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# THE INVARIANTS OF THE THIRD SYMMETRIC POWER REPRESENTATION OF $SL_2(\mathbb{F}_p)$

ASHLEY HOBSON AND R. JAMES SHANK

ABSTRACT. For a prime  $p > 3$ , we compute a finite generating set for the  $SL_2(\mathbb{F}_p)$ -invariants of the third symmetric power representation. The proof relies on the construction of an infinite SAGBI basis and uses the Hilbert series calculation of Hughes and Kemper.

## 1. INTRODUCTION

Consider the generic binary cubic over a field  $\mathbb{F}$  of characteristic not 3:

$$a_0X^3 + 3a_1X^2Y + 3a_2XY^2 + a_3Y^3.$$

Identifying

$$X = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \text{ and } Y = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

induces a left action of the general linear group  $GL_2(\mathbb{F})$  on the third symmetric power

$$V := \text{Span}_{\mathbb{F}}[Y^3, 3Y^2X, 3YX^2, X^3]$$

and a right action on the dual  $V^* = \text{Span}_{\mathbb{F}}[a_3, a_2, a_1, a_0]$ . For example

$$\sigma = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \text{ acts on } V^* \text{ as } \begin{bmatrix} 1 & 3 & 3 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

with  $a_3 = [1 \ 0 \ 0 \ 0]$ ,  $a_2 = [0 \ 1 \ 0 \ 0]$ ,  $a_1 = [0 \ 0 \ 1 \ 0]$ ,  $a_0 = [0 \ 0 \ 0 \ 1]$ . The action on  $V^*$  extends to an action by algebra automorphisms on the symmetric algebra  $\mathbb{F}[V] = \mathbb{F}[a_3, a_2, a_1, a_0]$ . For any subgroup  $G \leq GL_2(\mathbb{F})$ , we denote the subring of invariant polynomials by  $\mathbb{F}[V]^G$ .

Throughout we assume that  $\mathbb{F}$  has characteristic  $p > 3$ . Thus  $\mathbb{F}_p \subseteq \mathbb{F}$  and  $SL_2(\mathbb{F}_p) \leq GL_2(\mathbb{F})$ . The primary goal of this paper is to compute a finite generating set for  $\mathbb{F}[V]^{SL_2(\mathbb{F}_p)}$ . We note that  $V$  is the unique four-dimensional irreducible representation of  $SL_2(\mathbb{F}_p)$  (see, for example, [2, pp. 14–16]). Also, for  $p \neq 7$ ,  $\mathbb{F}[V]^{SL_2(\mathbb{F}_p)}$  is not Cohen-Macaulay and in fact has depth 3 [13, §5]. In the language of L.E. Dickson [6, Lecture III §9], we give a fundamental

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system for the formal modular invariants of the binary cubic. Dickson considered this problem but was only able to identify a few specific invariants. We proceed by constructing the required invariants and then proving that the given set generates  $\mathbb{F}[V]^{SL_2(\mathbb{F}_p)}$ . Our proof relies on the construction of an infinite SAGBI basis and uses the Hilbert series calculation of Hughes and Kemper [8]. Recall that a SAGBI basis is a **S**ubalgebra **A**nalog of a **G**röbner **B**asis for **I**deals. SAGBI bases were introduced independently by Robbiano-Sweedler [11] and Kapur-Madlener [9]; a useful reference is Chapter 11 of Sturmfels [15] (who uses the term *canonical subalgebra basis*). The ring of invariants of a modular representation of a  $p$ -group always has a finite SAGBI basis for an appropriate choice of term order, see [14]. A finite SAGBI basis for the ring of invariants of the Sylow  $p$ -subgroup of  $SL_2(\mathbb{F}_p)$  was computed in [12]. Extensive preliminary calculations for small primes, using MAGMA [4], involving SAGBI bases and the relative transfer map, lead to the given generating set (see [7]). We use the graded reverse lexicographic order with  $a_0 < a_1 < a_2 < a_3$ . For background material on term orders and Gröbner bases see Adams-Loustaunau [1]. For background material on the invariant theory of finite groups see Benson [3], Derksen-Kemper [5] or Neusel-Smith [10].

A classical example of an invariant of a binary form is the discriminant, which in this case can be written as

$$D := 3a_2^2a_1^2 - 4a_3a_1^3 - 4a_2^3a_0 + 6a_3a_2a_1a_0 - a_3^2a_0^2.$$

Following Lecture III of L. E. Dickson's Madison Colloquium [6] we identify the  $SL_2(\mathbb{F}_p)$ -invariant

$$L := 3(a_2^p a_1 - a_2 a_1^p) - (a_3^p a_0 - a_3 a_0^p).$$

Let  $B$  denote the Borel subgroup of  $SL_2(\mathbb{F}_p)$  consisting of upper triangular matrices and let  $P$  denote the unique Sylow  $p$ -subgroup of  $B$ . Observe that  $P$  is cyclic of order  $p$  and is also a Sylow  $p$ -subgroup of  $SL_2(\mathbb{F}_p)$ . Define

$$N := \prod_{\tau \in P} (a_3)\tau.$$

By Corollary 2.4,  $N \cdot a_0$  is  $SL_2(\mathbb{F}_p)$ -invariant (or see [6]).

For a subgroup  $H$  of a group  $G$ , choose coset representatives  $G/H$  and define the relative transfer

$$\begin{aligned} \mathrm{tr}_H^G : \mathbb{F}[V]^H &\rightarrow \mathbb{F}[V]^G \\ f &\mapsto \sum_{\tau \in G/H} (f)\tau. \end{aligned}$$

The transfer,  $\mathrm{tr}^G$ , is the special case when  $H$  is the trivial group. Define

$$K := -\mathrm{tr}^{SL_2(\mathbb{F}_p)}(a_1^{p-1}).$$

We show in Lemma 2.10 that  $K$  is non-zero with lead monomial  $a_2^{p-1}$ .

For  $\omega \in \mathbb{F}_p^*$ , the diagonal matrix

$$\rho_\omega = \begin{bmatrix} \omega & 0 \\ 0 & \omega^{-1} \end{bmatrix} \text{ acts on } V^* \text{ as } \begin{bmatrix} \omega^3 & 0 & 0 & 0 \\ 0 & \omega & 0 & 0 \\ 0 & 0 & \omega^{-1} & 0 \\ 0 & 0 & 0 & \omega^{-3} \end{bmatrix}.$$

This motivates the definition of a multiplicative weight function on monomials by

$$\text{wt}(a_i) = 2i - 3.$$

Thus for any monomial  $\beta$ , we have  $(\beta)\rho_\omega = \omega^{\text{wt}(\beta)}\beta$ . Since  $\omega^{p-1} = 1$ , it is convenient to assume that the weight function takes values in  $\mathbb{Z}/(p-1)\mathbb{Z}$ . Since  $B$  is generated by elements of  $P$  and  $\rho_\omega$  for  $\omega \in \mathbb{F}_p^*$ , it is clear that the  $B$ -invariants are precisely the isobaric  $P$ -invariants of weight zero (modulo  $p-1$ ).

We show in Lemma 2.1 that  $N$  is isobaric of weight 3 (modulo  $p-1$ ). Let  $c$  denote the smallest positive integer satisfying  $3c \equiv_{(p-1)} 0$ . Thus  $c = (p-1)/3$  if  $p \equiv_{(3)} 1$  and  $c = p-1$  if  $p \equiv_{(3)} -1$ . Then  $N^c$  is  $B$ -invariant and

$$\delta := \text{tr}_B^{SL_2(\mathbb{F}_p)}(N^c)$$

is  $SL_2(\mathbb{F}_p)$ -invariant. It follows from Theorem 2.5 that the lead monomial of  $\delta$  is  $a_3^{pc}$ . We show in Theorem 2.12 that  $\{D, K, Na_0, \delta\}$  forms a homogeneous system of parameters, i.e., the set is algebraically independent and  $\mathbb{F}[V]^{SL_2(\mathbb{F}_p)}$  is a finite module over  $\mathbb{F}[D, K, Na_0, \delta]$ .

It is easily verified that  $d := a_1^2 - a_2a_0$  and  $e := 2a_1^3 + a_0(a_3a_0 - 3a_2a_1)$  are isobaric  $P$ -invariants of weight  $-2$  and  $-3$  respectively. Define

$$\tilde{e} := \text{tr}_B^{SL_2(\mathbb{F}_p)}(Ne).$$

We will show, see Theorem 3.1, that for  $p \equiv_{(3)} 1$ , the  $SL_2(\mathbb{F}_p)$ -invariants are generated by

$$D, K, L, Na_0, \delta, \tilde{e}$$

and an explicitly described finite subset of the image of the transfer. For  $p \equiv_{(3)} -1$  the additional invariant

$$\tilde{d} := \text{tr}_B^{SL_2(\mathbb{F}_p)}(N^{\frac{p+1}{3}}d)$$

is required.

