarded as a convenient approximation, valid when the potential well in figure 2(*b*) is sufficiently deep and narrow.

Having set up our Hamiltonian, we can now use the standard method reviewed by Randeria [38], which has been applied to specific models by several authors [14, 20, 21, 28], to discuss the ground state at $T = 0$ and the superconducting instability at $T_c$. In short, we introduce bosonic fields $\Delta_{l,m,\mathbf{q}}(\tau)$, $\Delta^*_{l,m,\mathbf{q}}(\tau)$ coupling to the pair creation and annihilation operators $\hat{b}^+_{l,m,\mathbf{q}}$, $\hat{b}_{l,m,\mathbf{q}}$, respectively. In terms of these fields one can define an effective action $S_{\mathrm{eff}}[\Delta^*, \Delta]$ that determines the partition function of the system:

$$Z = \int D[\Delta^*, \Delta]\, \mathrm{e}^{-S_{\mathrm{eff}}[\Delta^*,\Delta]}. \tag{15}$$

We then expand it to the lowest non-trivial (Gaussian) order:

$$S_{\mathrm{eff}}[\Delta^*, \Delta] \approx S_{\mathrm{eff}}[\Delta^{*(0)}, \Delta^{(0)}] + S^{(2)}_{\mathrm{eff}}[\Delta^*, \Delta]. \tag{16}$$

Here $\Delta^{*(0)}_{l,m,\mathbf{q}}(\tau)$, $\Delta^{(0)}_{l,m,\mathbf{q}}(\tau)$ is the configuration of the pairing fields at the saddle point and the Gaussian contribution, $S^{(2)}_{\mathrm{eff}}[\Delta^*, \Delta]$, takes into account fluctuations of the fields about that saddle point.

## 4. Exotic pairs in the ground state

In the ground state, the pairing fields become 'frozen' in their saddle-point configurations [38]. $S_{\mathrm{eff}}^{(2)}[\Delta^*, \Delta]$ acquires a trivial form such that $\int D[\Delta^*, \Delta] \exp\{-S_{\mathrm{eff}}^{(2)}[\Delta^*, \Delta]\} = 1$, equation (15) thus becoming $Z = \exp\{-S_{\mathrm{eff}}[\Delta^{*(0)}, \Delta^{(0)}]\}$. As usual we look for a stationary and homogeneous saddle point:

$$\Delta_{l,m,\mathbf{q}}^{(0)}(\tau) = \Delta_{l,m}^{(0)} \delta_{\mathbf{q},0} \qquad \Delta_{l,m,\mathbf{q}}^{*(0)}(\tau) = \Delta_{l,m}^{(0)*} \delta_{\mathbf{q},0}$$

which yields the BCS ground state [2]. As is well known this state can describe, at least variationally, the BCS to Bose crossover at $T = 0$ [7, 8, 31]. The amplitudes $\Delta_{l,m}^{(0)}$ are related to the expectation values of the pair annihilation operators by $\Delta_{l,m}^{(0)} = V_l \langle \hat{b}_{l,m,0} \rangle$. The equations determining the saddle point take the following form:

$$\Delta_{l,m}^{(0)} = -V_l \sum_{l',m'} \left\{ \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3} \frac{\phi_{l,m,\mathbf{k}}^* \phi_{l',m',\mathbf{k}}}{2E_{\mathbf{k}}} \right\} \Delta_{l',m'}^{(0)} \tag{17}$$

where $E_{\mathbf{k}} \equiv \sqrt{\varepsilon_{\mathbf{k}}^2 + |\Delta_{\mathbf{k}}|^2}$ with $\Delta_{\mathbf{k}} \equiv \sum_{l,m} \Delta_{l,m}^{(0)} R_l(|\mathbf{k}|) Y_{l,m}(\hat{\mathbf{k}})$ the usual BCS 'gap function'. Evidently this non-linear system of equations may have many different solutions, each corresponding to a different superposition of angular momentum states. To determine their relative stability one has to evaluate the appropriate potential. If, as usual, we fix the density $n$, treating the chemical potential $\mu$ as a parameter to be determined self-consistently [8], the energy $\mathcal{U}_0 = \langle \hat{H} \rangle$ has to be calculated. Quite generally, it is given by

$$\frac{1}{L^3} \mathcal{U}_0 = \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3} \frac{\hbar^2 |\mathbf{k}|^2}{2m^*} \left(1 - \frac{\varepsilon_{\mathbf{k}}}{E_{\mathbf{k}}}\right) + \sum_{l,m} \frac{\left|\Delta_{l,m}^{(0)}\right|^2}{V_l} \tag{18}$$

which results from taking the $T \to 0$ limit of $L^{-3} \langle \hat{H} \rangle = -L^{-3} \beta^{-1} \ln Z + \mu n$.

