VECTOR INVARIANTS FOR THE TWO DIMENSIONAL MODULAR REPRESENTATION OF A CYCLIC GROUP OF PRIME ORDER

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Abstract. In this paper, we study the vector invariants of the 2-dimensional indecomposable representation V_2 of the cylic group, C_p, of order p over a field $\mathbf{F}$ of characteristic p, $\mathbf{F}[m\,V_2]^{C_p}$. This ring of invariants was first studied by David Richman [21] who showed that the ring required a generator of degree $m(p-1)$, thus demonstrating that the result of Noether in characteristic 0 (that the ring of invariants of a finite group is always generated in degrees less than or equal to the order of the group) does not extend to the modular case. He also conjectured that a certain set of invariants was a generating set with a proof in the case $p = 2$. This conjecture was proved by Campbell and Hughes in [3]. Later, Shank and Wehlau in [24] determined which elements in Richman's generating set were redundant thereby producing a minimal generating set.

We give a new proof of the result of Campbell and Hughes, Shank and Wehlau giving a minimal algebra generating set for the ring of invariants $\mathbf{F}[m\,V_2]^{C_p}$. In fact, our proof does much more. We show that our minimal generating set is also a SAGBI basis for $\mathbf{F}[m\,V_2]^{C_p}$. Further, our results provide a procedure for finding an explicit decomposition of $\mathbf{F}[m\,V_2]$ into a direct sum of indecomposable C_p-modules. Finally, noting that our representation of C_p on V_2 is as the p-Sylow subgroup of $SL_2(\mathbf{F}_p)$, we describe a generating set for the ring of invariants $\mathbf{F}[m\,V_2]^{SL_2(\mathbf{F}_p)}$ and show that $(p+m-2)(p-1)$ is an upper bound for the Noether number, for $m > 2$.

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1. Introduction

We suppose G is a group represented on a vector space V over a field $\mathbf{F}$. If $\{x_1, x_2, \ldots, x_n\}$ is a basis for the hom-dual, $V^* = \hom_{\mathbf{F}}(V, \mathbf{F})$, of V, then we denote the symmetric algebra on V^* by

$$\mathbf{F}[V] = \mathbf{F}[x_1, x_2, \ldots, x_n]$$

and we note that G acts on $f \in \mathbf{F}[V]$ by the rule

$$\sigma(f)(v) = f(\sigma^{-1}(v)).$$

As an aside, the notation $\mathbf{F}[V]$ is often used in the literature to denote the ring of regular functions on V. Our notation coincides with the usual notion when the field $\mathbf{F}$ is infinite. However, for example, if $\mathbf{F} = \mathbf{F}_p$, the prime field, then the functions x_1 and x_1^p coincide on V.

The ring of functions left invariant by this action of G is denoted $\mathbf{F}[V]^G$. Invariant theorists often seek to relate algebraic properties of the invariant ring to properties of the representation. For example, when G is finite of order $|G|$ and the characteristic p of $\mathbf{F}$ does *not* divide $|G|$ – the *non-modular* case – then $\mathbf{F}[V]^G$ is a polynomial algebra if and only if G is generated by reflections (group elements fixing a hyperplane of V). This is a famous result due to Coxeter [8], Shephard and Todd [26], Chevalley [6], and Serre[22]. For another example in the non-modular case, it is known by work of Noether [19] (when $p = 0$), Fogarty [12] and Fleischmann [13] (when $p > 0$), that $\mathbf{F}[V]^G$ is generated in degrees less than or equal to $|G|$. And, in the non-modular case with G finite, it is well-known that $\mathbf{F}[V]^G$ is always Cohen-Macaulay.

The case when $p > 0$, G is finite, V is finite dimensional and p does divide $|G|$ is that of modular invariant theory. Many results that are well understood in the non-modular case are not yet understood or even within reach in the modular case. For example, in the modular case it is known that if $\mathbf{F}[V]^G$ is a polynomial algebra then G must be generated by reflections, but this is far from sufficient. For another example, in the modular case $\mathbf{F}[V]^G$ is "most often" not Cohen-Macaulay. Finally, in the modular case, there are examples where $\mathbf{F}[V]^G$ requires generators of degrees (much) larger than $|G|$, see below: this paper re-examines the first known such example in considerable detail.

There are now several references for modular invariant theory, see Benson [1], Smith[27], Neusel and Smith[18], Derksen and Kemper[9], Campbell and Wehlau[3].

Invariant theorists also seek to determine generators for $\mathbf{F}[V]^G$ and, if possible, relations among those generators. A famous example is the case of *vector invariants*, see Weyl [28]. Here we consider the vector space

$$m\,V = \overbrace{V \oplus V \oplus \cdots \oplus V}^{m\ summands}$$

with G acting diagonally. The invariants $\mathbf{F}[m\,V]^G$ are called vector invariants, and in this case, we seek to describe, determine or give constructions for, the generators of this ring, a *first main theorem for* (G, V). Once this is done a theorem determining the relations among the generators is referred to as a *second main theorem for* (G, V).

The cyclic group C_p has exactly p inequivalent indecomposable representations over a field $\mathbf{F}$ of characteristic p. There is one indecomposable V_n of dimension n for each $1 \le n \le p$. To see this choose a basis for V_n with respect to which a fixed generator, σ, of C_p is represented by a matrix in Jordan Normal form. Since V_n is indecomposable this matrix has a single Jordan block and since σ has order p the common eigenvalue must be 1, the only p^{th} root of unity in a field of characteristic p. Thus σ is represented on V_n by the matrix

$$\begin{pmatrix} 1 & 0 & 0 & \dots & 0 & 0 \\ 1 & 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & 0 & \dots & 1 & 1 \end{pmatrix}.$$

In order that this matrix have order p (or 1) we must have $n \le p$. We call such a basis of V_n for which σ is in (lower triangular) Jordan Normal form a *triangular* basis.

Observe the following chain of inclusions:

$$V_1 \subset V_2 \subset \cdots \subset V_p.$$

If V is any finite dimensional C_p-module then V can be decomposed into a direct sum of indecomposable C_p-modules:

$$V \cong m_1\,V_1 \oplus m_2\,V_2 \cdots \oplus m_p\,V_p$$

where $m_i \in \mathbb{N}$ for all i. This decomposition is far from unique but does have the property that the *multiplicities* m_ℓ are unique.

We are interested in the representation $m\,V_2$ and the action of C_p on $\mathbf{F}[m\,V_2]$. The ring of invariants $\mathbf{F}[m\,V_2]^{C_p}$ was first studied by David Richman [21]. He showed that this ring required a generator of degree $m(p-1)$, showing that the result of Noether in characteristic 0 did not extend to the modular case. He also conjectured that a certain set of invariants was a generating set with a proof in the case $p=2$. This conjecture was proved by Campbell and Hughes in [3]: the proof is long, complex, and counter-intuitive in some respects. Later, Shank and Wehlau in [24] determined which elements in Richman's generating set were redundant thereby producing a minimal generating set.

We will show later (and the proof is not difficult), that $\mathbf{F}[m\,V_2]^{C_p}$ is not Cohen-Macaulay, or see Ellingsrud and Skjelbred [11].

In this paper, we give a new proof of the result of Campbell and Hughes, Shank and Wehlau giving a minimal algebra generating set for the ring of invariants $\mathbf{F}[m\,V_2]^{C_p}$. In fact, our proof does much more. We show that our minimal generating set is also a SAGBI basis for $\mathbf{F}[m\,V_2]^{C_p}$. In our view, this result is extraordinary. Further, our techniques also yield a procedure for finding a decomposition of $\mathbf{F}[m\,V_2]$ into a direct sum of indecomposable C_p-modules.

Our paper is organised as follows. In the second section of our paper, Preliminaries, we provide more details on the the representation theory of C_p, our use of graded reverse lexicographical ordering on the monomials in $\mathbf{F}[m\,V_2]^{C_p}$, and define the term *SAGBI* basis. In the next section, Polarisation, we define the polarisation map $\mathbf{F}[V] \to \mathbf{F}[m\,V]$, its (roughly speaking) inverse, known as restitution, and we note that these maps are G-equivariant, hence map G-invariants to G-invariants. These techniques allow us to focus our attention on multi-linear invariants. The next section, Partial Dyck Paths, describes a concept arising in the study of lattices in the plane, see, for example the book by Koshy [17, p. 151], and is followed by a section on Lead Monomials. Here we show that there is a bijection between the set of lead monomials of multi-linear invariants and certain collections of Partial Dyck Paths. This work requires us to count the number of indecomposable C_p summands in

$$\overset{m}{\otimes} V_2 = \overbrace{V_2 \otimes V_2 \otimes \cdots \otimes V_2}^{m\ copies},$$

and in fact we are able to determine a decomposition of $\overset{m}{\otimes} V_2$ as a C_p-module, see Theorem 5.5. Following this, in section § 6, we prove that we have a generating set for our ring of invariants. The next section describes how our techniques provide a procedure for finding a decomposition of $\mathbf{F}[m\,V_2]$ as a C_p-module. In the final section, noting that our representation of C_p on V_2 is as the p-Sylow subgroup of $SL_2(\mathbf{F}_p)$, we are able to describe a generating set for the ring of invariants $\mathbf{F}[m\,V_2]^{SL_2(\mathbf{F}_p)}$.

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2. Preliminaries

Suppose $\{e_1, e_2, \ldots, e_n\}$ is a triangular basis for V_n. Note that the C_p-module generated by e_1 is all of V_n. We also note that the indecomposable module $V_n^* = \hom(V_n, \mathbf{F})$ is isomorphic to V_n since $\dim(V_n^*) = \dim(V_n)$. Because of our interest in invariants we often focus on the C_p action on V_n^* rather than on V_n itself. Therefore we will choose the dual basis $\{x_1, x_2, \ldots, x_n\}$ for V^* to the basis $\{e_1, e_2, \ldots, e_n\}$. With this choice of basis the matrices representing G are upper-triangular on V^*. We note that $\sigma^{-1}(x_1) = x_1$ and $\sigma^{-1}(x_i) = x_i + x_{i-1}$ for $2 \leq i \leq n$: for convenience, and since σ^{-1} also generates C_p, we will change notation and write σ instead of σ^{-1} for the remainder of this paper. With this convention, we note that $(\sigma - 1)^r(x_n) = x_{n-r}$ for $r < n$ and $\dim(V_n) = n$ is the largest value of r such that $x_1 \in (\sigma - 1)^{r-1}(V_n^*)$. We say that the invariant x_1 has *length* n in this case and write $\ell(x_1) = n$. We observe that the socle of V_n is the line $V_n^{C_p}$ spanned by $\{e_n\}$. Similarly $(V_n^*)^{C_p}$ has basis $\{x_1\}$.

Note that the kernel of $\sigma - 1 : V_i \to V_i$ is $V_i^{C_p}$ which is one dimensional for all i. Thus

$$\dim((\sigma - 1)^j(V_i)) = \begin{cases} 0 & \text{if } j - 1 \geq i; \\ i - j & \text{if } j - 1 < i. \end{cases}$$

For

$$V \cong m_1\,V_1 \oplus m_2\,V_2 \cdots \oplus m_p\,V_p$$

this gives $(p-j)m_p + (p-1-j)m_{p-1} + \cdots + (i-j)m_i = \dim((\sigma-1)^j(V))$ for all $0 \leq j \leq p-1$ and this system of equations uniquely determines the coefficients $m_1, m_2, \ldots, m_p$.

Each indecomposable C_p-module, V_n, satisfies $\dim(V_n)^{C_p} = 1$. Therefore the number of summands occurring in a decomposition of V is given by $m_1 + m_2 + \cdots + m_p = \dim V^{C_p}$.

Consider $\mathrm{Tr} := \sum_{\tau \in C_p} \tau$, an element of the group ring of C_p. If W is any finite dimensional C_p-representation, we also use Tr to denote the corresponding $\mathbf{F}[W]^{C_p}$-module homomorphism,

$$\mathrm{Tr} : \mathbf{F}[W] \to \mathbf{F}[W]^{C_p}.$$

Similarly we define

$$\mathrm{N} : \mathbf{F}[W] \to \mathbf{F}[W]^{C_p}$$

by $\mathrm{N}(w) = \prod_{\tau \in C_p} \tau(w)$.

