Citation for published version

Scaparra, Maria Paola and Losada, Chaya and Church, Richard L. and Daskin, Mark S. (2008) Modeling Approaches for the Multi-source Interdiction Median Problem. Working paper. University of Kent Canterbury, Canterbury

DOI

Link to record in KAR

https://kar.kent.ac.uk/25486/

Document Version

UNSPECIFIED

ISSN 1748-7595 (Online)

Kent Business School

Working Paper Series

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Working Paper No.187
March 2010

The Multiple Resource Probabilistic Interdiction Median Problem

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June 13, 2009

Abstract

Interdiction is the deliberate act of attempting to destroy or damage a set of components of an infrastructure system to degrade its overall performance. A variety of mathematical interdiction models have been proposed in the literature to identify critical assets in supply systems. In this paper, we present an interdiction model for median systems. In this model, the outcome of an attack is uncertain, i.e. an attack is successful only with a given probability, and the probability of success depends upon the amount of resources invested in the attack. The objective is to allocate the interdiction resources among the system facilities to maximize the expected disruption. We study three modeling alternatives for this problem. We present a computational comparison of the three formulations, an analysis of the solutions obtained, and a study that identifies those parameters that influence the time performance the most. We also test the robustness of the models to different probability distributions. Finally, we present results that demonstrate that the new model is more versatile than

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previously proposed interdiction models which do not consider probabilistic attack outcomes and/or multiple offensive resources.

1 Introduction

Critical infrastructures are vital to the welfare of the society. The USA PATRIOT Act[1] defined critical infrastructures as "systems and assets, whether physical or virtual, so vital to the United States that the incapacity or destruction of such systems and assets would have a debilitating impact on the security, national economic security, national public health or safety, or any combination of those matters". The terms *interdiction* and *attack* are used interchangeably and are referred to as the deliberate act of attempting to destroy or damage a set of components of an infrastructure system to degrade its overall performance. An exhaustive survey of interdiction models published before 2004 can be found in Church et al. [8]. The majority of these models assume the existence of an *interdictor* who wants to degrade a system performance the most using a limited amount of interdiction resources. Interdiction models can be used to identify critical assets as well as estimate the worst-case scenario due to disruption, be it intentional or not.

During the last five years, many other network interdiction models have been proposed in the literature to capture more complex issues. Extensions to deterministic network interdiction models, include those by Khachiyan et al. [21], Bayrak and Bailey [3], Royset and Wood [30], Lim and Smith [23], and Murray et al. [26]. The stochastic interdiction literature is much more limited than its deterministic counterpart. Some recent stochastic network interdiction models are those by Janjarassuk and Linderoth [20], Hemmecke et al. [19], Held et al. [18] and Held and Woodruff [17]. On the front of applications of network interdiction models to real world problems, some noteworthy studies are found in Salmeron et al. [32], Salmerón et al. [31] and

[1] *Public Law Pub.L. 107-56*, 2001

Arroyo and Galiana [1].

Interdiction applied to *facility location* differs from network interdiction in that arcs and their characteristics are not considered, and only facilities/nodes and their services can be interdicted. To the best of the authors´ knowledge, the first interdiction models for facility location date back to 2004 (Church et al. [8]). These models, referred to as r-Interdiction Median Problem (RIM) and r-Interdiction Covering Problem (RIC), are built on the underlying p-median problem (Hakimi [16]) and the maximal covering problem (Church and ReVelle [9]), respectively. In both models, interdiction is deterministic and complete. In RIM, the objective is to maximize the demand weighted distance between customers and the nearest working facilities after the loss of r facilities. The RIC objective maximizes the decrease in coverage due to the removal of r facilities. A probabilistic version of RIM, in which the outcome of an attack is uncertain, is discussed in Church and Scaparra [11]. In this model, called the Probabilistic Interdiction Median Problem (PIM), the attack is successful with a given probability which is known to the interdictor but cannot be controlled by him. The model proposed in this paper is a further extension of RIM and PIM, in which the interdictor can, to a given extent, increase his probability of success through the use of additional limited resources.

Interdiction models allow analysists and modelers to estimate the impact of disruptions in supply systems. The logical next step is to develop strategies that mitigate the consequences of disruptions and increase the system´s overall reliability. This can be accomplished by devising a protection plan for an existing system ($fortification$) or creating an inherently reliable system from scratch ($design$), where both $fortification$ and $design$ models, commonly have an embedded interdiction model. *Fortification* is concerned with the optimal investment of efforts to thwart the attacker's best strategies. Church and Scaparra [7] showed that the most cost effective allocation of protective resources does not necessarily involve the protection of all the facilities in the critical set, the set of sites an interdictor would attack in the absence of any fortification. However, at least one site in the critical set must be in the optimal fortification plan. Thus, the solution to a protection plan is neither obvious nor easy to identify. Church and Scaparra [7] proposed the first fortification model for facilities in a supply/service system, specifically for the p-median system (Hakimi [16]). This model assumes sufficient resources to protect no more than q facilities, which are declared out of the interdictor's reach and are always operational. Two different solution approaches to solve this problem can be found in Scaparra and Church [34] and Scaparra and Church [35]. Piyade et al. [28] present an extension to this model that contemplates dynamic enlargement of the capacity of the facilities at some pre-defined costs and considers nonuniform fortification costs instead of fixing the number of protected facilities to a specific quantity. In a similar line, Scaparra and Church [36] consider capacity restrictions on the system facilities and introduce penalty costs that are incurred when the total customer demand is not met after a disruption. Liberatore et al. [22] propose a fortification/interdiction uncapacitated median problem where the number of attacks is uncertain.

Design models are used to identify the optimal location of a set of facilities by taking into account not only the day-to-day operating expenses but also the costs associated with potential disruptions (Snyder and Daskin [37], Qi et al. [29]). O'Hanley and Church [27] proposed a design model for the maximal coverage problem. This model locates facilities to maximize both the initial demand coverage and the coverage after the loss of r facilities. Bailey et al. [2] present a *design* problem with a second-stage adversarial subproblem modeled as a Markov Decision Process.

In this paper, we extend the small collection of interdiction models for *facility supply systems* by introducing the Multiple Resource Probabilistic Interdiction Median Problem (MRPIM). The model assumes that a supply system, where the demand of each customer is entirely served by their closest facility, is vulnerable to malicious attacks. Attacks, if successful, render the interdicted facilities completely inoperable and force the reassignment of the customers to further facilities, thus increasing the overall operating costs of the system. In contrast with previous research, an attack is successful with a given probability and this probability depends on the amount of resources employed in the attack. In other words, we assume that the interdictor, by distributing a limited amount of offensive resources among the facilities, can control his probability of success and, consequently, the overall expected damage caused to the system.

We study three modeling alternatives for MRPIM. Our initial formulation is a non-linear formulation which includes a new form of closest assignment (CA) constraints. These constraints were devised to correctly compute the expected distances between customers and facilities in the presence of probabilistic assignment variables, (See Gerrard and Church [14] for an in-depth discussion of other closest assignment constraints in location problems). We show how this formulation, called MIP-MRPIM, can be linearized so that it can be solved directly by commercial MIP optimizers. Unfortunately, due to the structure of the CA constraints, solving this formulation requires a considerable amount of computing time even for small problem instances. We therefore propose an alternative bilevel model, BI-MRPIM, in which the probabilistic assignments of customers to their closest non-interdicted facilities are enforced in the lower level problem. We show how this model can be reduced to a single level model which is amenable to solution by MIP solvers. The bi-level formulation significantly increases the scope and applicability of MRPIM to larger problem instances. Finally, we provide a third formulation (NET-MRPIM) in which the probabilities of assigning customers to their closest facilities, in accordance with the interdictions made, are defined as flow variables on an appropriately defined network. This network formulation is very effective and optimally solves all the instances attempted in negligible computing time.

The major contributions of this paper fall into two areas: first, it enhances the interdiction modeling literature by introducing a more realistic model in which the success of an attack is not certain and different amounts of offensive resources can be used to disrupt the facilities. Second, from a modeling prospective, it introduces the definition of new closest assignment constraints which can be employed in combination with continuous assignment variables; the use of bilevel programs to model complex constraint structures and a non-obvious network representation of a probabilistic interdiction problem.