The above equations are entirely equivalent to the usual mean-field theory [43] for a sufficiently general interaction potential (at least, when only pairing correlations are taken into account). In particular, in the limit of very weak attraction (BCS limit), the argument of section 2 applies. Thus we expect that, for the type of interaction that we are interested in, represented in figure 2(*b*), pairing with $l = 2$ will be preferred to $l = 0$ for a range of values of the density, $n = k_F^3 / 3\pi^2$, such that $k_F r_0 \sim 3$. In the extreme case in which the attraction takes place only exactly at $r = r_0$ one expects the densities at which the preferred value $l_{\max}$ of the angular momentum quantum number changes to be given by

$$j_0(k_F r_0)^2 = j_2(k_F r_0)^2. \tag{19}$$

This result, which is degenerate in the quantum number $m$, is exact for the DSM (13), and it gives the boundaries of a *quantum phase transition* in which the ground state changes the rotational symmetry.

On the other hand in the dilute, strong coupling limit (BE limit), there is no longer a well-defined Fermi surface and approximation (3) ceases to be useful. The gap equation (17) describes a two-body bound state with energy $\varepsilon_b^l = 2\mu$ and wavefunction $\psi_{\mathbf{k}} = \Delta_{\mathbf{k}}/2E_{\mathbf{k}}$ [7, 8, 27] and so, evidently, the full functional form of the interaction potential $V(\mathbf{r} - \mathbf{r}')$ has to be taken into account. Since $l$ is a good quantum number for the Schrödinger equation, a well-defined angular momentum is a shared characteristic of the BCS and BE limits. On the other hand it is easy to show, using $\mu \ll -\left|\Delta_{l,m}^{(0)}\right|$, that in the BE limit equation (18)

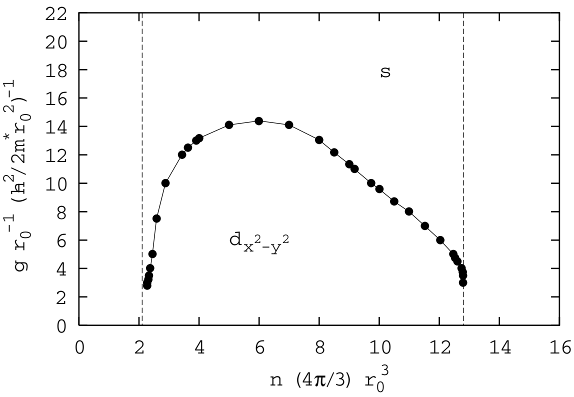


**Figure 3.** Phase diagram of the relative stability of the s and $d_{x^2-y^2}$ trial ground states of the delta-shell model, taken from [48]. In the limit of small coupling constant $g$, the phase transition takes place precisely at the values of the density $n = k_F^3/3\pi^2$ for which condition (19) holds. These are marked by the dashed vertical lines.

becomes

$$\frac{1}{L^3}\mathcal{U}_0 = \frac{1}{2}n\varepsilon_b^l. \tag{20}$$

This equation has a simple interpretation: in the dilute, strong-coupling limit, the system is a BE condensate of non-overlapping pairs. Each pair has $\hbar\mathbf{q} = 0$ and they are all in the same internal state with energy $\varepsilon_b^l$. It follows that the energy of the system is given simply by the formation energies of the individual pairs. Since, for a central potential, the bound state with lowest energy always has $l = 0$, one expects that *rotational symmetry is never broken in the BE limit*.

