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Incorporating Value Judgments in DEA Models

Wenbin Liu, KBS, University of Kent, UK

Wei Meng, Institute of Physics Science, Chinese Academy of Sciences, China

T.Q. Zhang, Institute of Policy and Management Science, Chinese Academy of Sciences, China

1. Introduction

The issue of reflecting value judgments very frequently arises in both theoretical investigations and practical applications of DEA. As Allen et al. (1997, p.2) presented: the concept of value judgments, albeit frequently discussed, is lacking a formal definition in the context of DEA, and therefore value judgments are considered as *“logical constructs, incorporated within an efficiency assessment study, reflecting the Decision Makers’ (DMs) preferences in the process of assessing efficiency.* The immediate intention of incorporating value judgments into DEA methodology is to reflect prior views or information in efficiency assessments. This prior information can be incorporated in different ways, and then has different implications on the outcomes of the assessments.

This chapter examines the issue of incorporating value judgments in DEA, some available approaches and their advantages and drawbacks. The chapter is organized as follows: The next section highlights motivations of incorporating value judgments in DEA models, and reviews some existing approaches. In Section 3, we further discuss the approaches related to weights restrictions. In Section 4, methods associated with preferences changes are introduced from a viewpoint of changing the default preference in DEA models. Relationships and interpretations of these approaches are discussed in Section 5. In the sixth section, we conclude this chapter.

2. Value Judgments in DEA: purposes and motivations

Value judgments are often closely associated with preferences used in performance evaluation processes, as explained in Liu et al. (2006). Preferences are a set of rules that are explicitly or implicitly assumed in a performance evaluation in order to judge whether or not some input-output data sets (thus DMUs) are superior to others. For instance, under Pareto preference, which is widely used in performance evaluations

and assumed in the classic DEA models, one desirable output vector is said to be higher (or better) than another one only if each component of the former is not lower than that of the latter respectively. Thus under Pareto preference, one questionable but legitimate strategy for a DMU to become efficient is to maximize its efficiency only over a subset of the output components (e.g., any single output), while totally ignoring its performance on the other ones, because then no other DMUs can possibly beat it over this subset of the outputs thus in Pareto preference. For instance let us assume that for the same amount of time spent on their studies, chemistry students A, B, and C have the examination results (71, 2, 3), (70, 70, 70), and (50, 50, 99) on mathematics, physics, and chemistry respectively. Most teachers may well think that the student B is the most efficient, but under Pareto preference that is used in standard DEA models, they are all efficient as no one is better than the others on all outputs i.e. A outperforms B and C on mathematics, B outperforms A and C on physics, and C outperforms A and B on chemistry. Such an assessment may not be desirable for the DMs or for the objectives of evaluation, as most of our universities will not allow too low results across two subjects, while some chemistry teachers may think that the student C is very talented. Such relevant a priori information may often need to be incorporated in DEA so that the value judgments will be reflected in the outcomes of the assessments.

Next, we discuss why it may be necessary to incorporate value judgments in real-life applications of classical DEA models.

2.1 Incorporating prior knowledge of the variables

There are several inter-related problems associated with the default preference of DEA models when applying them in performance evaluations, which frequently occur in the real-life applications. Among them is the problem of unrealistic weight distribution, where some DMUs are identified as efficient simply because they have relatively very large (or small) weights in a single output (or input) while these extreme weights may be practically undesirable. This is illustrated through the following multiplier DEA model and the results are shown in Table 1.

$$\begin{aligned}
& \text{Max} && \sum_{r=1}^s u_r y_{r0} \\
& \text{Subject to:} && \sum_{r=1}^s u_r y_{rj} - \sum_{i=1}^m v_i x_{ij} \leq 0 \\
& && \sum_{i=1}^m v_i x_{i0} = 1 \quad (2.1) \\
& && u_r, v_i \geq 0, \\
& && i = 1, \dots, m, r = 1, \dots, s, j = 1, \dots, n
\end{aligned}$$

Table 1

Student	Mathematics	Physics	Chemistry	Score	u_1^*	u_2^*	u_3^*
A	71	2	3	1	0.0141	0	0
B	70	70	70	1	0	0.0099	0.0044
C	50	50	99	1	0	0	0.0101

where x_{ij}, y_{rj}, v_i, u_r are the inputs, outputs and the corresponding weights respectively.

Clearly some outputs are ignored in this DEA assessment by assigning zero weight, such as physics and chemistry for student A. This phenomenon often occurs in practical applications of DEA. As matter of fact, in the first DEA paper on performance evaluation of the “program follow through” in the USA, Charnes et al. (1978) observed that many DMUs were rated efficient by putting their output weights solely on “self esteem” and ignoring performance on mathematics and verbal reasoning.

The default preference in standard DEA models such as CCR and BCC is Pareto preference, assuming no prior preference and knowledge on any of the inputs and outputs, so that a DMU has the complete freedom to select the weights that are most favorable for its assessment to achieve the maximum efficiency score. This full flexibility of selecting weights is important in the identification of inefficient DMUs. Nevertheless, weights with full flexibility may not always be suitable in practical applications when prior information or value judgments of DMs need to be incorporated on performance evaluation, such as the marginal rates of substitution between the inputs and/or outputs, or relative importance of the inputs and outputs. Still taking the chemistry students as an example, if DMs believe that the students must make good progress in a broad spectral of subjects, so that the weight for each subject in the DEA model should not be allowed too low, say, lower than 0.0045. If these constraints are added into Model 2.1, then only student B is rated as efficient. Other the other hand, if the chemistry department believes that the weights for the

chemistry are more important than those of physics and mathematics, then one may add other weights constraints to incorporate such value judgments in the multiplier DEA model as seen in the next sections. In fact weights restriction is an easy-to-use and yet powerful approach in incorporating prior information and value judgments in DEA models, and has been extensively studied in the DEA literature, such as Allen et al. (1997), Thanassoulis et al. (2004). Weights restrictions can be used to reflect prior knowledge on marginal rates of substitution and/or transformation of the factors of production in Thanassoulis (1995), or to capture special interdependencies between the inputs and outputs of the production process being modeled in Beasley (1990), Thanassoulis et al. (1995), or to incorporate price information in Charnes et al. (1990). In the next section we will further discuss how to set up appropriate weights restrictions according to application needs.