## 2. PRELIMINARIES, LEAD MONOMIALS AND TÊTE-À-TÊTES

For the remainder of the paper we use  $G$  to denote  $SL_2(\mathbb{F}_p)$ . The following generalises [13, 2.4].

**Lemma 2.1.** *If  $f$  is an isobaric polynomial of weight  $\lambda$ , then  $\text{tr}^P(f)$  is isobaric of weight  $\lambda$ . Furthermore  $N$  is isobaric of weight 3.*

*Proof.* The result follows from the fact that  $P$  is normal in  $B$ . For  $\omega \in \mathbb{F}_p^*$

$$\begin{aligned} (\mathrm{tr}^P(f)) \rho_\omega &= \sum_{\tau \in P} (f) \tau \rho_\omega = \sum_{\tau' \in P} (f) \rho_\omega \tau' \\ &= \sum_{\tau' \in P} \omega^\lambda (f) \tau' = \omega^\lambda \mathrm{tr}^P(f). \end{aligned}$$

Thus  $\mathrm{tr}^P(f)$  is isobaric of weight  $\lambda$ . A similar calculation gives  $\mathrm{wt}(N) = \mathrm{wt}(a_3) = 3$ .  $\square$

Let  $Q$  denote the subgroup generated by the transpose of  $\sigma$ , i.e., the lower triangular Sylow  $p$ -subgroup, and define

$$\eta := \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

**Lemma 2.2.**  $Q \cup \{\eta\}$  is a set of coset representatives for  $B$  in  $SL_2(\mathbb{F}_p)$ .

*Proof.* Since the index of  $B$  in  $SL_2(\mathbb{F}_p)$  is  $p + 1$ , we have the right number of elements. To show that the cosets  $(\sigma^T)^n B$  are distinct for  $n = 1, \dots, p$ , it is sufficient to show that  $(\sigma^T)^n B \neq B$  for  $n < p$ ; this is clear. To show that  $\eta B \neq (\sigma^T)^n B$ , it is sufficient to show that  $\eta^{-1}(\sigma^T)^n \notin B$ ; this is a straight forward calculation.  $\square$

**Lemma 2.3.**  $Na_0 = -a_3 \prod_{\tau \in Q} (a_0) \tau$ .

*Proof.* Consider the orbits

$$a_3 P = \{a_3 + 3sa_2 + 3s^2a_1 + s^3a_0 \mid s \in \mathbb{F}_p\}$$

and

$$a_0 Q = \{s^3a_3 + 3s^2a_2 + 3sa_1 + a_0 \mid s \in \mathbb{F}_p\}.$$

Thus

$$\begin{aligned} Na_0 &= a_0 \prod_{s \in \mathbb{F}_p} (a_3 + 3sa_2 + 3s^2a_1 + s^3a_0) = a_0 a_3 \prod_{s \in \mathbb{F}_p^*} (a_3 + 3sa_2 + 3s^2a_1 + s^3a_0) \\ &= a_0 a_3 \prod_{s \in \mathbb{F}_p^*} s^3 \left( (s^{-1})^3 a_3 + 3(s^{-1})^2 a_2 + 3s^{-1} a_1 + a_0 \right) \\ &= a_3 \left( \prod_{s \in \mathbb{F}_p^*} s^3 \right) \prod_{\tau \in Q} (a_0) \tau = -a_3 \prod_{\tau \in Q} (a_0) \tau \end{aligned}$$

$\square$

Since  $\{\sigma, \sigma^T\}$  generates  $SL_2(\mathbb{F}_p)$ , any polynomial which is both  $P$ -invariant and  $Q$ -invariant is  $SL_2(\mathbb{F}_p)$ -invariant, giving the following corollary (see also Lecture III §9 of [6]).

**Corollary 2.4.**  $Na_0$  is  $SL_2(\mathbb{F}_p)$ -invariant.

**Theorem 2.5.** *Suppose  $f$  is an isobaric  $P$ -invariant with  $\text{wt}(N \cdot f) = 0$ . Then  $a_0$  divides  $\text{tr}_B^G(N \cdot f) - N \cdot f$  and, if  $a_0$  does not divide  $f$ , the lead terms of  $\text{tr}_B^G(N \cdot f)$  and  $N \cdot f$  are equal.*

*Proof.* Using the fact that  $Na_0$  is  $SL_2(\mathbb{F}_p)$ -invariant we see that

$$\begin{aligned} \text{tr}_B^G(N \cdot f) - N \cdot f &= Na_0 (\text{tr}_B^G(fa_0^{-1})) - N \cdot f \\ &= N (a_0 \text{tr}_B^G(fa_0^{-1}) - f). \end{aligned}$$

Observe that  $(a_0)\eta = -a_3$ . Thus, using the coset representatives from Lemma 2.2, we have

$$a_0 \text{tr}_B^G(fa_0^{-1}) - f = a_0 \left( \sum_{\tau \in Q \setminus \{1\}} \frac{(f)\tau}{(a_0)\tau} - \frac{(f)\eta}{a_3} \right).$$

From Lemma 2.3,  $N$  is a least common multiple of  $\{a_3\} \cup \{(a_0)\tau \mid \tau \in Q \setminus \{1\}\}$ . Taking  $N$  as the common denominator in the above sum gives

$$a_0 \text{tr}_B^G(fa_0^{-1}) - f = \frac{a_0 J}{N}$$

for some polynomial  $J$ . Therefore  $\text{tr}_B^G(N \cdot f) - N \cdot f = a_0 J$ . If  $a_0$  does not divide  $f$ , then the lead term of  $N \cdot f$  is not divisible by  $a_0$  and is also the lead term of  $\text{tr}_B^G(N \cdot f)$ .  $\square$

We use LM to denote lead monomial and LT to denote lead term. It is clear that  $\text{LM}(N) = a_3^p$ . In the following lemmas, we use the lead monomial calculations from [12]. Note that although the basis used in [12] is different from the one used here, the change of basis is upper triangular and so the lead monomial calculations still apply.

**Lemma 2.6.** *For  $m = 2 + \lfloor 3j/(p-1) \rfloor$ ,*

$$\text{LM} \left( \text{tr}_B^G \left( N^j \text{tr}^P \left( a_2^{(m-1)(p-1)-3j} a_3^{p-1} \right) \right) \right) = a_3^{pj} a_2^{m(p-1)-3j} =: \gamma_j.$$

*Proof.* We know from [12, 3.3] that  $\text{tr}^P(a_2^b a_3^{p-1})$  has lead monomial  $a_2^{b+p-1}$  if  $1 \leq b \leq p-1$ . Since  $m = 2 + \lfloor 3j/(p-1) \rfloor$ , we have  $3j/(p-1) - 1 < m - 2 \leq 3j/(p-1)$ , which simplifies to  $0 < (m-1)(p-1) - 3j \leq p-1$ . The result then follows from Lemma 2.1 and Theorem 2.5.  $\square$

**Lemma 2.7.** *For  $0 \leq j \leq (p-1)/2$ ,*

$$\text{LM} \left( \text{tr}_B^G \left( N^j \text{tr}^P \left( a_3^{p-1-j} \right) \right) \right) = a_3^{pj} a_2^{p-1-2j} a_1^j =: \beta_j.$$

*Proof.* From [12, 3.2],  $\text{tr}^P(a_3^b)$  has lead monomial  $a_2^{2b-(p-1)} a_1^{p-1-b}$  if  $(p-1)/2 \leq b \leq p-1$ . Simplifying  $(p-1)/2 \leq p-1-j \leq p-1$  gives  $0 \leq j \leq (p-1)/2$ . The result then follows from Lemma 2.1 and Theorem 2.5.  $\square$

**Lemma 2.8.** *For  $m = 2 + \lfloor 3j/(p-1) \rfloor$  and  $j \neq \lceil (m-2)(p-1)/3 \rceil$ ,*

$$\text{LM} \left( \text{tr}_B^G \left( N^j \text{tr}^P \left( a_3^{p-2} a_2^{(m-1)(p-1)+3-3j} \right) \right) \right) = a_3^{pj} a_2^{m(p-1)+1-3j} a_1 =: \Delta_j.$$