Figure 3 shows a specific result for the DSM [48] that illustrates the above points. It is a phase diagram for the relative stability of ground states in which Cooper pairs have s and $d_{x^2-y^2}$ symmetries. At weak coupling, pairing with $l = 0$ is preferred at low and high densities, with a quantum phase transition leading to the onset of $d_{x^2-y^2}$ pairing in the intermediate-density regime. The location of the phase boundary is accurately predicted by equation (19). In contrast, as the attraction is made stronger the range of densities over which the state with $l = 2$ is preferred becomes narrower until, above some critical value of the coupling constant, rotational symmetry is restored for all values of the density. The $d_{x^2-y^2}$ state is thus confined to a relatively small 'island' in parameter space. This result not only confirms our expectation that pairing should take place in the s state in the BE limit, but in fact suggests that $l = 0$ is preferred at all densities, provided that the attraction is sufficiently strong. However note that the phase boundary, at finite values of the coupling constant $g$, is not necessarily degenerate in $m$ (unlike at weak coupling). Thus investigations for more general trial ground states (allowing, for example, for mixing of s and d symmetries) have to be carried out.

## 5. Superconducting critical temperature for exotic pairs

When the attraction is strong, fluctuations around the saddle-point configurations of the fields are important to determine the low-lying excitations of the BCS ground state [15, 21] as well as to describe the superconducting instability at $T_c$ [14, 20, 28]. Our interest here is in the latter,

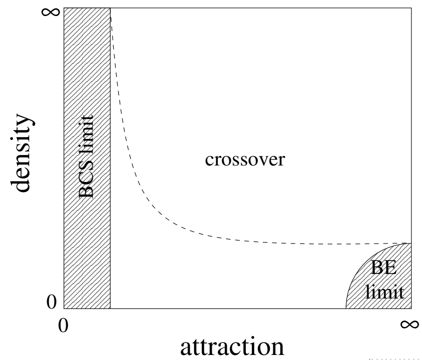


**Figure 4.** Validity of the Gaussian description of the superconducting instability [38]. The theory is deemed correct in the shaded areas: BCS limit (weak-coupling) and BE limit (strong-coupling *and* low densities). For the crossover region it can be regarded as a convenient interpolation scheme at low densities (below the dashed line), while at larger densities it fails (see the text).

as we wish to see whether such instability can correspond, in the strong-coupling, low-density limit, to the formation of a BE condensate of *exotic* pairs.

Just above $T_c$, the partition function (15) is given by

$$Z = Z_0 \times \delta Z \tag{21}$$

where $Z_0 = \exp\{-S[0,0]\}$ corresponds to a free electron gas while $\delta Z = \int D[\Delta^*, \Delta] \exp\left\{S^{(2)}_{\text{eff}}[\Delta^*, \Delta]\right\}$ is the contribution from PP just above $T_c$. The normal state is composed of a mixture of two gases, one made out of free electrons and the other one consisting of PP. In effect, on account of (21) the total electron density $n(\beta, \mu) = L^{-3} k_B T \partial \ln Z / \partial \mu$ can be written as

$$n(\beta, \mu) = n_0(\beta, \mu) + \delta n(\beta, \mu) \tag{22}$$

where the first and second terms on the right-hand side come from $Z_0$ and $\delta Z$, respectively. In particular, the contribution from free electrons has the familiar form $n_0(\beta, \mu) = 2L^{-3} \sum_{\mathbf{k}} f(\beta \varepsilon_{\mathbf{k}})$ where $f(x) \equiv (\mathrm{e}^x + 1)^{-1}$ is the Fermi distribution function. As is well known in the BCS limit $\mu\beta \to \infty$ all electrons are unpaired just above $T_c$ so we have $n \approx n_0$. Conversely, in a dilute system with strong attraction (BE limit) we have $n \approx \delta n$: the normal state just above $T_c$ is composed exclusively of preformed pairs, whose BE condensation leads to superconductivity.