The paper is organized as follows: In Section 2 we describe the problem and present the first attempt at modeling MRPIM (MIP-MRPIM). In Section 3 and Section 4, the BI-MRPIM and NET-MRPIM formulations are presented, respectively. We test the performance of the three formulations and analyze the solutions obtained in Section 5. Finally, we discuss some conclusions in Section 6.

2 Initial Model

2.1 Problem description, input data, assumptions

Our model assumes the existence of a p-median system composed of a set of customers N ($|N| = n$), indexed by i, and a set of facilities F ($|F| = p$), indexed by j, where h_i is the level of demand of customer i, and d_{ij} is the shortest distance between each customer-facility pair (i, j). As in most p-median problems, each customer is served by its closest open facility (CA rule) since every facility is assumed to have unlimited capacity.

Our objective is to maximally disrupt this system by interdiction where an attack on a facility may be successful or not with a given probability. This probability can be manipulated (increased or decreased) according to the amount of resources invested in the attack. For simplicity, we use a set of discrete values that reflect the possible amounts of offensive resource invested on each facility and denominate them levels of attack. Let $K = \{0, ..., k_{max}\}$, indexed by k, be the set of all the levels of attack permitted and sorted in increasing order of attack intensity, and let c_{jk} equal the cost to attack site j at level k. We denote by p_{jk} the working probability of a facility j after an attack of level k , where $p_{jk} > p_{jk+1}$ and $c_{jk} < c_{jk+1}$, $\forall j, k$. For convenience, we will define the lowest level k as no attack so that $p_{j0} = 1$ and $c_{j0} = 0$.

The objective of the attacker is to maximize the overall expected traveling distance for serving all customers by disrupting some facilities with a limited offensive budget, r. To ensure service to all demand points even when all the facilities may be disrupted, we use a dummy facility d that cannot be interdicted. Its working probability is always one and there is a penalty $d_{id} = g_i$ for assigning a customer i to it. From now onward, we denote by $\bar{F}$ the set of facilities which includes the dummy facility as well. Finally, to model the CA rule, we define the set $T_{ij} = \{l \in F \,|d_{il} \leq d_{ij},\, j \neq l\}$, for every customer-facility pair $(i,\, j)$. T_{ij} represents the set of facilities closer to i than j.

Some of the assumptions made in modeling MRPIM are that the attacks on the facilities are independent, and that the attacker has perfect information, e.g., attacks success probabilities, about all the input parameters involved in the problem.

2.2 A MIP formulation (MIP-MRPIM)

In this section, we present a single-level formulation of MRPIM, which uses the following two sets of binary variables:

$$y_{jk} = \begin{cases} 1 & \text{if facility } j \text{ is interdicted at level } k \\ 0 & \text{otherwise} \end{cases}$$

$$\theta_{ij} = \begin{cases} 1 & \text{if customer } i \text{ can receive service from facility } j \\ 0 & \text{otherwise} \end{cases}$$

In addition, for each costumer i and each facility j, we define a continuous decision variable z_{ij}, which represents the probability that customer i receives service from facility j.

The single-level formulation of MRPIM is:

$$Max\, Z = \sum_{i \in N} \sum_{j \in \bar{F}} h_i d_{ij} z_{ij} \tag{1}$$

Subject to

$$\sum_{j \in \bar{F}} z_{ij} = 1 \quad \forall i \in N \tag{2}$$

$$\sum_{k \in K} y_{jk} = 1 \quad \forall j \in F \tag{3}$$

$$\sum_{j \in F} \sum_{k \in K} c_{jk} y_{jk} \leq r \tag{4}$$

$$z_{ij} \leq \left(1 - \sum_{l \in T_{ij}} z_{il}\right) \left(\sum_{k \in K} p_{jk} y_{jk}\right) \quad \forall i \in N, j \in F \tag{5}$$

$$\sum_{l \in T_{ij}} \left(\left(1 - \sum_{w \in T_{il}} z_{iw}\right) \sum_{k \in K} p_{lk} y_{lk} - z_{il} \right) \leq M(1 - \theta_{ij}) \quad \forall i \in N, j \in \bar{F} \tag{6}$$

$$0 \leq z_{ij} \leq \theta_{ij} \quad \forall i \in N, j \in \bar{F} \tag{7}$$

$$y_{jk} = 0, 1 \quad \forall j \in F, k \in K \tag{8}$$

$$\theta_{ij} = 0, 1, \quad \forall i \in N, j \in \bar{F} \tag{9}$$

The objective function maximizes the expected weighted distance. Equation (2) ensures that each customer is served with probability one, either by a unique facility or through a combination of many, including the dummy facility. The fact that there is a single level of attack to be executed over a particular facility is accounted for by (3). Note that within the attack levels considered it is $k = 0$, representing that if selected, no attack is accomplished and therefore $p_{j0} = 1$. Constraint (4) prevents the overall cost of the attacks from exceeding the available budget. Constraints (5) set upper bounds on the probabilities z_{ij}. Namely, they state that the probability of assigning i to j is at most equal to the probability that i is not assigned to any facility closer than j $(1 - \sum_{l \in T_{ij}} z_{il})$ multiplied by the probability that facility j is working $(\sum_{k \in K} p_{jk} y_{jk})$. Note that the value of z_{ij} is strictly less than this upper bound only if by setting it exactly equal to the upper bound, the total probability of assigning i exceeds one. Constraints (6) and (7) force closest assignments. Without these two set of constraints, the program would allocate customers to the farthest facilities, regardless of the operational status of the closer ones. The idea behind the CA constraints is the following: for any given customer i and facility j, we check whether all the variables associated with facilities closer to i than j (z_{il}, $l \in T_{ij}$) take on the maximum value given by equation (5). If they do, then θ_{ij} can take value 1 to indicate that customer i can be assigned to facility j with some positive probability ($z_{ij} \geq 0$). Otherwise, if there is a closer facility with some unused probability of service, θ_{ij} is forced to zero and, because of constraint (7), z_{ij} is forced to zero as well. Note that a possible value for the big M used in these constraints is simply one. Finally, equations (8) and (9) state the domain of the binary decision variables. Note that the variables z_{ij} and θ_{ij} are defined also for the dummy facility, whereas the interdiction variables y_{jk} are only defined for the actual facilities.

A drawback of this model is the presence of the quadratic terms in the constraints (5) and (6). These can be linearized with the following transformation. For each i, j, $l \in T_{ij}$ and k, we define a variable $v_{ijlk} \geq 0$. Then the product $z_{il}y_{jk}$ is replaced by v_{ijlk}, and the correct behavior of the newly defined variables is imposed by the inclusion of two additional constraints:

$$v_{ijlk} \leq M_1 y_{jk} \quad \forall i, j, l \in T_{ij}, k \tag{10}$$

$$v_{ijlk} \leq z_{il} + M_1(1 - y_{jk}) \quad \forall i, j, l \in T_{ij}, k \tag{11}$$

In fact, when y_{jk} is zero, then constraints (10) force v_{ijlk} to be zero as well. If y_{jk} equals one, then v_{ijlk} takes at most the value of z_{il} (11). As we are dealing with a maximization problem, v_{ijlk} takes the maximum value possible, i.e., z_{il}. M_1 is an upper bound for the variables v_{ijlk} and a possible value for it is 1.

Note that constraints (5) could have been written in a more intuitive way as:

$$z_{ij} \leq \prod_{l \in T_{ij}} \left(1 - \sum_k p_{lk} y_{lk}\right) \left(\sum_k p_{jk} y_{jk}\right) \quad \forall i, j \tag{12}$$

Here the right hand side of the inequality represents the probability that all the facilities closer to i than j are not working and that facility j is working. However these constraints have a higher degree of non-linearity and cannot be easily linearized through a variable replacement.