For many applications, the dual DEA models can be modified to effectively incorporate prior information on some variables, such as the marginal rates of substitution between the inputs and/or outputs and relative importance of the inputs and outputs. The most well known example is probably the Cone-Ratio Model, see Charnes et al. (1989, 1990), and Brockett et al. (1997). This approach can be interoperated from a point view of date transformations, see Thanassoulis et al. (2004). In Liu et al. (2006), it was regarded as a special case of another approach: Preferences Changes – to change the default preference of DEA models to reflect such information, since Pareto preference does not allow substitutions between inputs or outputs. These will be further explained below and in Section 4.

2.2 Incorporating preferences on inputs and outputs

Pareto preference implies that all the components of inputs and outputs are equally important in the evaluations, but this may not be desirable in some real-life applications. In fact the DMs often have some preferences on relative importance of the input or output components. For instance in assessing scientific research, often quality of research is regarded as of dominate importance over the other outputs. In the above simple example in Table 1, since the students are in a chemistry department, the department may put more emphasis on the examination results of chemistry, and wishes to incorporate this value judgment into DEA evaluation. One of the most difficult tasks of incorporating such value judgments is how to quantify them into DEA models. Usually these preferences are expressed in vague daily languages like

“more important” or “very valuable”, and have to be translated into precise mathematical relationship in DEA models. One widely used approach is again weights restrictions. In fact since the weights in the DEA models represent relative values attached to inputs and outputs, to some extent putting extra weights restrictions in the DEA models can reflect preferences on relative importance of the inputs and the outputs. In our students’ example, if one adds the weights restrictions like: $v_3 \geq v_2 \geq v_1$ in the multiplier DEA model to reflect the DMs’ preference on importance of the marks of the subjects, then students B and C are now classified as efficient. This approach will be further discussed in Section 3.

On the other hand, value judgments can be incorporated into DEA directly by changing the default preference in dual DEA models as discussed recently in Liu et al. (2006). Let us explain this approach by examining the following CCR model:

$$\begin{array}{ll} \text{Max} & \theta \\ \text{S.t.} & X\lambda \leq X_0, \\ & Y\lambda \geq \theta Y_0, \\ & \lambda \geq 0 \end{array}$$

Basically this approach (to be referred to as the approach of Preferences Changes) replaces the original meanings of the inequalities (\geq or \leq) in the dual DEA models with something that can reflect the desired preferences in an application. Still take the students’ case as an example, the second inequality for the outputs in the above model originally means greater than or equal in all the marks of the three subjects. If one replaces the meaning of this inequality in the above model by a suitable preference so that “ \geq ” means “to have higher examination scores both in chemistry and in the total”, then student A is classified as inefficient by the modified DEA model. One can also replace it by “to have higher total examination scores”. This approach will be discussed in Section 4. Some existing DEA models like the Cone-model and DEA model with preferences in Zhu (1996) can be related to this approach.

2.3 Improving discrimination between DMUs

When the number of DMUs under performance evaluation is small compared to the total number of inputs and outputs, the problem of weak discriminating power often occurs. In this situation, classical DEA models, such as CCR, BCC models, often identify too many DMUs as efficient. This is again partly due to the fact that DEA

models like CCR, BCC, allow full flexibility in the selection of weights so that many DMUs will be able to achieve the maximum DEA efficiency score, especially if the number of DMUs is relatively small to the total number of inputs and outputs since it is then more likely that each DMU specializes on a specific input-output mix not directly comparable with that of the other DMUs. Thus weights restrictions can much increase discrimination power of a DEA model, and the example in Table 1 has demonstrated this. Thompson et al. (1986) and Cook et al. (1991) addressed the issue of discriminating between efficient DMUs through weights restrictions. Similarly the preferences in DEA dual models affect discrimination power of the models. For instance, by replacing the Pareto preference with the preference of total average in the dual CCR DEA model, only student B is classified as efficient. Some DEA models with the preferences of total or part averaging have been successfully applied on smaller data sets, for example, in Zhu (1996) and Meng et al. (2005).

Discrimination between efficient DMUs can also be addressed by other methods like the super-efficiency DMU model in Andresen and Petersen (1993), the cross-evaluation procedure in Green et al. (1996), and multiple-objective programs in Li and Reeves (1999).

2.4 Methods for incorporating Value Judgments

The issue of incorporating value judgments should be considered even before DEA models are selected and computations start. In fact careful studies should be carried out on selections of all the possible input-output indicator combinations, and their implications on value judgments should be highlighted and discussed with all the stakeholders in an evaluation. Agreed value judgments should be borne in mind throughout the processes of indicator selection. Furthermore some outlets can be removed from an evolution during these processes according to the agreed value judgments. For example, if research quality is the key objective of a research evaluation, then maybe only publications in the very top academic journals should be counted when selecting output data for that evaluation.

