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# THE INVARIANTS OF THE SECOND SYMMETRIC POWER REPRESENTATION OF $SL_2(\mathbb{F}_q)$

ASHLEY HOBSON AND R. JAMES SHANK

ABSTRACT. For a prime  $p > 2$  and  $q = p^n$ , we compute a finite generating set for the  $SL_2(\mathbb{F}_q)$ -invariants of the second symmetric power representation, showing the invariants are a hypersurface and the field of fractions is a purely transcendental extension of the coefficient field. As an intermediate result, we show the invariants of the Sylow  $p$ -subgroups are also hypersurfaces.

## 1. INTRODUCTION

Consider the generic binary quadratic form over a field  $\mathbb{F}$  of characteristic not 2:

$$a_0X^2 + 2a_1XY + a_2Y^2.$$

Identifying

$$X = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \text{and} \quad Y = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

induces a left action of the general linear group  $GL_2(\mathbb{F})$  on the second symmetric power

$$V := \text{Span}_{\mathbb{F}}[Y^2, 2XY, X^2]$$

and a right action on the dual  $V^* = \text{Span}_{\mathbb{F}}[a_2, a_1, a_0]$ . For example

$$\sigma_c = \begin{bmatrix} 1 & c \\ 0 & 1 \end{bmatrix} \quad \text{acts on } V^* \quad \text{as} \quad \begin{bmatrix} 1 & 2c & c^2 \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}$$

with  $a_2 = [1 \ 0 \ 0]$ ,  $a_1 = [0 \ 1 \ 0]$ ,  $a_0 = [0 \ 0 \ 1]$ . The action on  $V^*$  extends to an action by algebra automorphisms on the symmetric algebra  $\mathbb{F}[V] = \mathbb{F}[a_2, a_1, a_0]$ . For any subgroup  $G \leq GL_2(\mathbb{F})$ , we denote the subring of invariant polynomials by  $\mathbb{F}[V]^G$ .

Throughout we assume that  $\mathbb{F}$  has characteristic  $p > 2$ ,  $q = p^n$  and  $\mathbb{F}_q \subseteq \mathbb{F}$ . Thus  $SL_2(\mathbb{F}_q) \leq GL_2(\mathbb{F})$ . Our primary goal is to describe

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$\mathbb{F}[V]^{SL_2(\mathbb{F}_q)}$ . Our work generalises and is inspired by L.E. Dickson's solution to the  $q = p$  case [4, Lecture II, §8-9].

Let  $P$  denote the subgroup  $\{\sigma_c \mid c \in \mathbb{F}_q\}$ .  $P$  is a Sylow  $p$ -subgroup of  $SL_2(\mathbb{F}_q)$ . The orbit products

$$\beta := \prod_{c \in \mathbb{F}_q} a_1 P = \prod_{c \in \mathbb{F}_q} (a_1 + ca_0) = a_1^q - a_0^{q-1} a_1$$

and, for  $k \in \mathbb{F}_q$ ,

$$\gamma_k := \prod_{c \in \mathbb{F}_q} (a_2 - ka_0) P = \prod_{c \in \mathbb{F}_q} (a_2 + 2ca_1 + (c^2 - k)a_0)$$

are clearly  $P$ -invariant. The discriminant,  $\Delta := a_1^2 - a_0 a_2$ , is a well-known  $SL_2(\mathbb{F}_q)$ -invariant. In Section 2, we show that  $\mathbb{F}[V]^P$  is the hypersurface generated by  $a_0, \Delta, \beta, \gamma_0$  subject to the relation

$$\beta^2 = a_0^q \gamma_0 + \Delta (\Delta^{\frac{q-1}{2}} - a_0^{q-1})^2.$$

Let  $\mathcal{Q}$  denote the set of quadratic residues in  $\mathbb{F}_q$  and let  $\overline{\mathcal{Q}}$  denote the set of quadratic nonresidues, i.e., if  $\omega$  is a generator for  $\mathbb{F}_q^*$ , then  $\mathcal{Q}$  consists of the even powers of  $\omega$  and  $\overline{\mathcal{Q}}$  consists of the odd powers. Define

$$\begin{aligned} \Gamma &:= \prod_{k \in \overline{\mathcal{Q}}} \gamma_k, \\ B &:= \beta \prod_{k \in \mathcal{Q}} \gamma_k, \\ J &:= a_0 \gamma_0. \end{aligned}$$

In Section 3, we show that  $\mathbb{F}[V]^{SL_2(\mathbb{F}_q)}$  is the hypersurface generated by  $\Delta, J, \Gamma, B$  subject to a relation of the form

$$B^2 = \Delta^q \Gamma^2 + J \Phi(\Delta, J, \Gamma)$$

for some polynomial  $\Phi$ .