Table 1: RIM vs. MIP-MRPIM: Size comparison

Ω	RIM	MIP-MRPIM
Cont. Vars	$np+p$	$n\,(p+1)+np\,(p-1)\,\|K\|/2$
Bin. Vars	p	$p\|K\|+n\,(p+1)$
Constraints	$np+n+1$	$3np+n\|K\|p\,(p-1)+p+n+1$

Table 2: RIM vs MIP-MRPIM: Size growth rate

	$\partial\Omega/\partial p$		$(\partial\Omega/\partial n)$	
Ω	RIM	MIP-MRPIM	RIM	MIP-MRPIM
Cont. Vars	$n+1$	$n+np\|K\|-(n\|K\|/2)$	p	$(p+1)+p\,(p-1)\,\|K\|/2$
Bin. Vars	1	$\|K\|+n$	0	$p+1$
Constraints	n	$3n+2n\|K\|p-n\|K\|+1$	$p+1$	$3p+\|K\|p\,(p-1)+1$

2.3 RIM vs MIP-MRPIM

We now briefly analyze the increased difficulty of the model MRPIM as compared to the deterministic model RIM (Church et al. [8]). Table 1 displays the size of the two models for three categories (rows under the Ω heading), namely the total number of continuous variables, binary variables and constraints. Table 2 shows the partial derivatives of each element with respect to n and p for the two models. These values express the rate at which each quantity of interest varies for a unit increase in the number of customers and facilities, respectively. Besides having much more complex CA constraints, it is evident that the size of MIP-MRPIM is considerably greater than the size of RIM for each of the three categories. Moreover, the size of MIP-RIM increases significantly faster as n and p increase.

Preliminary attempts at solving this formulation revealed that for many real sized problems was not possible to prove the solution optimality within a few hours of computing time. We believe that the slow convergence was mainly to be imputed to the constraints (6) and (7). To overcome this difficulty, we propose a bi-level formulation where the objective of the lower level problem enforces the CA rule.

3 A Bi-level formulation (BI-MRPIM)

Multi-level formulations are hierarchical problems in which the objective function and the set of feasible decisions made at one level are in part determined by decisions made at other levels (Bracken and McGill [4]). Multi-level problems are very common in the literature of *interdiction* and *fortification* where the programs have as many levels as the number of agents with different objectives, e.g., systems planners and defenders, attackers, system users and operators (Brown et al. [5], Cormican et al. [12], Scaparra and Church [34], Cappanera and Scaparra [6]). Although conceptually straightforward, bi-level problems are burdensome to solve, especially if binary variables are present in the lower level problem (See Moore and Bard [24] or Dempe [13] for an in-depth discussion of bi-level mixed-integer models). Even bi-level problems with only continuous variables can become intractable as the problems grow in size (Lim and Smith [23]). Here, we propose a bi-level formulation for MRPIM with binary interdiction variables that, after simple manipulations, can be efficiently solved by all-purpose optimization software.

The bi-level formulation of MRPIM is as follows:

$$\max \quad H(y) \tag{13}$$

$$\text{s.t.} \quad \sum_{k \in K} y_{jk} = 1 \quad \forall j \in F \tag{14}$$

$$\sum_{j \in F} \sum_{k \in K} c_{jk} y_{jk} \leq r \tag{15}$$

$$y_{jk} \in \{0,1\} \quad \forall j \in F, \forall k \in K \tag{16}$$

where

$$H(y) = \min \sum_{i \in N} \sum_{j \in \bar{F}} h_i d_{ij} z_{ij} \tag{17}$$

$$\sum_{j \in \bar{F}} z_{ij} = 1 \quad \forall i \in N \tag{18}$$

$$z_{ij} \leq \left(1 - \sum_{l \in T_{ij}} z_{il}\right) \left(\sum_{k \in K} p_{jk} y_{jk}\right) \quad \forall i \in N, j \in F \tag{19}$$

$$z_{id} \leq 1 \quad \forall i \in N \tag{20}$$

$$z_{ij} \geq 0 \quad \forall i \in N, \forall j \in \bar{F} \tag{21}$$

Bi-level problems make use of the idea of two-player or leader-follower games (Stackelberg [38]). Here, the upper level program, (13)-(16) plays the role of the interdictor who leads the game by choosing which facilities to attack and at which level (fixes y_{jk}) in order to maximize the system disruption, which is evaluated in the user problem. The lower level problem (17)-(21) defines the strategy of the follower or system user. The objective minimizes the expected demand-weighted total distance expressed in terms of the user variables only (z_{ij}), but the feasible region is defined by the upper level interdiction variables. Note that the constraints involved in both the upper and lower level programs are the same constraints that were in the single-level formulation. For this reason, we omit any further explanation. Note that the closest assignment constraints of MIP-MRPIM are not necessary as the objective of the lower level problem will ensure this property in conjunction with constraints (19).

To solve this problem it is necessary to transform it into a single level formulation (see for example Wood [40]). For this purpose, we take the dual of the lower level linear sub-problem in the z_{ij} variables.

Let α_i, β_{ij} and $\boldsymbol{\gamma}_i$ be the dual variables associated with constraints (18), (19) and

(20), respectively. The dual of the lower level sub-problem (17)-(21) is:

$$\max \quad \sum_{i \in N} (\alpha_i - \gamma_i) - \sum_{i \in N} \sum_{j \in F} \beta_{ij} \sum_k p_{jk} y_{jk} \tag{22}$$

$$\alpha_i - \beta_{ij} - \sum_{l \in T_{ij}} \beta_{il} \sum_k p_{lk} y_{lk} \leq h_i d_{ij} \quad \forall i \in N, \forall j \in F \tag{23}$$

$$\alpha_i - \gamma_i \leq h_i d_{id} \quad \forall i \in N \tag{24}$$

$$\beta_{ij} \geq 0 \quad \forall i \in N, \forall j \in F \tag{25}$$

$$\alpha_i \quad \text{unrestricted} \quad \forall i \in N \tag{26}$$

$$\gamma_i \geq 0 \quad \forall i \in N \tag{27}$$

Note that there are quadratic terms in both the objective function and constraints (23). These quadratic terms are linearized as follows. Let $v_{ijk} \geq 0$ be the product of $\beta_{ij} y_{jk}$; then, we insert into the model the linearization constraints displayed below:

$$v_{ijk} \leq M y_{jk}, \quad \forall i, \forall j, \forall k \tag{28}$$

$$v_{ijk} \geq \beta_{ij} - M(1 - y_{jk}), \quad \forall i, \forall j, \forall k \tag{29}$$

The use of constraints (28) and (29) ensures that if $y_{jk} = 0$, then $v_{ijk} = 0$ for each i; whereas if $y_{jk} = 1$ then $v_{ijk} = \beta_{ij}$ for each i. M is a valid upper limit of v_{ijk}. A possible value for M is $\max_{ij} (h_i d_{ij})$.

The resulting single-level MIP linear problem is:

$$\max \quad \sum_{i \in N} (\alpha_i - \gamma_i) - \sum_{i \in N} \sum_{j \in \bar{F}} \sum_{k} p_{jk} v_{ijk} \tag{30}$$

$$\text{s.t.} \quad \sum_{k \in K} y_{jk} = 1 \quad \forall j \in F \tag{31}$$

$$\sum_{j \in F} \sum_{k \in K} c_{jk} y_{jk} \le r \tag{32}$$

$$\alpha_i - \beta_{ij} - \sum_{l \in T_{ij}} \sum_{k \in K} p_{lk} v_{ilk} \le h_i d_{ij} \quad \forall i \in N, \forall j \in F \tag{33}$$

$$v_{ijk} \le M y_{jk}, \quad \forall i \in N, \forall j \in F, \forall k \in K \tag{34}$$

$$v_{ijk} \ge \beta_{ij} - M(1 - y_{jk}), \quad \forall i \in N, \forall j \in F, \forall k \in K \tag{35}$$

$$\alpha_i - \gamma_i \le h_i d_{id} \quad \forall i \in N \tag{36}$$

$$y_{jk} \in \{0, 1\} \quad \forall j \in F, \forall k \in K \tag{37}$$

$$\beta_{ij} \ge 0 \quad \forall i \in N, \forall j \in F \tag{38}$$

$$\alpha_i \quad \text{unrestricted} \quad \forall i \in N \tag{39}$$

$$\gamma_i \ge 0 \quad \forall i \in N \tag{40}$$

$$v_{ijk} \ge 0 \quad \forall i \in N, \forall j \in F, \forall k \in K \tag{41}$$

Although this formulation is more efficient than the previous one, it still experiences some convergence difficulty, especially for large values of the interdictor budget. We therefore provide a third formulation for MRPIM which proved to be more scalable across larger parameter ranges.