It is important to note that the above functional-integral theory is only valid either at weak coupling or for sufficiently dilute systems with strong attraction. The problem is (leaving aside the fact that it relies on an expansion in powers of the *amplitude* of the pairing fields, while neglecting other fluctuations due exclusively to their *phase*, which are essential in two dimensions [44]) that as the density is increased interactions between the PP, which the Gaussian theory neglects, become important. Similarly, when the interaction becomes weaker the PP increase their radius and begin to overlap. Quite generally, at low densities and intermediate coupling the Gaussian theory of $T_c$ must be regarded only as a convenient interpolation scheme [8, 38], while it fails completely at higher densities (as is evidenced for example in the negative bosonic mass obtained for the DSM at sufficiently large values of the density and the coupling constant [48]). Such limitations (illustrated in figure 4) have been, at least partially, addressed [16, 18, 45]; however, for simplicity our discussion of $T_c$ below refers only to the regions of parameter space in which the standard theory [38] can be safely applied at that temperature. These are illustrated in figure 4.

Let us now see what form the above standard equations [38] take in the case of our fairly general model. In terms of the bosonic Matsubara frequencies $\omega_\nu = 2\nu\pi/\beta$ the quadratic contribution to the effective action has the form

$$S_{\text{eff}}^{(2)}[\Delta^*, \Delta] = \beta \sum_{\mathbf{q}\nu} \sum_{l,m,l',m'} \Delta^*_{l,m,\mathbf{q}}(\omega_\nu)\Gamma^{-1}_{l,m,l',m'}(\mathbf{q}, \mathrm{i}\omega_\nu)\Delta_{l',m',\mathbf{q}}(\omega_\nu) \tag{23}$$

where the sums on $l$ and $l'$ extend only over values of the angular momentum quantum number having the same parity (both even or both odd). As usual, we start by writing out the inverse bosonic propagator explicitly. It is

$$\Gamma^{-1}_{l,m,l',m'}(\mathbf{q}, \mathrm{i}\omega_\nu) = -\frac{L^3}{V_l}\delta_{l,l'}\delta_{m,m'} - \sum_{\mathbf{k}} \mathrm{i}^{l'-l}\phi^*_{l,m,\mathbf{k}}\phi_{l',m',\mathbf{k}} \times \frac{1}{\beta}\sum_n G_0\left(\frac{\mathbf{q}}{2}+\mathbf{k}, \mathrm{i}\omega_n\right) G_0\left(\frac{\mathbf{q}}{2}-\mathbf{k}, \mathrm{i}\omega_\nu - \mathrm{i}\omega_n\right) \tag{24}$$

where $G_0(\mathbf{k}, \mathrm{i}\omega_n) \equiv (\varepsilon_{\mathbf{k}} - \mathrm{i}\omega_n)^{-1}$ is a free-electron Green's function and $\omega_n = (2n+1)\pi/\beta$ a fermion Matsubara frequency. This is a slightly more general form of the similar expression found in the literature [8, 14, 20, 28, 48]. Such fairly general bosonic propagator, like the one obtained in [48] for the DSM, describes not only the centre-of-mass motion of the PP existing above $T_{\mathrm{c}}$ (represented by their total momentum $\hbar\mathbf{q}$) but also the freedom that they have to change their angular momentum (represented by the labels $l, m$). On the other hand, in [8, 14, 20, 28] these internal degrees of freedom were not taken into account. In effect, the interaction potentials employed in [8, 14, 20] had the form (3) with $l_{\max} = 0$ and, therefore, could only lead to pairing in the s state; similarly, the potential in [28] was chosen so that it could only lead to pairing in a particular d-wave state, with $\mathrm{d}_{x^2-y^2}$ symmetry. But for strong *central* attraction one expects that, just above $T_{\mathrm{c}}$, preformed pairs exist with all values of the angular momentum quantum number, as reflected in equation (24). The question that we are trying to answer here is what pairs will form a BE condensate at $T_{\mathrm{c}}$; in particular, whether they can have $l > 0$.