*Proof.* Using [12, 3.4],  $\text{LM}(\text{tr}^P(a_3^{p-2}a_2^b)) = a_2^{b+p-3}a_1$  for  $2 \leq b \leq p-1$ . As in the proof of Lemma 2.6, we have  $0 < (m-1)(p-1) - 3j \leq p-1$ . Therefore  $3 < (m-1)(p-1) + 3 - 3j \leq p+2$ . Thus the lead monomial calculation is valid as long as  $(m-1)(p-1) + 3 - 3j \notin \{p, p+1, p+2\}$ . This simplifies to  $j \notin \{(m-2)(p-1)/3 + \varepsilon/3 \mid \varepsilon \in \{0, 1, 2\}\}$ , i.e.,  $j \neq \lceil (m-2)(p-1)/3 \rceil$ . The result then follows from Lemma 2.1 and Theorem 2.5.  $\square$

**Lemma 2.9.** For  $p \equiv_{(3)} -1$  and  $j = (2p-1)/3, \dots, p-2$ ,

$$\text{LM}\left(\text{tr}_B^G\left(N^j \text{tr}^P\left(a_3^{\frac{5p-7}{3}-j}a_2^j\right)\right)\right) = a_3^{pj}a_2^{\frac{7p-5}{3}-2j}a_1^{j-\frac{2p-4}{3}} =: \phi_j.$$

*Proof.* From [12, 3.5],  $\text{LM}(\text{tr}^P(a_3^b a_2^a)) = a_2^{2b-p+3}a_1^{p-1-b}$  for  $(p-2)/3 \leq b \leq p-1$ . The inequalities  $(p-2)/3 \leq (5p-7)/3 - j \leq p-1$  simplify to  $(2p-4)/3 \leq j \leq (7p-5)/6 = p-1 + (p+1)/6$ . Thus the lead monomial calculation is valid for the given range of  $j$ . The result then follows from Lemma 2.1 and Theorem 2.5.  $\square$

Define  $\xi = 3a_2^2 - 4a_3a_1$ .

**Lemma 2.10.**  $K = -\text{tr}^P(a_3^{p-1}) - a_0^{p-1} \equiv_{(a_0)} (3\xi)^{\frac{p-1}{2}} + a_1^{p-1}$ .

*Proof.* A simple calculation gives  $\text{tr}^P(a_1^{p-1}) = -a_0^{p-1}$  (or see [12, 3.2]). Since  $\text{wt}(a_0^{p-1}) = 0$  and the index of  $P$  in  $B$  is  $p-1$ , we have  $\text{tr}^B(a_1^{p-1}) = a_0^{p-1}$ . Using the coset representatives from Lemma 2.2 gives

$$\begin{aligned} -K &= \text{tr}^G(a_1^{p-1}) = \text{tr}_B^G(a_0^{p-1}) = ((a_0)\eta)^{p-1} + \text{tr}^Q(a_0^{p-1}) = a_3^{p-1} + \text{tr}^Q(a_0^{p-1}) \\ &= a_3^{p-1} + \sum_{s \in \mathbb{F}_p} (s^3 a_3 + 3s^2 a_2 + 3s a_1 + a_0)^{p-1} \\ &= a_3^{p-1} + a_0^{p-1} + \sum_{s \in \mathbb{F}_p^*} (s^3 a_3 + 3s^2 a_2 + 3s a_1 + a_0)^{p-1} \\ &= a_3^{p-1} + a_0^{p-1} + \sum_{s \in \mathbb{F}_p^*} s^{3(p-1)} (a_3 + 3s^{-1} a_2 + 3(s^{-1})^2 a_1 + (s^{-1})^3 a_0)^{p-1} \\ &= a_0^{p-1} + \sum_{t \in \mathbb{F}_p} (a_3 + 3ta_2 + 3t^2 a_1 + t^3 a_0)^{p-1} = a_0^{p-1} + \text{tr}^P(a_3^{p-1}) \\ &\equiv_{(a_0)} \sum_{t \in \mathbb{F}_p} (a_3 + 3ta_2 + 3t^2 a_1)^{p-1} \\ &\equiv_{(a_0)} \sum_{t \in \mathbb{F}_p} \sum_{a+b+c=p-1} \binom{p-1}{a, b, c} t^{b+2c} a_3^a (3a_2)^b (3a_1)^c. \end{aligned}$$

It is well known that  $\sum_{t \in \mathbb{F}_p} t^i$  is  $-1$  if  $i$  is a positive multiple of  $p-1$  and  $0$  otherwise. Thus, for  $a, b, c$  non-negative with  $a+b+c = p-1$ , we see that  $\sum_{t \in \mathbb{F}_p} t^{b+2c}$  is non-zero only when  $b+2c = p-1$  or  $b+2c = 2(p-1)$ . If  $b+2c = 2(p-1)$  then  $c = p-1$  and  $a = b = 0$ . If  $b+2c = p-1$  then  $a = c$ .

Therefore

$$\begin{aligned} -K &\equiv_{(a_0)} \binom{p-1}{0, 0, p-1} (-1) (3a_1)^{p-1} - \sum_{c=0}^{\frac{p-1}{2}} \binom{p-1}{c, b, c} (3a_2)^{p-1-2c} (3a_1a_3)^c \\ -K &\equiv_{(a_0)} -a_1^{p-1} - 3^{\frac{p-1}{2}} \sum_{c=0}^{\frac{p-1}{2}} \binom{p-1}{c, b, c} (3a_2^2)^{\frac{p-1}{2}-c} (a_1a_3)^c. \end{aligned}$$

Simplifying binomial coefficients modulo  $p$  gives

$$\binom{p-1}{c, p-1-2c, c} = \binom{2c}{c} = (-4)^c \binom{\frac{p-1}{2}}{c}.$$

Thus

$$K \equiv_{(a_0)} a_1^{p-1} + 3^{\frac{p-1}{2}} (3a_2^2 - 4a_1a_3)^{\frac{p-1}{2}},$$

as required.  $\square$

A similar calculation using the identity

$$\binom{p-2}{a, p-3-2a, a+1} \equiv_{(p)} -2(a+1) \binom{2a+1}{a} \equiv_{(p)} -2(-4)^a \binom{\frac{p-3}{2}}{a}$$

gives the following lemma.

**Lemma 2.11.**  $\text{tr}^P(a_3^{p-2}) \equiv_{(a_0)} 6a_1(3\xi)^{\frac{p-3}{2}}.$

**Theorem 2.12.** *The set  $\{D, K, Na_0, \delta\}$  is a homogeneous system of parameters.*

*Proof.* With out loss of generality, we may assume  $\mathbb{F}$  is algebraically closed. We will show that the variety associated to  $(D, K, Na_0, \delta)\mathbb{F}[V]$ , say  $\mathcal{V}$ , consists of the zero vector.

Suppose  $v \in \mathcal{V}$ . Since  $Na_0(v) = 0$ , there exists  $g \in SL_2(\mathbb{F}_p)$  such that  $a_0g(v) = 0$ . Replacing  $v$  with  $g(v)$  if necessary, we may assume  $a_0(v) = 0$ . Note that  $D \equiv_{(a_0)} a_1^2\xi$ . From Lemma 2.10,  $K \equiv_{(a_0)} (3\xi)^{\frac{p-1}{2}} + a_1^{p-1}$ . Thus  $a_1^2K - 3(3\xi)^{\frac{p-3}{2}}D \equiv_{(a_0)} a_1^{p+1}$ . Therefore  $a_1(v) = 0$ . Since  $\text{LM}(K) = a_2^{p-1}$  in the grevlex order, we have  $a_2(v) = 0$ . Since  $\text{LM}(\delta) = a_3^{pc}$ , we have  $a_3(v) = 0$ . Therefore  $v$  is the zero vector.  $\square$

If  $f$  and  $h$  are polynomials with  $\text{LT}(f) = \text{LT}(h)$ , we refer to  $f - h$  as a tête-à-têtes (see [11] or [12]).

**Theorem 2.13.** *There is an infinite family of tête-à-têtes in  $\mathbb{F}[V]^{SL_2(\mathbb{F}_p)}$ , defined as follows:*

$$\begin{aligned} h_1 &= K \cdot \text{tr}_B^{SL_2(\mathbb{F}_p)}(Ne) - D \cdot \text{tr}_B^{SL_2(\mathbb{F}_p)}(N\text{tr}^P(a_3^{p-2})), \\ h_2 &= K \cdot h_1 - (3D)^{\frac{p-1}{2}} \cdot \text{tr}_B^{SL_2(\mathbb{F}_p)}(Ne), \\ h_i &= K \cdot h_{i-1} - (3D)^{\frac{p-1}{2}} \cdot h_{i-2} \text{ for } i \geq 3, \end{aligned}$$

with  $\text{LT}(h_i) = 2a_3^p a_1^{p+2+(i-1)(p-1)}$  for  $i \geq 1$ .