Throughout we use the graded reverse lexicographic (grevlex) order with  $a_0 < a_1 < a_2$ . We will see that the given generating sets for  $\mathbb{F}[V]^P$  and  $\mathbb{F}[V]^{SL_2(\mathbb{F}_q)}$  are SAGBI bases with respect to this order. A SAGBI basis is the **S**ubalgebra **A**nalogue of a **G**röbner **B**asis for **I**deals. The concept was introduced independently by Robbiano-Sweedler [9] and Kapur-Madlener [6]; a useful reference is Chapter 11 of Sturmfels [10] (who uses the term *canonical subalgebra basis*). For background material on the invariant theory of finite groups, see Benson [1], Derksen-Kemper [3] or Neusel-Smith [8].

2.  $P$ -INVARIANTS

For  $\omega \in \mathbb{F}_q^*$ , the diagonal matrix

$$\rho_\omega = \begin{bmatrix} \omega & 0 \\ 0 & 1 \end{bmatrix} \text{ acts on } V^* \text{ as } \begin{bmatrix} \omega^2 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

This motivates the definition of a multiplicative weight function on monomials by

$$\text{wt}(a_i) = i.$$

Thus for any monomial  $\beta$ , we have  $(\beta)\rho_\omega = \omega^{\text{wt}(\beta)}\beta$ . Since  $\omega^{q-1} = 1$ , it is convenient to assume that the weight function takes values in  $\mathbb{Z}/(q-1)\mathbb{Z}$ .

**Lemma 2.1.** *If  $f$  is an isobaric polynomial of weight  $\lambda$  and  $|fP| = |P|$  (i.e., the stabiliser subgroup of  $f$  is trivial), then  $\prod fP$  is isobaric of weight  $\lambda$ .*

*Proof.* Note that  $P$  is normal in the subgroup of upper-triangular matrices. Thus, for  $\omega \in \mathbb{F}_q^*$ ,

$$\begin{aligned} \left(\prod fP\right)\rho_\omega &= \prod_{\sigma \in P} f\sigma\rho_\omega = \prod_{\sigma' \in P} f\rho_\omega\sigma' \\ &= \prod_{\sigma' \in P} \omega^\lambda f\sigma' = \omega^\lambda \prod fP. \end{aligned}$$

Thus  $\prod fP$  is isobaric of weight  $\lambda$ . □

It is clear that  $\Delta$  is isobaric of weight 2. From the lemma,  $\gamma_0$  is isobaric of weight 2 and  $\beta$  is isobaric of weight 1. Thus our proposed generators for  $\mathbb{F}[V]^P$  are all isobaric.

**Lemma 2.2.** *The  $P$ -invariants  $a_0, \Delta, \beta$  and  $\gamma_0$  satisfy the relation*

$$\beta^2 = a_0^q \gamma_0 + \Delta(\Delta^{\frac{q-1}{2}} - a_0^{q-1})^2.$$

*Proof.* Define  $\zeta = a_0^q \gamma_0 + \Delta(\Delta^{\frac{q-1}{2}} - a_0^{q-1})^2$ . We first show that  $\zeta|_{a_1=0} = 0$ , which implies  $a_1$  divides  $\zeta$ .

Substituting  $a_1 = 0$  in  $\gamma_0$  gives

$$\begin{aligned} \gamma_0|_{a_1=0} &= \prod_{t \in \mathbb{F}_q} (t^2 a_0 + a_2) = a_2 \prod_{t \in \mathbb{F}_q^*} (t^2 a_0 + a_2) \\ &= a_2 a_0^{q-1} \prod_{t \in \mathbb{F}_q^*} \left( t^2 + \frac{a_2}{a_0} \right) = a_2 a_0^{q-1} \prod_{s \in \mathcal{Q}} \left( \frac{-a_2}{a_0} - s \right)^2 \\ &= a_2 a_0^{q-1} \left( \left( \frac{-a_2}{a_0} \right)^{\frac{q-1}{2}} - 1 \right)^2 = a_2 \left( (-a_2)^{\frac{q-1}{2}} - a_0^{\frac{q-1}{2}} \right)^2. \end{aligned}$$