As usual the '$T_{\mathrm{c}}$ equation' is found as the temperature at which the system becomes unstable with respect to pairing fluctuations describing a homogeneous, static field:

$$\beta \sum_{l,m,l',m'} \Delta^*_{l,m,0}(0)\Gamma^{-1}_{l,m,l',m'}(0,0)\Delta_{l',m',0}(0) = 0. \tag{25}$$

We obtain

$$1 = -\frac{V_l}{(2\pi)^3}\int_0^\infty \mathrm{d}|\mathbf{k}||\mathbf{k}|^2|R_l(|\mathbf{k}|)|^2\frac{1-2f(\beta_c\varepsilon_{\mathbf{k}})}{2\varepsilon_{\mathbf{k}}} \tag{26}$$

which is diagonal in $l$ and degenerate in $m$. In general (26) has several solutions $\beta_c \equiv 1/k_B T_{\mathrm{c}}$, corresponding to the formation of a superconducting state with different values of the angular momentum quantum number $l = 0, 2, 4, \ldots$. Evidently the highest $T_{\mathrm{c}}$ corresponds to the true superconducting instability, and it gives the angular momentum quantum number of the Cooper pairs in the superconducting state, just below $T_{\mathrm{c}}$.

As expected [38], equation (26) has the same form as in the mean-field theory and so in the weak-coupling limit the Gaussian theory reduces to it. In particular, the argument for exotic pairing in the BCS limit that we recalled above applies also to the critical temperature: thus at intermediate densities we expect pairs to form and condense simultaneously, at $T_{\mathrm{c}}$, with angular momentum quantum number $l_{\max} = 2$.

Let us now focus on the BE limit. For sufficiently strong coupling and low densities the inverse propagator $\Gamma^{-1}_{l,m,l',m'}(\mathbf{q}, \mathrm{i}\omega_\nu)$ can be expanded to the lowest non-trivial order in the pair's total momentum $\hbar\mathbf{q}$ and frequency $\omega_\nu$ [8]. Such expansion was carried out in [48] for the DSM, following closely the procedure of [14, 28]. The derivation is entirely analogous in the present slightly more general case. In short we find that, after appropriate rescaling of the fields,

$$\Gamma^{-1}_{l,m,l',m'}(\mathbf{q}, \mathrm{i}\omega_\nu) = [-\mathrm{i}\omega_\nu - \mu^b_l(\beta, \mu)]\delta_{l,l'}\delta_{m,m'} + \sum_{i,j=x,y,z} \frac{\hbar^2 \mathbf{q}_i \mathbf{q}_j}{2m^{b,ij}_{l,m,l',m'}(\beta, \mu)}. \tag{27}$$

Thus the action $S^{(2)}_{\mathrm{eff}}[\Delta^*, \Delta]$ describes an ideal Bose gas made out of bosons that propagate with some effective mass $m^{b,ij}_{l,m,l',m'}(\beta, \mu)$ and chemical potential $\mu^b_l(\beta, \mu)$ (these quantities are given by explicit formulae which we shall omit, for brevity). In general, the effective masses $m^{b,ij}_{l,m,l',m'}(\beta, \mu)$ represent anisotropic dispersion relations that are different for PP in distinct internal states. Moreover, their off-diagonal values ($l \neq l'$, $m \neq m'$) may be finite, reflecting hybridization between such states. Nevertheless, in the BE limit we have

$$m^{b,ij}_{l,m,l',m'}(\beta, \mu) \to \begin{cases} 2m^* & \text{if} \quad l = l' \quad \text{and} \quad m = m' \\ \infty & \text{otherwise} \end{cases}$$

indicating that the tightly-bound PP propagate freely as particles of mass $2m^*$ without changing their angular momentum. This simplifies (23) to

$$S^{(2)}_{\mathrm{eff}}[\Delta^*, \Delta] = \beta \sum_{\mathbf{q}\nu} \sum_{l,m} \Delta^*_{l,m,\mathbf{q}}(\omega_\nu) \left( -\mathrm{i}\omega_\nu + \frac{\hbar^2 |\mathbf{q}|^2}{4m^*} - \mu^b_l(\beta, \mu) \right) \Delta_{l,m,\mathbf{q}}(\omega_\nu) \tag{28}$$

in that limit. The effective chemical potentials $\mu^b_l(\beta, \mu)$, on the other hand, are different for bosons with different values of the angular momentum quantum number $l$. The condition for an instability of the gas of preformed pairs to a superconducting state with angular momentum quantum number $l$, equation (26), corresponds to the BE condensation of the corresponding PP: $\mu^b_l(\beta_c, \mu_c) = 0$. Typically, other PP with angular momentum quantum number $l' \neq l$ are also present in the normal state, just above $T_{\mathrm{c}}$, however their chemical potential is $\mu^b_{l'}(\beta_c, \mu_c) < 0$ at the transition so, unless they are also close to their own critical temperature, they are only present in small number. One can thus neglect such additional PP, and check the assumption *a posteriori* by ensuring that the critical temperatures are quite different. Thus we eliminate the sum on $l$ from equation (28). We can now very easily deduce the explicit form of the density equation (22) in this case. At the critical temperature, it is

