Fermions, Skyrmions and the 3-Sphere

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August 14, 2009

Abstract

This paper investigates a background charge one Skyrme field chirally coupled to light fermions on the 3-sphere. The Dirac equation for the system commutes with a generalised angular momentum or grand spin. It can be solved explicitly for a Skyrme configuration given by the hedgehog form. The energy spectrum and degeneracies are derived for all values of the grand spin. Solutions for non-zero grand spin are each characterised by a set of four polynomials. The paper also discusses the energy of the Dirac sea using zeta function regularization.

1 Introduction

The Skyrme model is a nonlinear $SU(2)$ field theory which gives a good description of atomic nuclei and their low energy interactions [1]. In addition to the fundamental pion excitations the theory also has topological soliton solutions, known as Skyrmions. These are labelled by a topological charge or generalised winding number B, which can be interpreted as the baryon number of the configuration. On quantization, Skyrmions are found to describe nuclei, Δ-resonance [2] and also bound states of nuclei, see [3, 4, 5, 6, 7, 8] for the quantization of multi-Skyrmions and [9, 10, 11] for recent quantitative predictions of the Skyrme model. It is well known that Skyrmions can be quantized as fermions [12, 13]. Therefore, when the Skyrme field is coupled to a fermion field, there are two different ways of describing fermions in the same model. The fermion field can then be thought of as light quarks in the presence of atomic nuclei, [14]. In the presence of a Skyrme field the energy spectrum of the Dirac operator shows a curious behaviour, namely, a mode crosses from the positive to the negative spectrum as the coupling constant is increased, [15]. In a very similar model, Kahana and Ripka calculate the baryon density in the

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one-loop approximation [16] and the energy of the Dirac sea quarks [17]. Recently, these calculations have been extended to multi-Skyrmions, [18, 19].

Static field configurations in the original Skyrme model in flat space are given by maps $\mathbb{R}^3 \to SU(2)$. By using the boundary condition for the Skyrme field to unify the domain of such a map with infinity we make the domain compact and equivalent to the 3-sphere S^3. If we also consider that S^3 is the group manifold of $SU(2)$, we can see that the field configurations are topologically equivalent to maps $S^3 \to S^3$ and, because of this, the model can be generalized to the base space being a sphere of radius L, [20, 21]. In the limit $L \to \infty$, the original model is recovered. The advantage of working on S^3 is that the Bogomoln'yi equation can be solved for $B = 1$, and the solution is given by the identity map [20]. This enables us to calculate the energy spectrum and the corresponding fermion wave functions explicitly.

In [22] a system of light fermions, on S^3, coupled to a spherically symmetric background Skyrme field was studied for grand spin $G = 0$. In this paper we consider the general case where the grand spin also takes positive integer values. The Dirac equation on S^3 is derived in section 2 through the use of stereographic coordinates. In section 2.1, the solution of the Dirac equation for $G = 0$ is reviewed. In section 3, the correct ansatz for the spin-isospin spinor for general G is deduced using parity arguments. We then present the general solution. Plots of energy against fermion-Skyrmion coupling constant are also given. In section 3.2, we discuss the degeneracy of energy eigenvalues. In section 4, we address the problem of calculating the energy of the Dirac sea using zeta function regularization. We end with a conclusion.

2 The Dirac equation on S^3

Following [22], we now recall the Dirac equation on a 3-sphere of radius $L = 1$. Consider the stereographic projection from the north pole N to the plane through the equator. Let S^3 be embedded in $\mathbb{R}^4$ with coordinates (x_1, x_2, x_3, w). As a result of projecting from N onto the equatorial $\mathbb{R}^3$ plane, points of S^3 can be labelled with coordinates X_i. The chart is defined everywhere apart from the projection point, N. The coordinates X_i can be written in terms of $\mathbb{R}^4$ coordinates as

$$X_i = \frac{x_i}{1-w}. \tag{1}$$

We define $R^2 = X_1^2 + X_2^2 + X_3^2$. Then the metric can be written as

$$g_{\mathbb{R}\times S^3} = \text{diag}\ \left(1, -\frac{4}{(1+R^2)^2}, -\frac{4}{(1+R^2)^2}, -\frac{4}{(1+R^2)^2}\right). \tag{2}$$

We now choose the non-coordinate basis

$$\hat{e}_\alpha = e_\alpha{}^\mu \partial_{X_\mu}. \tag{3}$$

It is convenient to choose diagonal vierbeins $e_\alpha{}^\mu$, such that

$$e_0{}^0 = 1, \quad e_i{}^i = -\frac{1+R^2}{2}, \tag{4}$$

where all other components vanish. With our choice of vierbeins, we can calculate the matrix valued connection 1-form $\omega_{\alpha\beta}$. The 1-form $\omega_{\alpha\beta}$ satisfies the metric compatibility condition $\omega_{\alpha\beta} = -\omega_{\beta\alpha}$, and the torsion-free condition

$$d\hat{\theta}^\alpha + \omega^\alpha{}_\beta \wedge \hat{\theta}^\beta = 0, \tag{5}$$

where $\hat{\theta}^\alpha = e^\alpha{}_\mu dX^\mu$ is the dual basis of $\hat{e}_\alpha$. After a short calculation we find

$$\omega^{\alpha\beta} = \begin{cases} \frac{2}{1+R^2}(X^\alpha dX^\beta - X^\beta dX^\alpha) & \alpha, \beta = 1, 2, 3, \\ 0 & \text{otherwise.} \end{cases} \tag{6}$$

The spin connection Ω_μ can now be expressed as

$$\Omega_\mu dX^\mu = -\tfrac{i}{2}\omega^{\alpha\beta}\Sigma_{\alpha\beta}, \tag{7}$$

where $\Sigma_{\alpha\beta} = \frac{i}{4}[\gamma_\alpha, \gamma_\beta]$ and the components of the commutator are the standard gamma-matrices, satisfying $\{\gamma_\alpha, \gamma_\beta\} = 2\eta_{\alpha\beta}$. We work with the following representation of gamma-matrices

$$\gamma^0 = \begin{pmatrix} 1_2 & 0 \\ 0 & -1_2 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}, \quad \gamma^5 = \begin{pmatrix} 0 & 1_2 \\ 1_2 & 0 \end{pmatrix}, \tag{8}$$

because we will be working with parity eigenfunctions. Here σ_i denotes the set of three Pauli matrices, defined by

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{9}$$

For massless fermions in curved space-time, the Lagrangian is

$$\mathcal{L}_{fermion} = \bar{\psi}(i\gamma^\alpha e_\alpha{}^\kappa(\partial_\kappa + \Omega_\kappa))\psi. \tag{10}$$

With our choice of coordinates and vierbeins, we obtain

$$\mathcal{L}_{fermion} = \bar{\psi}(X_i, t)\left(i\gamma^0\partial_t - i\gamma^i\left(\frac{1+R^2}{2}\partial_{X_i} - X_i\right)\right)\psi(X_i, t). \tag{11}$$