Thus

$$\zeta|_{a_1=0} = a_0^q a_2 \left( (-a_2)^{\frac{q-1}{2}} - a_0^{\frac{q-1}{2}} \right)^2 + (-a_2 a_0) \left( (-a_0 a_2)^{\frac{q-1}{2}} - a_0^{q-1} \right)^2 = 0.$$

Therefore  $a_1$  divides  $\zeta$ . However,  $\zeta$  is isobaric of weight 2 and  $a_1$  is the only variable of odd weight. Hence  $a_1^2$  divides  $\zeta$ .

Suppose  $a_1$  divides  $f \in \mathbb{F}[V]^P$ . Then  $a_1 \sigma_c = a_1 + c a_0$  divides  $f = f \sigma_c$  for every  $c \in \mathbb{F}_q$ . Therefore  $\beta = \prod a_1 P$  divides  $f$ . Since  $a_1^2$  divides  $\zeta$ , we see that  $\beta^2$  divides  $\zeta$ . By comparing degrees and lead terms, we conclude that  $\beta^2 = \zeta$ , as required.  $\square$

**Lemma 2.3.**  $\mathbb{F}(V)^P = \mathbb{F}(a_0, \beta, \Delta)$ .

*Proof.* It is easy to verify that  $\mathbb{F}[a_0, a_1]^P = \mathbb{F}[a_0, \beta]$  (see, for example, [3, Theorem 3.7.5]). Since  $\Delta$  has degree 1 as a polynomial in  $a_2$ , applying [2, Theorem 2.4] gives  $\mathbb{F}(V)^P = \mathbb{F}(a_0, a_1)^P(\Delta) = \mathbb{F}(a_0, \beta, \Delta)$  (see also [5]).  $\square$

**Lemma 2.4.**  $\{a_0, \Delta, \gamma_0\}$  is a homogeneous system of parameters.

*Proof.* Using grevlex with  $a_0 < a_1 < a_2$ , the lead monomials are  $a_0$ ,  $a_1^2$  and  $a_2^q$ . Thus  $(a_0, \Delta, \gamma_0)\mathbb{F}[V]$  is a zero-dimensional ideal and  $\{a_0, \Delta, \gamma_0\}$  is a homogeneous system of parameters.  $\square$

**Theorem 2.5.**  $\mathcal{B} := \{a_0, \Delta, \beta, \gamma_0\}$  is a generating set, in fact a SAGBI basis, for  $\mathbb{F}[V]^P$ .

*Proof.* Let  $R$  denote the algebra generated by  $\mathcal{B}$ . Using grevlex with  $a_0 < a_1 < a_2$ , there is a single non-trivial tête-à-tête,  $\beta^2 - \Delta^q$ , which, using the relation given in Lemma 2.2, subducts to 0. Thus  $\mathcal{B}$  is a SAGBI basis for  $R$ .

Using Lemmas 2.3 and 2.4,  $\mathbb{F}[V]^P$  is an integral extension of  $R$  with the same field of fractions. Thus to show  $R = \mathbb{F}[V]^P$ , it is sufficient to show that  $R$  is normal, i.e., integrally closed in its field of fractions. Unique factorisation domains are normal; therefore it is sufficient to show  $R$  is a UFD.

Using the relation, we see that  $R[a_0^{-1}] = \mathbb{F}[a_0, a_0^{-1}][\Delta, \beta]$ , with  $a_0, \Delta, \beta$  algebraically independent. Thus  $R[a_0^{-1}]$  is a UFD. It follows from [7, Theorem 20.2] (or [1, Lemma 6.3.1]) that if  $a_0R$  is a prime ideal,  $R$  is a UFD.