4 A Network-like formulation (NET-MRPIM)

The second reformulation of MRPIM is based on a network representation of the problem similar to the one used in Morton et al. [25] and Scaparra [33]. We assume that each customer i performs a walk through a path including $P + 1$ nodes. Each node along the path corresponds to a facility (including the dummy facility) and the nodes are sorted in increasing order of distance from i. Let i_j represent the j^{th} closest node to customer i. Node i_j is connected to the next node i_{j+1} through k_{max} arcs, but at most one of these arcs is enabled. Namely, an arc from i_j to i_{j+1} at level k is enabled only if the facility corresponding to node i_j is interdicted with k resources, where k ranges between 1 and k_{max}. Customer i visits each node along the path until he reaches either a node i_j with none of its outgoing arcs enabled (i.e., a node corresponding to a non-interdicted facility), or the terminal node in his path (i.e., the node corresponding to the dummy facility).

In this formulation, the interdictor decides which arcs to enable (i.e., which facilities to interdict and at which level), so that the overall demand-weighted distance is maximized. Note that in this formulation, the decision variables y_{jk} determine whether an arc leaving node j at level k is enabled or not.

In order to capture the probability that each customer reaches a given node in this model, we define the following additional variables.

δ_{ij} : probability that customer i stops at facility j. This quantity is positive only if facility j is not interdicted.

w_{ijk}: if facility j is interdicted at level k, this is the probability that customer i reaches facility j. This is equivalent to the probability that all the facilities closer to i than j are not working. If facility j is not attacked with k resources, this quantity is zero.

x_i: probability that customer i reaches the dummy facility.

MRPIM can then be formulated as follows:

$$\text{Max} \quad \sum_i h_i \left(\sum_{j=1}^{p} d_{ii_j} \left(\delta_{ii_j} + \sum_k p_{jk} w_{ii_jk} \right) + g_i x_i \right)$$

s.t.

$$w_{ijk} \leq y_{jk} \quad \forall i \in N, j = 1, ..., p, k = 1, ..., k_{max} \tag{42}$$

$$\sum_k y_{jk} \leq 1, \quad \forall j = 1, ..., p \tag{43}$$

$$\sum_j \sum_k c_{jk} y_{jk} = r \tag{44}$$

$$\sum_k w_{ii_1k} + \delta_{ii_1} = 1, \quad \forall i \in N \tag{45}$$

$$\sum_k (1 - p_{i_{j-1}k}) w_{ii_{j-1}k} = \sum_k w_{ii_jk} + \delta_{ii_j}, \quad \forall i \in N, j = 2, ..., p \tag{46}$$

$$x_i = \sum_k (1 - p_{i_pk}) w_{ii_pk}, \quad \forall i \in N \tag{47}$$

$$y_{jk} \in \{0, 1\}, \quad \forall i \in N, k = 1, ..., k_{max} \tag{48}$$

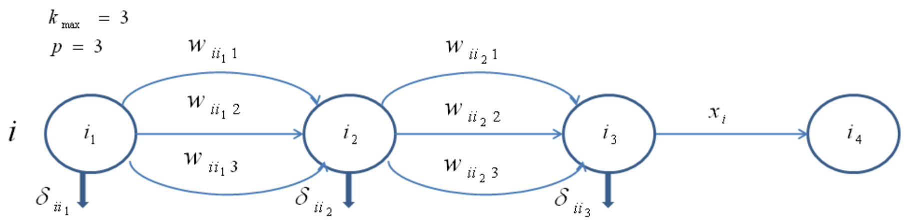
$$w_{ijk} \geq 0, \quad \forall i \in N, j = 1, ..., p, k = 1, ..., k_{max} \tag{49}$$

$$x_i \geq 0, \quad \forall i \in N \tag{50}$$

$$\delta_{ij} \geq 0, \quad \forall i \in N, j = 1, ..., p \tag{51}$$

The objective function maximizes the sum of expected distances traveled by all the customers. Constraints (42) state that if a facility j is not attacked with k resources ($y_{jk} = 0$), the probability that any customer i uses arc k to travel from j to his next closest facility must be zero; else, the probability w_{ijk} is at most one. The interdictor can only use one level of resources to interdict each facility (43) and cannot exceed an overall budget of r resources (44). Constraints (45)-(47) can be seen as flow balance constraints. For each customer i, constraints (45) force one unit of flow out of the initial node in i's path. This unit can either flow to the next closest facility if one of the outgoing arcs from node i_1 is enabled ($\sum_k w_{ii_1k} = 1$) or stop at the first node ($\delta_{i1} = 1$), meaning that customer i's demand can be fully served by his first closest facility. Constraints (45) enforce flow conservation at all intermediate nodes along the path. More specifically, they state that the probability that a customer i either stops at his j^{th} closest facility ($\delta_{ii_j} > 0$) or proceeds to the next closest facility using one of the k arcs ($\sum w_{ii_jk} > 0$) is equal to the probability that i arrives at i_j, which in turn is the probability that i arrives at his $j^{th} - 1$ closest facility through some enabled arc k multiplied by the probability that facility i_j is not working, $(1 - p_{i_{j-1}k})$. Note that the maximization nature of the problem inherently enforces the decisions of leaving or stopping at a given node j to be exclusive, i.e., if $\sum_k w_{ii_jk} > 0$ then $\delta_{ii_j} = 0$, else if $\delta_{ii_j} > 0$, then $\sum w_{ii_jk} = 0$. Finally, constraints (47) enforce the probability that each customer reaches the terminal node to be equal to the probability that the customer leaves the former node, i_p, multiplied by the probability of finding that node inoperative. Constraints (48) through (51) define the domain of the decision variables. Figure 1 displays a visual representation of the NET-MRPIM formulation for a given customer i.

Figure 1: Visual representation of the NET-MRPIM formulation



4.1 Remarks and Extensions

- The three formulations for MRPIM are a generalization of RIM and PIM. That is, by setting $K = \{0, 1\}$, RIM and PIM can be obtained by choosing $p_{j1} = 0$ and $p_{j1} = a$ respectively, where $0 < a < 1$.

- In any of the presented three formulations, cardinality constraints or budget constraints on the interdiction resources can be used interchangeably. Using cardinality constraints is equivalent to assuming that each attack level corresponds to the number of resource units used on a facility, each unit of resource has a unit cost and r is the total number of resources available.

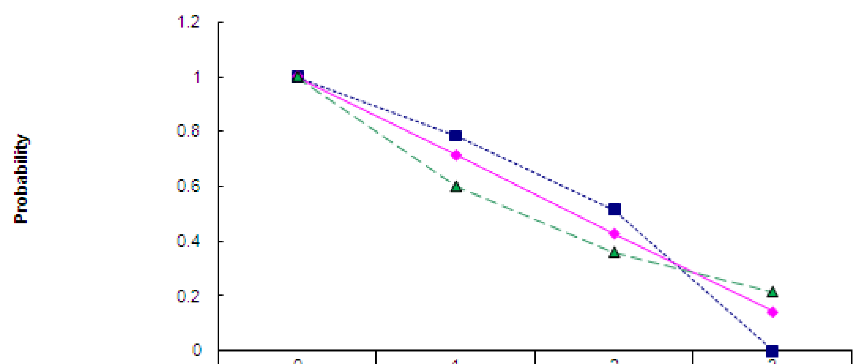
- The MRPIM model can be extended to tackle the case where the facilities can be interdicted with different types of offensive resources (Wood [40]). Several variations of this extension may be considered. These include: 1) models where the resources are independent and only one type of resource can be used on a given facility; 2) models where different types of resources can be employed to interdict the same facility, and the probability of a successful attack depends upon both the type and the amount of resources employed; 3) models where at least one unit of each type of resource must be employed to have a positive probability of success.