*Proof.* The proof is by induction on  $i$ . Recall that  $\text{LT}(D) = 3a_1^2a_2^2$ . From Lemma 2.10,  $\text{LT}(K) = a_2^{p-1}$ . Using Theorem 2.5 and Lemma 2.11, we have  $\text{LT}(\text{tr}_B^G(N \text{tr}^P(a_3^{p-2}))) = \frac{2}{3}a_1a_2^{p-3}a_3^p$  and  $\text{LT}(\text{tr}_B^G(Ne)) = 2a_1^3a_3^p$ . Thus  $h_1$  is indeed a tête-à-tête. Since  $\text{LT}((3D)^{(p-1)/2}) = (a_1a_2)^{p-1}$ , it is sufficient to prove  $\text{LT}(h_i) = 2a_3^pa_1^{p+2+(i-1)(p-1)}$  for  $i \geq 1$ .

Define

$$\begin{aligned} r_1 &= K \cdot e - D \cdot \text{tr}^P(a_3^{p-2}), \\ r_2 &= K \cdot r_1 - (3D)^{\frac{p-1}{2}} \cdot e, \\ r_i &= K \cdot r_{i-1} - (3D)^{\frac{p-1}{2}} \cdot r_{i-2} \text{ for } i \geq 3. \end{aligned}$$

Since  $K$  and  $D$  are  $G$ -invariant, we have  $h_i = \text{tr}_B^G(Nr_i)$ . Thus, using Theorem 2.5, it is sufficient to prove  $\text{LT}(r_i) = 2a_1^{p+2+(i-1)(p-1)}$  for  $i \geq 1$ .

Note that  $e \equiv_{(a_0)} 2a_1^3$  and  $D \equiv_{(a_0)} a_1^2\xi$ . Thus, using Lemma 2.10 and Lemma 2.11,

$$r_1 \equiv_{(a_0)} ((3\xi)^{\frac{p-1}{2}} + a_1^{p-1}) \cdot 2a_1^3 - a_1^2\xi \cdot 2(3^{\frac{p-1}{2}})a_1\xi^{\frac{p-3}{2}} = 2a_1^{p+2}.$$

Similarly

$$r_2 \equiv_{(a_0)} ((3\xi)^{\frac{p-1}{2}} + a_1^{p-1}) \cdot 2a_1^{p+2} - (3a_1^2\xi)^{\frac{p-1}{2}} \cdot 2a_1^3 = 2a_1^{(p+2)+(p-1)}.$$

Using the induction hypothesis,

$$\begin{aligned} r_i &\equiv_{(a_0)} ((3\xi)^{\frac{p-1}{2}} + a_1^{p-1}) \cdot 2a_1^{p+2+(i-2)(p-1)} - (3a_1^2\xi)^{\frac{p-1}{2}} \cdot 2a_1^{p+2+(i-3)(p-1)} \\ &\equiv_{(a_0)} 2a_1^{p+2+(i-1)(p-1)}, \end{aligned}$$

as required.  $\square$

### 3. GENERATORS AND HILBERT SERIES

This section is devoted to the proof of the main theorem.

**Theorem 3.1.** *For  $p > 3$ ,  $\mathbb{F}[V]^{SL_2(\mathbb{F}_p)}$  is generated by*

- elements from the image of the transfer
- $D, K, L, \delta, Na_0, \tilde{e}$  and
- for  $p \equiv -1 \pmod{3}$ ,  $\tilde{d}$ .

*The generators from the image of the transfer fall into three families:*

$$(1) \text{tr}^{SL_2(\mathbb{F}_p)}(N^j a_2^{(m-1)(p-1)-3j} a_3^{p-1}) \text{ where}$$

$$j = \begin{cases} 1, \dots, (p-4)/3 & \text{for } p \equiv 1 \pmod{3} \\ 1, \dots, p-2 & \text{for } p \equiv -1 \pmod{3} \end{cases}$$

$$\text{and } m = 2 + \lfloor 3j/(p-1) \rfloor;$$

(2)  $\text{tr}^{SL_2(\mathbb{F}_p)}(N^j a_3^{p-1-j})$  where

$$j = \begin{cases} 1, \dots, (p-4)/3 & \text{for } p \equiv 1 \pmod{3} \\ 1, \dots, (p-2)/3 & \text{for } p \equiv -1 \pmod{3}; \end{cases}$$

(3) and  $\text{tr}^{SL_2(\mathbb{F}_p)}(N^j a_3^{p-2} a_2^{(m-1)(p-1)+3-3j})$  where

$$j = \begin{cases} 2, \dots, (p-4)/3 & \text{for } p \equiv 1 \pmod{3} \\ 2, \dots, p-2 \text{ with } j \neq (p+1)/3, (2p-1)/3 & \text{for } p \equiv -1 \pmod{3} \end{cases}$$

and  $m = 2 + \lfloor 3j/(p-1) \rfloor$ .

For  $p \equiv -1 \pmod{3}$ , we have the further family of invariants:

$$\text{tr}^{SL_2(\mathbb{F}_p)}(N^j a_3^{\frac{5p-7-3j}{3}} a_2^2), \quad j = \frac{2p-1}{3}, \dots, p-2.$$

Let  $\mathcal{C}$  denote the proposed generating set and let  $R$  denote the algebra generated by  $\mathcal{C}$ . Since the elements of  $\mathcal{C}$  are homogeneous invariants,  $R$  is a graded subalgebra of  $\mathbb{F}[V]^G$ . Recall that the *Hilbert Series* of a graded vector space  $M = \bigoplus_{\ell=0}^{\infty} M_{\ell}$  is the formal power series  $HS(M, t) = \sum_{\ell=0}^{\infty} \dim(M_{\ell}) t^{\ell}$ . Since  $R$  is a graded subalgebra of  $\mathbb{F}[V]^G$ , we have  $HS(R, t) \leq HS(\mathbb{F}[V]^G, t)$ . We prove the theorem by showing these series are equal.

Define  $\mathcal{G} := \mathcal{C} \cup \{h_i, \forall i \geq 1\}$  and let  $\text{LT}(\mathcal{G})$  denote the subalgebra generated by the lead monomials of the elements of  $\mathcal{G}$ . In each of the two cases,  $p \equiv 1 \pmod{3}$  and  $p \equiv -1 \pmod{3}$ , we choose a graded subspace  $Z$  of  $\text{LT}(\mathcal{G})$ , giving a chain of inequalities:

$$HS(Z, t) \leq HS(\text{LT}(\mathcal{G}), t) \leq HS(\text{LT}(R), t) = HS(R, t) \leq HS(\mathbb{F}[V]^G, t).$$

We calculate  $HS(Z, t)$  and compare with Hughes-Kemper [8] to show  $HS(Z, t) = HS(\mathbb{F}[V]^G, t)$ . This proves that  $\mathcal{C}$  is a generating set and  $\mathcal{G}$  is a SAGBI basis.

The invariants  $D, K, Na_0$ , and  $\delta$  have lead monomials  $LM(D) = a_2^2 a_1^2$ ,  $LM(K) = a_2^{p-1}$ ,  $LM(Na_0) = a_3^p a_0$  and  $LM(\delta) = a_3^c$ , where  $c = (p-1)/3$  if  $p \equiv_{(3)} 1$  and  $a = p-1$  if  $p \equiv_{(3)} -1$ . Define

$$A := \mathbb{F}[a_2^2 a_1^2, a_2^{p-1}, a_3^p a_0, a_3^c],$$

the algebra generated by  $LM(D), LM(K), LM(Na_0)$  and  $LM(\delta)$ . In each of the two cases we will define  $Z$  as an  $A$ -submodule of  $\text{LT}(\mathcal{G})$ . For a monomial  $a_3^{e_3} a_2^{e_2} a_1^{e_1} a_0^{e_0}$  we assign a *parity* ( $e_2 \pmod{2}, e_1 \pmod{2}$ ) and observe that the action of  $A$  preserves parity.