In the rest of the chapter we will concentrate on the issue of how to incorporate value judgments through DEA models. In the DEA literature there exist several types of approaches to incorporate value judgments in DEA models, some of which have been discussed above, such as

1. **Weights Restrictions (WRs).** This approach incorporates value judgments by adding suitable weights restrictions on the weights in the multiplier DEA models as explained above;
2. **Data Transformations (DTs).** This approach incorporates value judgments by translating the original input-output data sets into new ones and then applying the standard DEA models;
3. **Preferences changes (PCs).** This approach incorporates value judgments by replacing the default Pareto preference in the dual DEA models with some more suitable preferences.

There exist several other interesting ideas used to incorporate value judgments in DEA, such as, restricting (extending) facets of the efficient frontier in Bessent et al. (1988) and Olesen and Petersen (1996), and formulating DMs preferred value functions in Halme et al. (1999). According to the discussions in Liu et al. (2006), at least value judgments can be reflected also through expanding the PPS set (see Thanassoulis et al. (2004) for adding extra DMUs), and selecting suitable merit measurements. These approaches have been used to incorporate prior information in DEA models, although we will not discuss them in further details.

In the next section, we will focus on the DEA models with the weights restrictions. In the fourth section, we will discuss the DEA models with preferences different from the Pareto's. For simplicity, we will often restrict the discussion to the classical CCR DEA model (Charnes et al. 1978), which implicitly assumes that the DMUs are constant returns to scale. All the inputs and outputs are assumed to be desirable and positive.

3. DEA models with weights restrictions

3.1 Constraint forms of weights restrictions

Generally weights restrictions incorporate value judgments in DEA through the models of multiplier type, such as Model 3.1 below. In order to make the coming explanations simpler, CCR multiplier models with a constant return to scale are to be used throughout this section.

Model 3.1(multiplier model)

Model 3.2(dual model)

$$\begin{array}{ll}
\text{Max} & \sum_{r=1}^s u_r y_{r0} \\
\text{Subject to:} & \sum_{r=1}^s u_r y_{rj} - \sum_{i=1}^m v_i x_{ij} \leq 0 \\
& \sum_{i=1}^m v_i x_{i0} = 1 \\
& u_r, v_i \geq 0, \\
& i = 1, \dots, m, r = 1, \dots, s, j = 1, \dots, n
\end{array}
\qquad
\begin{array}{ll}
\text{Min} & \theta \\
\text{Subject to:} & \sum_{j=1}^n x_{ij} \lambda_j \leq \theta x_{i0} \\
& \sum_{j=1}^n y_{rj} \lambda_j \geq y_{r0} \\
& \lambda_j \geq 0, \\
& i = 1, \dots, m, r = 1, \dots, s, j = 1, \dots, n
\end{array}$$

In a CCR multiplier model like Model 3.1, the weights (u_r and v_i) assigned to DMU₀ originally have full flexibility as long as the linear programming constraints are satisfied. In some ways, the weights u_r and v_i in Model 3.1 reflect relative importance of the inputs and outputs for each assessed DMU, although the precise meanings could be quite subtle and are context dependent as to be seen later. Allen et al. (1997) classified weights restrictions on the multiplier models into three major categories as: 1) absolute weights restriction, 2) assurance regions of type I: to represent the relative weights relationship within inputs variables or outputs variables respectively. And assurance region of type II: to represent the relative relationship between inputs and outputs. 3) Restrictions on virtual inputs and outputs. Table X below summaries them:

Categories	Formulas	
	WRs On inputs	WRs on outputs
Absolute WRs	$\delta_i \leq v_i \leq \tau_i$ (a ₁)	$\rho_r \leq u_r \leq \eta_r$ (a ₂)
Assurance region of type I	$k_i v_i + k_{i+1} v_{i+1} \leq v_{i+2}$ (b1)	$w_r u_r + w_{r+1} u_{r+1} \leq u_{r+2}$ (b2)
	$\alpha_i \leq \frac{v_i}{v_{i+1}} \leq \beta_i$ (c1)	$\theta_r \leq \frac{u_r}{u_{r+1}} \leq \phi_r$ (c2)
Assurance region of type II	$\gamma_i v_i \leq u_r$ (d)	
WRs on virtual input and outputs	$\psi_i \leq \frac{v_i x_{ij}}{\sum_{i=1}^m v_i x_{ij}} \leq \varrho_i$ (e1)	$\varpi_r \leq \frac{u_r y_{rj}}{\sum_{r=1}^s u_r y_{rj}} \leq \sigma_r$ (e2)

* The Greek letters ($\delta_i, \tau_i, \rho_r, \eta_r, \kappa_i, \omega_r, \alpha_i, \beta_i, \theta_r, \phi_r, \gamma_i, \psi_i, \theta_i, \varpi_r, \sigma_r$) are user-specified constants to reflect value judgment of DMs in the formulae (a)-(e).

1. Absolute weights restrictions

Absolute weights restrictions directly provide lower and/or upper bounds for input

weights v_i and/or output weights u_r , as the formulas (a1) and (a2) below show:

$$\delta_i \leq v_i \leq \tau_i \quad (a_1) \qquad \rho_r \leq u_r \leq \eta_r \quad (a_2)$$

This type of weights restrictions was first introduced by Dyson and Thanassoulis (1988) on an application to rate departments. Cook et al. (1991) presented an application of this type on evaluation of highway maintenance patrols. Allen et al. (1997) explained that initially this type of weights restrictions was mainly introduced to prevent the inputs or outputs from being over emphasised or ignored in the assessment. It is straightforward to include the constraints of type (a1) and/or (a2) into Model 3.1. However, how to interpret the meanings of these bounds parameters in efficiency assessment is still a challenge, since weights in DEA models are significant on a relative basis. In general the values of the bounds are context dependent. Another difficulty is associated with the potential infeasibility of DEA models with absolute weights restrictions; see for example, Podinovski and Athanassopoulos (1998) and Podinovski (1999, 2004). It was also noted that switching from an input orientation to an output orientation might produce different efficiency scores for DEA models with absolute WRs, see Allen et al. (1997).