5 Results and analysis

5.1 Test settings and Input parameters

The tests were conducted on an Intel Core 2 CPU 6700 @ 2.66 GHz, with 2 GB of RAM. The formulations were implemented in C++ and compiled using Microsoft Visual C++.NET 2003. Then, the optimizer was called from our source code. We used CPLEX 9.1. Two data sets were used for the tests: the Swain data set with 55 nodes (Swain [39]), and the London Ontario data set with 150 nodes (Goodchild and Noronha [15]). In our experiments, we chose $k_{max} = 3$ in order to have three levels of attack: a high level, a medium level and a low level. Also, we used three different probability functions to model the impact of the amount of offensive resources on the working probability of a facility. Specifically, we assumed that for each facility j, the probability p_{jk} is equal to: 1) $p_{jk} = ((k_{max} - k)/k_{max})^{\phi}$ (*concave function*); 2) $p_{jk} = 1 - k/(k_{max} + 0.5)$ (*linear function*); 3) $p_{jk} = \phi^k$ (*convex function*). We assumed that $p_{j0} = 1$ for each function, and set $\phi = 0.6$. A discretized representation of the three probability functions and the corresponding p_{jk} values for each value of k are displayed in Figure 2.

Figure 2: Convex, Linear and Concave Probability Functions



	0	1	2	3
Concave	1	0.784	0.517	0
Linear	1	0.714	0.429	0.143
Convex	1	0.6	0.36	0.216

Figure 3: Reduction of the working probability from level $k-1$ to level k

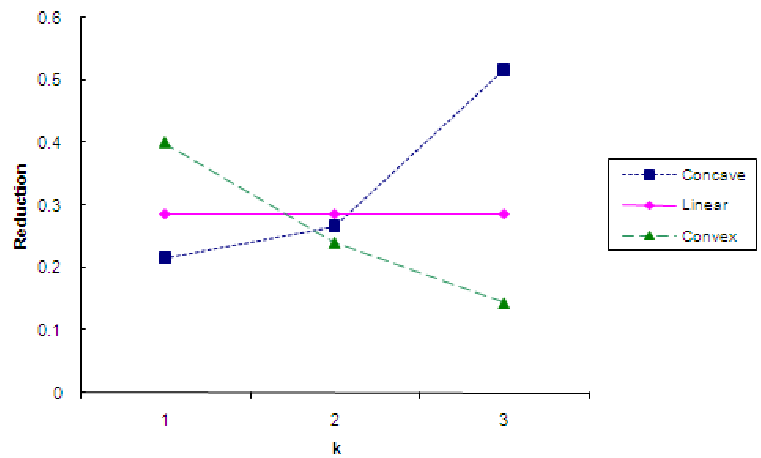


Figure 3 displays the drop of the working probability after an attack at each level k for the three different and non-dominated functions. The convex distribution achieves a faster reduction of the working probability for low level attacks. The concave function produces smaller reductions for low values of k, but for the largest value $k = 3$ it reduces the working probability to zero (Figure 2).

In our initial testing, we use cardinality constraints where r ranges between 1 and 4, as in Church and Scaparra [10]. The number of facilities in the system, p, is equal to 5, 10 and 15 for the Swain data set, and to 10, 20, 30, and 50 for the London data set. Finally, in our algorithmic implementation, we set a time limit of 3 hours for the branch and bound procedure and an optimality tolerance of 0.1%.

5.2 Analysis of Performance

5.2.1 Efficiency comparison of the three formulations

Table 3 displays the results for the Swain data set and the convex probability function obtained with the three proposed models. Each problem in this data set was solved to optimality by all the three formulations. The optimal objective function value for each instance is displayed in the right most column. On average, MIP-MRPIM, BI-MRPIM and NET-MRPIM spent 2569.53, 15.49 and 0.4 seconds, respectively, to find the optimal solutions. The network formulation is significantly faster than the other two formulations.

These results were corroborated when solving larger instances. Table 4 displays the results for the London data set and the convex probability function. The performance of MIP-MRPIM and BI-MRPIM is clearly inferior on this set of instances, in terms of both time and solution quality. MIP-MRPIM found the optimal solution for only 17 out of the 40 instances and for 8 instances (denoted with an asterisk) could not find a feasible solution within the 3 hours of computing time allowed. The average percentage distance from the optimal solution, displayed in the column GAP(%), was 2.22, and the average time was 6714.3 seconds. The bilevel formulation produced better results, by solving all but 2 instances to optimality, with an average gap of only 0.06%. The computing times though were significant (almost 2500 seconds on average). The NET-MRPIM is the fastest formulation and always reached the optimum with an average solution time of 5.6 seconds. The most difficult instance ($p = 30$, $k = 3$, $r = 4$) was solved in only 38.86 seconds. Similar improvements from one formulation to another were obtained with the other probability functions.

5.2.2 Insights on NET-MRPIM performance

Given the efficiency of the NET-MRPIM formulation, we put it to the test on the London data set for larger values of the budget r, ranging from 1 up to 12. All the instances tested were solved to optimality with an average computing time of 75.25 seconds and a worst-case time of 3802.95 seconds, obtained for the instance with the concave probability function, $p = 50$, $k = 3$, and $r = 10$.

In the Tables 5, 6 and 7 we study the sensitivity of the computing effort to the parameters r, p and k, and to the probability distributions. In each table, we report the average time across different values of the other parameters while maintaining

Table 3: Performance Comparison of MIP/BI/NET for **Swain**

			TIME(s)			OBJ.VALUE
p	k	r	MIP	BI	NET	
5	1	1	3284.88	0.67	0.02	3437.78
5	1	2	1887.89	0.5	0.05	3894.97
5	1	3	1791.15	0.42	0.05	4344.62
5	1	4	1611.19	0.36	0.03	4712.21
5	2	2	2063.83	0.67	0.09	3894.97
5	2	3	1645.31	1.11	0.2	4344.62
5	2	4	2017.37	1.25	0.27	4799.65
5	3	3	2107.86	1.2	0.34	4344.62
5	3	4	2208.87	1.48	0.55	4799.65
10	1	1	4.33	2.06	0.03	2065.05
10	1	2	24.73	2.92	0.03	2202.33
10	1	3	41.11	3.45	0.01	2339.54
10	1	4	20.2	3.03	0.03	2468.99
10	2	2	4960.63	3.59	0.19	2202.33
10	2	3	3603.45	6.78	0.36	2339.54
10	2	4	3052.9	11.47	0.8	2468.99
10	3	3	10800.74	7.69	0.8	2339.54
10	3	4	10800.76	15.78	1.63	2468.99
15	1	1	6.14	8.31	0.03	1473.88
15	1	2	44.55	16.05	0.03	1580.2
15	1	3	64.08	26.41	0.05	1675.16
15	1	4	64.02	30.85	0.03	1772.21
15	2	2	273.57	9.47	0.22	1580.2
15	2	3	1532.93	31.91	0.48	1675.16
15	2	4	708.76	65.69	0.75	1777.78
15	3	3	4493.3	34.86	1.38	1675.16
15	3	4	10262.8	130.26	2.48	1777.78
Average			*2569.53*	*15.49*	*0.40*	