Note that $(\sigma-1)^{p-1} = \sum_{i=0}^{p-1}(-1)^i\binom{p-1}{i}\sigma^i = \sum_{i=0}^{p-1}\sigma^i = \mathrm{Tr}$. It follows that $\mathrm{Tr}(v) = 0$ if $v \in V_n$ for $n < p$, while $\mathrm{Tr}(x_p) = x_1$ in V_p.

It is also the case that $V_p \cong \mathbf{F}C_p$ is the only free C_p-module and hence also the only projective.

The next theorem plays an important role in our decomposition of $\mathbf{F}[V]_{(d_1,d_2,\ldots,d_m)}$ as a C_p-module (modulo projectives). A proof in the case $V = V_n$ may be found in Hughes and Kemper [14, section 2.3], and a proof of the version cited here is in Shank and Wehlau [25, section 2]

Theorem 2.1 (Periodicity Theorem)**.** *Let $V = V_{n_1} \oplus V_{n_2} \oplus \cdots \oplus V_{n_m}$. Let $d_1, d_2, \ldots, d_m$ be non-negative integers and write $d_i = q_i p + r_i$ where $0 \le r_i \le p-1$ for $i = 1, 2, \ldots, m$. Then*

$$\mathbf{F}[V]_{(d_1,d_2,\ldots,d_m)} \cong \mathbf{F}[V]_{(r_1,r_2,\ldots,r_m)} \oplus t\,V_p$$

as C_p-modules for some non-negative integer t.

Comparing dimensions shows that in the above theorem

$$t = \left(\prod_{i=1}^{m}\binom{n_i+d_i-1}{d_i} - \prod_{i=1}^{m}\binom{n_i+r_i-1}{r_i}\right)/p\ .$$

In this paper, we are primarily interested in the case $V = m\,V_2$. We denote the basis for the i^{th}-copy of V_2^* in this direct sum by $\{x_i, y_i\}$ and we have $\sigma(x_i) = x_i$ and $\sigma(y_i) = y_i + x_i$.

For this representation of C_p, there is another "obvious" family of invariants, namely the

$$u_{ij} = x_i y_j - x_j y_i = \begin{vmatrix} x_i & y_i \\ x_j & y_j \end{vmatrix}$$

for $m \ge 2$.

2.2. **Relations involving the u_{ij}.** We will consider now two important families of relations involving the invariants $u_{ij} = x_i y_j - y_i x_j$. First we consider algebraic dependencies among the u_{ij}. Suppose $m \ge 4$ and let $1 \le i < j < k < \ell \le m$. It is easy to verify that $0 = u_{ij}u_{k\ell} - u_{ik}u_{j\ell} + u_{i\ell}u_{jk}$. It can be shown that these relations generate all the algebraic relations among the u_{st}.

It is useful to represent products of the various u_{st} graphically as follows. We consider the vertices of a regular m-gon and label them clockwise by $1, 2, \ldots, m$. We represent the factor u_{ij} by an arrow or directed edge from vertex i to vertex j. Thus a product of various u_{st} is represented by a number of directed edges joining the labelled vertices of the regular m-gon.

Returning to the relation $u_{ij}u_{k\ell} - u_{ik}u_{j\ell} + u_{i\ell}u_{jk}$, we say that the middle product in this relation, $u_{ik}u_{j\ell}$, is a *crossing* since the arrows representing the two factors u_{ik} and $u_{j\ell}$ cross (intersect). Rewriting the relation as $u_{ik}u_{j\ell} = u_{ij}u_{k\ell} + u_{i\ell}u_{jk}$, we see that we may replace a crossing with a sum of two non-crossings. As observed by Kempe [16], since the length of two (directed) diagonals representing u_{ik} and $u_{j\ell}$ exceeds both the lengths represented by the sides u_{ij} and $u_{k\ell}$ and the two sides $u_{i\ell}$ and u_{jk}, we may repeatedly use"uncrossing" relations to rewrite any product of u_{st}'s by a sum of such products without any crossings. Thus the space generated by the monomials in the u_{st} of degree d has a basis represented by planar directed graphs on m vertices having d directed edges. Here we allow multiple (or weighted) edges to represent powers such as u_{ij}^a for $a \geq 2$.

Now we consider another important class of relations, this time involving the u_{st} and the x_r. Take $m \geq 3$, let $1 \leq i < j < k \leq m$ and consider the matrix

$$\begin{pmatrix} x_i & x_j & x_k \\ x_i & x_j & x_k \\ y_i & y_j & y_k \end{pmatrix}.$$

Computing the determinant by expanding along the first row we find $x_iu_{jk} - x_ju_{ik} + x_ku_{ij} = 0$. Since x_1, x_2, x_3 is a partial homogeneous system of parameters in $\mathbf{F}[m\,V_2]$ consisting of invariants it is a partial homogeneous system of parameters in $\mathbf{F}[m\,V_2]^{C_p}$. The relation $x_1u_{23} - x_2u_{13} + x_3u_{12} = 0$ shows that the product x_3u_{12} represents 0 in the quotient ring $\mathbf{F}[m\,V_2]^{C_p}/(x_1, x_2)$. Considering degrees, it is easy to see that u_{12} and x_3 do not lie in the ideal of $\mathbf{F}[m\,V_2]^{C_p}$ generated by (x_1, x_2). Thus x_3 represents a zero divisor in the quotient ring $\mathbf{F}[m\,V_2]^{C_p}/(x_1, x_2)$. This shows that the partial homogeneous system of parameters x_1, x_2, x_3 in $\mathbf{F}[m\,V_2]^{C_p}$ does not form a regular sequence. Therefore $\mathbf{F}[m\,V_2]^{C_p}$ is not a Cohen-Macaulay ring when $m \geq 3$. For $m \leq 2$ the ring of invariants $\mathbf{F}[m\,V_2]^{C_p}$ is Cohen-Macaulay since $\mathbf{F}[V_2]^{C_p} = \mathbf{F}[x_1, \mathrm{N}(y_1)]$ is a polynomial ring and $\mathbf{F}[2\,V_2]^{C_p} = \mathbf{F}[x_1, x_2, u_{12}, \mathrm{N}(y_1), \mathrm{N}(y_2)]$ is a hypersurface ring.

Throughout this paper we will use graded reverse lexicographic term orders. We write $\mathrm{LM}(f)$ for the lead monomial of f and $\mathrm{LT}(f)$ for the lead term of f. We follow the convention that monomials are power products of variables and terms are scalar multiples of power products of variables. If $S = \oplus_{d=0}^{\infty} S_d$ is a graded subalgebra of a polynomial ring then we say a set B is *a SAGBI basis for S in degree d* if for every $f \in S_d$

we can write $\mathrm{LM}(f)$ as a product $\mathrm{LM}(f) = \prod_{g \in B} \mathrm{LM}(g)^{e_g}$ where each e_g is a non-negative integer. If B is a SAGBI basis for S in degree d for all d then we say that B is a SAGBI basis for S. If B is a SAGBI basis for S then B is an algebra generating set for S. The word SAGBI is an acronym for "sub-algebra analogue of Gröbner bases for ideals", and was introduced by Robbianno and Sweedler in [20] and (independently) by Kapur and Madlener in [15]. For a detailed discussion of term orders we direct the reader to Chapter 2 of Cox, Little and O'Shea [7]. For a discussion and application of SAGBI bases in modular invariant theory, we recommend Shank's paper [23].

Given a sequence of variables $z_1, z_2, \ldots, z_m$ and an element $E = (e_1, e_2, \ldots, e_m)$ we write z^E to denote the monomial $z_1^{e_1} z_2^{e_2} \cdots z_m^{e_m}$ and we denote the degree $e_1 + e_2 + \cdots + e_m$ of this monomial by $|E|$.

The following well-known lemma is very useful for computing the lead term of a transfer.

Lemma 2.3. *Let t be a positive integer. Then*

$$\sum_{i=0}^{p-1} i^t = \begin{cases} -1, & \text{if } p-1 \text{ divides } t; \\ 0, & \text{if } p-1 \text{ does not divide } t. \end{cases}$$

For a proof of this lemma see for example, [5, Lemma 9.4].

3. Polarisation

Let V be a representation of a group G and let $r \in \mathbb{N}$ with $r \geq 2$ and consider the map of G-representations

$$\nabla^* : r\,V \longrightarrow (r-1)\,V$$

defined by $\nabla^*(v_1, v_2, \ldots, v_r) = (v_1, v_2, \ldots, v_{r-2}, v_{r-1} + v_r)$. We also consider the morphism

$$\Delta^* : (r-1)\,V \longrightarrow r\,V$$

given by $\Delta^*(v_1, v_2, \ldots, v_{r-1}) = (v_1, v_2, \ldots, v_{r-2}, v_{r-1}, v_{r-1})$. Dual to these two maps we have the corresponding algebra homomorphisms:

$$\nabla : \mathbf{F}[(r-1)\,V] \longrightarrow \mathbf{F}[r\,V]$$

and

$$\Delta : \mathbf{F}[r\,V] \longrightarrow \mathbf{F}[(r-1)\,V]$$

given by $\nabla(f) = f \circ \nabla^*$ and $\Delta(F) = F \circ \Delta^*$. We also define $\nabla_r^* = (\nabla^*)^{r-1} : r\,V \to V$ and $\Delta_r^* = (\Delta^*)^{r-1} : V \to r\,V$.

Thus $\nabla_r : \mathbf{F}[V] \longrightarrow \mathbf{F}[r\,V]$ is given by $(\nabla_r(f))(v_1, v_2, \ldots, v_r) = f(v_1 + v_2 + \cdots + v_r)$ and $\Delta_r : \mathbf{F}[r\,V] \longrightarrow \mathbf{F}[V]$ is given by $(\Delta_r(F))(v) =$

$F(v, v, \ldots, v)$. The homomorphism ∇_r is called *(complete) polarisation* and the homomorphism Δ_r is called *restitution*.

The algebra $\mathbf{F}[r\,V]$ is naturally $\mathbb{N}^r$ graded by multi-degree:

$$\mathbf{F}[r\,V] = \bigoplus_{(\lambda_1,\lambda_2,\ldots,\lambda_r)\in\mathbb{N}^r} \mathbf{F}[r\,V]_{(\lambda_1,\lambda_2,\ldots,\lambda_r)}$$

where

$$\mathbf{F}[r\,V]_{(\lambda_1,\lambda_2,\ldots,\lambda_r)} \cong \mathbf{F}[V]_{\lambda_1} \otimes \mathbf{F}[V]_{\lambda_2} \otimes \cdots \otimes \mathbf{F}[V]_{\lambda_r}\ .$$

For each multi-degree, $(\lambda_1, \lambda_2, \ldots, \lambda_r) \in \mathbb{N}^r$ we have the projection $\pi_{(\lambda_1,\lambda_2,\ldots,\lambda_r)} : \mathbf{F}[r\,V] \to \mathbf{F}[r\,V]_{(\lambda_1,\lambda_2,\ldots,\lambda_r)}$. Given a homogeneous element $f \in \mathbf{F}[V]$ of total degree d, i.e., $f \in \mathbf{F}[V]_d$, its *full polarisation* is the multi-linear function $\mathcal{P}(f) = \pi_{(1,1,\ldots,1)}(\nabla_d(f)) \in \mathbf{F}[d\,V]_{(1,1,\ldots,1)}$. Thus $\mathcal{P} : \mathbf{F}[V]_d \to \mathbf{F}[d\,V]_{(1,1,\ldots,1)}$.

More generally, we may use isomorphisms of the form $\mathbf{F}[V \oplus W] \cong \mathbf{F}[V] \otimes \mathbf{F}[W]$ to define

$$\nabla_{r_1,r_2,\ldots,r_m} = \nabla_{r_1} \otimes \nabla_{r_2} \otimes \cdots \otimes \nabla_{r_m} : \mathbf{F}[\oplus_{i=1}^m W_i] \longrightarrow \mathbf{F}[\oplus_{i=1}^m r_i W_i]\ .$$

Again, for ease of notation, if $f \in \mathbf{F}[\oplus_{i=1}^m W_i]_{(\lambda_1,\lambda_2,\ldots,\lambda_m)}$ we write $\mathcal{P}(f) = \pi_{(1,1,\ldots,1)}(\nabla_{\lambda_1,\lambda_2,\ldots,\lambda_m}(f)) \in \mathbf{F}[\oplus_{i=1}^m \lambda_i W_i]_{(1,1,\ldots,1)}$. Here again we call the multi-linear function $\mathcal{P}(f)$ the full polarisation of f.