It is important to realise that there is an implicit interdependence between the bounds on different weights when applying absolute weight restrictions. For example, in our student example above, there is only one input and it is assumed to be the same for all the DMUs. Thus the total virtual outputs have to be less than or equal to the unit, and when we impose a lower bound for one output weight, we have also implicitly set the upper bounds for the other output weights. In this example, if the absolute lower bounds are set to be lower than 0.004 then these restrictions have no effects to the model. However, if the lower bounds are higher than 0.005, then the constrained models are infeasible, as explained above. Thus it is neither straightforward nor intuitive to set suitable absolute WRs for this case. Sometimes, it is helpful to first examine the unbounded the solutions in model 2.1, and then to decide the appropriate bound levels. Therefore we can set the lower bounds to be 0.0045. Then only student B is rated as efficient, and the computation results are as follows.

Table2

Student	Score	u_1^*	u_2^*	u_3^*
A	0.397786	0.0053	0.0045	0.0045
B	1	0.0045	0.0045	0.0053

C	0.973286	0.0045	0.0045	0.0053
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Roll and Golany (1993) studied the approach of using unbounded DEA models to decide the weights restrictions systematically. When absolute weights restrictions are used in the DEA models of multiplier form, new variables are introduced in the dual models not only on the constraints but also the objective function. Customary interpretation of radial measurement in standard DEA models may break down when absolute weights restrictions are incorporated. How to test the validation of a model with absolute weights restrictions and how to interpret the results clearly and make it is meaningful for DMs and managers have been extensively discussed. Podinovski (2001, 2004) proposed to replace the traditional DEA objective functions (which measure absolute efficiency) by max-min objective functions to measure relative efficiency in order to avoid mis-representation of unit relative efficiency.

2. Assurance region method

(1) Assurance regions of type I (WRs within inputs or outputs)

$$k_1 v_i + k_{i+1} v_{i+1} \leq v_{i+2} \quad (b_1) \qquad w_r u_r + w_{r+1} u_{r+1} \leq u_{r+2} \quad (b_2)$$

$$\alpha_i \leq \frac{v_i}{v_{i+1}} \leq \beta_i \quad (c_1) \qquad \theta_r \leq \frac{u_r}{u_{r+1}} \leq \phi_r \quad (c_2)$$

These types of restrictions impose relative lower and/or upper bounds for either inputs weights (b_1 and c_1) or outputs weights (b_2 and c_2), or both to incorporate value judgments in the models. They were first used by Thompson et al. (1986) to improve discrimination on laboratory site selections. Since then, such weights restrictions have been applied in various applications, and among them type (c) restrictions were used most frequently. ARIs are particularly suitable when there is a priori information on marginal rates of substitution between inputs and/or outputs. For instance, when all the weights are set to be the same in (c), then the total sums of the inputs or outputs replace the virtual inputs or outputs. Generally, this type of weights restrictions shares similarity with assessing trade-off in multi-criteria decision analysis. The difference between multi-criteria decision analysis and DEA is that the former aims to identify the trade-off exactly, while DEA leaves some weight flexibility, see, e.g. Dyson et al. (2001).

In many applications, these types of weights restrictions are explicit and meaningful to DMs and managers, although the bound values for ARI are dependent on the scaling of the inputs or outputs, and are sensitive to non-commensurable indicators, as discussed in Allen et al. (1997). Take the chemistry students' case as example; if we add the constraints $u_3 \geq u_2 \geq u_1 \geq 0$ into the multiplier model to reflect the value judgments that the marks of chemistry are more important than those of physics and the latter are important than those of mathematics, and then we will have the following results:

Table3

Students	Score	u_1^*	u_2^*	u_3^*
A	0.3619	0.0048	0.0048	0.0048
B	1	0.0026	0.0058	0.0058
C	1	0.0000	0.0067	0.0067

Furthermore unlike absolute weights restrictions, Charnes et al. (1990) and Thompson et al. (1990) pointed out that different oriented DEA models produce consistent results.

Bounds in ARIs generally reflect information on marginal rates of substitution between inputs and/or outputs, and such information can be obtained by summarizing expert opinions on the comparative important between inputs or outputs, see Beasley (1990) and Takamura (2003), or by combining expert opinions with price or cost information as presented in Thompson et al. (1990). Usually such price information is not accurate and a range of prices is used instead, see Camanho and Dyson (2005).

Meanwhile, when weights restrictions of ARI are incorporated in the multiplier DEA model, their managerial meanings may need to be clarified and resolved by using its dual model. However, when complex weights restrictions are used, its dual model may have complicated structures so it may be very difficult to give clear interpretations for their managerial meanings, as discussed in Allen et al (1997) and Thanassoulis et al. (2004).

(2) Assurance regions of type II (WRs between Inputs and Outputs):

$$\gamma_i v_i \leq u_r \quad (d)$$

In Thompson et al. (1990) both ARIs and ARIIs were proposed, although only ARIs

were actually used to assess the efficiency of Kansas farming. As a matter of fact, they already realised several problematic issues in applying ARIIs, see, e.g., Thanassoulis et al. (2004) for more details. This type of weights restrictions has mainly been applied to profit efficiency analysis, since the ratios like $\frac{u_i}{v_i}$ are naturally associated with profit efficiency. The applications of ARIIs with the traditional DEA models are not very easily found in the DEA literature. One interesting example is Thanassoulis et al. (1995), where they assessed the efficiency of perinatal care units in England, and required the weight on “babies at risk” (input) to be linked with the weight on “number of survivals” (output). Otherwise, a DMU could be efficient by assigning very high weights on survivals or very lower weights on babies at risk, while actual survival rate is ignored. How to meaningfully interpret the results have also been discussed in, for example, Thompson and Thrall (1994) and Thompson et al. (1995).