Table 4: Comparison Performance MIP/BI/NET-MRPIM for **London**

				TIME(s)		OBJ.VALUE	GAP(%)	
p	k	r	MIP	BI	NET		MIP	BI
10	1	1	19.31	19.23	0.09	201739.29	0.00	0.00
10	1	2	70.25	24.33	0.14	214437	0.00	0.00
10	1	3	91.49	27.34	0.14	228097.15	0.00	0.00
10	1	4	287.84	27.22	0.13	241318.44	0.00	0.00
10	2	2	9235.72	22.42	1.08	214437	0.00	0.00
10	2	3	5436.53	42.45	2.28	228097.15	0.00	0.00
10	2	4	6300.4	71.74	2.83	241318.44	0.00	0.00
10	3	3	10701.64	47.81	4.61	228097.15	0.00	0.00
10	3	4	10815.35	94.66	8.38	241318.44	0.00	0.00
20	1	1	156.16	15.49	0.09	130190.89	0.00	0.00
20	1	2	1293.6	498.47	0.11	137365.91	0.00	0.00
20	1	3	1548.6	1272.4	0.17	141875.34	0.00	0.00
20	1	4	2180.13	354.73	0.22	146333.56	0.00	0.00
20	2	2	8721.38	160.87	0.22	137365.12	0.00	0.00
20	2	3	10804.99	405.22	5.14	142062.98	0.00	0.00
20	2	4	10804.76	2444.78	5.49	146572.34	3.54	0.00
20	3	3	10805.36	949.34	11.17	142063.01	5.26	0.00
20	3	4	10806.37	3628.37	19.86	146572.56	5.24	0.00
30	1	1	710.74	55.61	0.16	93169.71	0.00	0.00
30	1	2	10808.65	158.43	0.14	97236.04	1.26	0.00
30	1	3	10807.17	647.99	0.17	100388.3	11.86	0.00
30	1	4	10807.62	2208.51	0.17	103490.37	14.51	0.00
30	2	2	10812.12	324.9	0.63	97235.49	3.82	0.00
30	2	3	10812.14	7129.07	3.13	100387.75	5.01	0.00
30	2	4	10812.14	6691.73	8.25	103522.06	5.89	0.00
30	3	3	10816.91	1354.55	17.63	100387.75	2.70	0.00
30	3	4	10815.73	8082.4	38.86	103522.18	2.97	0.00
50	1	1	716.35	62.85	0.25	53452.66	0.00	0.00
50	1	2	*	354.11	0.25	55636.11	*	0.00
50	1	3	*	5449.38	0.25	57569.56	*	0.00
50	1	4	*	10807.98	0.25	59423.29	*	0.00
50	2	2	*	721.98	1.38	55634.92	*	0.00
50	2	3	*	5632.72	5.78	57568.37	*	0.00
50	2	4	*	10813.38	8.13	59422.09	*	1.02
50	3	3	*	8007.06	19.16	57568.37	*	0.00
50	3	4	*	10801.96	34.78	59422.09	*	1.18
Average			*6714.27*	*2483.65*	*5.60*		*2.22*	*0.06*

Table 5: Mean computing time for some fixed values of the parameter r

r	1	2	3	4	5	6	7	8	9	10	11	12
Time	6.87	4.9	9.02	19.02	27.09	35.11	76.63	73.52	103.89	204.79	169.26	162.91

Table 6: Mean computing time for some fixed values of the parameter p

p	10	20	30	40	50
Time	8.5	25.38	50.4	110.54	177.27

one of the parameters fixed.

Note that the largest variation in time is observed for the parameter k, when its value increases from 2 to 3 (Table 7 (left)). As expected, increasing values of the parameters r and p result in more difficult problems to solve. Finally, the instances using a concave probability distribution seem to be more time-consuming than the ones with linear and convex distributions.

5.3 Choosing the probability distribution

In this section, we analyze how the probability functions affect the number of resources which are employed to hit each facility in the optimal solutions. We then analyze the robustness of the solutions found to possible misestimations of the probability functions.

5.3.1 Analysis of the solutions obtained

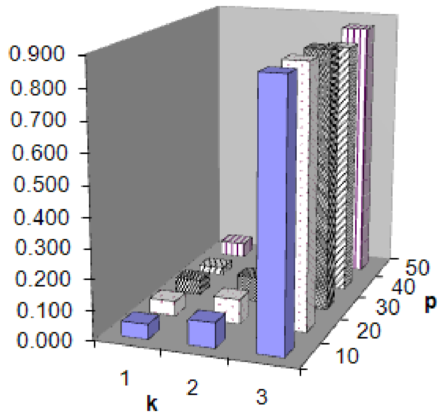
Figures 4, 5 and 6 display the allocation of resources, averaged across different budget amounts (r), for different values of the parameters k and p. The charts show

Table 7: Mean computing time for fixed values of the parameter k (left) and probability function (right)

k	***Time***
1	2.51
2	6.73
3	214.01

probabiliy func.	***Time***
Concave	154.7
Linear	22.99
Convex	45.56

Figure 4: Distribution of resources for London and the concave function

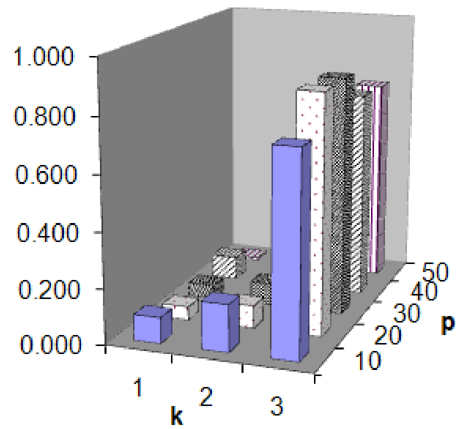


	1	2	3
10	0.049	0.083	0.868
20	0.049	0.083	0.868
30	0.049	0.083	0.868
40	0.025	0.177	0.838
50	0.049	0.083	0.868

that the probability functions have a big impact on the distribution of the offensive resources among the facilities. In fact, for the concave, linear and convex distributions, respectively, the attacker spends on average 82.6%, 79.6% and 3.3% of its resources in attacks of level 3. This confirms a quite intuitive result. Namely, when we use the concave function, which reduces to zero the working probability of a facility attacked at level 3, the attacker tends to concentrate his offensive resources on a few key facilities to increase his chances of completely disabling them. On the other side, when we use the convex probability function, where the marginal working probability decreases with increasing attack's levels, the attacks are more spread out across the facilities and the majority of the facilities are attacked with only one unit of offensive resources. The behavior of the linear function is similar to that of the concave function, although less pronounced, especially in systems with a small number of facilities (e.g., $p = 10$).

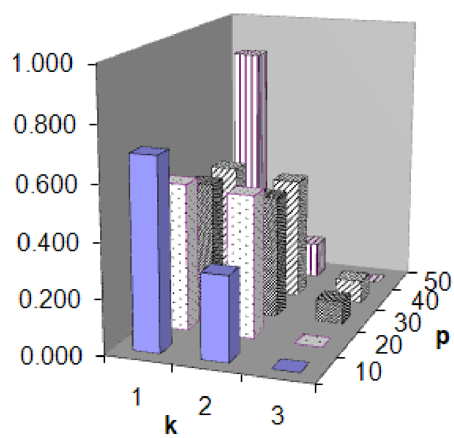
Tables 8 and 9 display the usage rate of each attack level k given a particular

Figure 5: Distribution of resources for London and the linear function



	1	2	3
10	0.094	0.172	0.734
20	0.049	0.083	0.868
30	0.049	0.083	0.868
40	0.085	0.155	0.759
50	0.010	0.240	0.750

Figure 6: Distribution of resources for London and the convex function



	1	2	3
10	0.692	0.308	0.000
20	0.534	0.516	0.000
30	0.473	0.445	0.082
40	0.473	0.445	0.082
50	0.869	0.131	0.000

Table 8: Expected usage of level k for **London** data set

	$k=1$	$k=2$	$k=3$
Concave	0.044	0.115	0.993
Linear	0.058	0.168	0.922
Convex	0.608	0.413	0.037

Table 9: Expected usage of level k for **Swain** data set

	$k=1$	$k=2$	$k=3$
Concave	0.061	0.201	0.981
Linear	0.190	0.450	0.819
Convex	0.480	0.523	0.028

probability distribution for the London and Swain data sets, respectively. The tables display the average values obtained across a set of solutions with different combinations of the parameters p and r. We calculate the usage rate of k for a given solution as the total number of facilities hit at level k divided by the maximum number of facilities that could have been attacked at k ($\max_k$), e.g., for $r = 5$, $\max_1 = 5$, $\max_2 = 2$, $\max_3 = 1$. Again, it can be noticed that the choice of the level of attack is highly dependent on the probability distribution: for the concave distribution, the usage rate of attacks at level 3 is 0.993, whereas for the convex distribution is only 0.037 (Table 8). Similar results hold for the Swain data set, shown in Table 9.