### The $p \equiv 1 \pmod{3}$ Case

Recall from Theorem 2.13 that the lead monomials of the tête-à-têtes  $h_i$  are  $LM(h_i) = a_3^p a_1^{p+2+(i-1)(p-1)}$  for  $i \geq 1$ . By Lemma 2.5 the lead monomial of the invariant  $\tilde{e} = \text{tr}_B^{SL_2(\mathbb{F}_p)}(Ne)$  is equal to  $a_3^p a_1^3$ . Hence we have

$$n_i := a_3^p a_1^{3+i(p-1)} \text{ for } i \geq 0$$

as the lead monomials of  $\tilde{e}$  and  $h_i$ . Denote

$$\alpha_{ij} := n_0^{j-1} n_i = a_3^{pj} a_1^{3j+(p-1)i}, \quad 1 \leq j \leq (p-1)/3, \quad i \geq 0$$

and

$$\epsilon_{ij} := LM(L)\alpha_{ij} = a_3^{pj} a_2^p a_1^{1+3j+(p-1)i}, \quad 1 \leq j \leq (p-1)/3, \quad i \geq 0.$$

Define  $Z$  to be the  $A$ -module generated by the monomials

$$\mathcal{B} := \{1, LM(L), \gamma_j, \beta_j, \Delta_j, \alpha_{ij}, \epsilon_{ij} \mid i \in \mathbb{N}\}.$$

where  $1 \leq j \leq (p-1)/3$  for the  $\alpha$  and  $\epsilon$  families,  $1 \leq j < (p-1)/3$  for the  $\gamma$  and  $\beta$  families, and  $1 < j < (p-1)/3$  for the  $\Delta$  family; see Lemma 2.6, Lemma 2.7 and Lemma 2.8 for the definition of  $\gamma_j$ ,  $\beta_j$  and  $\Delta_j$ , and compare with the range of  $j$  for the families of transfers in Theorem 3.1.

The action of  $LM(Na_0)$  and  $LM(\delta)$  on  $Z$  is essentially free: every monomial in  $Z$  with a factor of  $a_0^{e_0}$  is divisible by  $LM(Na_0)^{e_0}$  and the remaining power of  $a_3$  determines the power of  $LM(\delta)$ . Let  $\tilde{Z}$  denote the span of the monomials in  $Z$  which are reduced with respect to  $LM(Na_0)$  and  $LM(\delta)$ . Then

$$HS(Z, t) = \frac{HS(\tilde{Z}, t)}{(1-t^{p+1})(1-t^{p(p-1)/3})}.$$

Define  $\tilde{Z}_j$  to be the span of the monomials in  $\tilde{Z}$  of the form  $a_3^{pj} a_2^{e_2} a_1^{e_1}$ . Then

$$\tilde{Z} = \bigoplus_{j=0}^{(p-1)/3} \tilde{Z}_j.$$

We proceed by computing  $HS(\tilde{Z}_j, t)$  for  $j = 0, 1, \dots, (p-1)/3$ . For fixed  $j$ , we determine the monomials  $a_3^{pj} a_2^x a_1^y \in \tilde{Z}_j$ . This set can be identified with a subset of the integral lattice in the  $xy$ -plane. Each element of  $\mathcal{B}$  gives rise to a  $\mathbb{F}[LM(D), LM(K)]$ -submodule corresponding to a cone in the  $xy$ -plane. The monomials in  $\tilde{Z}_j$  correspond to the union of these cones. The cones corresponding to elements of  $\mathcal{B}$  of different parity are disjoint.

For  $j = 0$ , the only elements of  $\mathcal{B}$  are 1 and  $LM(L) = a_2^p a_1$ , of parity  $(0, 0)$  and  $(1, 1)$  respectively. Thus

$$HS(\tilde{Z}_0, t) = \frac{1 + t^{p+1}}{(1-t^4)(1-t^{p-1})}.$$

For  $j = (p-1)/3 = c$ , the elements of  $\mathcal{B}$  fall into two families:

- $\alpha_{ic} = a_3^{pc} a_1^{p-1+i(p-1)}$  for  $i \in \mathbb{N}$ , with parity  $(0, 0)$ ;
- $\epsilon_{ic} = a_3^{pc} a_2^p a_1^{p+i(p-1)}$  for  $i \in \mathbb{N}$ , with parity  $(1, 1)$ .

For parity  $(0, 0)$ : Note that  $\alpha_{0c} \text{LM}(K) = \text{LM}(\delta) \text{LM}(D)^{\frac{p-1}{2}} \notin \tilde{Z}$ . Furthermore, for  $i > 0$ , we have  $\alpha_{ic} \text{LM}(K) = \alpha_{i-1,c} \text{LM}(D)^{\frac{p-1}{2}}$ . Thus it is sufficient to count the monomials  $\alpha_{ic} \text{LM}(D)^\ell$  with  $i, \ell \in \mathbb{N}$ .

For parity  $(1, 1)$ : Note that  $\epsilon_{0c} \text{LM}(K) = \text{LM}(\delta) \text{LM}(L) \text{LM}(D)^{\frac{p-1}{2}} \notin \tilde{Z}$ . Furthermore, for  $i > 0$ , we have  $\epsilon_{ic} \text{LM}(K) = \epsilon_{i-1,c} \text{LM}(D)^{\frac{p-1}{2}}$ . Thus it is sufficient to count the monomials  $\epsilon_{ic} \text{LM}(D)^\ell$  with  $i, \ell \in \mathbb{N}$ .

Counting monomials and identifying the appropriate geometric series gives

$$HS(\tilde{Z}_c, t) = \frac{t^{pc}(t^{p-1} + t^{2p})}{(1-t^4)(1-t^{p-1})} = \frac{t^{pc+p-1}(1+t^{p+1})}{(1-t^4)(1-t^{p-1})}.$$

In the case  $j = 1$ , we have the following elements of  $\mathcal{B}$ :

- $\alpha_{i1} = a_3^p a_1^{3+i(p-1)}$  for  $i \in \mathbb{N}$ , with parity  $(0, 1)$ ;
- $\beta_1 = a_3^p a_2^{p-3} a_1$ , with parity  $(0, 1)$ ;
- $\gamma_1 = a_3^p a_2^{2p-5}$ , with parity  $(1, 0)$ ;
- $\epsilon_{i1} = a_3^p a_2^p a_1^{4+i(p-1)}$  for  $i \in \mathbb{N}$ , with parity  $(1, 0)$ .

For Parity  $(0, 1)$ : Since  $\alpha_{01} \text{LM}(K) = \beta_1 \text{LM}(D)$  and  $\alpha_{i1} \text{LM}(K) = \alpha_{i-1,1} \text{LM}(D)^{\frac{p-1}{2}}$ , for  $i > 0$ , it is sufficient to count the monomials  $\alpha_{i1} \text{LM}(D)^\ell$  and  $\beta_1 \text{LM}(K)^i \text{LM}(D)^\ell$ .

For Parity  $(1, 0)$ : Since  $\epsilon_{01} \text{LM}(K) = \gamma_1 \text{LM}(D)$  and  $\epsilon_{i1} \text{LM}(K) = \epsilon_{i-1,1} \text{LM}(D)^{\frac{p-1}{2}}$ , for  $i > 0$ , it is sufficient to count the monomials  $\epsilon_{i1} \text{LM}(D)^\ell$  and  $\gamma_1 \text{LM}(K)^i \text{LM}(D)^\ell$ .

Counting monomials and identifying the appropriate geometric series gives

$$HS(\tilde{Z}_1, t) = \frac{t^p(t^3 + t^{p-2} + t^{p+4} + t^{2p-5})}{(1-t^4)(1-t^{p-1})}.$$

We now consider the case where  $j = 2k$  is even and  $2 \leq j < \frac{p-1}{3}$ . The relevant monomials are:

- $\alpha_{ij} = a_3^{pj} a_1^{3j+i(p-1)}$  for  $i \in \mathbb{N}$ , with parity  $(0, 0)$ ;
- $\beta_j = a_3^{pj} a_2^{p-1-2j} a_1^j$ , with parity  $(0, 0)$ ;
- $\gamma_j = a_3^{pj} a_2^{2p-2-3j}$ , with parity  $(0, 0)$ ;
- $\Delta_j = a_3^{pj} a_2^{2p-1-3j} a_1$ , with parity  $(1, 1)$ ;
- $\epsilon_{ij} = a_3^{pj} a_2^p a_1^{3j+1+i(p-1)}$  for  $i \in \mathbb{N}$ , with parity  $(1, 1)$ .

For parity  $(0, 0)$ : Observe that  $\beta_j \text{LM}(K) = \gamma_j \text{LM}(D)^k$ ,  $\alpha_{0j} \text{LM}(K) = \beta_j \text{LM}(D)^j$  and  $\alpha_{ij} \text{LM}(K) = \alpha_{i-1,j} \text{LM}(D)^{(p-1)/2}$  for  $i > 0$ . Thus it is sufficient to count the monomials  $\alpha_{ij} \text{LM}(D)^\ell$ ,  $\beta_j \text{LM}(D)^\ell$  and  $\gamma_j \text{LM}(D)^\ell \text{LM}(K)^i$ , for  $i, \ell \in \mathbb{N}$ .