3. Restricting virtual inputs and outputs

Apart from direct restrictions on weights, there exists an implicit type of weights restrictions on virtual inputs and/or outputs, as formulae (e1) and (e2) shown.

$$\psi_i \leq \frac{v_i X_{ij}}{\sum_{i=1}^m v_i X_{ij}} \leq \rho_i \quad (\text{e1}) \qquad \omega_r \leq \frac{u_r y_{rj}}{\sum_{r=1}^s u_r y_{rj}} \leq \sigma_r \quad (\text{e2})$$

Restrictions on virtual inputs and outputs were first investigated in Wong and Beasley (1990). The virtual inputs/outputs of a DMU reveal the relative contribution of each input/output to its efficiency rating, and therefore can be quite helpful in identifying strong and weak areas of performance. Restrictions on the virtual inputs and outputs impose implicit weights restrictions and received relatively little attention in the DEA literature. One advantage of virtual weights restrictions is that virtual inputs and outputs are independent on units of measurement. Furthermore virtual weights restrictions are often more intuitive for DMs. However, as Thanassoulis et al. (2004) pointed out, DEA models with such constraints may become computationally expensive, and sensitive to the model orientation. Also they have the same complexity as ARI and ARII on results interpretations. To decide the bounds, it is useful to have some prior view on the relative importance of the individual inputs and outputs. Sarrico and Dyson (2004) extended proportional virtual weights restrictions in Wong and Beasley (1990) to non-proportional, and categorised them as simple virtual

weights restrictions, and virtual assurance regions of type I and II.

4. Cone-Ratio approach

Charnes et al. (1989) proposed the following Cone-Ratio model, which confines weights in cones, V and U :

$$\begin{aligned} \text{Max} \quad & u^T y_0 \\ \text{Subject to:} \quad & u^T y_j - v^T x_j \leq 0, \\ & v^T x_0 = 1, j = 1, \dots, n \\ & v \in V \subset R_m^+, u \in U \subset R_s^+. \end{aligned} \quad (3.3)$$

When $V = R_m^+$ and $U = R_s^+$, it is back to the classic CCR DEA model. In practical applications, value judgments are incorporated into the models by defining suitable polyhedral cones V and U . There are two different ways to represent such cones: Either, we can say that a cone $V \subset R^m$ is polyhedral cone, if it has the so-called half space form:

$V = \{v \mid Cv \geq 0\}$, where C is a $m \times s$ matrix. Or if it is generated by a finite set of vectors with the form:

$$\hat{V} = \text{cone}(\{a_1, \dots, a_s\}) = \left\{ v \mid v = \sum_{j=1}^s \mu_j a_j, \mu_j \geq 0, j = 1, \dots, s \right\},$$

where (a_1, \dots, a_s) are some vectors in R^m , and s is a positive integer. Let $m \times s$ matrix A^T be formed from the vectors (a_1, \dots, a_s) . Then we can say the cone \hat{V} is generated by matrix A^T .

Generally the constraints of ARIs can be written as follows: $V = \{v \mid Cv \geq 0\}$, and $U = \{u \mid Du \geq 0\}$. Thus DEA models with weights restriction of ARI are clearly a specific case of Cone-Ratio models.. Charnes et al. (1991) discussed relationships between A and C : which are very useful in applying Cone-Ratio model Then we can transform the cone-ratio model 3.3 into the following programming problems (Charnes et al. 1990):

Cone-Ratio Prime Model 3.4	Cone-Ratio dual Model 3.5
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Maximize $\beta^T (BY_0)$ s.t. $\alpha^T (AX_0) = 1,$ $-\alpha^T (AX) + \beta^T (BY) \leq 0,$ $\alpha \geq 0, \beta \geq 0.$	Minimize θ s.t. $(AX)\lambda \leq \theta (AX_0),$ $(BY)\lambda \geq (BY_0),$ $\lambda \geq 0$
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Let $\hat{X} = AX$, and $\hat{Y} = BY$ in models 3.4 and 3.5. Then it is clear that the cone-ratio model is consistent with a CCR model, but with the transformed inputs and outputs \hat{X} and \hat{Y} . Thus the cone-ratio model can be viewed as a special case of the approach of data transformations, see Thanassoulis et al. (2004) for the details. In the next section we will see a different interpolation of the cone-ratio model.

There are several different ways to construct suitable cones to incorporate values judgments. One possible approach uses expert opinions through, for example, the analytic hierarchy process (AHP) or Delphi first. Then we may be able to decide the constraints of ARI and the then the matrixes C , D . One can also use the DMUs optimal weights to construct the cones. Generally, one first solves the original CCR model and then selects the preferable DMUs among the efficient ones. Then we can use the set of the optimal weights v^* and u^* of the preferable DMUs as the elements of matrix A^T , B^T , see Cooper et al. 2000. Charnes et al. (1990) and Brockett et al. (1997) used this approach to set the admissible directions for the cone to analyze efficiency for some banks. In the next section, we will explain another possible approach to construct the matrixes A and B .