Similarly we define the restitution map

$$\Delta_{r_1,r_2,\ldots,r_m} = \Delta_{r_1} \otimes \Delta_{r_2} \otimes \cdots \otimes \Delta_{r_m} : \mathbf{F}[\oplus_{i=1}^m r_i W_i] \longrightarrow \mathbf{F}[\oplus_{i=1}^m W_i]\ .$$

In this setting, if we have co-ordinate variables such as x_i, y_i, z_i for W_i we will use the symbols x_{ij}, y_{ij}, z_{ij} with $1 \le j \le r_i$ to denote the co-ordinate variables for $r_i W_i$. In this notation, restitution is the algebra homomorphism determined by $\Delta_{r_1,r_2,\ldots,r_m}(x_{ij}) = x_i$, $\Delta_{r_1,r_2,\ldots,r_m}(y_{ij}) = y_i$, $\Delta_{r_1,r_2,\ldots,r_m}(z_{ij}) = z_i$, etc. For this reason, restitution is sometimes referred to as *erasing subscripts*. For ease of notation, we will write $\mathcal{R}$ to denote the algebra homomorphism $\Delta_{\lambda_1,\lambda_2,\ldots,\lambda_m}$ when restricted to $\mathbf{F}[\oplus_{i=1}^m \lambda_i W_i]_{(1,1,\ldots,1)}$. Thus if $F \in \mathbf{F}[\oplus_{i=1}^m \lambda_i W_i]_{(1,1,\ldots,1)}$ then $\mathcal{R}(F) \in \mathbf{F}[\oplus_{i=1}^m W_i]_{(\lambda_1,\lambda_2,\ldots,\lambda_m)}$. (However, we will sometimes abuse notation and use $\mathcal{R}$ to denote $\Delta_{\lambda_1,\lambda_2,\ldots,\lambda_m}$ when the indices $\lambda_1, \lambda_2, \ldots, \lambda_m$ are clear from the context.)

It is a relatively straightforward exercise to verify that for any $f \in \mathbf{F}[\oplus_{i=1}^m W_i]_{(\lambda_1,\lambda_2,\ldots,\lambda_m)}$ we have $\mathcal{R}(\mathcal{P}(f)) = (\lambda_1!, \lambda_2!, \ldots, \lambda_m!)f$.

Since ∇^* and Δ^* are both G-equivariant, so are all the homomorphisms $\nabla_{r_1,r_2,\ldots,r_m}$ and $\Delta_{r_1,r_2,\ldots,r_m}$. In particular, if f is G-invariant then so is $\mathcal{P}(f)$. Similarly, $\mathcal{R}(F)$ is G-invariant if F is. We also note that since the action of G preserves degree an element f is invariant if and only if all its homogeneous components are invariant.

4. Partial Dyck Paths

In this section we consider a generalization of Dyck paths (see the book by Koshy [17, p. 151] for an introduction to Dyck paths). For us, a lattice path is a finite sequence of steps in the first quadrant of the xy-plane starting from the origin. Each step in the path is given by either the vector (1,0) (an x-step) or the vector (0,1) (a y-step). The number of steps in the path is called its *length*. The path is said to have height h if h is the largest integer such that the path touches the line $y = x - h$. A lattice path has *finishing height* h if the final step ends at a point on the line $y = x - h$.

Associated to each lattice path of length d is a word of length d , i.e., an ordered sequence of d symbols each either an x or a y. This word encodes the path as follows: the i^{th} symbol of the word is x if the i^{th} step of the path is an x-step and the i^{th} symbol of the word is a y if the i^{th} step is a y-step.

We will consider two types of lattice paths: (i) partial Dyck paths and (ii) initial Dyck paths of escape height $p-1$.

Definition 4.1. A *partial Dyck path* is a lattice path which stays on or below the diagonal (the line with equation $y = x$). A partial Dyck path of finishing height 0, i.e., which finishes on the diagonal, is called a Dyck path.

Definition 4.2. An *initial Dyck path of escape height* t is a lattice path of height at least t and which if it crosses above the diagonal does so only after it touches the line $y = x - t$. Expressed another way, these are paths which consist of an partial Dyck path of finishing height t followed by an entirely arbitrary sequence of x-steps and y-steps.

Clearly there are 2^d lattice paths of length d. We may associate these paths with the 2^d monomials in $\mathbf{F}[d\,V_2]_{(1,1,\ldots,1)} \cong \otimes^d V_2$. The lattice path γ of length d is associated to the word $\gamma_1\gamma_2\cdots\gamma_d$ and is associated to the multi-linear monomial $\Lambda(\gamma) = z_1z_2\cdots z_d$ where $\begin{cases} z_i = x_i, & \text{if } \gamma_i = x; \\ z_i = y_i, & \text{if } \gamma_i = y. \end{cases}$

We let $\mathrm{PDP}^d_{\leq q}$ denote the set of all partial Dyck paths of length d and height at most q. Furthermore, we will denote by $\mathrm{PDP}^d_{\leq q}(h)$ the set of partial Dyck paths of length d, height at most q and finishing height h. We write IDP^d_q to denote the set of all initial Dyck paths of escape height q and length d.

5. Lead Monomials

We wish to consider the C_p-representation $\mathbf{F}[d\,V_2]_{(1,1,\dots,1)} \cong \otimes^d V_2$. We consider a decomposition of $\otimes^d V_2$ into a direct sum of indecomposable C_p-representations. Each summand V_h has a one dimensional socle spanned by an element f and we associate to this summand the monomial $\mathrm{LM}(f)$. We say that the invariant f has *length* h and we write $\ell(f) = h$. In general a non-zero invariant has length h if $h-1$ is the maximal value of r for which f lies in the image of $(\sigma-1)^r$.

In order to study $\mathbf{F}[d\,V_2]^{C_p}_{(1,1,\dots,1)} \cong (\otimes^d V_2)^{C_p}$ we use the graded reverse lexicographic order determined by $y_1 > x_1 > y_2 > x_2 \cdots > y_d > x_d$ and consider

$$M = \{\mathrm{LM}(f) \mid f \in (\otimes^d V_2)^{C_p}\}\ .$$

We will show that the set map

$$\Lambda : \mathrm{PDP}^d_{\le p-2} \sqcup \mathrm{IDP}^d_{p-1} \longrightarrow M$$

is a bijection.

We begin by showing that the image of Λ lies inside M. In fact we will show that if $\gamma \in \mathrm{PDP}^d_{\le p-2}(h)$ then $\Lambda(\gamma)$ is the lead monomial of an invariant of length $h+1$. Furthermore if $\gamma \in \mathrm{IDP}^d_{p-1}$ then $\Lambda(\gamma)$ is the lead monomial of an invariant of length p, i.e, an invariant lying in $\mathrm{Tr}(\otimes^d V_2)$.

Consider a path $\gamma \in \mathrm{PDP}^d_{\le p-1}(h)$ and let $\gamma_1\gamma_2\cdots\gamma_d$ be the associated word. We wish to match each symbol γ_j which is a y with an earlier symbol $\gamma_{\rho(j)}$ which is an x. We do this recursively as follows. Choose the smallest value j such that $\gamma_j = y$ and for which we have not yet found a matching x. Take i to be maximal such that $i < j$, $\gamma_i = x$ and $i \ne \rho(s)$ for all $s < j$. Then we define $\rho(j) = i$. Let $I_1 = \{j \mid \gamma_j = y\}$, $I_2 = \rho(I_1)$ and $I_3 = \{1,2,\dots,d\}\setminus(I_1 \sqcup I_2)$. Then $|I_1| = |I_2| = (d-h)/2$, $|I_3| = h$ and $\gamma_i = x$ for all $i \in I_3$.

Define

$$\theta_0(\gamma) = \left(\prod_{j\in I_1} u_{\rho(j),j}\right)\prod_{i\in I_3} x_i \text{ and } \theta_0'(\gamma) = \left(\prod_{j\in I_1} u_{\rho(j),j}\right)\prod_{i\in I_3} y_i\ .$$

Then $\theta_0(\gamma) \in (\otimes^d V_2)^{C_p}$ and

$$\mathrm{LM}(\theta_0(\gamma)) = \prod_{j\in I_1}\mathrm{LM}(u_{\rho(j),j})\prod_{i\in I_3} x_i = \prod_{j\in I_1} x_{\rho(j)}y_j \prod_{i\in I_3} x_i = \Lambda(\gamma)\ .$$

Lemma 5.1. $(\sigma-1)^h(\theta_0'(\gamma)) = h!\,\theta_0(\gamma)$ *and thus* $\ell(\theta_0(\gamma)) \ge h+1$.

Proof. We will prove a more general statement. We will show that

$$(\sigma-1)^{|E|}(y^E) = |E|!\,x^E.$$

Note that this also implies that $(\sigma-1)^{|E|+1}(y^E)=0$. We proceed by induction on $|E|$. The result is clear for $|E|=0,1$. Assume, without loss of generality, that $e_i \geq 1$ for all i and define $Z_i \in \mathbb{N}^d$ by $x_i = x^{Z_i}$. For $|E| \geq 2$ we have

$$\begin{aligned}
(\sigma-1)^{|E|}(y^E) &= (\sigma-1)^{|E|-1}(\sigma-1)(y^E)\\
&= (\sigma-1)^{|E|-1}\left(\sum_i e_i x_i y^{E-Z_i} + \text{ terms divisible by some } x_k x_\ell\right)\\
&= (\sigma-1)^{|E|-1}\left(\sum_i e_i x_i y^{E-Z_i}\right)\\
&\quad \text{since the other terms lie in the kernel of } (\sigma-1)^{|E|-1}\\
&= \sum_i e_i x_i (\sigma-1)^{|E|-1}\left(y^{E-Z_i}\right)\\
&= \sum_i e_i x_i (|E|-1)!\, x^{E-Z_i} \text{ by induction}\\
&= \sum_i e_i (|E|-1)!\, x^E = \left(\sum_i e_i\right)(|E|-1)!\, x^E\\
&= |E|(|E|-1)!\, x^E = |E|!\, x^E
\end{aligned}$$

□

If $\gamma \in \mathrm{PDP}^d_{\leq p-2}$ then we define $\theta(\gamma)=\theta_0(\gamma)$ and $\theta'(\gamma)=\theta'_0(\gamma)$.

Suppose instead that $\gamma \in \mathrm{IDP}^d_{p-1}$ and let $\gamma_1\gamma_2\cdots\gamma_d$ be the word associated to γ. Take s minimal such that the path γ' associated to $\gamma_1\gamma_2\cdots\gamma_s$ is a partial Dyck path of finishing height $p-1$. Since $\gamma' \in \mathrm{PDP}^s_{\leq p-1}(p-1)$, from the above we have $I_1 = \{j \leq s \mid \gamma_j = y\}$, $I_2 = \rho(I_1)$ and $I_3 = \{1,2,\ldots,s\}\backslash(I_1 \sqcup I_2)$ with $|I_1|=|I_2|=(s-p+1)/2$, $|I_3| = p-1$ and $\gamma_i = x$ for all $i \in I_3$. We further define $I_4 = \{i > s \mid \gamma_i = x\}$ and $I_5 = \{i > s \mid \gamma_i = y\}$. Define

$$\theta'(\gamma) = \theta'_0(\gamma')\prod_{i\in I_5} y_i \prod_{i \in I_4} x_i = \prod_{j \in I_1} u_{\rho(j),j} \prod_{i \in I_3 \cup I_5} y_i \prod_{i\in I_4} x_i$$

and

$$\theta(\gamma) = \mathrm{Tr}\left(\theta'_0(\gamma')\right)\prod_{i\in I_5} y_i \prod_{i \in I_4} x_i = \mathrm{Tr}\left(\prod_{i \in I_3 \cup I_5} y_i\right)\prod_{j \in I_1} u_{\rho(j),j} \prod_{i\in I_4} x_i$$

Then $\theta(\gamma) \in \mathrm{Tr}(\otimes^d V_2) \subset (\otimes^d V_2)^{C_p}$ and $\ell(\theta(\gamma)) = p$.