Suppose  $f, g \in R$  with  $fg \in a_0R$ . Since  $R$  is graded, we may assume  $f$  and  $g$  are homogeneous. Clearly  $a_0\mathbb{F}[V]$  is prime. Therefore, without loss of generality, we may assume  $f \in a_0\mathbb{F}[V]$ . Hence the lead monomial  $\text{LM}(f)$  is divisible by  $a_0$ .  $\mathcal{B}$  is a SAGBI basis for  $R$  and  $f \in R$ . Thus  $f$  subducts to 0. Using the grevlex order with  $a_0$  small, every monomial of degree  $\deg(f)$ , less than  $\text{LM}(f)$ , is divisible by  $a_0$ . Thus at each stage of the subduction, there is a factor of  $a_0$ . Hence  $f \in a_0R$  and  $a_0R$  is prime.  $\square$

### 3. $SL_2(\mathbb{F}_q)$ -INVARIANTS

The group element

$$\tau = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \text{ acts on } V^* \text{ as } \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

It is well-known and easily verified that  $\{\tau\} \cup P$  generates  $SL_2(\mathbb{F}_q)$ . Thus to show that  $f \in \mathbb{F}[V]^P$  is  $SL_2(\mathbb{F}_q)$ -invariant, it is sufficient to show  $(f)\tau = f$ .

**Lemma 3.1.**  *$J, \Gamma$  and  $B$  are  $SL_2(\mathbb{F}_q)$ -invariant.*

*Proof.* By construction,  $J, \Gamma$  and  $B$  are  $P$ -invariant. A relatively straightforward calculation shows that each of these polynomials is fixed by  $\tau$  and is therefore  $SL_2(\mathbb{F}_q)$ -invariant. It is perhaps more instructive to note that  $SL_2(\mathbb{F}_q)$  permutes the lines in  $V^*$  and that each of  $J, \Gamma$ , and  $B$  is a projective orbit product. For example, the stabiliser of the line  $a_0\mathbb{F}_q$  has order  $q(q-1)$  and  $J$  is a product of  $q+1$  linear factors, one taken from each line in the orbit of  $a_0\mathbb{F}_q$ . Similarly, the stabiliser of  $a_1\mathbb{F}$  has order  $2(q-1)$  and  $B$  is the product of  $q(q+1)/2$  linear factors, each representing a line in the orbit. The linear factors of  $\Gamma$  are of the form  $a_2 + 2ca_1 + (c^2 - k)a_0$  for  $c \in \mathbb{F}_q$  and  $k \in \overline{\mathbb{Q}}$ . Applying  $\tau$  gives

$$\begin{aligned} (a_2 + 2ca_1 + (c^2 - k)a_0)\tau &= a_0 - 2ca_1 + (c^2 - k)a_2 \\ &= (c^2 - k) \left( a_2 + 2a_1 \frac{-c}{c^2 - k} + a_0 \frac{1}{c^2 - k} \right). \end{aligned}$$

However

$$\frac{1}{c^2 - k} = \left( \frac{-c}{c^2 - k} \right)^2 - \frac{k}{(c^2 - k)^2}$$

with  $k/(c^2 - k)^2 \in \overline{\mathbb{Q}}$ . Thus  $\tau$  permutes the lines in  $V^*$  corresponding to the linear factors of  $\Gamma$ .

Since  $SL_2(\mathbb{F}_q)$  acts on  $J$ ,  $\Gamma$  and  $B$  by permuting the linear factors, up to scalar multiplication, the action on each of these polynomials is by a multiplicative character. However, for  $q \notin \{2, 3\}$ ,  $SL_2(\mathbb{F}_q)$  is simple (see, for example [11, 4.5]); hence the character is trivial and the polynomials are invariant. The case  $q = 2$  is inconsistent with our hypothesis  $\text{char}(\mathbb{F}) > 2$ . The  $q = 3$  case was covered by Dickson's work [4] (and can be easily verified by computer).  $\square$

**Lemma 3.2.**  $\{\Delta, J, \Gamma\}$  is a homogeneous system of parameters for  $\mathbb{F}[V]^{SL_2(\mathbb{F}_q)}$ .

*Proof.* Without loss of generality, we may assume  $\mathbb{F}$  is algebraically closed. We will show that the variety associated to  $(\Delta, J, \Gamma)\mathbb{F}[V]$ , say  $\mathcal{V}$ , consists of the zero vector.

Suppose  $v \in \mathcal{V}$ . Since  $J(v) = 0$ , there exists  $g \in SL_2(\mathbb{F}_q)$  such that  $a_0g(v) = 0$ . Replacing  $v$  with  $g(v)$  if necessary, we may assume  $a_0(v) = 0$ . Thus  $\Delta(v) = a_1^2(v)$ , giving  $a_1(v) = 0$ . Since  $\Gamma \in a_2^{q(q-1)/2} + (a_0, a_1)\mathbb{F}[V]$ , we have  $\Gamma(v) = a_2^{q(q-1)/2}(v)$ , giving  $a_2(v) = 0$ .  $\square$

Define  $A := \mathbb{F}[\Delta, J, \Gamma]$ .