In this paper, we investigate fermions coupled to Skyrmions on S^3. We consider a background $B = 1$ Skyrme field coupled to the fermions. The full Lagrangian $\mathcal{L}$ is the sum of the fermion Lagrangian $\mathcal{L}_{fermion}$, the Skyrmion Lagrangian $\mathcal{L}_{Skyrmion}$ and the interaction Lagrangian $\mathcal{L}_{int}$. We consider fermions in the background of a static Skyrme field and neglect the backreaction. Therefore, we no longer discuss the Skyrmion Lagrangian, and the interested reader is referred to [21]. $\mathcal{L}_{int}$ is derived in [23], namely

$$\mathcal{L}_{int} = -g\bar{\psi}(\sigma + i\gamma_5 \boldsymbol{\tau}\cdot\boldsymbol{\pi})\psi, \tag{12}$$

where $U = \sigma + i\boldsymbol{\tau}\cdot\boldsymbol{\pi}$ is a parametrization of the Skyrme field and g is the coupling constant. ψ is a spin-isospin spinor. It is convenient to split the spinor into two 2×2 spin-isospin matrices ψ_1 and ψ_2 such that

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}. \tag{13}$$

Since any complex 2×2 matrix can be expressed as a linear combination of the Pauli matrices and the identity, it is convenient to choose these four as a basis of $SU(2)$. The spin-isospin matrices can then be written as $\psi_i = a_0^{(i)} 1_2 + i a_k^{(i)} \sigma_k$. With this notation spin operators act on ψ by left multiplication, $\sigma_k \psi_i$, whereas the isospin matrices act on ψ by right multiplication,

$$\begin{aligned} \tau_k \psi_i &= \psi_i \sigma_k^T \\ &= -\psi_i \sigma_2 \sigma_k \sigma_2. \end{aligned} \tag{14}$$

In this paper, we only consider spherically symmetric Skyrmions. The $B = 1$ Skyrmion on S^3 is spherically symmetric [20], but for $B > 1$ this is no longer true. Spherically symmetric Skyrme fields are best expressed in terms of polar coordinates,

$$U = \exp(i f(\mu) \boldsymbol{e}_\mu \cdot \boldsymbol{\tau}), \tag{15}$$

where $f(\mu)$ is the "radial" shape function and $\mathbf{e}_\mu$ is the unit vector in the μ direction, see equation (19). Using (11) and (12) we can write down the Dirac equations for fermions coupled to a spherically symmetric background Skyrmion. We obtain

$$\left(i\gamma^0 \partial_t - i\gamma^i \left(\frac{1+R^2}{2} \partial_{X_i} - X_i \right) - g U^{\gamma_5} \right) \psi(X_i, t) = 0, \tag{16}$$

where

$$U^{\gamma_5} = \cos f(\mu) + i\gamma_5 \boldsymbol{e}_\mu \cdot \boldsymbol{\tau} \sin f(\mu). \tag{17}$$

2.1 Solutions of the Dirac Equation for $G = 0$

The above Dirac equation (16) and the ansatz $\psi(X_i, t) = \mathrm{e}^{iEt} \psi(X_i)$ lead us to the time-independent Dirac equation

$$E\psi = \begin{pmatrix} g \cos f(\mu) & \boldsymbol{\sigma} \cdot \boldsymbol{p} + i g \boldsymbol{e}_\mu \cdot \boldsymbol{\tau} \sin f(\mu) \\ \boldsymbol{\sigma} \cdot \boldsymbol{p} - i g \boldsymbol{e}_\mu \cdot \boldsymbol{\tau} \sin f(\mu) & -g \cos f(\mu) \end{pmatrix} \psi, \tag{18}$$

where

$$\boldsymbol{e}_\mu = \begin{pmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{pmatrix}, \quad \boldsymbol{e}_\theta = \begin{pmatrix} \cos\theta\cos\phi \\ \cos\theta\sin\phi \\ -\sin\theta \end{pmatrix}, \quad \boldsymbol{e}_\phi = \begin{pmatrix} -\sin\phi \\ \cos\phi \\ 0 \end{pmatrix}, \tag{19}$$

and

$$\boldsymbol{\sigma} \cdot \mathbf{p} = -i \left(\boldsymbol{e}_\mu \cdot \boldsymbol{\sigma} \left(\partial_\mu + \tfrac{\sin\mu}{1-\cos\mu} \right) - \tfrac{1}{\sin\mu} \boldsymbol{e}_\theta \cdot \boldsymbol{\sigma} \partial_\theta - \tfrac{1}{\sin\mu \sin\theta} \boldsymbol{e}_\phi \cdot \boldsymbol{\sigma} \partial_\phi \right). \tag{20}$$

The elements of the matrix in (18) commute with the total angular momentum operator $\mathbf{G} = \mathbf{L} + \mathbf{S} + \mathbf{I}$ where $\mathbf{L}$ is the orbital angular momentum, $\mathbf{S} = \frac{1}{2}\boldsymbol{\sigma}$ is the spin operator and $\mathbf{I} = \frac{1}{2}\boldsymbol{\tau}$ is the isospin operator.

The above equation (18) is invariant under parity $\hat{P}$ where

$$\hat{P}\psi(X_i) = \gamma_0 \psi(-X_i), \quad \hat{P} X_i \hat{P}^{-1} = -X_i. \tag{21}$$

The $G = 0$ case is treated in [22]. There the ansatz for ψ gives rise to a system of two first order ODEs, which can be expressed as a second order ODE. This equation can be solved analytically for $f(\mu) = 0$ and $f(\mu) = \mu$. In [22] the following energy spectrum was derived for $f(\mu) = 0$,

$$E = \pm\sqrt{g^2 + \left(N + \tfrac{3}{2}\right)^2} \quad \text{for} \quad N = 0, 1, 2, \ldots \tag{22}$$

Setting $u = \cos\mu$, the eigenfunctions $G_N(u)$ were found to be given by Jacobi polynomials. The shape function $f(\mu) = \mu$ was also considered in [22]. This leads to a second order Fuchsian equation with four regular singular points, two at $u = \pm 1$, one at infinity and one depending on E and g. The equation could still be solved in terms of polynomials. The following energy spectrum was derived,

$$E_0 = \tfrac{3}{2} - g, \quad E_n^{\pm} = \tfrac{1}{2} \pm \sqrt{n^2 + 2n + (g-1)^2} \quad \text{for} \quad n = 1, 2, \ldots \tag{23}$$

with eigenfunctions

$$G_n(u) = \sum_{j=0}^{n} a_j (u+1)^j, \tag{24}$$

where

$$a_j = \tfrac{(-1)^j (E+g-\frac{3}{2})(E-g+\frac{2j+1}{2})}{j!(2j+1)!!} \prod_{i=1}^{j-1} (E^2 - E + 2g - g^2 + \tfrac{1}{4} - (i+1)^2), \tag{25}$$

for $j = 1, 2, \ldots$ and $a_0 = 1$. Here $(2j+1)!! = 1 \cdot 3 \cdot \cdots \cdot (2j+1)$ is the product of odd integers. Using $E = \frac{1}{2} \pm \sqrt{(n+1)^2 - 2g + g^2}$ in the product in (25), a_j can be written as

$$a_j = \tfrac{(-1)^j (E+g-\frac{3}{2})(E-g+\frac{2j+1}{2})}{j!(2j+1)!!} \prod_{i=1}^{j-1} ((n-i)(n+i+2)). \tag{26}$$

Expanding the product in (26) we obtain

$$a_j = (-1)^j \begin{pmatrix} n \\ j \end{pmatrix} \tfrac{(n+j+1)!}{(n+1)!(2j+1)!!} \tfrac{(E+g-\frac{3}{2})(E-g+\frac{2j+1}{2})}{n(n+2)}. \tag{27}$$