For parity  $(1, 1)$ : Since  $\epsilon_{0j} \text{LM}(K) = \Delta_j \text{LM}(D)^{3k}$  and  $\epsilon_{ij} \text{LM}(K) = \epsilon_{i-1,j} \text{LM}(D)^{\frac{p-1}{2}}$  for  $i > 0$ , it is sufficient to count the monomials  $\epsilon_{ij} \text{LM}(D)^\ell$  and  $\Delta_j \text{LM}(K)^i \text{LM}(D)^\ell$ .

Counting monomials and identifying the appropriate geometric series gives

$$HS(\tilde{Z}_{2k}, t) = t^{2kp} \left( \frac{t^{6k} + t^{2p-2-6k} + t^{p+6k+1} + t^{2p-6k}}{(1-t^4)(1-t^{p-1})} + \frac{t^{p-1-2k}}{1-t^4} \right)$$

for  $k = 1, \dots, \frac{p-7}{6}$ .

For  $j = 2k + 1$  odd with  $1 < j < (p-1)/3$ , the elements of  $\mathcal{B}$  are:

- $\alpha_{ij} = a_3^{pj} a_1^{3j+i(p-1)}$  for  $i \in \mathbb{N}$ , with parity  $(0, 1)$ ;
- $\beta_j = a_3^{pj} a_2^{p-1-2j} a_1^j$  with parity  $(0, 1)$ ;
- $\Delta_j = a_3^{pj} a_2^{2p-1-3j} a_1$  with parity  $(0, 1)$ ;
- $\gamma_j = a_3^{pj} a_2^{2p-2-3j}$  with parity  $(1, 0)$ ;
- $\epsilon_{ij} = a_3^{pj} a_2^p a_1^{3j+1+i(p-1)}$  for  $i \in \mathbb{N}$ , with parity  $(1, 0)$ .

For parity  $(0, 1)$ : Observe that  $\beta_j \text{LM}(K) = \Delta_j \text{LM}(D)^k$ ,  $\alpha_{0j} \text{LM}(K) = \beta_j \text{LM}(D)^j$  and  $\alpha_{ij} \text{LM}(K) = \alpha_{i-1,j} \text{LM}(D)^{(p-1)/2}$  for  $i > 0$ . Thus it is sufficient to count the monomials  $\alpha_{ij} \text{LM}(D)^\ell$ ,  $\beta_j \text{LM}(D)^\ell$  and  $\Delta_j \text{LM}(D)^\ell \text{LM}(K)^i$ , for  $i, \ell \in \mathbb{N}$ .

For parity  $(1, 0)$ : Since  $\epsilon_{0j} \text{LM}(K) = \gamma_j \text{LM}(D)^{3k}$  and  $\epsilon_{ij} \text{LM}(K) = \epsilon_{i-1,j} \text{LM}(D)^{\frac{p-1}{2}}$  for  $i > 0$ , it is sufficient to count the monomials  $\epsilon_{ij} \text{LM}(D)^\ell$  and  $\gamma_j \text{LM}(K)^i \text{LM}(D)^\ell$ .

Counting monomials and identifying the appropriate geometric series gives

$$HS(\tilde{Z}_{2k+1}, t) = t^{(2k+1)p} \left( \frac{t^{6k+3} + t^{2p-2-6k-3} + t^{p+6k+4} + t^{2p-6k-3} + t^{p-2-2k}}{(1-t^4)(1-t^{p-1})} + \frac{t^{p-2-2k}}{1-t^4} \right)$$

for  $k = 1, \dots, \frac{p-7}{6}$ .

The even and odd formulae can be put in a common form: for  $1 < j < (p-1)/3$ ,

$$HS(\tilde{Z}_j, t) = \frac{t^{jp} (t^{3j} + t^{2p-2-3j} + t^{p+1+3j} + t^{2p-3j} + t^{p-1-j}(1-t^{p-1}))}{(1-t^4)(1-t^{p-1})}.$$

Summing over  $j$  and simplifying gives

$$HS(Z, t) = \frac{\text{Numer}(t)}{\text{Denom}(t)}$$

where

$$\begin{aligned} \text{Numer}(t) &= (1 + t^{p+1} + t^{p+3} + t^{2p-2} + t^{2p+4} + t^{3p-5} + t^{p-1}(t^{2p-2} - t^{(p-1)(p-1)/3}) \\ &\quad + t^{\frac{p(p-1)}{3}+p-1} + t^{\frac{p(p-1)}{3}+2p})(1-t^{p-3})(1-t^{p+3}) \\ &\quad + (t^{2p-2} + t^{2p})(t^{2p-6} - t^{(p-3)(p-1)/3})(1-t^{p+3}) \\ &\quad + (1 + t^{p+1})(t^{2p+6} - t^{(p+3)(p-1)/3})(1-t^{p-3}) \end{aligned}$$

and

$$\text{Denom}(t) = (1-t^4)(1-t^{p-3})(1-t^{p-1})(1-t^{p+1})(1-t^{p+3})(1-t^{\frac{p(p-1)}{3}}).$$

This agrees with the calculation of  $HS(\mathbb{F}[V]^G, t)$  by Hughes-Kemper [8, 2.7(d)].

**The  $p \equiv -1 \pmod 3$  Case**

In this case the lead monomial of  $\delta = \text{tr}_B^G(N^c)$  is  $a_3^{p(p-1)}$  and the generators of  $Z$  will be monomials divisible by  $a_3^{pj}$  for  $j \leq p-1$ . Using Lemma 2.5 the lead monomial of  $\tilde{d}$  is  $a_3^{(p+1)/3} a_1^2$ . As in the proof of the  $p \equiv_{(3)} 1$  case, we denote the lead monomials of  $\tilde{e}$  and  $h_i$  by  $n_i = a_3^p a_1^{3+i(p-1)}$  for  $i \geq 0$ . Define  $s := \lfloor 3j/(p-1) \rfloor$ ,

$$\alpha_{ij} := \text{LM}(\tilde{d})^s n_i n_0^{j-1-s(p-1)/3} = a_3^{pj} a_1^{3j+(p-1)(i-s)}, \quad 1 \leq j \leq (p-1), \quad i \in \mathbb{N}$$

and

$$\epsilon_{ij} := \text{LM}(L)\alpha_{ij} = a_3^{pj} a_2^p a_1^{3j+(p-1)(i-s)+1}, \quad 1 \leq j \leq (p-1), \quad i \in \mathbb{N}.$$

Further, we assign the following notation:

$$\lambda := \text{LM}(\tilde{d})\gamma_{\frac{p-2}{3}} = a_3^{p\frac{2p-1}{3}} a_2^p a_1^2,$$

$$\mu := \beta_1 \cdot \gamma_{\frac{p-2}{3}} = a_3^{p\frac{p+1}{3}} a_2^{2p-3} a_1,$$

$$\eta_j := \text{LM}(\tilde{d})\beta_{j-(p+1)/3} = a_3^{pj} a_2^{\frac{5p-1}{3}-2j} a_1^{j-\frac{p-5}{3}} \quad \text{for } \frac{p+4}{3} \leq j \leq \frac{2p-1}{3}.$$

Define  $Z$  to be the  $A$ -module generated by

$$\mathcal{B} := \{1, \text{LM}(L), \alpha_{i,j}, \epsilon_{i,j}, \gamma_j, \beta_j, \Delta_j, \phi_j, \lambda, \mu, \eta_j \mid i \in \mathbb{N}\}.$$

where the ranges in  $j$  are given above or in the statement of Theorem 3.1

As in the  $p \equiv_{(3)} 1$  case, the action of  $\text{LM}(Na_0)$  and  $\text{LM}(\delta)$  on  $Z$  is essentially free. Let  $\tilde{Z}$  denote the span of the monomials of  $Z$  which are reduced with respect to  $\text{LM}(Na_0)$  and  $\text{LM}(\delta)$ . Then

$$HS(Z, t) = \frac{HS(\tilde{Z}, t)}{(1-t^{p+1})(1-t^{p(p-1)})}.$$

Define  $\tilde{Z}_j$  to be the span of the monomials in  $\tilde{Z}$  of the form  $a_3^{pj} a_2^x a_1^y$ . Then

$$\tilde{Z} = \bigoplus_{j=0}^{p-1} \tilde{Z}_j.$$

The calculation of  $HS(\tilde{Z}_j, t)$  for  $j < (p-1)/3$  is precisely as in the  $p \equiv_{(3)} 1$  case.