$$n(\beta_c, \mu) = \sum_{m=-l}^{l} \delta n_{l,m}(\beta_c) \tag{29}$$

where $\delta n_{l,m}(\beta_c) = 2L^{-3} \sum_{\mathbf{q}} g(\beta_c \hbar^2 |\mathbf{q}|^2 / 4m^*)$ is the density of a Bose gas that is exactly at its BE condensation temperature, $T_{\mathrm{c}}$ ($g(x) \equiv (\mathrm{e}^x - 1)^{-1}$ is the Bose distribution function). Evidently the fact that there are $2l + 1$ values of $m$ lowers the critical temperature: it is given by

$$k_B T_{\mathrm{c}} = 3.315 \frac{\hbar^2}{2m^*} \left[ \frac{n}{2(2l+1)} \right]^{2/3} \tag{30}$$

i.e. it is the BE condensation temperature for $n/[2(2l + 1)]$ bosons of mass $2m^*$ each. This reduces to the usual result [8, 14, 20, 28] only for bosons with internal angular momentum $l = 0$:

$$k_B T_c = 3.315 \frac{\hbar^2}{2m^*} \left(\frac{n}{2}\right)^{2/3} \qquad \text{for} \quad l = 0 \tag{31}$$

For higher values of the angular momentum, the bosons still condense all at the same temperature, but they do so as $2l + 1$ independent Bose gases, each one corresponding to one of the degenerate angular momentum states consistent with that value of $l$. Obviously, the crucial consequence of this is that *the superconducting instability in the low-density, strong-coupling limit always corresponds to the BE condensation of pairs with* $l = 0$. Evidently, this result is quite generic since the degeneracy in $m$ is an unavoidable feature of any central potential[4].

## 6. Conclusion

In these pages we have revisited the old problem [3, 4] of exotic pairing via a central potential. According to a well-known argument, a central potential can lead to a superconducting ground state in which Cooper pairing takes place with a finite value of the angular momentum quantum number $l > 0$. We have demonstrated that the natural framework for that to happen is provided by an interaction in which the distance between the paired electrons is 'locked' to some finite value $r_0$. We have then used the well-known functional integral formulation of the BCS to Bose crossover [38] to explore the behaviour of such models away from the original weak-coupling limit. We have found evidence that a *quantum phase transition*, in which the symmetry of the superconducting order parameter changes, is associated, quite generally, with this type of interactions. The phase transition may occur at weak coupling as the density is varied, making the preferred pairing channel change, or on approaching the BE limit, where rotational symmetry is always restored. The latter is a consequence of very elementary energetic considerations, related to the fact that the two-particle bound state with $l = 0$ always has lower energy than those with finite angular momentum quantum number. Finally, we have discussed the BE limit of the critical temperature. By neglecting preformed pairs with all but one value of $l$, we have been able to estimate the BE limit of $T_c$ for different values of the angular momentum quantum number. For $l > 0$, we have found the surprising result that $T_c$ is not given by the usual, simple formula for BE condensation, but instead it is considerably suppressed due to the $2l + 1$ degeneracy of the corresponding bound state. This may be an interesting example of the effect of internal degrees of freedom of the constituent bosons on the properties of BE condensates, as discussed by Nozières [46].

Our remarks may serve as preliminary steps for a systematic exploration of the possibility of rotational symmetry breaking in the possible BCS state of degenerate Fermi gases [26, 35–37], where the distance $r_0$ could be related to the shape of the interatomic interaction. They also provide an indication that some of the features that we have identified [41, 47, 48] in the new, delta shell model may be relevant to a larger, and important, class of interaction potentials.

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[4] In contrast, the anisotropic effective interaction considered in [28] has, by definition, a single, non-degenerate bound state. Hence the result that the BE limit of the critical temperature is given by (31), in spite of the d-wave nature of the superconducting state.

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