3 Solutions of the Dirac equation for general G

In the following we derive the solution of the Dirac equation (18) for general G. As a starting point, we construct the total angular momentum operator eigenstates $|jm\rangle_1$ and $|jm\rangle_2$ in terms of angular momentum and spin states. They are expressed as

$$|jm\rangle_1 = \sqrt{\tfrac{j-m}{2j}} Y_{j-\frac{1}{2}, m+\frac{1}{2}} \left|\tfrac{1}{2} - \tfrac{1}{2}\right\rangle_S + \sqrt{\tfrac{j+m}{2j}} Y_{j-\frac{1}{2}, m-\frac{1}{2}} \left|\tfrac{1}{2}\tfrac{1}{2}\right\rangle_S, \tag{28}$$

$$|jm\rangle_2 = \sqrt{\tfrac{j+m+1}{2j+2}} Y_{j+\frac{1}{2}, m+\frac{1}{2}} \left|\tfrac{1}{2} - \tfrac{1}{2}\right\rangle_S - \sqrt{\tfrac{j-m+1}{2j+2}} Y_{j+\frac{1}{2}, m-\frac{1}{2}} \left|\tfrac{1}{2}\tfrac{1}{2}\right\rangle_S, \tag{29}$$

where $Y_{j,m}$ is a spherical harmonic and $\left|\frac{1}{2} \pm \frac{1}{2}\right\rangle_S$ is a spin state. For general G we consider four eigenstates, each of which can be written in terms of $|jm\rangle_1$ and $|jm\rangle_2$.

These are

$$|GM\rangle_{a,c} = \sqrt{\tfrac{G-M}{2G}}\left|j = G - \tfrac{1}{2}, m = M + \tfrac{1}{2}\right\rangle_{1,2}\left|\tfrac{1}{2} - \tfrac{1}{2}\right\rangle_I$$
$$+\sqrt{\tfrac{G+M}{2G}}\left|j = G - \tfrac{1}{2}, m = M - \tfrac{1}{2}\right\rangle_{1,2}\left|\tfrac{1}{2}\tfrac{1}{2}\right\rangle_I, \tag{30}$$

and

$$|GM\rangle_{b,d} = \sqrt{\tfrac{G+M+1}{2G+2}}\left|j = G + \tfrac{1}{2}, m = M + \tfrac{1}{2}\right\rangle_{1,2}\left|\tfrac{1}{2} - \tfrac{1}{2}\right\rangle_I$$
$$-\sqrt{\tfrac{G-M+1}{2G+2}}\left|j = G + \tfrac{1}{2}, m = M - \tfrac{1}{2}\right\rangle_{1,2}\left|\tfrac{1}{2}\tfrac{1}{2}\right\rangle_I, \tag{31}$$

where $\left|\frac{1}{2} \pm \frac{1}{2}\right\rangle_I$ is an isospin state. To carry out our analysis, an ansatz in terms of the G-eigenstates must be found. For each row of our ansatz for the spin-isospin spinor the states must be of the same parity. Under parity

$$Y_{l,m} \to (-1)^l Y_{l,m}, \tag{32}$$

so that

$$|jm\rangle_{1,2} \to (-1)^{j\mp\frac{1}{2}}\,|jm\rangle_{1,2}. \tag{33}$$

It follows that

$$\begin{array}{ll} |GM\rangle_a \to (-1)^{G-1}\,|GM\rangle_a, & |GM\rangle_b \to (-1)^{G}\,|GM\rangle_b, \\ |GM\rangle_c \to (-1)^{G}\,|GM\rangle_c, & |GM\rangle_d \to (-1)^{G+1}\,|GM\rangle_d. \end{array} \tag{34}$$

We see that $|GM\rangle_b$ and $|GM\rangle_c$ both have parity $(-1)^G$ and that $|GM\rangle_a$ and $|GM\rangle_d$ both have parity $-(-1)^G$. Hence the spinor

$$\psi = \begin{pmatrix} b(u)\,|GM\rangle_b + c(u)\,|GM\rangle_c \\ d(u)\,|GM\rangle_d + a(u)\,|GM\rangle_a \end{pmatrix}, \tag{35}$$

will have overall parity $(-1)^G$. Clearly exchanging the upper and lower rows will change the parity by a factor -1. A short calculation shows that this is equivalent to making the transformation

$$g \to -g \tag{36}$$

in the resulting equations. For general G and parity $(-1)^G$ we make the ansatz

$$\psi = \begin{pmatrix} \sqrt{1-u}\sqrt{1-u^2}^{\,G}G_2(u)\,|GM\rangle_b + \sqrt{1+u}\sqrt{1-u^2}^{\,G-1}G_3(u)\,|GM\rangle_c \\ i\sqrt{1+u}\sqrt{1-u^2}^{\,G}G_4(u)\,|GM\rangle_d + i\sqrt{1-u}\sqrt{1-u^2}^{\,G-1}G_1(u)\,|GM\rangle_a \end{pmatrix}, \tag{37}$$

where $G_1(u)$, $G_2(u)$, $G_3(u)$ and $G_4(u)$ are functions of u to be found, and the normalization factors are chosen for later convenience. Substituting this state into (18), we obtain a system of four coupled first order differential equations in $G_1(u)$, $G_2(u)$, $G_3(u)$ and $G_4(u)$. We will solve for the case $f(\mu) = \mu$.

We need to know how the operator $\boldsymbol{e}_\mu \cdot \boldsymbol{\sigma}$ acts on the G-eigenstates in (30) and (31). The results we require are

$$\boldsymbol{e}_\mu \cdot \boldsymbol{\sigma}\,|GM\rangle_a = -\,|GM\rangle_c, \quad \boldsymbol{e}_\mu \cdot \boldsymbol{\sigma}\,|GM\rangle_c = -\,|GM\rangle_a, \tag{38}$$

$$\boldsymbol{e}_\mu \cdot \boldsymbol{\sigma}\,|GM\rangle_b = -\,|GM\rangle_d, \quad \boldsymbol{e}_\mu \cdot \boldsymbol{\sigma}\,|GM\rangle_d = -\,|GM\rangle_b. \tag{39}$$

These results can be derived by first deducing that

$$\begin{aligned} \boldsymbol{e}_\mu \cdot \boldsymbol{\sigma} \, |jm\rangle_1 &= -\, |jm\rangle_2 \, , \qquad (40) \\ \boldsymbol{e}_\mu \cdot \boldsymbol{\sigma} \, |jm\rangle_2 &= -\, |jm\rangle_1 \, . \qquad (41) \end{aligned}$$

To obtain the relations (40) and (41) the operator $\boldsymbol{e}_\mu \cdot \boldsymbol{\sigma}$ is expressed in terms of spherical harmonics. Formulae for products of spherical harmonics are then needed. The required results can be found in [24], (38) and (39) then follow.