For  $j = \frac{p+1}{3}$  the elements of  $\mathcal{B}$  are:

- $\alpha_{i, \frac{p+1}{3}} = a_3^{p\frac{p+1}{3}} a_1^{2+i(p-1)}$  for  $i \in \mathbb{N}$ , with parity  $(0, 0)$ ;

- $\gamma_{\frac{p+1}{3}} = a_3^{p\frac{p+1}{3}} a_2^{2p-4}$  with parity  $(0, 0)$ ;
- $\epsilon_{i, \frac{p+1}{3}} = a_3^{p\frac{p+1}{3}} a_2^p a_1^{3+i(p-1)}$  for  $i \in \mathbb{N}$ , with parity  $(1, 1)$ ;
- $\mu = a_3^{p\frac{p+1}{3}} a_2^{2p-3} a_1$  with parity  $(1, 1)$ .

For parity  $(0, 0)$ : Observe that  $\text{LM}(D)\gamma_{\frac{p+1}{3}} = \text{LM}(K)^2\alpha_{0, \frac{p+1}{3}}$  and  $\alpha_{ij} \text{LM}(K) = \alpha_{i-1, j} \text{LM}(D)^{(p-1)/2}$  for  $i > 0$ . Thus it is sufficient to count the monomials  $\alpha_{i+1, (p+1)/3} \text{LM}(D)^\ell$ ,  $\alpha_{0, (p+1)/3} \text{LM}(D)^\ell \text{LM}(K)^i$ , and  $\gamma_{(p+1)/3} \text{LM}(K)^i$  for  $i, \ell \in \mathbb{N}$ .

For parity  $(1, 1)$ : Observe that  $\text{LM}(D)\mu = \text{LM}(K)\epsilon_{0, \frac{p+1}{3}}$  and  $\epsilon_{ij} \text{LM}(K) = \epsilon_{i-1, j} \text{LM}(D)^{(p-1)/2}$  for  $i > 0$ . Thus it is sufficient to count the monomials  $\mu \text{LM}(K)^i$  and  $\epsilon_{i, (p+1)/3} \text{LM}(D)^\ell$ .

Counting monomials and identifying the appropriate geometric series gives

$$HS\left(\tilde{Z}_{\frac{p+1}{3}}, t\right) = t^{p(p+1)/3} \left( \frac{t^2 + t^{p+1} + t^{p+3} + t^{2p-2}}{(1-t^4)(1-t^{p-1})} + \frac{t^{2p-4}}{1-t^{p-1}} \right).$$

We now consider the range  $\frac{p+4}{3} \leq j \leq \frac{2p-4}{3}$ . The following table indicates the monomials and their respective parities:

	Monomial	Parity $j$ even	Parity $j$ odd
$\alpha_{i, j}$	$a_3^{pj} a_1^{3j-p+1+i(p-1)}$ , $i \in \mathbb{N}$	$(0, 0)$	$(0, 1)$
$\eta_j$	$a_3^{pj} a_2^{\frac{5p-1-6j}{3}} a_1^{\frac{3j-p+5}{3}}$	$(0, 0)$	$(0, 1)$
$\gamma_j$	$a_3^{pj} a_2^{3p-3-3j}$	$(0, 0)$	$(1, 0)$
$\Delta_j$	$a_3^{pj} a_2^{3p-2-3j} a_1$	$(1, 1)$	$(0, 1)$
$\epsilon_{i, j}$	$a_3^{pj} a_2^p a_1^{3j-p+2+i(p-1)}$ , $i \in \mathbb{N}$	$(1, 1)$	$(1, 0)$

For  $j$  even, parity  $(0, 0)$ : We have  $\eta_j \text{LM}(K) = \gamma_j \text{LM}(D)^{(3j-p+5)/6}$ ,  $\alpha_{0j} \text{LM}(K) = \eta_j \text{LM}(D)^{j-(p+1)/3}$  and  $\alpha_{ij} \text{LM}(K) = \alpha_{i-1, j} \text{LM}(D)^{(p-1)/2}$  for  $i > 0$ . Thus we need to count  $\alpha_{ij} \text{LM}(D)^\ell$ ,  $\eta_j \text{LM}(D)^\ell$  and  $\gamma_j \text{LM}(K)^i \text{LM}(D)^\ell$ .

For  $j$  even, parity  $(1, 1)$ :  $\epsilon_{ij} \text{LM}(K) = \epsilon_{i-1, j} \text{LM}(D)^{(p-1)/2}$  and  $\epsilon_{0j} \text{LM}(K) = \Delta_j \text{LM}(D)^{(3j-p+1)/2}$ . Thus we need to count  $\epsilon_{ij} \text{LM}(D)^\ell$  and  $\Delta_j \text{LM}(K)^i \text{LM}(D)^\ell$ .

Counting monomials and identifying the appropriate geometric series gives:

$$HS(\tilde{Z}_j, t) = t^{jp} \left( \frac{t^{3j+2} + t^{3p-3-3j} + t^{3p-1-3j} + t^{3j-p+1}}{(1-t^4)(1-t^{p-1})} + \frac{t^{4(p+1)/3-j}}{1-t^4} \right).$$

For  $j$  odd, the calculations are analogous with the roles of  $\gamma_j$  and  $\Delta_j$  reversed. The contribution to  $HS(\tilde{Z}_j, t)$  is the same for both  $j$  even and  $j$  odd. Thus for  $(p+1)/3 < j < (2p-1)/3$  we have:

$$HS(\tilde{Z}_j, t) = t^{jp} \left( \frac{t^{3j+2} + t^{3p-3-3j} + t^{3p-1-3j} + t^{3j-p+1} + t^{4(p+1)/3-j}(1-t^{p-1})}{(1-t^4)(1-t^{p-1})} \right).$$

For  $j = \frac{2p-1}{3}$  the monomials to consider are:

- $\alpha_{i, \frac{2p-1}{3}} = a_3^{\frac{p}{3}} a_1^{p+i(p-1)}$  for  $i \in \mathbb{N}$ , with parity  $(0, 1)$ ;
- $\phi_{\frac{2p-1}{3}} = a_3^{\frac{p}{3}} a_2^{p-1} a_1$  with parity  $(0, 1)$ ;
- $\eta_{\frac{2p-1}{3}} = a_3^{\frac{p}{3}} a_2^{\frac{p+1}{3}} a_1^{\frac{p+4}{3}}$  with parity  $(0, 1)$ ;
- $\gamma_{\frac{2p-1}{3}} = a_3^{\frac{p}{3}} a_2^{2p-3}$  with parity  $(1, 0)$ ;
- $\epsilon_{i, \frac{2p-1}{3}} = a_3^{\frac{p}{3}} a_2^p a_1^{p+1+i(p-1)}$  for  $i \in \mathbb{N}$ , with parity  $(1, 0)$ ;
- $\lambda = a_3^{\frac{p+1}{3}} a_2^p a_1^2$  with parity  $(1, 0)$ .

For parity  $(0, 1)$ :  $\alpha_{ij} \text{LM}(K) = \alpha_{i-1,j} \text{LM}(D)^{(p-1)/2}$  for  $i > 0$ ,  $\alpha_{0j} \text{LM}(K) = \eta_j \text{LM}(D)^{(p-2)/3}$  and  $\eta_j \text{LM}(K) = \phi_j \text{LM}(D)^{(p+1)/6}$ . Thus we need to count  $\alpha_{ij} \text{LM}(D)^\ell$ ,  $\eta_j \text{LM}(D)^\ell$  and  $\phi_j \text{LM}(K)^i \text{LM}(D)^\ell$ .

For parity  $(1, 0)$ :  $\epsilon_{ij} \text{LM}(K) = \epsilon_{i-1,j} \text{LM}(D)$  for  $i > 0$ ,  $\epsilon_{0j} \text{LM}(K) = \lambda \text{LM}(D)^{(p-1)/2}$  and  $\lambda \text{LM}(K) = \gamma_j \text{LM}(D)$ . Thus we need to count  $\epsilon_{ij} \text{LM}(D)^\ell$ ,  $\lambda \text{LM}(D)^\ell$  and  $\gamma_j \text{LM}(K)^i \text{LM}(D)^\ell$ .