By Lemma 2.3

$$\begin{aligned}\mathrm{LM}(\theta(\gamma)) &= \left(\prod_{i\in I_4} x_i \prod_{j\in I_1} \mathrm{LM}(u_{\rho(j),j})\right) \mathrm{LM}(\mathrm{Tr}(\prod_{i\in I_3\cup I_5} y_i))\\ &= \left(\prod_{i\in I_4} x_i \prod_{j\in I_1} x_{\rho(j)}y_j\right) \prod_{i\in I_3} x_i \prod_{i\in I_5} y_i = \Lambda(\gamma)\end{aligned}$$

In summary, if $\gamma \in \mathrm{PDP}^d_{\le p-2}(h)$ then $\theta(\gamma)$ is an invariant of length at least $h+1$ and lead monomial $\Lambda(\gamma)$. If $\gamma \in \mathrm{IDP}^d_{p-1}$ then $\theta(\gamma)$ is an invariant of length p and with lead monomial $\Lambda(\gamma)$. Note that since these lead monomials are all distinct, the maps θ and Λ are injective.

It remains to show that Λ is onto M and to determine the exact length of the invariants $\theta(\gamma)$ when $\gamma \in \mathrm{PDP}^d_{\le p-2}$. We will show that Λ is onto by showing $|M| = |\mathrm{PDP}^d_{\le p-2} \sqcup \mathrm{IDP}^d_{p-1}|$. To determine $|M|$ we examine the number of indecomposable summands in the decomposition of $\otimes^d V_2$.

Define non-negative integers $\mu^d_p(h)$ by the direct sum decomposition of the C_p-module $\otimes^d V_2$ over $\mathbf{F}$:

$$\bigotimes^d V_2 \cong \bigoplus_{h=1}^p \mu^d_p(h)\, V_h\ .$$

Using the convention $\otimes^0 V_2 = V_1$, we have the following lemma.

Lemma 5.2. *Let $p \ge 3$. Then*

$$\mu^0_p(h) = \delta^1_h \text{ and } \mu^1_p(h) = \delta^2_h,$$

and

$$\mu^{d+1}_p(h) = \begin{cases} \mu^d_p(2), & \text{if } h = 1;\\ \mu^d_p(h-1) + \mu^d_p(h+1), & \text{if } 2 \le h \le p-2;\\ \mu^d_p(p-2), & \text{if } h = p-1;\\ \mu^d_p(p-1) + 2\mu^d_p(p), & \text{if } h = p;\end{cases}$$

for $d \ge 1$.

Proof. The initial conditions are clear. The recursive conditions follow immediately from the following three equations which may be found for example in Hughes and Kemper [14, Lemma 2.2]:

$$\begin{aligned}&V_1 \otimes V_2 \cong V_2\\ &V_h \otimes V_2 \cong V_{h-1} \oplus V_{h+1} \text{ for all } 2 \le h \le p-1\\ &V_p \otimes V_2 \cong 2\, V_p.\end{aligned}$$

□

Next we count lattice paths. Let $\nu_q^d(h) = |\mathrm{PDP}^d_{\le q}(h)|$ for $1 \le h \le q$. We also define $\bar{\nu}_q^d = |\mathrm{IDP}^d_q|$. With this notation we have the following lemma.

Lemma 5.3. *Let $q \ge 2$. Then*

$$\nu_q^0(h) = \delta_h^0 \text{ and } \nu_q^1(h) = \delta_h^1 \ ,$$

$$\bar{\nu}_q^0 = 0 \text{ and } \bar{\nu}_q^1 = 0 \ ,$$

and

$$\nu_q^{d+1}(h) = \begin{cases} \nu_q^d(1), & \text{if } h = 0; \\ \nu_q^d(h-1) + \nu_q^d(h+1), & \text{if } 1 \le h \le q-1; \\ \nu_q^d(q-1), & \text{if } h = q; \end{cases}$$

and

$$\bar{\nu}_q^{d+1} = \nu_{q-1}^d(q-1) + 2\bar{\nu}_q^d$$

for all $d \ge 1$.

Proof. All of these equations are easily seen to hold except perhaps the final one. Its left-hand term $\bar{\nu}_q^{d+1} = |\mathrm{IDP}^{d+1}_q|$ is the number of initial Dyck paths of length $d+1$ and escape height q. We divide such paths into two classes: those which first achieve height q on their final step and those which achieve height q sometime during the first d steps. Paths in the first class are partial Dyck paths of length d, height at most $q-1$ and finishing height $q-1$ followed by an x-step for the $(d+1)^{\mathrm{st}}$ step. There are $\nu_{q-1}^d(q-1) = |\mathrm{PDP}^d_{\le q-1}(q-1)|$ such paths. The second class consists of initial Dyck paths of escape height q and length d followed by a final step which may be either an x-step or a y-step. Clearly there are $2|\mathrm{IDP}^d_q| = 2\bar{\nu}_q^d$ paths of this kind. □

Corollary 5.4. *For all $d \in \mathbb{N}$, all primes p and all $h = 1, 2, \ldots, p-1$ we have*

$$\mu_p^d(h) = \nu_{p-2}^d(h-1) \quad \text{and} \quad \mu_p^d(p) = \bar{\nu}_{p-1}^d \ .$$

Proof. Comparing the recursive expressions and initial conditions for $\mu_p^d(h)$ and $\nu_{p-2}^d(h-1)$ and for $\mu_p^d(p)$ and $\bar{\nu}_{p-1}^d$ given in the previous two lemmas makes the result clear for $p \ge 5$.

For $p = 2$ it is easy to see that $\mu_2^d(1) = \nu_0^d(0) = \delta_d^0$ for $d \ge 0$ and $\mu_2^d(2) = 2^{d-1} = \bar{\nu}_1^d$ for $d \ge 1$.

For $p = 3$ and $h = 1, 2$ we have

$$\mu_3^d(h) = \nu_1^d(h-1) = \begin{cases} 1, & \text{if } h+d \text{ is odd;} \\ 0, & \text{if } h+d \text{ is even.} \end{cases}$$

Hence $\mu_3^d(3) = \lfloor \frac{2^d-1}{3} \rfloor$ for $d \geq 0$. From the recursive relation $\bar{\nu}_2^{d+1} = \nu_1^d(1) + 2\bar{\nu}_2^d$ it is easy to see that $\bar{\nu}_2^d = \lfloor \frac{2^d-1}{3} \rfloor = \mu_3^d(3)$. □

This corollary implies that the map Λ is a bijection. Furthermore for all d, every element of $\{\mathrm{LM}(f) \mid f \in (\otimes^d V_2)^{C_p}\}$ may be written as a product with factors from the set $\{\mathrm{LM}(g) \mid g \in B\}$ where

$$B := \{x_i \mid 1 \leq i \leq d\} \cup \{u_{ij} \mid 1 \leq i < j \leq d\}$$
$$\cup \{\mathrm{Tr}(\prod_{i=1}^{d} y_i^{e_i}) \mid 0 \leq e_i \leq 1, \forall\, i = 1, 2, \ldots, d\}\ .$$

We record and extend these results in the following theorem.

Theorem 5.5. *Let p be a prime, let $d \in \mathbb{N}$ and suppose $0 \leq h \leq p-2$. Let $\gamma \in PDP^d_{\leq p-2} \cup IDP^d_{p-1}$. Then*

(1) $\mathrm{LM}(\theta(\gamma)) = \Lambda(\gamma)$.

(2) *If $\gamma \in PDP^d_{\leq p-2}(h)$ then the invariant $\theta(\gamma)$ lies in*
$$\mathbf{F}[d\,V_2]^{C_p}_{(1,1,\ldots,1)} \cong (\otimes^d V_2)^{C_p}$$
and has length $h+1$.

(3) *If $\gamma \in IDP^d_{p-1}$ then the invariant $\theta(\gamma)$ lies in*
$$\mathbf{F}[d\,V_2]^{C_p}_{(1,1,\ldots,1)} \cong (\otimes^d V_2)^{C_p}$$
and has length p.

(4) *B is a SAGBI basis in multi-degree $(1,1,\ldots,1)$ for $\mathbf{F}[d\,V_2]^{C_p}$.*

Furthermore, we have the following decomposition of the C_p representation $\otimes^d V_2$ into indecomposable summands:

$$\bigotimes^d V_2 \cong \bigoplus_{\gamma \in PDP^d_{\leq p-2} \cup IDP^d_{p-1}} V(\gamma)$$

where $V(\gamma) \cong V_{h+1}$ is a C_p-module generated by $\theta'(\gamma)$, with socle spanned by $\theta(\gamma)$ and

$$h = \ell(\theta(\gamma)) - 1 = \begin{cases} \text{the finishing height of } \gamma; & \text{if } \gamma \in PDP^d_{\leq p-2}(h); \\ p-1 & \text{if } \gamma \in IDP^d_{p-1}. \end{cases}$$

Proof. The assertions (1) and (3) have already been proved.

To prove the other assertions we consider the C_p-module

$$W = \sum_{\gamma \in \mathrm{PDP}^d_{\leq p-2} \cup \mathrm{IDP}^d_{p-1}} V(\gamma)$$

generated by the set $\{\theta'(\gamma) \mid \gamma \in \mathrm{PDP}^d_{\leq p-2} \cup \mathrm{IDP}^d_{p-1}\}$. The set of vectors $\{\theta(\gamma) \mid \gamma \in \mathrm{PDP}^d_{\leq p-2} \cup \mathrm{IDP}^d_{p-1}\}$ spanning the socles of the $V(\gamma)$ is linearly independent since these vectors have distinct lead monomials. This implies that the above sum is direct:

$$W = \bigoplus_{\gamma \in \mathrm{PDP}^d_{\leq p-2} \cup \mathrm{IDP}^d_{p-1}} V(\gamma) \ .$$

Thus $\dim W = (\sum_{h=0}^{p-2} (h+1) \cdot \nu_p^d(h)) + p \cdot \bar{\nu}_p^d$. Applying Corollary 5.4, yields $\dim W = \dim \otimes^d V_2$. Since W is a submodule of $\otimes^d V_2$ we see that $W = \otimes^d V_2$. Furthermore, any set of (spanning vectors for the) socles in any direct sum decomposition of $\otimes^d V_2$ there will be exactly $\nu_p^d(h)$ invariants of length $h+1$ for $0 \leq h \leq p-2$ (and $\bar{\nu}_p^d$ of length p). Combining this fact with $\ell(\theta(\gamma)) \geq h+1$ for all $\gamma \in \mathrm{PDP}^d_{\leq p-2}(h)$, we get $\ell(\theta(\gamma)) = h+1$ for all $\gamma \in \mathrm{PDP}^d_{\leq p-2}(h)$, completing the proof of assertion (2) as well as the final assertion of the theorem. Assertion(4) also follows now since we have $\{\mathrm{LM}(f) \mid f \in (\otimes^d V_2)^{C_p}\} = \{\mathrm{LM}(\theta(\gamma)) \mid \gamma \in \mathrm{PDP}^d_{\leq p-2} \cup \mathrm{IDP}^d_{p-1}\}$ and each of these lead monomials may be factored into a product of lead monomials of elements of B. $\square$

6. A Generating Set

Consider the set

$$\begin{aligned} \mathcal{B} = \{x_i, \mathrm{N}(y_i) \mid 1 \leq i \leq m\} \cup \{u_{ij} \mid 1 \leq i < j \leq m\} \\ \cup \{\mathrm{Tr}(y^E) \mid 0 \leq e_i \leq p-1\} \ . \end{aligned}$$

We will show that $\mathcal{B}$ is a generating set, in fact a SAGBI basis for $\mathbf{F}[m\,V_2]^{C_p}$. Let $f \in \mathbf{F}[m\,V_2]^{C_p}$ be monic and multi-homogeneous, of multi-degree$(\lambda_1, \lambda_2, \ldots, \lambda_m)$. Let A denote the subalgebra $\mathbf{F}[\mathcal{B}]$. We proceed by induction on the total degree $d = \lambda_1 + \lambda_2 + \cdots + \lambda_m$ of f. Clearly if f has total degree 0 then f is constant, $f \in A$ and $\mathrm{LM}(f) = 1$ and there is nothing more to prove.