We also need to know how the operator $(-\boldsymbol{e}_\theta \cdot \boldsymbol{\sigma}\partial_\theta - \frac{1}{\sin\theta}\boldsymbol{e}_\phi \cdot \boldsymbol{\sigma}\partial_\phi)$ acts on the G-eigenstates. The results are

$$(-\boldsymbol{e}_\theta \cdot \boldsymbol{\sigma}\partial_\theta - \tfrac{1}{\sin\theta}\boldsymbol{e}_\phi \cdot \boldsymbol{\sigma}\partial_\phi)\, |GM\rangle_a = -(G-1)\, |GM\rangle_c, \qquad (42)$$

$$(-\boldsymbol{e}_\theta \cdot \boldsymbol{\sigma}\partial_\theta - \tfrac{1}{\sin\theta}\boldsymbol{e}_\phi \cdot \boldsymbol{\sigma}\partial_\phi)\, |GM\rangle_b = -G\, |GM\rangle_d, \qquad (43)$$

$$(-\boldsymbol{e}_\theta \cdot \boldsymbol{\sigma}\partial_\theta - \tfrac{1}{\sin\theta}\boldsymbol{e}_\phi \cdot \boldsymbol{\sigma}\partial_\phi)\, |GM\rangle_c = (G+1)\, |GM\rangle_a, \qquad (44)$$

$$(-\boldsymbol{e}_\theta \cdot \boldsymbol{\sigma}\partial_\theta - \tfrac{1}{\sin\theta}\boldsymbol{e}_\phi \cdot \boldsymbol{\sigma}\partial_\phi)\, |GM\rangle_d = (G+2)\, |GM\rangle_b. \qquad (45)$$

In order to derive equations (42)-(45) we note that

$$2\mathbf{L} \cdot \mathbf{S} = S_+ L_- + S_- L_+ + 2 S_3 L_3, \qquad (46)$$

where S_+ and S_- are defined as $S_+ = S_1 + iS_2$ and $S_- = S_1 - iS_2$ and (S_1, S_2, S_3) are a set of generators of the Lie algebra of $SU(2)$ and are related to the Pauli matrices via $S_i = \frac{1}{2}\sigma_i$. L_+, L_- and L_3 are the orbital angular momentum operators. We note the following result

$$(-\boldsymbol{e}_\theta \cdot \boldsymbol{\sigma}\partial_\theta - \tfrac{1}{\sin\theta}\boldsymbol{e}_\phi \cdot \boldsymbol{\sigma}\partial_\phi) = (\boldsymbol{e}_\mu \cdot \boldsymbol{\sigma})(2\mathbf{L} \cdot \mathbf{S}), \qquad (47)$$

which can easily be proved by multiplying out. Then

$$2\mathbf{L} \cdot \mathbf{S}\, |jm\rangle_1 = (j - \tfrac{1}{2})\, |jm\rangle_1 \, , \qquad (48)$$

$$2\mathbf{L} \cdot \mathbf{S}\, |jm\rangle_2 = -(j + \tfrac{3}{2})\, |jm\rangle_2 \, , \qquad (49)$$

can be proved by considering how L_+, L_- and L_3 act on the spherical harmonics. The necessary formulae can be found in [24]. These two equations also follow from $2\mathbf{L} \cdot \mathbf{S} = \mathbf{J}^2 - \mathbf{L}^2 - \mathbf{S}^2$.

It can then be seen that

$$(-\boldsymbol{e}_\theta \cdot \boldsymbol{\sigma}\partial_\theta - \tfrac{1}{\sin\theta}\boldsymbol{e}_\phi \cdot \boldsymbol{\sigma}\partial_\phi)\, |jm\rangle_1 = -(j - \tfrac{1}{2})\, |jm\rangle_2 \, , \qquad (50)$$

$$(-\boldsymbol{e}_\theta \cdot \boldsymbol{\sigma}\partial_\theta - \tfrac{1}{\sin\theta}\boldsymbol{e}_\phi \cdot \boldsymbol{\sigma}\partial_\phi)\, |jm\rangle_2 = (j + \tfrac{3}{2})\, |jm\rangle_1 \, . \qquad (51)$$

This leads to (42)-(45).

The operator $\boldsymbol{e}_\mu \cdot \boldsymbol{\tau}$ acts on the G-eigenstates to give

$$\boldsymbol{e}_\mu \cdot \boldsymbol{\tau}\, |GM\rangle_a = -\frac{2\sqrt{G(G+1)}}{2G+1}\, |GM\rangle_b - \frac{1}{2G+1}\, |GM\rangle_c \, , \qquad (52)$$

$$\boldsymbol{e}_\mu \cdot \boldsymbol{\tau}\, |GM\rangle_b = -\frac{2\sqrt{G(G+1)}}{2G+1}\, |GM\rangle_a + \frac{1}{2G+1}\, |GM\rangle_d \, , \qquad (53)$$

$$\boldsymbol{e}_\mu \cdot \boldsymbol{\tau}\, |GM\rangle_c = -\frac{1}{2G+1}\, |GM\rangle_a - \frac{2\sqrt{G(G+1)}}{2G+1}\, |GM\rangle_d \, , \qquad (54)$$

$$\boldsymbol{e}_\mu \cdot \boldsymbol{\tau}\, |GM\rangle_d = \frac{1}{2G+1}\, |GM\rangle_b - \frac{2\sqrt{G(G+1)}}{2G+1}\, |GM\rangle_c \, . \qquad (55)$$

These equations can be proved by expanding $\boldsymbol{e}_\mu \cdot \boldsymbol{\tau}$ then multiplying this, from the right, by each G-eigenstate. The right hand side of each equation is then computed and matrix components are compared.

Substituting (37) into (18) and using the identities (38), (39), (42)-(45) and (52)-(55) we are led to two equations, one in $|GM\rangle_a$ and $|GM\rangle_d$ and one in $|GM\rangle_b$ and $|GM\rangle_c$. We equate coefficients of the G-eigenstates to obtain four ODEs in $G_1(u)$, $G_2(u)$, $G_3(u)$ and $G_4(u)$. These ODEs are shown below,

$$\begin{aligned}
(1-u)\frac{dG_1}{du} &= \left(G+\tfrac{1}{2}-\frac{g(1-u)}{2G+1}\right)G_1+(E-gu)G_3-\frac{2g\sqrt{G(G+1)}(1-u^2)}{2G+1}G_4, && (56)\\
(1-u)\frac{dG_2}{du} &= \left(G+\tfrac{3}{2}-\frac{g(1-u)}{2G+1}\right)G_2-(E+gu)G_4+\frac{2g\sqrt{G(G+1)}}{2G+1}G_3, && (57)\\
(1+u)\frac{dG_3}{du} &= -\left(G+\tfrac{1}{2}-\frac{g(1+u)}{2G+1}\right)G_3-(E+gu)G_1+\frac{2g\sqrt{G(G+1)}(1-u^2)}{2G+1}G_2, && (58)\\
(1+u)\frac{dG_4}{du} &= -\left(G+\tfrac{3}{2}-\frac{g(1+u)}{2G+1}\right)G_4+(E-gu)G_2-\frac{2g\sqrt{G(G+1)}}{2G+1}G_1. && (59)
\end{aligned}$$

These equation give the solutions for states with parity $(-1)^G$. Due to the symmetry (36) states with parity $(-1)^{G+1}$ are obtained by replacing g by $-g$ in the equations above.

3.1 The energy spectrum

In this section, we derive the energy spectrum of the time-independent Dirac equation (18) and the corresponding eigenfunctions. We discuss a useful symmetry of our system of equations (56)-(59) and also comment on associated second order and fourth order equations. In order to derive the spectrum, we use the theory of Fuchsian differential equations, and in particular, regular singular points and their exponents, see [25]. Finally, we present the explicit solution.