4. Incorporating value judgments in DEA by changing preferences

As mentioned in the Section 2 of this chapter, one difficult task in incorporating value judgments in DEA is how to quantify them. Usually they are expressed in vague daily languages like “more important” or “very valuable”, and have to be translated into precise mathematical relationship in DEA models. In many applications, value judgments can be reflected by preference that has been one of the bases of multiple criteria decision-making theory. Preference can be viewed as an order relation, and aims to clarify the precise meanings of our fuzzy concepts like “higher”, or “lower”; “better”, or “worse”. With a preference selected, we can unambiguously state that outcome of one DMU is greater or better than another.

4.1 Preferences and properties

A preference is a relationship defined for some pairs (x, y) on a set X , which can be denoted by $\{>\}$ and $\{<\}$ to represent “better than”, and “worse than”. That is, for $\forall x, y \in X$, if $x > y$, then “ x is at least as good as y ”; if $x < y$, then “ x is at most as good as y ”. The definition of preference looks slightly abstract, but essentially it just clarifies the precise meanings for the vague expressions like “better, worse”. Clearly one should have some agreements on these meanings before an evaluation is carried out. The most classic example is the numerical order (preference) for the real numbers like “ $5 > 3$ ” and “ $4 < 6$ ”. Such an order can be generalized to a column or a table of real numbers – like the Pareto preference to be discussed below. However unlike the real number preference, a pair (x, y) generally may not have such a relationship under these generalized preferences –many pairs may not be comparable under these preferences.

When there exist no other element in X better than an element x , then it will be considered as “optimal, or non-dominant ” solution in X , although this does not really mean that it better than the others in assigned preference like in the real numbers, since this could only mean there are many elements incomparable with it. In a sense, DEA is to find “optimal” DMUs in PPS under Pareto preference, see, Cooper et al. (2004) and Liu et al. (2006).

All the preferences discussed here are **Reflexive**: For all x in X , $x > x$, and most of them are **Transitive**: For all x, y , and z in X , if $x > y$ and $y > z$, then $x > z$. These properties usually hold in the real-life applications of DEA, and they are defined to make sure the mathematical summaries of the value judgments of decision makers are consistent and rational. Let us first examine some preferences frequently used.

Example 1: Pareto preference

The Pareto preference is by far the most widely used one in economical and management areas. We will keep using the usual inequality symbols \geq, \leq for this preference.

Let $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n)$ be two outputs. Then in Pareto preference, $x > y$ ($x < y$), or x is better than (worse than) y , if and only if $x_i \geq y_i$ ($x_i \leq y_i$) for

$i = 1, \dots, n$. Thus if $n=1$, then $\{>\}$ in Pareto preference is just the standard numeral symbol “ \geq ”. However when $n > 1$, this standard “ \geq ” is no longer meaningful, and Pareto preference is just a natural generalization of this symbol to higher dimensions.

Pareto’s “better than” requires each of the outputs is better. Therefore for the same inputs, if one DMU has achieved the maximum in any one of the outputs, then it becomes an “optimal” solution in this preference, as no other DMUs can be better than it in Pareto preference. Thus students A, B, and C having the examination results (71, 2, 3), (70, 70, 70), and (50, 50, 99) are all efficient. Pareto preference also implies **no-substitutions** between outputs and **equal-importance** of all components. This is quite different if we define the preference using the total of the marks, as then only student B will be efficient due to substitutions between marks of different subjects.

For DMUs (X, Y) with desirable inputs and outputs, $DMU_1 (X_1, Y_1)$ is better than $(>)$ $DMU_2 (X_2, Y_2)$ in the standard DEA means $X_1 \leq X_2$, and $Y_1 \geq Y_2$ in Pareto preference. DEA is to find the “optimal” DMUs in Possible Production Sets under this preference, see, Liu, et al. 2006.

Example 2: (A, B) matrix preference

Let we re-examine the chemistry students example. Let 3-dimensional vector X_i denote the examination marks of math, physics and chemistry of student i respectively. In the example above, all the subject marks are regarded as equally important and non-substitutable so that the students A, B, C are not comparable. This may not be the case as the students are in a chemistry department. For example, the department may think student- i is better than student- j if his or her chemistry marks **and** marks in total are higher, that is:

$$\begin{aligned} X_{3i} &\geq X_{3j} \\ X_{1i} + X_{2i} + X_{3i} &\geq X_{1j} + X_{2j} + X_{3j} \end{aligned}$$

If the students are however in a mathematics department, then this could be changed as: a student- i is better than student- j if

$$\begin{aligned} X_{1i} &\geq X_{1j} \\ X_{1i} + X_{2i} + X_{3i} &\geq X_{1j} + X_{2j} + X_{3j} \end{aligned}$$

This time still no students are directly comparable. If the university think all the subject marks are equally important and substitutable, then the totals are the most

important so that student-i is better than student-j if

$$x_{1i} + x_{2i} + x_{3i} \geq x_{1j} + x_{2j} + x_{3j}$$

The above preferences can be conveniently presented with matrixes. For the first case

we can let $A = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}$, so performance of student i is better than that of student j if

and only if $AX_i \geq AX_j$ in Pareto preference. For the second and third cases we can

let $A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}$, and $A = (1 \ 1 \ 1)$. Finally, if we let $A = I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, then the

matrix preference $X_i \succ X_j$ is just the Pareto preference:

$$\begin{aligned} x_{1i} &\geq x_{1j} \\ x_{2i} &\geq x_{2j} \\ x_{3i} &\geq x_{3j} \end{aligned}$$

In general let A be a $n \times m$ matrix with non-negative entries. A preference on X can be defined via the matrix A so that $X \succ W$ if and only if $AX \geq AW$ in the Pareto preference. Particularly, when A is the unit matrix, then we can see the matrix preference is just the standard Pareto Preference. When A is row vector with all “1” elements, then the preference is just to compare the totals. From the examples above, we can see many preferences can be obtained in this way.