**Corollary 3.3.**  $\mathbb{F}[V]^{SL_2(\mathbb{F}_q)}$  is a free  $A$ -module of rank 2.

*Proof.* It is well known that the ring of invariants of a 3 dimensional representation is Cohen-Macaulay (see [3, 3.4.2] or [8, 5.6.10]), i.e., a free module over any homogeneous system of parameters (hsop). For a faithful action, the rank is given by the order of the group divided by the product of the degrees of the elements in the hsop (see [3, 3.7.1] or [8, 5.5.8]).  $SL_2(\mathbb{F}_q)$  acts on  $V$  with kernel generated by  $-I$  and

$$\deg(\Delta)\deg(J)\deg(\Gamma) = 2(q+1)\frac{q(q-1)}{2} = |SL_2(\mathbb{F}_q)|.$$

Thus  $\mathbb{F}[V]^{SL_2(\mathbb{F}_q)}$  has rank 2 over  $A$ .  $\square$

**Theorem 3.4.**  $\mathbb{F}[V]^{SL_2(\mathbb{F}_q)}$  is generated by  $\Delta$ ,  $J$ ,  $\Gamma$  and  $B$  subject to a relation of the form

$$B^2 = \Delta^q\Gamma^2 + J\Phi(\Delta, J, \Gamma)$$

for some polynomial  $\Phi$ . Furthermore, this generating set is a SAGBI basis using the grevlex order with  $a_0 < a_1 < a_2$ .

*Proof.* For any  $f \in \mathbb{F}[V]$  we can write  $f = f_e + f_o$  where  $f_e$  is a sum of terms of even weight and  $f_o$  is a sum of terms of odd weight. If  $f \in \mathbb{F}[V]^P = \mathbb{F}[a_0, \Delta, \gamma_0] \oplus \beta\mathbb{F}[a_0, \Delta, \gamma_0]$ , then  $f_e \in \mathbb{F}[a_0, \Delta, \gamma_0]$  and

$f_o \in \beta\mathbb{F}[a_0, \Delta, \gamma_0]$ . It is clear that  $\tau$  preserves weight-parity. Thus if  $f$  is  $SL_2(\mathbb{F}_q)$ -invariant  $(f_e)\tau = f_e$  and  $(f_o)\tau = f_o$ , giving  $f_e, f_o \in \mathbb{F}[V]^{SL_2(\mathbb{F}_q)}$ . Every odd-weight term is divisible by  $a_1$ . Hence, every odd-weight  $SL_2(\mathbb{F}_q)$ -invariant is divisible by  $B$ . Thus  $\mathbb{F}[V]^{SL_2(\mathbb{F}_q)} = E \oplus BE$ , where  $E$  denotes the subalgebra of even-weight  $SL_2(\mathbb{F}_q)$ -invariants. Note that  $A \subseteq E$ .

Using Corollary 3.3, there exists a homogeneous  $SL_2(\mathbb{F}_q)$ -invariant, say  $\delta$ , such that  $\mathbb{F}[V]^{SL_2(\mathbb{F}_q)} = A \oplus \delta A$ . If  $\deg(\delta) < \deg(B)$ , then  $\delta \in E$ ; hence  $\mathbb{F}[V]^{SL_2(\mathbb{F}_q)} \subseteq E$ , giving a contradiction. If  $\deg(\delta) > \deg(B)$ , then  $B \in A$ ; hence  $\mathbb{F}[V]^{SL_2(\mathbb{F}_q)} \subseteq A \subseteq E$ , again giving a contradiction. Therefore  $\deg(\delta) = \deg(B)$ . Comparing Hilbert series, i.e., dimensions of homogeneous components, we see that  $A = E$  and  $\mathbb{F}[V]^{SL_2(\mathbb{F}_q)} = A \oplus BA$ . Therefore  $\mathbb{F}[V]^{SL_2(\mathbb{F}_q)}$  is generated by  $\Delta, J, \Gamma$  and  $B$ .

Since  $B^2$  has even weight, we have  $B^2 \in A$ . Furthermore,  $B^2 - \Delta^q \Gamma^2 \in \mathbb{F}[V]^{SL_2(\mathbb{F}_q)}$  is zero modulo  $a_0$ . Thus  $J$  divides  $B^2 - \Delta^q \Gamma^2$ . The quotient is of even weight and hence is an element of  $A$ , say  $\Phi(\Delta, J, \Gamma)$ . Therefore  $B^2 = \Delta^q \Gamma^2 + J\Phi(\Delta, J, \Gamma)$ .