Suppose then that the total degree d of f is positive. First suppose that $\lambda_i \geq p$ for some i. We consider f as a polynomial in y_i and write $f = \sum_{j=0}^{\lambda_i} f_j y_i^j$ where f_j is a polynomial which is homogeneous of degree $\lambda_i - j$ in x_i. Dividing f by $\mathrm{N}(y_i)$ in $\mathbf{F}[m\,V_2]$ yields $f = q\,\mathrm{N}(y_i) + r$ where the remainder r is a polynomial whose degree in y_i is at most $p-1$. Applying σ we have $f = \sigma(f) = \sigma(q)\,\mathrm{N}(y_i) + \sigma(r)$. Since applying σ cannot increase the degree in y_i, we see that $\sigma(r)$ also has degree at most $p-1$ in y_i. By the uniqueness of remainders and quotients we must have $\sigma(r) = r$ and $\sigma(q) = q$, i.e., $q, r \in \mathbf{F}[m\,V_2]^{C_p}$. Since $\lambda_i \geq p$, we see that x_i divides r and so we have $f = q\,\mathrm{N}(y_i) + x_i r'$ with

$q, r' \in \mathbf{F}[m\,V_2]^{C_p}$. By induction $q, r' \in A$ and thus $f \in A$. Also by induction we have that $\mathrm{LM}(q)$ and $\mathrm{LM}(r')$, hence also $\mathrm{LM}(f)$ may be written as products with factors from $\mathrm{LM}(\mathcal{B})$.

Therefore, we may assume that $\lambda_i < p$ for all $i = 1, 2, \ldots, m$. Then $\kappa = \lambda_1!\lambda_2! \cdots \lambda_m! \neq 0$. Define

$$F = \mathcal{P}(f) \in \mathbf{F}[d\,V_2]^{C_p}_{(1,1,\ldots,1)} = (\otimes_{i=1}^d V_2)^{C_p}.$$

At this point we want to fix some notation. We will use $\{x_{ij}, y_{ij} \mid 1 \leq i \leq m, 1 \leq j \leq \lambda_i\}$ as co-ordinate variables for $\lambda_1 V_2 \oplus \lambda_2 V_2 \oplus \cdots \oplus \lambda_m V_2$. We write $u_{ij,k\ell} = x_{ij}y_{k\ell} - x_{k\ell}y_{ij}$. We use a graded reverse lexicographic order on $\mathbf{F}[\oplus_{i=1}^m \lambda_i V_2]$ after ordering these variables such that the following conditions hold

- $y_{ij} > x_{ij}$,
- if $i < k$ then $y_{ij} > y_{k\ell}$ and $x_{ij} > x_{k\ell}$,
- if $j < \ell$ then $y_{ij} > y_{i\ell}$ and $x_{ij} > x_{i\ell}$.

We will first show that $\mathcal{B}$ generates $\mathbf{F}[m\,V_2]^{C_p}$ as an $\mathbf{F}$-algebra and then show that it is a SAGBI basis. Of course, the former statement follows from the latter but we include a separate proof of the former since the proof is short and illustrates the main idea we will need for the latter proof.

By Theorem 5.5, we may write

$$F = \sum_I \alpha_I \prod_{ij} x_{ij} \prod_{ij,k\ell} u_{ij,k\ell} \prod_E \mathrm{Tr}(\textstyle\prod_{ij} y_{ij}^{e_{ij}}) \ .$$

Let $e_i = \sum_j e_{ij}$.

$$\begin{aligned}
f &= \kappa^{-1}\mathcal{R}(\mathcal{P}(f)) = \kappa^{-1}\mathcal{R}(F) \\
&= \kappa^{-1}\mathcal{R}\left(\sum_I \alpha_I \prod_{ij} x_{ij} \prod_{ij,k\ell} u_{ij,k\ell} \prod_E \mathrm{Tr}(\textstyle\prod_{ij} y_{ij}^{e_{ij}})\right) \\
&= \kappa^{-1}\sum_I \alpha_I \prod_{ij} \mathcal{R}(x_{ij}) \prod_{ij,k\ell} \mathcal{R}(u_{ij,k\ell}) \prod_E \mathcal{R}(\mathrm{Tr}(\prod_{ij} y_{ij}^{e_{ij}})) \\
&= \kappa^{-1}\sum_I \alpha_I \prod_{ij} x_i \prod_{ij,k\ell} u_{ik} \prod_E \mathrm{Tr}(\prod_i y_i^{e_i}) \in A
\end{aligned}$$

where the last equality follows from the following equalities

$$\begin{aligned}
\mathcal{R}(\mathrm{Tr}(y^E)) &= \mathcal{R}(\sum_{\tau \in C_p} \tau(y^E)) = \sum_{\tau \in C_p} \mathcal{R}(\tau(y^E)) = \sum_{\tau \in C_p} \tau(\mathcal{R}(y^E)) \\
&= \mathrm{Tr}(\mathcal{R}(y^E)) \ .
\end{aligned}$$

This completes the proof that $\mathcal{B}$ generates $\mathbf{F}[m\,V_2]^{C_p}$ as an $\mathbf{F}$-algebra. We continue with the proof that $\mathcal{B}$ is a SAGBI basis. First we prove a lemma relating our term orders and polarisation.

Lemma 6.1. *Suppose γ_1, γ_2 are two monomials in $\mathbf{F}[m\,V_2]_{(\lambda_1,\lambda_2,\ldots,\lambda_m)}$ with $\gamma_1 > \gamma_2$. Then* $\mathrm{LT}(\mathcal{P}(\gamma_1)) > \mathrm{LT}(\mathcal{P}(\gamma_2))$.

Proof. Write $\gamma_1 = \prod_{i=1}^m x_i^{a_i} y_i^{\lambda_i - a_i}$ and $\gamma_2 = \prod_{i=1}^m x_i^{b_i} y_i^{\lambda_i - b_i}$. Choose s such that $a_s \neq b_s$ but $a_{s+1} = b_{s+1}, \ldots, a_m = b_m$. Since $\gamma_1 > \gamma_2$ we must have $b_s > a_s$.

Now

$$\mathrm{LT}(\mathcal{P}(\gamma_1)) = \prod_{i=1}^{m}\prod_{j=1}^{a_i} x_{ij} \prod_{j=a_i+1}^{\lambda_i} y_{ij} \text{ and } \mathrm{LT}(\mathcal{P}(\gamma_2)) = \prod_{i=1}^{m}\prod_{j=1}^{b_i} x_{ij} \prod_{j=b_i+1}^{\lambda_i} y_{ij}\ .$$

Writing

$$\Gamma_1 = \prod_{i=1}^{s-1}\prod_{j=1}^{a_i} x_{ij} \prod_{j=a_i+1}^{\lambda_i} y_{ij}, \qquad \Gamma_2 = \prod_{i=1}^{s-1}\prod_{j=1}^{b_i} x_{ij} \prod_{j=b_i+1}^{\lambda_i} y_{ij}$$

$$\text{and} \qquad \Gamma_0 = \prod_{i=s+1}^{m}\prod_{j=1}^{a_i} x_{ij} \prod_{j=a_i+1}^{\lambda_i} y_{ij}$$

we have

$$\mathrm{LT}(\mathcal{P}(\gamma_1)) = \Gamma_0\Gamma_1 \prod_{j=1}^{a_s} x_{sj} \prod_{j=a_s+1}^{\lambda_s} y_{sj}$$

and

$$\mathrm{LT}(\mathcal{P}(\gamma_2)) = \Gamma_0\Gamma_2 \prod_{j=1}^{b_s} x_{sj} \prod_{j=b_s+1}^{\lambda_s} y_{sj}.$$

Since $a_s < b_s$ we see that $\mathrm{LT}(\mathcal{P}(\gamma_1)) > \mathrm{LT}(\mathcal{P}(\gamma_2))$. □

Write $f = \gamma_1 + \gamma_2 + \cdots + \gamma_s$ where each γ_i is a term and $\mathrm{LM}(f) = \mathrm{LT}(f) = \gamma_1$ since f was assumed to be monic. Define $F = \mathcal{P}(f)$. By Lemma 6.1, $\mathrm{LM}(F) = \mathrm{LM}(\mathcal{P}(\gamma_1))$. Furthermore, each monomial of $\mathcal{P}(\gamma_1)$ restitutes to γ_1. In particular, $\mathcal{R}(\Gamma_1) = \gamma_1$ where $\Gamma_1 = \mathrm{LM}(F)$. By Proposition 5.5(4), we may write

$$\begin{aligned}\Gamma_1 = \mathrm{LM}(F) &= \mathrm{LM}\left(\prod_{ij} x_{ij} \prod_{ij,k\ell} u_{ij,k\ell} \prod_{E} \mathrm{Tr}(\textstyle\prod_{ij} y_{ij}^{e_{ij}})\right)\\ &= \prod_{ij} x_{ij} \prod_{ij,k\ell} \mathrm{LM}(u_{ij,k\ell}) \prod_{E} \mathrm{LM}(\mathrm{Tr}(\prod_{ij} y_{ij}^{e_{ij}})).\end{aligned}$$

Restituting we find

$$\begin{aligned}\gamma_1 = \mathcal{R}(\Gamma_1) &= \mathcal{R}\left(\prod_{ij} x_{ij} \prod_{ij,k\ell} \mathrm{LM}(u_{ij,k\ell}) \prod_{E} \mathrm{LM}(\mathrm{Tr}(\textstyle\prod_{ij} y_{ij}^{e_{ij}}))\right) \\ &= \prod_{ij} \mathcal{R}(x_{ij}) \prod_{ij,k\ell} \mathcal{R}(\mathrm{LM}(u_{ij,k\ell})) \prod_{E} \mathcal{R}(\mathrm{LM}(\mathrm{Tr}(\textstyle\prod_{ij} y_{ij}^{e_{ij}}))) \\ &= \prod_{ij} x_i \prod_{ij,k\ell} \mathrm{LM}(u_{i,k}) \prod_{E} \mathrm{LM}(\mathrm{Tr}(\textstyle\prod_{i} y_i^{\sum_j e_{ij}}))\end{aligned}$$

where the last equality follows using Lemma 6.2 below. Thus $\mathrm{LM}(f)$ may be written as a product of factors from $\mathrm{LM}(\mathcal{B})$. This shows that $\mathcal{B}$ is a SAGBI basis for $\mathbf{F}[m\,V_2]^{C_p}$.