Under the transformation

$$(G_1(u),G_2(u),G_3(u),G_4(u))\mapsto(-G_3(-u),-G_4(-u),-G_1(-u),-G_2(-u)) \qquad (60)$$

followed by $u\mapsto -u$, (57) is mapped into (59) and (56) into (58), and vice versa. Hence the system of equations (56)-(59) remains invariant. Eliminating $G_1(u)$ and $G_3(u)$ from the system (56)-(59) results in a system of two second order equations, which again map into each other via the symmetry (60). These second order equations prove to be useful for deriving an ansatz for $G_2(u)$ and $G_4(u)$. Finally, we can also derive two fourth order ODEs, by eliminating $G_4(u)$ and $G_2(u)$, respectively, again related via (60). Hence, if we have solutions for $G_2(u)$ we can find solutions for $G_4(u)$. We can then use our system of first order ODEs to find solutions for $G_1(u)$ and $G_3(u)$.

In order to solve our equations we first have to derive the energy eigenvalues. Our system of equations (56)-(59) has regular singular points at $u=\pm1$ and an irregular singular point at infinity. For both regular singular points the exponents are

$$(0,0,-G-\tfrac{1}{2},-G-\tfrac{3}{2}). \qquad (61)$$

We require our solutions to be regular over the whole 3-sphere and, in particular, at the north and south poles, $u = \pm 1$. The solutions corresponding to the exponents $-G - \frac{1}{2}$ and $-G - \frac{3}{2}$ contain poles, so we can exclude these solutions. The fact that there are two exponents taking values of zero means that corresponding to each regular singular point is a solution with logarithmic terms and the solution is therefore singular. As a result, we can also exclude these solutions. The regular solution can therefore be expanded as a power series in $1+u$ around the south pole, and also a power series in $1-u$ around the north pole. These two expansions only agree for certain values of the energy E. It turns out that these energy eigenvalues can be calculated from the exponents at infinity of the fourth order ODE in $G_2(u)$, mentioned above, which arises by eliminating $G_1(u)$, $G_3(u)$, and $G_4(u)$ from our system of equations (56)-(59). This equation is of Fuchsian type and has three regular singular points at $u = \pm 1$ and infinity.

The solutions of Fuchsian differential equations can only have singularities at their singular points. According to our discussion above, we are interested in the solutions of (18) which are non-singular. Therefore, $G_2(u)$ has to be regular at $u = \pm 1$, and hence on the entire complex plane. So, $G_2(u)$ is an analytic function, in fact an integral function, on the complex plane. As it is the solution of a Fuchsian differential equation, it can only have poles at infinity, and it follows that $G_2(u)$ is a polynomial.

The exponents corresponding to $u = \infty$ can be found by setting $u = \frac{1}{z}$ then considering $z \to 0$. We obtain the exponents

$$\rho_s = 1 + G \pm \tfrac{1}{2}\sqrt{1 + 4E^2 + 4E - 4g^2}, \tag{62}$$

$$\rho_a = 1 + G \pm \tfrac{1}{2}\sqrt{1 + 4E^2 - 4E + 8g - 4g^2}. \tag{63}$$

As argued above $G_2(u)$ is a polynomial. Let its degree be denoted by k. Then the exponents at $u = \infty$ can be equated with $-k$. From the exponent ρ_s we obtain the following energy eigenvalues

$$E^{\pm}_{sym} = -\tfrac{1}{2} \pm \sqrt{(k + G + 1)^2 + g^2} \quad \text{for G} = 1, 2, \ldots, \ \text{k} = 0, 1, \ldots \tag{64}$$

$E^{\pm}_{sym}$ is a novel feature which arises for $G > 0$ only. Note that this energy is invariant under $g \mapsto -g$. From ρ_a in (63) we obtain another family of energy eigenvalues, namely,

$$E^{\pm}_{asym} = \tfrac{1}{2} \pm \sqrt{(k + G + 1)^2 - 2g + g^2} \quad \text{for G} = 0, 1, \ldots, \ \text{k} = 0, 1, \ldots, \tag{65}$$

where G and k are not both zero. This energy spectrum has already been obtained in [22] for the case $G = 0$. A slight subtlety occurs for $k = 0$ and $G = 0$. In this case, only

$$E_0 = \tfrac{3}{2} - g \tag{66}$$

leads to a regular solution. The energy level (66) is rather special as it crosses from the positive spectrum to the negative spectrum as the coupling constant g is varied, also see [22] for further details.

Now that we have derived the energy spectrum, we can solve the system (56)-(59) by first considering the fourth order ODE in $G_2(u)$. We make the ansatz that

$G_2(u)$ is a polynomial in $1+u$ and insert this into our ODE to find the polynomial coefficients. The symmetry (60) and the system of second order equations for $G_2(u)$ and $G_4(u)$ leads to a related expression for $G_4(u)$. Then the solution corresponding to $E^{\pm}_{asym}$ is found to be

$$G_2(u) = \sum_{j=0}^{k} a_j (1+u)^j \quad \text{and} \quad G_4(u) = (-1)^k \sum_{j=0}^{k} a_j (1-u)^j. \tag{67}$$

The general expression for a_j is

$$a_j = (-1)^j \left(\begin{array}{c} k \\ j \end{array} \right) \tfrac{(2G+k+j+1)!(2G+1)!!}{(2G+k+1)!(2G+2j+1)!!} \tfrac{(E+g-\frac{3}{2}-G)(E-g+\frac{2j+1}{2}+G)}{2G(k+g)+k(k+2)}. \tag{68}$$

If we set $G=0$, and hence $k=n$, the above formula leads us to (27) which is equivalent to the result from [22].

For $E^{\pm}_{sym}$ we find

$$G_2(u) = \sum_{j=0}^{k} a_j (1+u)^j \quad \text{and} \quad G_4(u) = (-1)^{k+1} \sum_{j=0}^{k} a_j (1-u)^j. \tag{69}$$

The general expression for a_j is now

$$a_j = (-1)^{j+1} \left(\begin{array}{c} k \\ j \end{array} \right) \tfrac{(2G+k+j+1)!(2G+1)!!}{(2G+k+1)!(2G+2j+1)!!} \tfrac{(E+g+\frac{3}{2}+G)(E-g-\frac{2j+1}{2}-G)}{2(G+1)(g-k)-k^2}. \tag{70}$$

We then use equations (59) and (57) to obtain $G_1(u)$ and $G_3(u)$, respectively, and it is easy to see that $G_1(u)$ and $G_3(u)$ are polynomials of order $k+1$.

We can carry out a consistency check on our solutions by setting $g=0$ in equations (56)-(59) and manipulating the equations to obtain two uncoupled second order ODEs. These are both Jacobi equations and have polynomial solutions which can be expressed in terms of hypergeometric functions (see [26]). For $g=0$, our solutions are the same.

3.2 Degeneracy of the energy spectrum

In order to discuss the degeneracy of the energy spectrum it is convenient to introduce $n = k+G$, where the integer n is analogous to the principal quantum number arising in the quantum mechanics of the hydrogen atom. Then the energy spectrum for states of parity $(-1)^G$ is given by the two families

$$\begin{aligned} E^{\pm}_{sym}(n) &= -\tfrac{1}{2} \pm \sqrt{(n+1)^2+g^2} \quad \text{for n} = 1,2,\ldots, & (71)\\ E^{\pm}_{asym}(n) &= \tfrac{1}{2} \pm \sqrt{(n+1)^2-2g+g^2} \quad \text{for n} = 1,2\ldots, & (72) \end{aligned}$$

and the special energy level (66),

$$E_0 = \tfrac{3}{2} - g.$$

Figure 1 shows the energy spectra for different values of n. There are two different ways of reading figure 1. The obvious interpretation is the energy spectrum of states of parity $(-1)^G$ as a function of the coupling constant $g \in \mathbb{R}$. For the second interpretation and in the following, we restrict our attention to $g \geq 0$. Then, the negative values of g corresponds to states with parity $(-1)^{G+1}$ due to symmetry (36), while positive values of g again correspond to states of parity $(-1)^G$. The latter interpretation is very useful for discussing the degeneracy of the spectrum.