The lead terms of the generators are  $\text{LT}(\Delta) = a_1^2$ ,  $\text{LT}(J) = a_0 a_2^q$ ,  $\text{LT}(\Gamma) = a_2^{q(q-1)/2}$  and  $\text{LT}(B) = a_1^q a_2^{q(q-1)/2}$ . Thus the only non-trivial tête-a-tête is given by  $B^2 - \Delta^q \Gamma^2$ . Hence  $\{\Delta, J, \Gamma\}$  is a SAGBI basis for  $A$ ,  $\Phi(\Delta, J, \Gamma)$  subducts to zero using  $\{\Delta, J, \Gamma\}$  and  $B^2 - \Delta^q \Gamma^2$  subducts to zero. Therefore  $\{\Delta, J, \Gamma, B\}$  is a SAGBI basis for  $\mathbb{F}[V]^{SL_2(\mathbb{F}_q)}$ .  $\square$

**Corollary 3.5.** *Define*

$$m := \left\lfloor \frac{1}{2}(q+1+q(q-1)/2) \right\rfloor \quad \text{and} \quad s := \left\lfloor \frac{1}{2}(1+q(q-1)/2) \right\rfloor.$$

Then  $\mathbb{F}(V)^{SL_2(\mathbb{F}_q)} = \mathbb{F}(B/\Delta^m, J/\Delta^{(q+1)/2}, \Gamma/\Delta^s)$ , a purely transcendental extension of  $\mathbb{F}$ .

*Proof.* Let  $\mathcal{F}$  denote the field generated by  $\{B/\Delta^m, J/\Delta^{(q+1)/2}, \Gamma/\Delta^s\}$ . Clearly  $\mathcal{F} \subseteq \mathbb{F}(V)^{SL_2(\mathbb{F}_q)}$ .

Suppose  $(q-1)/2$  even. Then  $m = \frac{1}{2}(q+1+q(q-1)/2)$  and  $s = q(q-1)/4$ . Dividing the homogeneous relation from Theorem 3.4 by  $\Delta^{2m-1}$  gives

$$\Delta \left( \frac{B}{\Delta^m} \right)^2 = \left( \frac{\Gamma}{\Delta^s} \right)^2 + \left( \frac{J}{\Delta^{(q+1)/2}} \right) \Phi(1, J/\Delta^{(q+1)/2}, \Gamma/\Delta^s).$$

Thus  $\Delta \in \mathcal{F}$ . Therefore  $J, \Gamma, B \in \mathcal{F}$ , giving  $\mathcal{F} = \mathbb{F}(V)^{SL_2(\mathbb{F}_q)}$ .

Suppose  $(q-1)/2$  is odd. Then  $m = \frac{1}{2}(q + \frac{q(q-1)}{2})$  and  $s = \frac{1}{2}(\frac{q(q-1)}{2} + 1)$ . Furthermore  $\Gamma$  is of odd degree while  $J$  and  $\Delta$  are of even degree. Thus  $\Gamma$  can not appear in  $\Phi$ . Dividing the homogeneous relation from



Theorem 3.4 by  $\Delta^{2m}$  gives

$$\left(\frac{B}{\Delta^m}\right)^2 = \Delta \left(\frac{\Gamma}{\Delta^s}\right)^2 + \left(\frac{J}{\Delta^{(q+1)/2}}\right) \Phi(1, J/\Delta^{(q+1)/2}).$$

Thus  $\Delta \in \mathcal{F}$ . Therefore  $J, \Gamma, B \in \mathcal{F}$ , giving  $\mathcal{F} = \mathbb{F}(V)^{SL_2(\mathbb{F}_q)}$ . □

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SCHOOL OF MATHEMATICS, STATISTICS & ACTUARIAL SCIENCE,  
UNIVERSITY OF KENT, CANTERBURY, CT2 7NF, UK  
*E-mail address:* ashleyghobson@googlemail.com

SCHOOL OF MATHEMATICS, STATISTICS & ACTUARIAL SCIENCE,  
UNIVERSITY OF KENT, CANTERBURY, CT2 7NF, UK  
*E-mail address:* R.J.Shank@kent.ac.uk