Lemma 6.2. *Let $y^E = \prod_{i=1}^{m} \prod_{j=1}^{\lambda_i} y_{ij}^{e_{ij}}$ where $e_{ij} \in \{0,1\}$ for all i, j. Let $e_i = \sum_{j=1}^{\lambda_i} e_{ij}$. If $e_i < p$ for all $i = 1, 2, \ldots, m$ then*

$$\mathcal{R}\left(\mathrm{LM}(\mathrm{Tr}(y^E))\right) = \mathrm{LM}\left(\mathrm{Tr}(\mathcal{R}(y^E))\right) .$$

Proof. Let s be minimal such that $e_1 + e_2 + \cdots + e_s \geq p-1$. (If no such s exists then $\mathrm{Tr}(y^E) = 0$ and $\mathrm{Tr}(\mathcal{R}(y^E)) = 0$.) Let r be minimal such that $e_1 + e_2 + \cdots + e_{s-1} + e_{s1} + e_{s2} + \cdots + e_{sr} = p-1$. By Lemma 2.3

$$\mathrm{LM}(\mathrm{Tr}(y^E)) = \left(\prod_{i=1}^{s-1} \prod_{j=1}^{\lambda_i} x_{ij}^{e_{ij}}\right) \prod_{j=1}^{r} x_{sj}^{e_{sj}} \prod_{j=r+1}^{\lambda_s} y_{sj}^{e_{sj}} \left(\prod_{i=s+1}^{m} \prod_{j=1}^{\lambda_i} y_{ij}^{e_{ij}}\right) .$$

Since $\mathcal{R}(y^E) = \prod_{i=1}^{m} y_i^{e_i}$, again using Lemma 2.3 we see that

$$\mathrm{LM}(\mathrm{Tr}(\mathcal{R}(y^E))) = \left(\prod_{i=1}^{s-1} x_i^{e_i}\right) x_s^t y_s^{e_s - t} \left(\prod_{i=s+1}^{m} y_i^{e_i}\right)$$

where $t = (p-1) - (e_1 + e_2 + \cdots + e_{s-1}) = \sum_{j=1}^{r} e_{ij}$. Thus

$$\mathcal{R}\left(\mathrm{LM}(\mathrm{Tr}(y^E))\right) = \mathrm{LM}\left(\mathrm{Tr}(\mathcal{R}(y^E))\right)$$

as required. □

Theorem 6.3. *The set*

$$\begin{aligned}\mathcal{B}' = \{x_i, \mathrm{N}(y_i) \mid 1 \leq i \leq m\} &\cup \{u_{ij} \mid 1 \leq i < j \leq m\} \\ &\cup \{\mathrm{Tr}(y^E) \mid 0 \leq e_i \leq p-1,\ 2(p-1) < |E|\}\end{aligned}$$

is both a minimal algebra generating set and a SAGBI basis for $\mathbf{F}[m\,V_2]^{C_p}$.

Proof. We start by showing $\mathcal{B}'$ is a SAGBI basis. We need to see why we do not need invariants of the form $\mathrm{Tr}(y^E)$ where $|E| \leq 2(p-1)$ as generators. To see this, consider such a transfer $\mathrm{Tr}(y^E)$. By Lemma 2.3

its lead term is $x_r^{p-1-t+e_r} y_r^{t-p+1} \prod_{i=1}^{r-1} x_i^{e_i} \prod_{i=r+1}^{d} y_i^{e_i}$ where r is minimal such that $t = \sum_{i=1}^{r} e_i \geq p-1$. (We may assume that r exists since if $|E| < p-1$ then $\mathrm{Tr}(y^E) = 0$.)

Write $\mathrm{LM}(\mathrm{Tr}(y^E)) = x_{i_1} x_{i_2} \cdots x_{i_{p-1}} y_{i_p} y_{i_{p+1}} \cdots y_{i_e}$ where $1 \leq i_1 \leq i_2 \leq \cdots \leq i_e \leq m$. Consider $f = \prod_{j=1}^{2p-2-|E|} x_{i_j} \prod_{j=1}^{|E|-(p-1)} u_{i_{p-j}, i_{p-1+j}}$. Then $\mathrm{LM}(f) = \mathrm{LM}(\mathrm{Tr}(y^E))$. Thus $\{\mathrm{LM}(f) \mid f \in \mathcal{B}'\}$ generates the same algebra as $\{\mathrm{LM}(f) \mid f \in \mathcal{B}\}$ which shows that $\mathcal{B}'$ is a SAGBI basis (and hence a generating set) for $\mathbf{F}[m\, V_2]^{C_p}$.

Now we show that $\mathcal{B}'$ is a minimal generating set. It is clear that the elements x_i and u_{ij} cannot be written as polynomials in the other elements of $\mathcal{B}'$. Furthermore, since $\mathrm{LM}(\mathrm{N}(y_i)) = y_i^p$ is the only monomial occuring in any element of $\mathcal{B}'$ which is a pure power of y_i, we see that $\mathrm{N}(y_i)$ is required as a generator. This leaves elements of the form $\mathrm{Tr}(y^E)$ with $|E| > 2(p-1)$. We proceed similarly to the proof of [24, Lemma 4.3]. Assume by way of contradiction that $\mathrm{Tr}(y^E) = \gamma_1 + \gamma_2 + \cdots + \gamma_r$ where each γ_i is a scalar times a product of elements from $\mathcal{B}' \setminus \{\mathrm{Tr}(y^E)\}$ and that $\mathrm{LM}(\gamma_1) \geq \mathrm{LM}(\gamma_2) \cdots \geq \mathrm{LM}(\gamma_r)$. Then $\mathrm{LM}(\mathrm{Tr}(y^E)) \leq \mathrm{LM}(\gamma_1)$. First we suppose that $\mathrm{LM}(\gamma_1) = \mathrm{LT}(\mathrm{Tr}(y^E))$. As above we have

$$\mathrm{LM}(\gamma_1) = \mathrm{LM}(\mathrm{Tr}(y^E)) = x^A y^B = x_r^{p-1-t+e_r} y_r^{t-p+1} \prod_{i=1}^{r-1} x_i^{e_i} \prod_{i=r+1}^{d} y_i^{e_i}$$

where r is minimal such that $t = \sum_{i=1}^{r} e_i \geq p-1$.

Since each $e_i < p$ and $\mathrm{LM}(\mathrm{N}(y_i)) = y_i^p$ we see that $\mathrm{N}(y_i)$ does not divide γ_1. But then since $|A| = p-1$ we see that $|A| < |E| - |A| = |B|$ and thus there must be at least one transfer which divides γ_1. Conversely since $|A| = p-1$ exactly one transfer (to the first power) may divide γ_1. But then the lead monomials of the other factors must divide y^B and no element of $\mathcal{B}'$ has a lead monomial satisfying this constraint. This shows that for $|E| > 2(p-1)$, the monomial $\mathrm{LM}(\mathrm{Tr}(y^E))$ cannot be properly factored using lead monomials from $\mathcal{B}'$.

Therefore we must have $\mathrm{LM}(\gamma_1) > \mathrm{LM}(\mathrm{Tr}(y^E))$ (and $\mathrm{LM}(\gamma_1) = \mathrm{LM}(\gamma_2)$). Since we may assume that each term of each γ_i is homogeneous of degree E, we may write $\mathrm{LM}(\gamma_1) = x^C y^D$ where $C + D = E$. But $\mathrm{LM}(\mathrm{Tr}(y^E)) = x^A y^B$ is the biggest monomial in degree E which satisfies $|A| \geq p-1$. Hence $\mathrm{LM}(\gamma_1) > \mathrm{LM}(\mathrm{Tr}(y^E))$ implies that $|C| < p-1$. Therefore γ_1 must be a product of elements of the form x_i, u_{ij} and $\mathrm{N}(y_i)$ from $\mathcal{B}'$. As above, since each $e_i < p$, no $\mathrm{N}(y_i)$ can divide γ_1. But then $\mathrm{LM}(\gamma_1)$ is a product of factors of the form x_i

and $\mathrm{LM}(u_{ij}) = x_i y_j$ and this forces $|C| \geq |D| = |E| - |C|$. Therefore $2(p-1) > 2|C| \geq |E|$. This contradiction shows that we cannot express $\mathrm{Tr}(y^E)$ as a polynomial in the other elements of $\mathcal{B}'$ when $|E| > 2(p-1)$. $\square$

7. Decomposing $\mathbf{F}[m\,V_2]$ as a C_p-module

In this section we show that our techniques give a decomposition of the homogeneous component

$$\mathbf{F}[m\,V_2]_{(d_1,d_2,\ldots,d_m)}$$

as a C_p-module. We will describe $\mathbf{F}[m\,V_2]_{(d_1,d_2,\ldots,d_m)}$ modulo projectives, i.e., we compute the multiplicities of the indecomposable summands V_k of this component for which $k < p$. Having done this, a simple dimension computation will give the complete decomposition.

By the Periodicity Theorem (Theorem 2.1), we may assume that each $d_i < p$. Let $d = d_1 + d_2 + \cdots + d_m$. The symmetric group on d letters, Σ_d, acts on $\otimes^d V_2$ by permuting the factors. This action commutes with the action of C_p (in fact with the action of all of $GL(V_2)$). The image of the polarization map consists of those tensors which are fixed by the Young subgroup $Y = \Sigma_{d_1} \times \Sigma_{d_2} \times \cdots \times \Sigma_{d_m}$ of Σ_d. Since each $d_i < p$, we see that Y is a non-modular group. Maschke's Theorem then implies that polarization embeds $\mathbf{F}[m\,V_2]_d$ into $\otimes^d V_2$ as a C_p-*summand*. Therefore $\ell(\mathcal{P}(f)) = \ell(f)$ for all $f \in \mathbf{F}[m\,V_2]^{C_p}_{(d_1,d_2,\ldots,d_m)}$ and $\ell(\mathcal{R}(F)) = \ell(F)$ for all $F \in (\otimes^d V_2)^{C_p \times Y}$.

Using the relations given in Section 2.2, it is straightforward to write down a basis, consisting of products of u_{ij}'s and x_i's, for the invariants in multi-degree $(d_1, d_2, \ldots, d_m)$ which lie in the subring generated by $\{x_i \mid 1 \leq i \leq m\} \cup \{u_{i,j} \mid 1 \leq i < j \leq m\}$. Associated to the lead term of each invariant in this basis is an indecomposable summand of $\mathbf{F}[m\,V_2]_{(d_1,d_2,\ldots,d_m)}$. The dimension of this summand may be found using Theorem 5.5. More directly, consider a product of u_{ij}'s and x_i's, say

$$f := \prod_{i=1}^{m} x_i^{a_i} \cdot \prod_{1 \leq i < j \leq m} u_{i,j}^{b_{i,j}} \in \mathbf{F}[m\,V_2]^{C_p} .$$

It is not too difficult to show that $\mathrm{LT}(f)$ is the lead term of an element of the transfer if and only if there exists r with $1 \leq r \leq m$ such that

$$\sum_{i=1}^{r} a_i + \sum_{\substack{1 \leq i \leq r \leq j \leq m \\ i<j}}^{r} b_{ij} \geq p - 1 .$$

If no such r exists then $\ell(f) = 1 + \sum_{i=1}^{m} a_i$ gives the dimension of the associated summand.

Rather than working with the invariants lying in $\mathbf{F}[m\,V_2]$ directly, one may instead use Theorem 5.5 to decompose $\otimes^d V_2$. It is then possible to perturb this decomposition so that it is a refinement of the splitting given by polarisation/restitution and thus gives a decomposition of $\mathbf{F}[m\,V_2]_{(d_1,\ldots,d_m)}$.

Example 7.1. As an example we compute the decomposition of

$$\mathbf{F}[4\,V_2]_{(p+1,1,1,p+2)}.$$

This space has dimension $(p+2)(2)(2)(p+3) = 4p^2 + 20p + 24$. By Theorem 2.1, we know

$$\mathbf{F}[4\,V_2]_{(p+1,1,1,p+2)} \cong \mathbf{F}[4\,V_2]_{(1,1,1,2)} \oplus (4p+20)V_p$$

and we need to compute the decomposition of

$$\mathbf{F}[4\,V_2]_{(1,1,1,2)} = V_2 \otimes V_2 \otimes V_2 \otimes S^2(V_2).$$

We have available the invariants x_1, x_2, x_3, x_4 and $u_{12}, u_{13}, u_{14}, u_{23}, u_{24}, u_{34}$. Suppose now that $p \geq 7$. The products of these 10 invariants which lie in degree $(1,1,1,2)$ are as follows (sorted by length):

$\ell = 2$: $x_4u_{12}u_{34},\ x_4u_{13}u_{24},\ x_4u_{14}u_{23},\ x_1u_{24}u_{34},\ x_2u_{14}u_{34},\ x_3u_{14}u_{24}$
$\ell = 4$: $x_3x_4^2u_{12},\ x_1x_4^2u_{23},\ x_1x_2x_4u_{34},\ x_2x_4^2u_{13},\ x_1x_3x_4u_{24},\ x_2x_3x_4u_{14}$
$\ell = 6$: $x_1x_2x_3x_4^2$

Consider the invariants of length 2. Among the available relations for those of length 2 we have:

$$\begin{aligned}
0 &= x_4(u_{12}u_{34} - u_{13}u_{24} + u_{14}u_{23}),\\
0 &= u_{34}(x_1u_{24} - x_2u_{14} + x_4u_{23}), \text{ and}\\
0 &= u_{14}(x_2u_{34} - x_3u_{24} + x_4u_{23}).
\end{aligned}$$

Using these three relations we see that the three invariants

$$x_4u_{13}u_{24}, \quad x_2u_{14}u_{34}, \quad x_3u_{14}u_{24}$$

may be expressed in terms of the other three invariants

$$x_4u_{12}u_{34}, \quad x_4u_{14}u_{23}, \quad x_1u_{24}u_{34}.$$

Furthermore there are no relations involving only these latter three invariants and thus they represent the socles of 3 summands isomorphic to V_2.