For input-output data, DMU(X, Y) is preferred to DMU(W, Z) under the (A, B) matrix preference, if and only if $AX \leq AW$ and $BY \geq BZ$ in the Pareto preference, where A and B are $n \times m$ and $n \times s$ matrices with non-negative entries. In the DEA models, the matrix preference plays a very important role in incorporating value judgments. The Cone-Ratio model is closely related to this preference.

Example 3: Lexicographic preference

A lexicographic ordering is sometimes very useful when the k-th component is overwhelmingly more important than the k+1-th component for $k = 1, \dots, n-1$. A chemistry department may think that the chemistry marks are the most important and then are physics and then mathematics. A lexicographic ordering preference is defined as follows: the outcome $y = (y_1, \dots, y_n)$ is preferred to $x = (x_1, \dots, x_n)$ (i.e. $y \succ x$) if and only if $y_1 > x_1$, or there is some $k \in (2, \dots, n)$ so that $y_k > x_k$ and $y_i = x_i$ for

$i = 1, \dots, k-1$. For the students A, B, C, we have then $C \succ B \succ A$ in Lexicographic preference. Lexicographic order is used in Olympic games to rank the countries' performance according to the medals they have obtained. Thus one country has better performance than another, if its total of gold medals is more than that of another's. If the gold medal numbers are equals, then we count the silver medal numbers that the countries have obtained, and so on.

4.2. DEA models with various preferences

4.2.1 Classical DEA models under Pareto preference

Let us first examine the standard additive DEA model:

$$\begin{aligned}
 \max \quad & \sum_0^m s_j^- + \sum_0^s s_k^+ \\
 \text{subject to: } & X_0 - \sum_{i=0}^n X_i \lambda_i = S^- \geq 0, \\
 & \sum_{i=0}^n Y_i \lambda_i - Y_0 = S^+ \geq 0, \\
 & \lambda_i \geq 0, \sum \lambda_i = 1,
 \end{aligned} \tag{4.1}$$

where $S^- = (s_1^-, s_2^-, \dots, s_m^-)$, $S^+ = (s_1^+, s_2^+, \dots, s_s^+)$, and the inequalities are understood in

Pareto preference. It is clear that $\sum_0^m s_j^- + \sum_0^s s_k^+$ is an indicator for how much better

of the virtual DMU $(\sum_{i=0}^n X_i \lambda_i, \sum_{i=0}^n Y_i \lambda_i)$ than (X_0, Y_0) in Pareto preference, see Liu et

al. (2006) for more details. Thus this DEA model is to identify the virtual DMU, which is better than (X_0, Y_0) and achieves the maximum performance with the

additive indicator. Let us then examine the output oriented CCR model

$$\begin{aligned}
 \max \quad & \theta \\
 \text{subject to: } & \sum_{i=0}^n X_i \lambda_i \leq X_0, \\
 & \sum_{i=0}^n Y_i \lambda_i \geq \theta Y_0, \\
 & \lambda_i \geq 0, \theta \geq 1,
 \end{aligned} \tag{4.2}$$

where the inequalities are again understood under Pareto preference, and θ is an output orientated radial measurement for how much better of the virtual DMU

$(\sum_{i=0}^n X_i \lambda_i, \sum_{i=0}^n Y_i \lambda_i)$ than (X_0, Y_0) in Pareto preference, see Liu et al. (2006). Let us

also note that an inequality like: $\sum_{i=0}^n Y_i \lambda_i \geq \theta Y_0$ can be equivalently expressed by:

$\sum_{i=0}^n Y_i \lambda_i - S^+ = \theta Y_0, S^+ \geq 0$, using the slacks S^+ . Thus the CCR with slacks reads:

$$\begin{aligned} \max \quad & \theta + \varepsilon \left(\sum_0^m s_j^- + \sum_0^s s_k^+ \right) \\ \text{subject to: } \quad & \sum_{i=0}^n X_i \lambda_i + S^- = X_0, \\ & \sum_{i=0}^n Y_i \lambda_i - S^+ = \theta Y_0, \\ & S^+ \geq 0, S^- \geq 0, \lambda_i, \theta \geq 0. \end{aligned} \quad (4.3)$$

According to Liu, et al. (2006), one can substitute the default preference with those suitable for an application to incorporate value judgments directly.

4.2.2 DEA models with matrix preferences

We start from a simple example and assume that a mathematical department wishes to investigate study efficiency of mathematics and physics for its students via DEA. Total time on study is used as the input, and examination marks of mathematics and physics are used as the outputs for each of its students. Let $X_j = x_{1j}$ be the inputs and $Y_j = (y_{1j}, y_{2j})^T$ denote the examination marks of mathematics and physics respectively as outputs for student- j in the department. As a mathematical department, it puts more emphasis on mathematics. Thus it adopts the Pareto preference for the inputs, and the following preference for the outputs: Y_i is better than Y_j if

$$\begin{aligned} y_{1i} & \geq y_{1j} \\ y_{1i} + y_{2i} & \geq y_{1j} + y_{2j} \end{aligned}$$

Using these preferences, then the output oriented CCR model reads:

$$\begin{aligned} \max \quad & \theta \\ \text{subject to: } \quad & \sum_{j=1}^n X_j \lambda_j \leq X_0, \\ & \sum_{j=0}^n Y_j \lambda_j \succ \theta Y_0, \\ & \lambda_i \geq 0, \theta \geq 1, \end{aligned} \quad (4.4)$$

where “ \succ ” is understood in the above preference. Thus for individual components of the inputs and outputs, it reads:

$$\begin{aligned}
& \max \quad \theta \\
& \text{subject to: } \sum_{j=1}^n x_{1j} \lambda_j \leq x_{10}, \quad \sum_{j=0}^n y_{1j} \lambda_j \geq \theta y_{10}, \\
& \sum_{j=0}^n y_{1j} \lambda_j + \sum y_{2j} \lambda_j \geq \theta (y_{10} + y_{20}), \\
& \lambda_i \geq 0, \theta \geq 1.
\end{aligned} \tag{4.5}$$