Counting monomials and identifying the appropriate geometric series gives:

$$(3.1) \quad HS\left(\tilde{Z}_{\frac{2p-1}{3}}, t\right) = t^{p(2p-1)/3} \left( \frac{2t^p + t^{2p-3} + t^{2p+1}}{(1-t^4)(1-t^{p-1})} + \frac{t^{p+2} + t^{(2p+5)/3}}{1-t^4} \right).$$

We now consider the range  $\frac{2p+2}{3} \leq j \leq p-2$ . The following table gives the relevant monomials and their parities:

Monomial		Parity $j$ even	Parity $j$ odd
$\alpha_{i,j}$	$a_3^{pj} a_1^{3j-2p+2+i(p-1)}$ , $i \in \mathbb{N}$	(0,0)	(0,1)
$\phi_j$	$a_3^{pj} a_2^{\frac{7p-5-6j}{3}} a_1^{\frac{3j-2p+4}{3}}$	(0,0)	(0,1)
$\gamma_j$	$a_3^{pj} a_2^{4p-4-3j}$	(0,0)	(1,0)
$\Delta_j$	$a_3^{pj} a_2^{4p-3-3j} a_1$	(1,1)	(0,1)
$\epsilon_{i,j}$	$a_3^{pj} a_2^p a_1^{3j-2p+3+i(p-1)}$ , $i \in \mathbb{N}$	(1,1)	(1,0)

For  $j$  even, parity  $(0, 0)$ : We have  $\phi_j \text{LM}(K) = \gamma_j \text{LM}(D)^{(3j-2p+4)/6}$ ,  $\alpha_{0j} \text{LM}(K) = \phi_j \text{LM}(D)^{j-(2p-1)/3}$  and  $\alpha_{ij} \text{LM}(K) = \alpha_{i-1,j} \text{LM}(D)^{(p-1)/2}$  for  $i > 0$ . Thus we need to count  $\alpha_{ij} \text{LM}(D)^\ell$ ,  $\phi_j \text{LM}(D)^\ell$  and  $\gamma_j \text{LM}(K)^i \text{LM}(D)^\ell$ .

For  $j$  even, parity  $(1, 1)$ :  $\epsilon_{ij} \text{LM}(K) = \epsilon_{i-1,j} \text{LM}(D)^{(p-1)/2}$  and  $\epsilon_{0j} \text{LM}(K) = \Delta_j \text{LM}(D)^{(3j-2p+2)/2}$ . Thus we need to count  $\epsilon_{ij} \text{LM}(D)^\ell$  and  $\Delta_j \text{LM}(K)^i \text{LM}(D)^\ell$ .

Counting monomials and identifying the appropriate geometric series gives:

$$HS(\tilde{Z}_j, t) = t^{jp} \left( \frac{t^{3j-2p+2} + t^{4p-4-3j} + t^{4p-4-3j} + t^{3j-p+3}}{(1-t^4)(1-t^{p-1})} + \frac{t^{5(p+1)/3-j-2}}{1-t^4} \right).$$



For  $j$  odd, the calculations are analogous with the roles of  $\gamma_j$  and  $\Delta_j$  reversed. The contribution to  $HS(\tilde{Z}, t)$  is the same for both  $j$  even and  $j$  odd. Thus for  $(2p-1)/3 < j < p-1$  we have:

$$HS(\tilde{Z}_j, t) = t^{jp} \left( \frac{t^{3j+2-2p} + t^{4p-4-3j} + t^{4p-2-3j} + t^{3j-p+3} + t^{5(p+1)/3-j-2}(1-t^{p-1})}{(1-t^4)(1-t^{p-1})} \right).$$

Finally, we consider the case  $j = p-1$ . The only monomials we have here are

- $\alpha_{i,p-1} = a_3^{p(p-1)} a_1^{p-1+i(p-1)}$  for  $i \in \mathbb{N}$ , with parity  $(0, 0)$ ;
- $\epsilon_{i,p-1} = a_3^{p(p-1)} a_2^p a_1^{p+i(p-1)}$  for  $i \in \mathbb{N}$ , with  $(1, 1)$ .

Note that  $\alpha_{0,p-1} \text{LM}(K) = \text{LM}(\delta) \text{LM}(D)^{(p-1)/2} \notin \tilde{Z}$  and, for  $i > 0$ , we have  $\alpha_{i,p-1} \text{LM}(K) = \alpha_{i-1,p-1} \text{LM}(D)^{(p-1)/2}$ . Similarly,

$$\epsilon_{0,p-1} \text{LM}(K) = \text{LM}(\delta) \text{LM}(L) \text{LM}(D)^{(p-1)/2} \notin \tilde{Z}$$

and, for  $i > 0$ ,  $\epsilon_{i,p-1} \text{LM}(K) = \epsilon_{i-1,p-1} \text{LM}(D)^{(p-1)/2}$ . Thus it is sufficient to count the monomials  $\alpha_{i,p-1} \text{LM}(D)^\ell$  and  $\epsilon_{i,p-1} \text{LM}(D)^\ell$  with  $i, \ell \in \mathbb{N}$ . Counting monomials and identifying the appropriate geometric series gives

$$HS(\tilde{Z}_{p-1}, t) = \frac{t^{p(p-1)}(t^{p-1} + t^{2p})}{(1-t^4)(1-t^{p-1})}.$$

Summing over  $j$  and simplifying gives

$$HS(Z, t) = \frac{\text{Numer}(t)}{\text{Denom}(t)}$$

where

$$\begin{aligned} \text{Numer}(t) &= \chi_1(t)(1-t^{p-3})(1-t^{p+3}) + \chi_2(t)(1-t^{p+3}) + \chi_3(t)(1-t^{p-3}), \\ \chi_1(t) &= 1 + t^{p+1} + t^{p(p+1)} + t^{(p+1)(p-1)} + t^p(t^3 + t^{p-2} + t^{p+4} + t^{2p-5}) \\ &\quad + t^{p(p+1)/3}(t^2 + t^{p+1} + t^{p+3} + t^{2p-4} + t^{2p-2} - t^{2p}) \\ &\quad + t^{p(2p-1)/3}(2t^p + t^{p+2} + t^{2p-3} + t^{(2p+5)/3}(1-t^{p-1})) \\ &\quad + t^{3(p-1)}(1-t^{(p-1)(p-5)/3})(1+t^{p(p-2)/3+3} + t^{2p(p-2)/3+2}), \\ \chi_2(t) &= t^{4(p-2)}(1-t^{(p-3)(p-5)/3})(1+t^2)(1+t^{p(p-2)/3+1} + t^{2p(p-2)/3+2}), \\ \chi_3(t) &= t^{2p+6}(1-t^{(p+3)(p-5)/3})(1+t^{p+1})(1+t^{p(p-2)/3-1} + t^{2p(p-2)/3-2}) \end{aligned}$$

and

$$\text{Denom}(t) = (1-t^4)(1-t^{p-3})(1-t^{p-1})(1-t^{p+1})(1-t^{p+3})(1-t^{p(p-1)}).$$

This agrees with the calculation of  $HS(\mathbb{F}[V]^G, t)$  by Hughes-Kemper [8, 2.7(d)].

#### 4. CONCLUDING REMARKS

We do not claim that the generating sets given in Theorem 3.1 are minimal. However, for  $p = 5$  and  $p = 7$ , MAGMA [4] calculations confirm that the given sets are minimal generating sets. Recall that the *Noether number* is the maximum degree of an element in a minimal homogeneous generating set. Thus the Noether number is 22 for  $p = 5$  and 16 for  $p = 7$ . Examining the degrees of the polynomials occurring in Theorem 3.1 gives the following.

**Corollary 4.1.** *The Noether number of  $\mathbb{F}[V]^{SL_2(\mathbb{F}_p)}$  is bounded above by*

- $p^2 - p + 4$  if  $p \equiv_{(3)} -1$ ,
- $\frac{p^2 - p + 12}{3}$  if  $p \equiv_{(3)} 1$ .

It follows from the proof of Theorem 3.1 that  $\mathcal{G}$  is a SAGBI basis for  $\mathbb{F}[V]^{SL_2(\mathbb{F}_p)}$ . This means that the set  $LM(\mathcal{G})$  generates the lead term algebra of  $\mathbb{F}[V]^{SL_2(\mathbb{F}_p)}$  and if  $f \in \mathbb{F}[V]^{SL_2(\mathbb{F}_p)}$  then  $LM(f)$  can be written as a product of elements from  $LM(\mathcal{G})$ .

**Corollary 4.2.**  *$\mathbb{F}[V]^{SL_2(\mathbb{F}_p)}$  does not have a finite SAGBI basis using the graded reverse lexicographical order with  $a_0 < a_1 < a_2 < a_3$ .*

*Proof.* Observe that if  $a_1^j \in LM(\mathcal{G})$  then  $j = 0$  and if  $m \in LM(\mathcal{G})$  with  $a_3$  dividing  $m$ , then  $a_3^p$  divides  $m$ . Thus  $LM(h_i) = a_3^p a_1^{p+2+(i-1)(p-1)}$  is indecomposable in the lead term algebra of  $\mathbb{F}[V]^{SL_2(\mathbb{F}_p)}$ .  $\square$

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