Among the available relations involving invariants of length 4 we have

$$\begin{aligned}
0 &= x_4^2(x_1u_{23} - x_2u_{13} + x_3u_{12}),\\
0 &= x_1x_4(x_2u_{34} - x_3u_{24} + x_4u_{23}), \text{ and}\\
0 &= x_3x_4(x_1u_{24} - x_2u_{14} + x_4u_{12}).
\end{aligned}$$

These allow us to express the three invariants

$$x_2x_4^2u_{13}, \quad x_1x_3x_4u_{24}, \quad x_2x_3x_4u_{14}$$

using only

$$x_3x_4^2u_{12}, \quad x_1x_4^2u_{23}, \quad x_1x_2x_4u_{34}.$$

Again these there are no relations involving only these latter 3 invariants and so they represent the socles of 3 summands isomorphic to V_4.

Since $x_1x_2x_3x_4^2$ spans the socle of a summand isomorphic to V_6 we conclude that

$$\mathbf{F}[4\,V_2]_{(1,1,1,2)} \cong 3\,V_2 \oplus 3\,V_4 \oplus V_6 \quad \text{for } p \geq 7.$$

For $p = 5$, the foregoing is all correct except that the lattice paths corresponding to $x_1x_2x_3x_4^2$ and $x_1x_2x_3x_4y_4 = \mathrm{LT}(x_1x_2u_{34}x_4)$ both attain height $p - 1 = 4$. Thus in this case these two invariants both represent a projective summand and we have the decomposition

$$\mathbf{F}[4\,V_2]_{(1,1,1,2)} \cong 3\,V_2 \oplus 2\,V_4 \oplus 2\,V_5 \quad \text{for } p = 5.$$

For $p = 2, 3$ all the relevant lattice paths attain height $p - 1$ and so the summand is projective. Thus

$$\begin{aligned}
\mathbf{F}[4\,V_2]_{(1,1,1,2)} &\cong 8\,V_3 \quad \text{for } p = 3, \text{ and}\\
\mathbf{F}[4\,V_2]_{(1,1,1,2)} &\cong 12\,V_2 \quad \text{for } p = 2.
\end{aligned}$$

We will also illustrate how to use the decomposition of $\otimes^5 V_2$ to find the decomposition of $\mathbf{F}[4\,V_2]_{(1,1,1,2)}$. By the results of Section 5, we have $\otimes^5 V_2 \cong 5\,V_2 \oplus 4\,V_4 \oplus V_6$ for $p \geq 7$. Here the lead monomials are

$\ell = 2$: $x_1y_2x_3y_4x_5,\ x_1x_2y_3y_4x_5,\ x_1y_2x_3x_4y_5,\ x_1x_2y_3x_4y_5,\ x_1x_2x_3y_4y_5$
$\ell = 4$: $x_1y_2x_3x_4x_5,\ x_1x_2y_3x_4x_5,\ x_1x_2x_3y_4x_5,\ x_1x_2x_3x_4y_5$
$\ell = 6$: $x_1x_2x_3x_4x_5$

and the corresponding invariants are

$\ell = 2$: $x_5u_{12}u_{34},\ x_5u_{14}u_{23},\ x_4u_{12}u_{35},\ x_1u_{23}u_{45},\ x_1u_{25}u_{34}$
$\ell = 4$: $x_3x_4x_5u_{12},\ x_1x_4x_5u_{23},\ x_1x_2x_5u_{34},\ x_1x_2x_3u_{45}$
$\ell = 6$: $x_1x_2x_3x_4x_5$

The Young subgroup $Y := \Sigma_1 \times \Sigma_1 \times \Sigma_1 \times \Sigma_2$ acts by simultaneously interchanging x_4 with x_5 and y_4 with y_5. Clearly the action preserves length. The $C_p \times Y$ invariants are

$\ell = 2$: $x_5u_{12}u_{34} + x_4u_{12}u_{35},\ x_5u_{14}u_{23} + x_4u_{15}u_{23},\ x_4u_{12}u_{35} + x_5u_{12}u_{34},$
$x_1u_{23}u_{45} + x_1u_{23}u_{54} = 0,\ x_1u_{25}u_{34} + x_1u_{24}u_{35}$

$\ell = 4$: $x_3x_4x_5u_{12},\ x_1x_4x_5u_{23},\ x_1x_2x_5u_{34} + x_1x_2x_4u_{35},$
$x_1x_2x_3u_{45} + x_1x_2x_3u_{54} = 0$

$\ell = 6$: $x_1x_2x_3x_4x_5$

We now restitute these $C_p \times Y$ invariants to $\mathbf{F}[4\,V_2]^{C_p}_{(1,1,1,2)}$. We find

$$\begin{aligned}
\mathcal{R}(x_5u_{12}u_{34} + x_4u_{12}u_{35}) &= 2x_4u_{12}u_{34},\\
\mathcal{R}(x_5u_{14}u_{23} + x_4u_{15}u_{23}) &= 2x_4u_{14}u_{23},\\
\mathcal{R}(x_1u_{25}u_{34} + x_1u_{24}u_{35}) &= 2x_1u_{24}u_{34}.
\end{aligned}$$

Thus we find 3 summands of $\mathbf{F}[4\,V_2]_{(1,1,1,2)}$ isomorphic to V_2.

Restituting the invariants of length 4 we find

$$\begin{aligned}
\mathcal{R}(x_3x_4x_5u_{12}) &= x_3x_4^2u_{12},\\
\mathcal{R}(x_1x_4x_5u_{23}) &= x_1x_4^2u_{23}, \text{ and}\\
\mathcal{R}(x_1x_2x_5u_{34} + x_1x_2x_4u_{35}) &= 2x_1x_2x_4u_{34}.
\end{aligned}$$

Thus we have 3 summands isomorphic to V_4. Since $\mathcal{R}(x_1x_2x_3x_4x_5) = x_1x_2x_3x_4^2$, we see that

$$\mathbf{F}[4\,V_2]_{(1,1,1,2)} \cong 3\,V_2 \oplus 3\,V_4 \oplus V_6 \quad \text{for } p \geq 7.$$

For $p = 2, 3, 5$, the lengths of the above invariants change and we must adjust our conclusions accordingly as we did earlier. For $p = 2$ we must also use the Periodicity Theorem again since $d_4 = 2 = p$.

8. A First Main Theorem for $SL_2(\mathbf{F}_p)$

The purpose of this section is to use the relative transfer homomorphism to describe the ring of vector invariants, $\mathbf{F}[m\,V_2]^{SL_2(\mathbf{F}_p)}$. Let P denote the upper triangular Sylow p-subgroup of $SL_2(\mathbf{F}_p)$, giving $\mathrm{N}(y) = \mathrm{N}^P(y) = y^p - yx^{p-1}$. The ring of invariants of the defining representation of $SL_2(\mathbf{F}_p)$ is generated by $L = x\,\mathrm{N}(y)$ and $D = \mathrm{N}(y)^{p-1} + x^{p(p-1)}$ (see Dickson [10], Wilkerson [29], or Benson [1, §8.1]). For $\lambda \in \mathbb{N}^m$, define $L_\lambda = \pi_\lambda\nabla_m(L)$ and $D_\lambda = \pi_\lambda\nabla_m(D)$, the multi-degree λ polarisations. Further define L_i to be the polarisation of L corresponding to $\lambda_i = p+1$ and $\lambda_j = 0$ for $j \neq i$. It is easy to verify that $L_i = x_iy_i^p - x_i^py_i$ is the Dickson invariant for the i^{th} summand.

Let L_{ij} denote the polarisation corresponding to $\lambda_i = 1$, $\lambda_j = p$, and $\lambda_k = 0$ otherwise. So, for example, $L_{32} = L_{(0,p,1,0,\ldots,0)}$. Define

$$\mathcal{D}_m = \left\{ \lambda \in \mathbb{N}^m \mid p \text{ divides } \lambda_i \text{ for all } i \text{ and } \sum_{i=1}^{m} \lambda_i = p(p-1) \right\}.$$

Further define

$$\mathcal{S}_m = \{u_{ij} \mid i < j \leq m\} \cup \{L_i, L_{ij} \mid i, j \in \{1, \ldots, m\}, i \neq j\} \\ \cup \{D_\lambda \mid \lambda \in \mathcal{D}_m\}.$$

Theorem 8.1. *The ring of vector invariants, $\mathbf{F}[m V_2]^{SL_2(\mathbf{F}_p)}$, is generated by $\mathcal{S}_m$ and elements from the image of the transfer.*

Note that the elements of $\mathcal{S}_m$ are clearly $SL_2(\mathbf{F}_p)$-invariant and include a system of parameters. Let A denote the algebra generated by $\mathcal{S}_m$ and let $\mathfrak{a}$ denote the ideal in $\mathbf{F}[m V_2]^P$ generated by $\mathcal{S}_m$. A basis for the finite dimensional vector space $\mathbf{F}[m V_2]^P/\mathfrak{a}$ lifts to a set of A-module generators for $\mathbf{F}[m V_2]^P$, say $\mathcal{M}$. Since the relative transfer homomorphism is a surjective A-module morphism, $\mathbf{F}[m V_2]^{SL_2(\mathbf{F}_p)}$ is generated by $\mathcal{S}_m \cup \mathrm{Tr}_P^{SL_2(\mathbf{F}_p)}(\mathcal{M})$. The elements of $\mathcal{M}$ may be chosen to be monomials in the generators of $\mathbf{F}[m V_2]^P$. Since we are working modulo the image of the transfer, it is sufficient to consider monomials of the form $\mathrm{N}(y)^\alpha x^\beta$.

Let $\mathfrak{u}$ denote the ideal in $\mathbf{F}[m V_2]^P$ generated by $\{u_{ij} \mid i < j \leq m\}$.

Lemma 8.2. *For $i < j \leq m$, $L_{ij} = x_i \mathrm{N}(y_j) + u_{ij} x_j^{p-1}$ and $L_{ji} = x_j \mathrm{N}(y_i) - u_{ij} x_i^{p-1}$, giving $L_{ij} \equiv_{\mathfrak{u}} x_i \mathrm{N}(y_j)$ and $L_{ji} \equiv_{\mathfrak{u}} x_j \mathrm{N}(y_i)$.*

Proof. Applying ∇_m to L gives

$$(x_1 + \cdots + x_m) \left(y_1^p + \cdots + y_m^p - (y_1 + \cdots + y_m)(x_1 + \cdots + x_m)^{p-1} \right).$$

Expanding gives

$$(x_1 + \cdots + x_m)(y_1^p + \cdots + y_m^p) - (y_1 + \cdots + y_m)(x_1 + \cdots + x_m)^p.$$

Collecting the appropriate multi-degrees gives $L_{ij} = x_i y_j^p - y_i x_j^p$ and $L_{ji} = x_j y_i^p - y_j x_i^p$. Using $u_{ij} = x_i y_j - x_j y_i$ and $\mathrm{N}(y) = y^p - x^{p-1} y$ gives

$$x_i \mathrm{N}(y_j) + u_{ij} x_j^{p-1} = x_i y_j^p - x_i y_j x_j^{p-1} + x_i y_j x_j^{p-1} - x_j^p y_i = L_{ij}$$

and

$$x_j \mathrm{N}(y_i) - u_{ij} x_i^{p-1} = x_j y_i^p - x_j y_i x_i^{p-1} - y_j x_i^p + x_i^{p-1} x_j y_i = L_{ji}.$$

□

Since $\mathfrak{u} \subset \mathfrak{a}$, the preceding lemma and the formula $L_i = x_i \,\mathrm{N}(y_i)$ show that it is sufficient to compute $\mathrm{Tr}_P^{SL_2(\mathbf{F}_p)}$ on monomials of the form $\mathrm{N}(y)^\alpha$ or x^β.