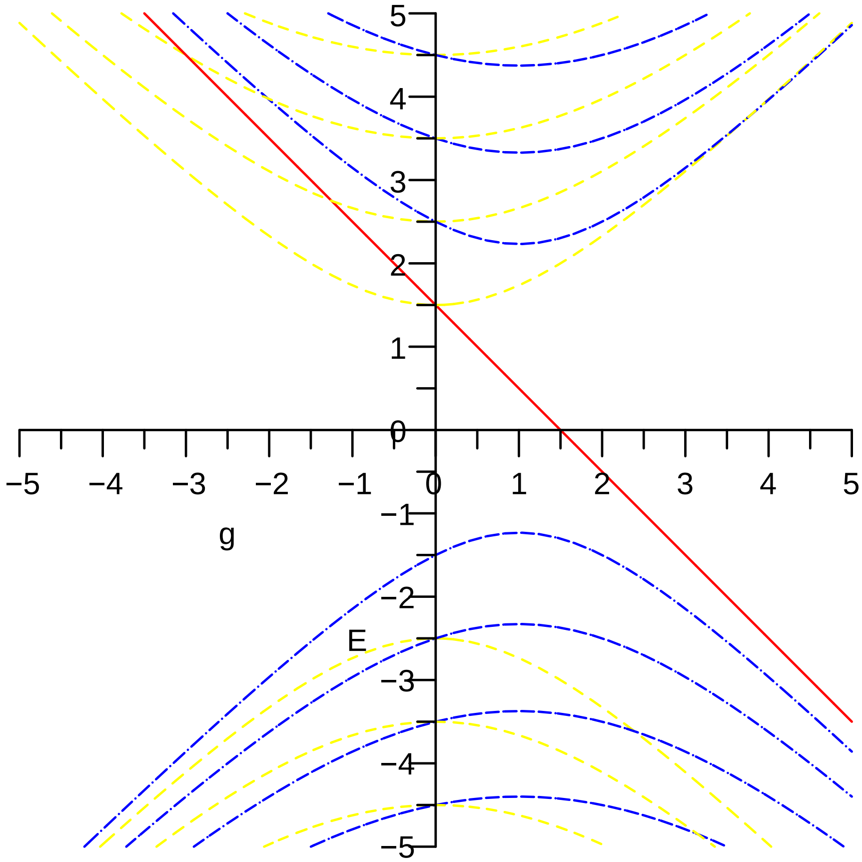

Figure 1: The energy E as a function of the coupling constant g for E_0 (solid red curve), $E^{\pm}_{asym}(n)$ (dotted blue curves) and $E^{\pm}_{sym}(n)$ (dashed yellow curves).

The energy level (66) only exists for $G = 0$. It gives rise to a positive parity state with energy $\frac{3}{2} - g$ and a negative parity state with energy $\frac{3}{2} + g$. Since the degeneracy of a state with grand spin G is $2G + 1$ these two states are non-degenerate for $g > 0$, and "parity doubling" occurs for $g = 0$, [14].

For $E^{\pm}_{asym}(n)$, positive and negative parity states will in general have different energy eigenvalues for a given value of the coupling constant g. Recall that $n = k + G$, hence G can vary from 0 to n. Therefore, the degeneracy is

$$D(E^{\pm}_{asym}(n)) = \sum_{G=0}^{n} (2G + 1) = (n + 1)^2. \tag{73}$$

For $E^{\pm}_{sym}(n)$, positive and negative parity states have the same energy for a given value of the coupling constant g. These states only exist for $G > 0$, hence G varies from 1 to n. Therefore, the degeneracy is

$$D(E^{\pm}_{sym}(n)) = 2\sum_{G=1}^{n}(2G+1) = 2n(n+2), \tag{74}$$

and the extra factor of 2 is due to parity.

We now consider the case of zero coupling ($g = 0$) which is equivalent to massless fermions on S^3. In this case there will clearly always be invariance under $g \to -g$, so

$$D_{g=0}\left(E_0\right) = 2, \quad D_{g=0}\left(E^{\pm}_{asym}(n)\right) = 2\left(n+1\right)^2, \quad D_{g=0}\left(E^{\pm}_{sym}(\tilde{n})\right) = 2\tilde{n}\left(\tilde{n}+2\right). \tag{75}$$

At $E = \frac{3}{2}$ the energy level E_0 and $E^{+}_{sym}(1)$ are degenerate, hence the degeneracy is

$$D_{g=0}(E = \tfrac{3}{2}) = 8. \tag{76}$$

The energy eigenvalue $E = -\frac{3}{2}$ only occurs for $E^{-}_{asym}(1)$, hence the degeneracy is again

$$D_{g=0}(E = -\tfrac{3}{2}) = 8. \tag{77}$$

The energy eigenvalue $E = N + \frac{3}{2}$, $N = 1, 2, \ldots$, is attained by $E^{+}_{asym}(N)$, and $E^{+}_{sym}(N+1)$, hence

$$D_{g=0}\left(E = N + \tfrac{3}{2}\right) = 2(N+1)(N+3) + 2(N+1)^2 = 4(N+1)(N+2). \tag{78}$$

Similarly, $E = -N - \frac{3}{2}$ is attained by $E^{-}_{asym}(N+1)$ and $E^{-}_{sym}(N)$, hence

$$D_{g=0}\left(E = -N - \tfrac{3}{2}\right) = 2(N+2)^2 + 2N(N+2) = 4(N+1)(N+2). \tag{79}$$

After considering the factor of 2 due to isospin and another factor of 2 due to parity, equations (78) and (79) are consistent with the results in [27].

So far, we have only considered generic degeneracies and the case $g = 0$. This energy spectrum is rather special in that we can also calculate all the accidental degeneracies for $g > 0$. These degeneracies all occur for rational values of g. For example, the negative parity state with energy $\frac{3}{2} + g$ is only degenerate with the states with $E^{+}_{sym}(n)$ for

$$g = \tfrac{1}{4}(n-1)(n+3) \tag{80}$$

and with the $(-1)^{G+1}$ parity states with $E^{-}_{asym}(n)$ (changing g to $-g$) for

$$g = \tfrac{1}{4}n(n+2). \tag{81}$$

The positive parity state with energy $\frac{3}{2} - g$ is always non-degenerate for $g > 0$. Similarly, $(-1)^G$ parity states of energy $E^{\pm}_{asym}(n)$ are degenerate with $(-1)^{G+1}$ parity states of energy $E^{\pm}_{asym}(\tilde{n})$ for

$$g = \tfrac{1}{4}n(n+2) - \tfrac{1}{4}\tilde{n}(\tilde{n}+2), \tag{82}$$

which is positive for $n > \tilde{n}$. Finally, states with energy $E^+_{asym}(n)$ and states with energy $E^+_{sym}(\tilde{n})$ are degenerate for

$$g = \frac{4(\tilde{n}+1)^2 - \left(1 + (\tilde{n}+1)^2 - (n+1)^2\right)^2}{4\left(1 + (\tilde{n}+1)^2 - (n+1)^2\right)}, \tag{83}$$

and a similar equation holds for $E^-_{asym}(n)$ and $E^-_{sym}(\tilde{n})$.