5.3.2 Robustness

Given the strong correlation between the probability functions used in the model and the distribution of the offensive resources among the facilities, we now analyze the robustness of the MRPIM optimal solutions to an uncertain type of probability distribution. Tables 11 and 10 display the average regret and maximum regret (in bold) of supposing a particular type of probability distribution that is not the one actually occurring. At the bottom of the tables (underscored numbers), we observe that on average the largest regret for the London data set amounts to 7.4% if a convex probability function is assumed. For the Swain data set, the largest average regret

Table 10: Robustess. Percentage difference in weighted distance for Swain (expected/**maximum**)

Swain — **Supposed** / *Actual*	**Concave**	**Linear**	**Convex**
Concave	*	0.039/**0.244**	0.183/**0.337**
Linear	0.016/**0.119**	*	0.046/**0.129**
Convex	0.107/**0.407**	0.057/**0.244**	*
Expected	0.061	0.048	0.115

Table 11: Robustess. Percentage difference in weighted distance for London (expected/**maximum**)

London — **Supposed** / *Actual*	**Concave**	**Linear**	**Convex**
Concave	*	0.005/**0.043**	0.112/**0.304**
Linear	0.002/**0.026**	*	0.036/**0.118**
Convex	0.041/**0.095**	0.036/**0.082**	*
Expected	0.022	0.020	0.074

is 11.5% and is also obtained when we assume a convex probability function. The safest choice is to assume a linear distribution of the working probabilities. For this distribution, the average regrets are 2.0% for London and 4.8% for Swain, while the maximum regrets are 8.2% and 24.4% respectively. The concave or convex functions can produce an error of up to 40% and 33.7% on the Swain data set. Finally, we observe that the instances solved for the larger data set (London) are generally less sensitive to misestimations of the probability distribution.

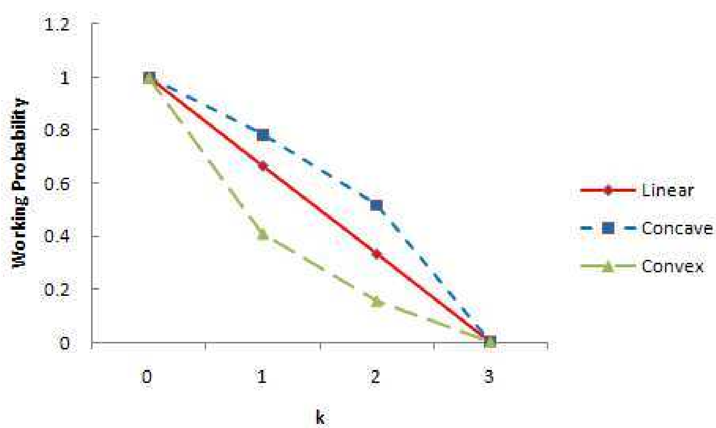
5.4 MRPIM Justification

In this subsection, we analyze the benefits of allowing the usage of multiple resources to attack a facility in a median interdiction problem. To this end, we compare the solutions obtained with MRPIM with the ones obtained with RIM and PIM.

5.4.1 MRPIM versus RIM

One of the basic assumptions of RIM is that an attack on a facility is always successful and entails a cost of one unit. To compare MRPIM and RIM, we changed the probability distributions so that an attack at the highest level guarantees the destruction of the facility, i.e., for $k = k_{max} = 3$, $p_{j3} = 0$ for each probability function. For each facility j, the modified probability functions, shown in Figure 7, are as follows: 1) $p_{jk} = (k_{max} - k/k_{max})^{0.6}$ (*concave function*), $p_{jk} = 1 - (k/3)$ (*linear function*); $p_{jk} = (k_{max} - k) / \left(k_{max} (k+1)^{0.7}\right)$ (*convex function*), $\forall k = 0, 1, 2, 3$. Then, an attack of level three in MRPIM generates the same disruption as an attack in RIM. However, the expenditure of resources is not the same ($c_{j3}^{MRIM} = 3 \neq c_j^{RIM} = 1$). Thus, to compare RIM with MRPIM, we assume that the budget available in each MRPIM instance is $|K|$ times the budget used in the corresponding RIM problem.

Figure 7: Probability distributions to compare RIM with MRPIM



Figures 8 and 9 display, for the London and Swain data sets respectively, the average objective function values for the three probability distributions and different values of p (chart on the right) and r (chart on the left). As expected, the convex distribution produces the best solutions for any value of r and p, since with this distribution the drop of working probability is the greatest for any k. For the London data set, MRPIM with the linear and concave distributions finds exactly the same

Figure 8: Objective Value of RIM vs MRPIM with different Probability Distributions for **London**

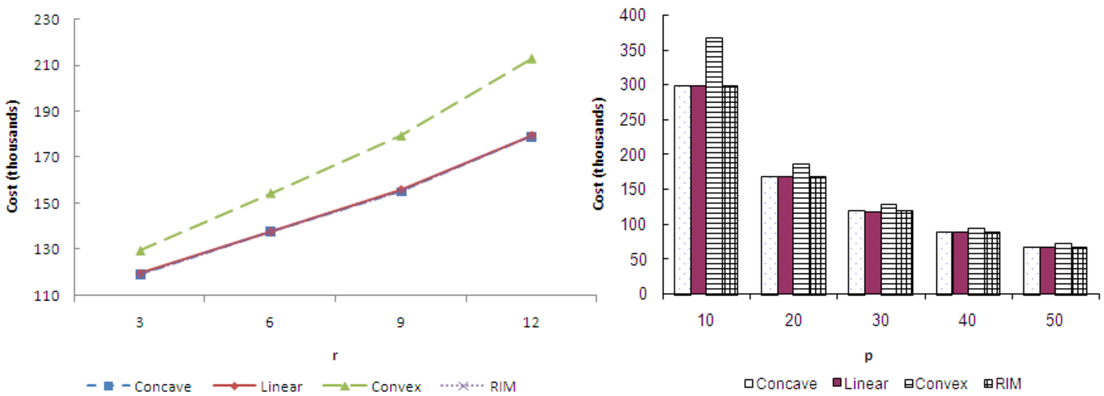


Figure 9: Objective Value of RIM vs MRPIM with different Probability Distributions for **Swain**

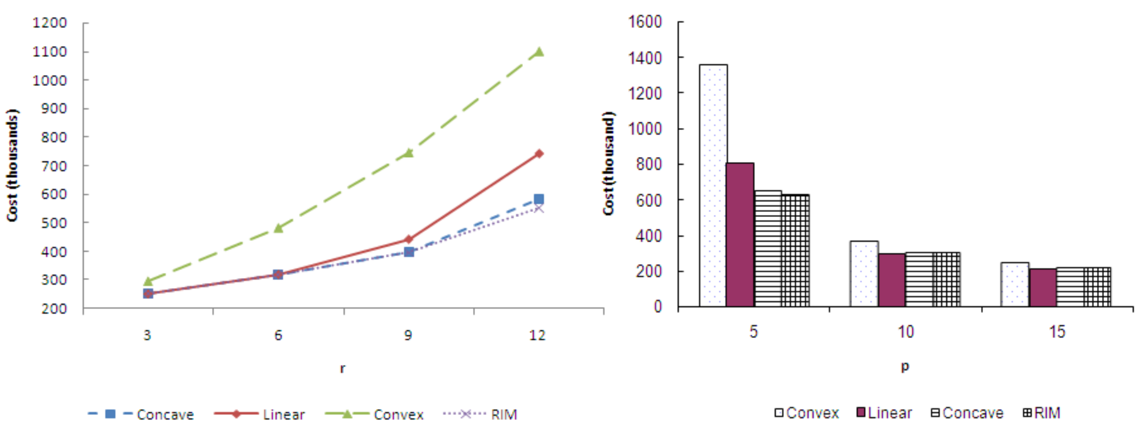
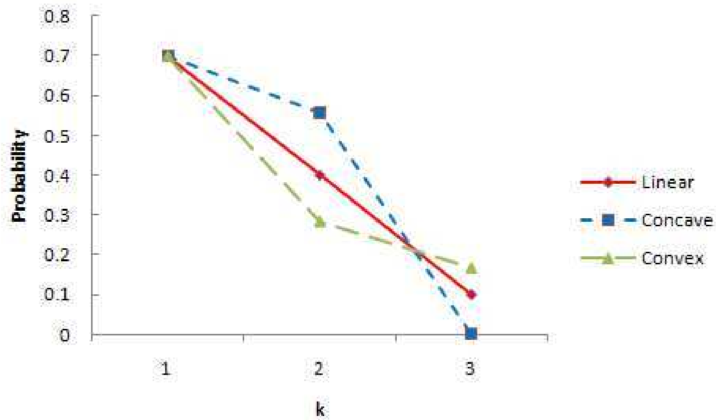