Let B denote the Borel subgroup containing P, i.e., the upper triangular elements of $SL_2(\mathbf{F}_p)$. Define a weight function on $\mathbf{F}[m\,V_2]$ by $\mathrm{wt}(x_i) \equiv_{(p-1)} 1$ and $\mathrm{wt}(y_i) \equiv_{(p-1)} -1$. Note that $\mathrm{N}(y_i)$ is isobaric of weight -1. Furthermore, $\mathbf{F}[m\,V_2]^B$ consists of the span of the the weight zero elements of $\mathbf{F}[m\,V_2]^P$. The relative transfer Tr_P^B is determined by weight:

$$\mathrm{Tr}_P^B\left(\mathrm{N}(y)^\alpha x^\beta\right) = \begin{cases} -\,\mathrm{N}(y)^\alpha x^\beta, & \text{if } (|\beta| - |\alpha|) \equiv_{(p-1)} 0; \\ 0, & \text{otherwise.} \end{cases}$$

Since $\mathrm{Tr}_P^{SL_2(\mathbf{F}_p)} = \mathrm{Tr}_B^{SL_2(\mathbf{F}_p)}\,\mathrm{Tr}_P^B$, it is sufficient to compute $\mathrm{Tr}_B^{SL_2(\mathbf{F}_p)}$ on $\mathrm{N}(y)^\alpha$ with $|\alpha|$ a multiple of $p-1$ and x^β with $|\beta|$ a multiple of $p-1$. However, if $|\beta| \geq p-1$, then $x^\beta \in \mathrm{Tr}^P(\mathbf{F}[m\,V_2])$ and $\mathrm{Tr}_P^{SL_2(\mathbf{F}_p)}(x^\beta) \in \mathrm{Tr}^{SL_2(\mathbf{F}_p)}(\mathbf{F}[m\,V_2])$. Thus is is sufficient to compute $\mathrm{Tr}_B^{SL_2(\mathbf{F}_p)}(\mathrm{N}(y)^\alpha)$ with $|\alpha|$ a multiple of $p-1$.

For $\lambda \in \mathcal{D}_m$, define

$$\mathrm{N}(y)^{\lambda/p} = \prod_{i=1}^{m} \mathrm{N}(y_i)^{\lambda_i/p}.$$

Lemma 8.3. *$\nabla_m(D) \equiv_{\mathfrak{u}} (\mathrm{N}(y_1)+\cdots+\mathrm{N}(y_m))^{p-1}+(x_1+\cdots+x_m)^{p(p-1)}$, giving $D_\lambda \equiv_{\mathfrak{u}} \binom{p-1}{\lambda/p}(\mathrm{N}(y)^{\lambda/p} + x^\lambda)$ and $\mathrm{N}(y)^{\lambda/p} \equiv_{\mathfrak{a}} -x^\lambda$.*

Proof. The proof is by induction on m. Note that $\nabla_m = \nabla^{m-1}$. Thus $\nabla_1(D) = \nabla^0(D) = D = \mathrm{N}(y)^{p-1} + x^{p(p-1)}$, as required. Recall that the action of ∇ on $\mathbf{F}[m\,V_2]$ is determined by $\nabla(x_m) = x_m + x_{m+1}$, $\nabla(y_m) = y_m + y_{m+1}$, $\nabla(x_i) = x_i$, and $\nabla(y_i) = x_i$ for $i < m$. Thus $\nabla(u_{ij}) = u_{ij}$ if $i < j < m$ and $\nabla(u_{im}) = y_i(x_m + x_{m+1}) - x_i(y_m + y_{m+1}) = u_{im} + u_{i,m+1}$. Therefore ∇ induces an algebra morphism on $\mathbf{F}[m\,V_2]^P/\mathfrak{u}$. Furthermore $\nabla(\mathrm{N}(y_i)) = \mathrm{N}(y_i)$ if $i < j < m$ and $\nabla(\mathrm{N}(y_m)) = y_m^p + y_{m+1}^p - (x_m + x_{m+1})^{p-1}(y_m + y_{m+1}) = \mathrm{N}(y_m) + \mathrm{N}(y_{m+1}) - u_{m,m+1}\sum_{j=0}^{p-2}(-x_m)^j x_{m+1}^{p-2-j}$. By induction,

$$\begin{aligned}\nabla_{m+1}(D) = \nabla(\nabla_m(D)) \in \nabla((&\mathrm{N}(y_1) + \cdots + \mathrm{N}(y_m))^{p-1} \\ &+ (x_1 + \cdots + x_m)^{p(p-1)} + \mathfrak{u}).\end{aligned}$$

Evaluating the algebra morphism ∇ gives

$$\begin{aligned}\nabla_{m+1}(D) &\in (\nabla(\mathrm{N}(y_1)) + \cdots + \nabla(\mathrm{N}(y_m)))^{p-1} + (x_1 + \cdots + x_{m+1})^{p(p-1)}\\ &\quad + \nabla(\mathfrak{u})\\ &\in (\mathrm{N}(y_1) + \cdots + \mathrm{N}(y_{m+1}))^{p-1} + (x_1 + \cdots + x_{m+1})^{p(p-1)} + \mathfrak{u},\end{aligned}$$

as required. $\square$

Using the lemma, if $p-1$ divides $|\alpha|$ then $\mathrm{Tr}_B^{SL_2(\mathbf{F}_p)}(\mathrm{N}(y)^\alpha)$ is decomposable modulo the image of the transfer, completing the proof of Theorem 8.1

To complete the calculation of a generating set for $\mathbf{F}[m\,V_2]^{SL_2(\mathbf{F}_p)}$ and compute an upper bound for the Noether number, we need only identify a set of A-module generators for $\mathbf{F}[m\,V_2]$. This can be done by applying the Buchberger algorithm to $\mathcal{S}_m$. For example, a Magma [2] calculation for $m = 3$ and $p = 3$, produces 522 A-module generators giving rise to 74 non-zero elements in the image of the transfer. Subducting the transfers against $\mathcal{S}_m$ gives 11 new generators and 29 in total. Magma's *MinimalAlgebraGenerators* command reduces the number of generators to 28, occuring in degrees 2, 4, 6 and 8. The same calculation for $p = 5$ and $m = 3$ gives a Noether number of 24. Thus for $p \in \{3, 5\}$ and $m = 3$, the Noether number is $(p + m - 2)(p - 1) = (p + 1)(p - 1)$.

Theorem 8.4. $\mathbf{F}[m\,V_2]^{SL_2(\mathbf{F}_p)}$ *is generated as an A-module in degrees less than or equal to $(p + m - 2)(p - 1)$.*

Proof. Define $\mathfrak{a}'$ to be the ideal in $\mathbf{F}[m\,V_2]$ generated by $\mathcal{S}_m$, i.e.,

$$\mathfrak{a}' = A^+\mathbf{F}[m\,V_2].$$

A basis for $\mathbf{F}[m\,V_2]/\mathfrak{a}'$ lifts to a set of A-module generators for $\mathbf{F}[m\,V_2]$. We may choose the A-module generators to be monomials, $y^\alpha x^\beta$, which are minimal representatives of their mod-$\mathfrak{a}'$ congruence class. For convenience, denote $d = |\alpha| + |\beta|$. For $i < j$, using $u_{ij} = x_i y_j - x_j y_i$, if x_i divides $y^\alpha x^\beta$, then y_j does not. For $j \le i$, using L_i and L_{ij}, if x_i divides $y^\alpha x^\beta$, then y_j^p does not. The remaining representatives fall into two classes: y^α and $y_1^{\alpha_1} \cdots y_k^{\alpha_k} x_k^{\beta_k} \cdots x_m^{\beta_m}$ with $\beta_k \ne 0$ and $\alpha_i \le p - 1$.

Case 1: y^α. Using D_λ with $\lambda \in \mathcal{D}_m$, we see that, for $|\gamma| \ge p-1$, $(y^\gamma)^p$ does not divide y^α. Write $\alpha_i = q_i p + r_i$ with $r_i < p$. Then $y^\alpha = (y^q)^p y^r$ with $|q| \le p - 2$. Thus $|\alpha| = p|q| + |r| \le p(p-2) + m(p-1) = (p + m - 1)(p - 1) - 1$. However, $\mathrm{Tr}(y^\alpha) = 0$ unless $p - 1$ divides $|\alpha|$. Therefore, the A-module generators of the form $\mathrm{Tr}(y^\alpha)$ satisfy $d = |\alpha| \le (p + m - 2)(p - 1)$.

Case 2: $y_1^{\alpha_1} \cdots y_k^{\alpha_k} x_k^{\beta_k} \cdots x_m^{\beta_m}$ with $\beta_k \ne 0$ and $\alpha_i \le p - 1$. For $i < j$, let x^γ be a monomial in $x_1, \ldots, x_{j-1}$. If $|\gamma| = p - 1$, then

$x^\gamma L_{ij} = x^\gamma(x_i y_j^p - x_j^p y_i) \equiv_{\mathfrak{u}} y^\gamma y_i x_j^p - x^\gamma y_i x_j^p$. Therefore, if $\beta_j \geq p$ for any $j > k$, then $|\alpha| < p$. If $|\gamma| \leq p-1$ then $x^\gamma L_j = x^\gamma(x_j y_j^p - x_j^p y_j) \equiv_{\mathfrak{u}} y^\gamma y_j^{p-|\gamma|} x_j^{|\gamma|+1} - x^\gamma y_j x_j^p$. Therefore, if $\beta_k \geq p$, we also have $|\alpha| < p$. If $|\beta_j| < p$ for all $j \geq k$, then $|\alpha| + |\beta| \leq (m-k+2)(p-1) \leq (p+m-2)(p-1)$ if $k > 1$. Hence it is sufficient to consider the case $|\alpha| < p$. However the transfer is zero unless $p-1$ divides $|\alpha|$ so we may assume $|\alpha| = p-1$. If $|\alpha| = p-1$, a straightforward calculation with binomial coefficients gives $\mathrm{Tr}^P(y^\alpha x^\beta) = -x^{\alpha+\beta}$. Furthermore, $\mathrm{Tr}_P^B(x^{\alpha+\beta}) = 0$ unless $p-1$ divides $|\alpha|+|\beta|$. Write $\alpha_i+\beta_i = q_i p + r_i$ with $r_i < p$. Then $x^{\alpha+\beta} = (x^q)^p x^r$. If $|q| \geq p-1$ and $|r| > 0$, we may choose i so that $r_i > 0$, choose $\lambda \in \mathcal{D}_m$ so that x^λ divides x^{pq} and choose j so that x_j divides x^λ. By Lemma 8.2, $x_i\,\mathrm{N}(y_j) \in \mathfrak{a}'$. Form the S-polynomial between D_λ and $x_i\,\mathrm{N}(y_j)$. Using Lemma 8.3, this S-polynomial reduces to $x_i x^\lambda$. Thus either $|q| < p-1$ or $|q| = p-1$ and $|r| = 0$. If $|r| = 0$ and $|q| = p-1$, then $d = (p-1)(p-1) \leq (p+m-2)(p-1)$. Suppose $|q| < p-1$. Then $d \leq m(p-1) + p(p-2) = (p+m-1)(p-1) - 1$. Since d must be a multiple of $p-1$, we have $d \leq (p+m-2)(p-1)$. □

Corollary 8.5. *For $m > 2$, the Noether number for $\mathbf{F}[m\,V_2]^{SL_2(\mathbf{F}_p)}$ is less than or equal to $(p+m-2)(p-1)$. For $m = 2$ and $p > 2$, the Noether number is $p(p-1)$ and for $m = 2$, $p = 2$, the Noether number is $p+1 = 3$.*

Proof. The elements of $\mathcal{S}_m$ lie in degrees 2, $p+1$ and $p(p-1)$. Clearly L_1 and $D_{(p(p-1),0,\ldots,0)}$ are indecomposable. □

For $p = 2$ and $m \in \{3, 4\}$, Magma calculations give the Noether number $(p+m-2)(p-1) = m$.

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