4 The Dirac Sea

In this section, we briefly comment on the zeta function regularization [28]. In order to calculate the energy of the Dirac sea,

$$E_{\text{Dirac}} = \sum_{N=0}^{\infty} D(N)E(N), \tag{84}$$

where $E(N)$ is the Nth negative energy eigenvalue, see e.g. [29], and $D(N)$ is its degeneracy, we define the zeta function

$$\zeta(s) = \sum_{N=0}^{\infty} D(N)E(N)^{-s}. \tag{85}$$

The expression (84) is clearly divergent. However, the expression (85) is convergent for large enough s. The Dirac sea energy (84) is then defined by the analytic continuation of (85) to $s = -1$. For example for $g = 0$ we have

$$E_{g=0} = -4 \sum_{N=0}^{\infty} (N+1)(N+2)(N+\tfrac{3}{2})^{-s}\Big|_{s=-1}. \tag{86}$$

Hence, the relevant zeta function is

$$\zeta_{g=0}(s) = -4 \sum_{N=0}^{\infty} ((N+\tfrac{3}{2})^2 - \tfrac{1}{4})(N+\tfrac{3}{2})^{-s}, \tag{87}$$

which can be rewritten as

$$\zeta_{g=0}(s) = -4\zeta_H(s-2, \tfrac{3}{2}) + \zeta_H(s, \tfrac{3}{2}), \tag{88}$$

where $\zeta_H(s, a)$ is the Hurwitz zeta function defined as

$$\zeta_H(s, a) = \sum_{n=0}^{\infty} (n+a)^{-s}. \tag{89}$$

Evaluating $\zeta_{g=0}(s)$ at $s = -1$ we obtain

$$E_{g=0} = \frac{17}{240}. \tag{90}$$

For massive fermions on S^3 corresponding to the case $f(\mu) = 0$ in Section 2.1, the energy is given by (22) and the degeneracy is

$$D(N) = 4(N+1)(N+2). \tag{91}$$

Hence the Dirac sea energy is given by

$$E_{f(\mu)=0} = -4 \sum_{N=0}^{\infty} (N+1)(N+2) \left(\left(N + \tfrac{3}{2}\right)^2 + g^2 \right)^{-s} \Big|_{s=-\frac{1}{2}}. \tag{92}$$

Zeta functions of generalized Epstein-Hurwitz type are of the form

$$F\left(s; a, b^2\right) = \sum_{n=0}^{\infty} \left((n+a)^2 + b^2\right)^{-s}. \tag{93}$$

Asymptotic expansions for (93) are discussed in [30, 28]. Here, we are concerned with a generalization of (93), namely,

$$F^{(m)}\left(s; a, b^2\right) = \sum_{n=0}^{\infty} (n+a)^m \left((n+a)^2 + b^2\right)^{-s}, \tag{94}$$

where we assume that $a > 0$ and $b \geq 0$. We follow [28] to derive a formula for $F^{(m)}(s = -\frac{1}{2}; a, b^2)$. We first perform a binomial expansion which is valid for $b < a$ and rewrite (94) as a contour integral

$$F^{(m)}\left(s; a, b^2\right) = \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} (-1)^k \frac{\Gamma(s+k)}{\Gamma(k+1)\Gamma(s)} b^{2k} (n+a)^{-2s-2k+m}, \tag{95}$$

$$= \sum_{n=0}^{\infty} \frac{1}{2\pi i} \int_C \frac{\Gamma(s+z) b^{2z} (n+a)^{-2s-2z+m}}{\Gamma(z+1)\Gamma(s)} \frac{\pi}{\sin(\pi z)} \mathrm{d}z. \tag{96}$$

Recall that

$$\frac{\pi}{\sin(\pi z)} = \frac{(-1)^k}{z-k} + O(z-k) \quad \text{for } k \in \mathbb{Z}. \tag{97}$$

The contour C encloses all the non-negative poles of $1/\sin(\pi z)$ with anti-clockwise orientation and can be split into a part

$$\int_{-z_0+i\infty}^{-z_0-i\infty},$$

where $0 < z_0 < \frac{1}{2}$, and a semi-circle at infinity. The latter does not contribute to the integral. Now, we can move the sum over n under the integral and use the definition of the Hurwitz zeta function (89) to obtain

$$F^{(m)}\left(s; a, b^2\right) = \frac{1}{2i} \int_{-z_0+i\infty}^{-z_0-i\infty} \frac{\Gamma(s+z)\zeta_H(2s+2z-m, a) b^{2z}}{\Gamma(z+1)\Gamma(s)\sin(\pi z)} \mathrm{d}z. \tag{98}$$

This can be evaluated by closing the contour again, and using Cauchy's theorem. This time the contribution of the integral over the semi-circle at infinity is non-zero. However, it was shown in [30] that the contribution is very small, so we neglect it in the following.

From now on, we focus on the physically relevant value $s = -\frac{1}{2}$. The integral (98) has poles at $z \in \mathbb{Z}$ due to $1/\sin(\pi z)$. Only the non-negative poles contribute, because of the contour. The gamma function $\Gamma(z - \frac{1}{2})$ has poles at $z - \frac{1}{2} = 0, -1, -2, \ldots$ Only the pole at $z = \frac{1}{2}$ lies inside the contour. Finally, there is a contribution from the pole of the Hurwitz zeta function at $2z - 1 - m = 1$. All the poles are simple unless the pole of ζ_H at $z = 1 + \frac{m}{2}$ is a non-negative integer. Hence, the integral in (98) becomes

$$F^{(m)}\left(-\tfrac{1}{2}; a, b^2\right) \approx \mathrm{Res}_{z=\frac{1}{2}} + \mathrm{Res}_{z=1+\frac{m}{2}} + \sum_{\substack{k=0 \\ k \neq 1+\frac{m}{2}}}^{\infty} (-1)^k \frac{\Gamma\left(k-\frac{1}{2}\right)}{k!\Gamma\left(-\frac{1}{2}\right)} \zeta_H(2k-1-m, a) b^{2k}, \quad (99)$$

where the sum arises from the simple poles of $1/\sin(\pi z)$. Note that

$$\Gamma(\epsilon) = \frac{1}{\epsilon} - \gamma + O(\epsilon) \quad (100)$$

and

$$\zeta_H(-k, a) = -\frac{B_{k+1}(a)}{k+1}, \quad \text{for } k \in \mathbb{N}, \quad (101)$$

where $B_m(a)$ are the Bernoulli polynomials and γ is the Euler-Mascheroni constant. Hence, the residue at $z = \frac{1}{2}$ gives

$$\mathrm{Res}_{z=\frac{1}{2}} = \frac{B_{m+1}(a)}{m+1} b. \quad (102)$$

Finally, for the residue at $z = 1 + \frac{m}{2}$, we note that

$$\zeta_H(1+\epsilon, a) = \frac{1}{\epsilon} - \Psi(a) + O(\epsilon), \quad (103)$$

where $\Psi(a) = \frac{d}{da} \ln \Gamma(a)$ is the digamma function, see [25, p. 271]. The behaviour depends on whether m is even or odd. For odd m this is just another simple pole, and we obtain

$$\mathrm{Res}_{z=1+\frac{m}{2}} = (-1)^{\frac{m-1}{2}} \frac{\sqrt{\pi}\Gamma(\frac{m+1}{2})}{4\Gamma(2+\frac{m}{2})} b^{2+m}. \quad (104)$$