Figure 10: Probability distribution used to compare MRPIM with PIM



solutions as RIM (Figure 8), i.e., the resources are not spread out. For the Swain data set, MRPIM with all three probability distributions produces better results than RIM (Figure 9). This indicates that MRPIM always finds solutions that are equal or better than those found by RIM. In other words, the increased flexibility captured by MRPIM allows the generation of more disruptive attacking strategies.

5.4.2 MRPIM versus PIM

In the PIM model, the facilities can only be attacked with one level of resources, and each attack has a given probability of success. MRPIM is a more versatile model than PIM in that it allows the use of different levels of resources to hit each facility and the probability of success is commensurate with the amount of resources employed in the attack.

To compare MRPIM and PIM, we use three non-dominated probability distributions where the values of the probability functions for $k = 1$ are equal to the probability used in the PIM model, which we assume to be 0.7. Specifically, the probability of each facility j working is given by: 1) $p_{jk} = ((k_{max} - k) \,/\, (k_{max} - 1))^{1/3} * 0.7$ (concave function); 2) $p_{jk} = 0.3 * (3 - k) + 0.1$ (linear function); 3) $p_{jk} = 0.7/k^{1.3}$ (convex function), where $k = 1, 2, 3$. The three functions are shown in Figure 10.

Figure 11: Objective Value of PIM vs MRPIM with different Probability Distributions for **London**

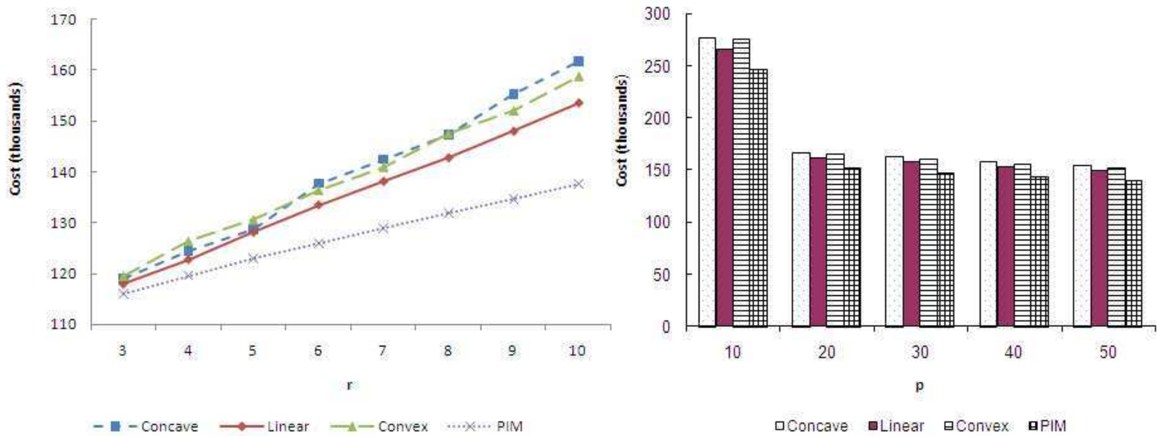
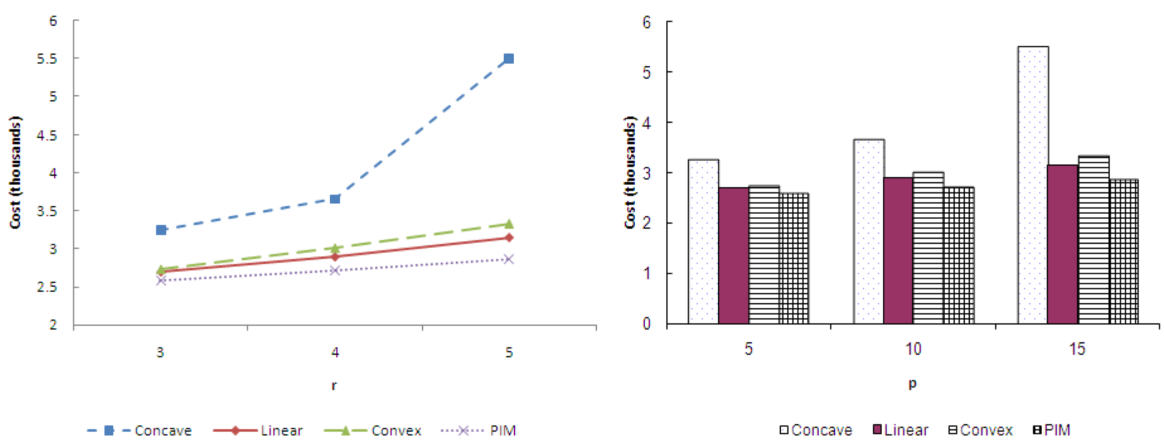


Figure 12: Objective Value of PIM vs MRPIM with different Probability Distributions for **Swain**



Figures 11 and 12 show, for the London and Swain data sets respectively, the average objective function values obtained with the three probability distributions for different values of the parameters p (graphs on the right) and r (graphs on the left). The MRPIM model always outperforms PIM with any probability function in both the London and Swain data sets. This is due to the greater flexibility of MRPIM which allows to concentrate more resources on one facility to increase its failure probability. As expected, the advantage of using MRPIM is more evident when the concave probability function is used, as in this case the interdiction resources are more concentrated on a few facilities.

6 Conclusions

Formulating the multi resource interdiction median problem in a computationally tractable way is a challenge. We have proposed three different ways of representing this problem mathematically.

We started with a non-linear formulation of the problem and showed how to linearize the resulting model so that it could be solved by commercial MIP solvers. This model, called MIP-MRPIM, however, only allowed us to solve problem instances of modest size. We then tried to improve this formulation by representing some problematic constraints (the ones enforcing closest assignments between customers and facilities) as a separate optimization problem, thus obtaining a bilevel formulation (BI-MRPIM). Although more efficient than the previous one, this model still required a considerable amount of computing time to solve some problem instances to optimality. Finally, we proposed a less intuitive formulation, based on a network representation of the problem, which proved to be extremely effective in solving problems of realistic size.

Overall, we obtained an efficiency improvement of about 76% when going from

MIP-MRPIM to BI-MRPIM and an additional 99% improvement when going from BI-MRPIM to NET-MRPIM. This last formulation allowed us to solve difficult problems, characterized by larger interdiction budgets. All problem instances were solved to optimality in a few seconds, with the largest instances requiring slightly more than a minute of computing time.

We provided a sensitivity analysis of the solution times to changes in the problem parameters, and discussed the robustness of the proposed models to possible misestimations of the probability functions which were used to represent the probability of a facility working probabilities as a function of the amount of resources employed in the attacks.

Finally, we demonstrated that MRPIM is a sensible extension of some previous interdiction models proposed in the literature, such as RIM and PIM, as it allows the identification of more cost effective interdiction strategies.

In the future, we plan to embed this new interdiction model within protection and design models so as to identify sound protection strategies and reliable system configurations in the face of malicious attacks.

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