However, for even m there is a double pole, and we have to use

$$\mathrm{Res}_{z=1+\frac{m}{2}} = \lim_{z \to 1+\frac{m}{2}} \frac{d}{dz} \left(\left(z - 1 - \tfrac{m}{2}\right)^2 \frac{\pi \Gamma(s+z) \zeta_H(2z-1-m, a)\ b^{2z}}{\Gamma(z+1)\Gamma(-\frac{1}{2}) \sin(\pi z)} \right) \quad (105)$$

to obtain

$$\begin{aligned} \mathrm{Res}_{z=1+\frac{m}{2}} &= (-1)^{\frac{m}{2}} \frac{b^{2+m}\Gamma(\frac{m+1}{2})}{4m(2+m)\sqrt{\pi}\Gamma(2+\frac{m}{2})} \quad (106) \\ &* \left(\left(\Psi(\tfrac{m+1}{2}) - \Psi(\tfrac{m}{2}) - 2\Psi(a) + 2\ln(b) \right) m(m+2) - 4(1+m) \right). \end{aligned}$$

We now use the same trick as in (87) to rewrite the regularized energy in (92) as

$$E_{f(\mu)=0} = -4F^{(2)}(-\tfrac{1}{2};\tfrac{3}{2},g^2) + F^{(0)}(-\tfrac{1}{2};\tfrac{3}{2},g^2). \tag{107}$$

The regularized energy is plotted in figure 2. As a consistency check it can be shown that $E_{f(\mu)=0}(g=0) = \frac{17}{240}$ as calculated in (90). It would be interesting to compare these results to other regularization methods.

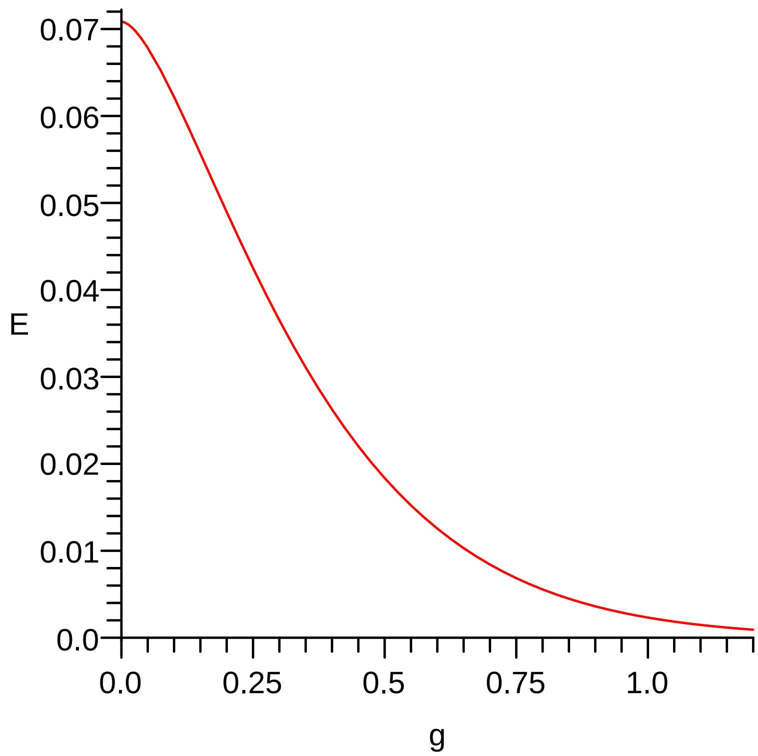

Figure 2: The Dirac sea energy $E_{f=0}$ as a function of the coupling constant g.

Finally, we address the problem of calculation the Dirac sea energy for fermions coupled to a $B=1$ background Skyrmion. In this case the Dirac Sea energy is given by

$$E_{f(\mu)=\mu} = -\sum_{n=1}^{\infty}\left(2n(n+2)\left(\tfrac{1}{2}+\sqrt{(n+1)^2+g^2}\right)^{-s}\right. \tag{108}$$
$$\left. +(n+1)^2\left(-\tfrac{1}{2}+\sqrt{(n+1)^2+2g+g^2}\right)^{-s} + (n+1)^2\left(-\tfrac{1}{2}+\sqrt{(n+1)^2-2g+g^2}\right)^{-s}\right)\Big|_{s=-1},$$

which is the sum of E_{sym} and E_{asym} for both parities. The energy of the "zero mode" $E_0 = \frac{3}{2} - g$ also needs to be taken into account, and we expect a similar picture as in [17].

Unfortunately, this is a much more complicated situation, and zeta functions of this type have not been discussed in the literature, to our knowledge. As a starting point, we could again perform a binomial expansion. We can then rewrite the energy $E_{f(\mu)=\mu}$ as an infinite sum of zeta functions $F^{(m)}(-\frac{1}{2};a;b^2)$. Unfortunately, the last term in (108) leads to $b^2 = -2g + g^2$ which is negative for small g, and our formula no longer converges. It would be interesting to derive alternative expressions for these types of zeta function.

5 Conclusion

In this paper we consider the Dirac equation for fermions on S^3 chirally coupled to a spherically-symmetric background Skyrmion with topological charge one. The time-independent Dirac equation commutes with the grand spin and parity, and these symmetries allow us to reduce the Dirac equation to a system of four linear ODEs. Making use of the theory of Fuchsian differential equations, we derive the complete energy spectrum and the corresponding eigenfunctions which are given by polynomials. There is a positive parity state with energy

$$E_0 = \tfrac{3}{2} - g$$

and a negative parity state with energy

$$E_0 = \tfrac{3}{2} + g.$$

Both states are generally non-degenerate. The energies

$$E^{\pm}_{asym}(n) = \tfrac{1}{2} \pm \sqrt{(n+1)^2 - 2g + g^2}$$

and

$$E^{\pm}_{asym}(n) = \tfrac{1}{2} \pm \sqrt{(n+1)^2 + 2g + g^2}$$

all have degeneracy $(n+1)^2$, and correspond to states with parity $(-1)^G$ and $(-1)^{G+1}$, respectively. For $G = 0$, these energies were found in [22]. Finally, the energies

$$E^{\pm}_{sym}(n) = -\tfrac{1}{2} \pm \sqrt{(n+1)^2 + g^2}$$

have degeneracy $2n(n+2)$. The factor of 2 arises because the energy of these states is independent of parity. Furthermore, these states only occur for $G > 0$.

For zero coupling ($g = 0$) the energy spectrum is $E = \pm(N + \frac{3}{2})$ and the degeneracy was found to be

$$D = 4(N+1)(N+2), \tag{109}$$

in agreement with [27]. We also found explicit formulae for accidental degeneracies which occur for special values of the coupling constant g.

The explicit formulae for the energy spectrum and its degeneracy enabled us to write down the zeta function related to the Dirac sea. For massive fermions on S^3, we were able to derive an asymptotic formula for a zeta function of generalized Epstein-Hurwitz type. The more interesting case of fermions coupled to Skyrmions on S^3 leads to an interesting novel type of zeta function. However, we were unable to evaluate it using our current approach. This is an interesting topic for further study.

Acknowledgements

The authors would like to thank N S Manton for fruitful discussions. SWG would like to gratefully acknowledge funding from EPSRC and from the School of Mathematics, Statistics and Actuarial Science at the University of Kent.

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