Now let us re-examine the chemistry students’ case and let $X_j = x_{1j}$ be the input and $Y_j = (y_{1j}, y_{2j}, y_{3j})^T$ denote the examination marks of mathematics, physics and chemistry respectively as outputs for student- j in a chemistry department. The department adopted the following preference:

$$\begin{aligned}
y_{3i} & \geq y_{3j} \\
y_{1i} + y_{2i} + y_{3i} & \geq y_{1j} + y_{2j} + y_{3j}
\end{aligned} \tag{4.6}$$

Using this preference, then the output oriented CCR model still reads:

$$\begin{aligned}
& \max \quad \theta \\
& \text{subject to: } \sum_{j=1}^n X_j \lambda_j \leq X_0, \\
& \sum_{j=0}^n Y_j \lambda_j \succ \theta Y_0, \\
& \lambda_i \geq 0, \theta \geq 1,
\end{aligned} \tag{4.7}$$

although here “ \succ ” is understood in the new preference. For individual components of the inputs and outputs, it now reads (adding the slacks):

$$\begin{aligned}
& \max \quad \theta + \varepsilon \left(\sum_0^m s_j^- + \sum_0^s s_k^+ \right) \\
& \text{subject to: } \sum_{j=1}^n x_{1j} \lambda_j + s_1^- = x_{10}, \quad \sum_{j=0}^n y_{3j} \lambda_j - s_1^+ = \theta y_{30}, \\
& \sum_{j=0}^n y_{1j} \lambda_j + \sum_{j=0}^n y_{2j} \lambda_j + \sum_{j=0}^n y_{3j} \lambda_j - s_2^+ = \theta (y_{10} + y_{20} + y_{30}), \\
& s_1^+, s_2^+ \geq 0, s_1^- \geq 0, \lambda_i \geq 0, \theta \geq 1.
\end{aligned} \tag{4.8}$$

The computational results using this DEA model is summarized in the following Table 4:

Student	θ	$1/\theta$	λ_1^*	λ_2^*	λ_3^*	Slacks-1 (Chem.)	Slacks-2 (Math. +Phys. +Chem.)
---------	----------	------------	---------------	---------------	---------------	---------------------	-----------------------------------

A	2.7632	0.3619	0	1	0	61.7105	0
B	1	1	0	1	0	0	0
C	1	1	0	0	1	0	0

It is interesting to know how to recover (4.8) by imposing ARI restrictions in the multiplier DEA model in the sense that the dual of the restricted model coincides with (4.8). These models can be conveniently summarized using the matrix preference:

$$\begin{aligned}
& \text{Max} && \theta \\
& \text{Subject to:} && \mathbf{X}\lambda \prec \mathbf{X}_0, \\
& && \mathbf{Y}\lambda \succ \theta\mathbf{Y}_0, \\
& && \lambda \geq 0, \theta \geq 1,
\end{aligned} \tag{4.9}$$

where the preferences are understood in the (A,B) metric preference. Using the standard inequalities:

$$\begin{aligned}
& \text{Max} && \theta \\
& \text{Subject to:} && (\mathbf{A}\mathbf{X})\lambda \leq \mathbf{A}\mathbf{X}_0, \\
& && (\mathbf{B}\mathbf{Y})\lambda \geq \theta(\mathbf{B}\mathbf{Y}_0), \\
& && \lambda \geq 0, \theta \geq 1
\end{aligned} \tag{4.10}$$

We have $\mathbf{A}=\mathbf{I}$ and $\mathbf{B}=\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, $\mathbf{A}=\mathbf{I}$ and $\mathbf{B}=\begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ for the first and second cases respectively.

Some of these models seem to coincide with the dual form of the well-known Cone-ratio model, although the preferences changes approach looks more general. For the reference change approach, the matrices A and B are decided directly from the value judgment information of an application, as we have seen above.

We can also replace the preference in the standard additive DEA model:

$$\begin{aligned}
& \text{max} && \sum_0^m s_j^- + \sum_0^s s_k^+ \\
& \text{subject to:} && \mathbf{X}_0 - \sum_{i=0}^n \mathbf{X}_i \lambda_i = \mathbf{S}^- \succ 0, \\
& && \sum_{i=0}^n \mathbf{Y}_i \lambda_i - \mathbf{Y}_0 = \mathbf{S}^+ \succ 0, \\
& && \lambda_i \geq 0, \sum \lambda_i = 1,
\end{aligned} \tag{4.11}$$

where the inequalities are understood in the (A,B) matrix preference so that using the standard symbols, it reads:

$$\begin{aligned}
& \max \quad \sum_0^m s_j^- + \sum_0^s s_k^+ \\
& \text{subject to: } \quad AX_0 - \sum_{i=0}^n AX_i \lambda_i = S^- \geq 0, \\
& \quad \quad \quad \sum_{i=0}^n BY_i \lambda_i - BY_0 = S^+ \geq 0, \\
& \quad \quad \quad \lambda_i \geq 0, \sum \lambda_i = 1,
\end{aligned} \tag{4.12}$$

The computational results using this model are presented as follows:

Student	Score	λ_1^*	λ_2^*	λ_3^*	Slacks-1 (Chem.)	Slacks-2 (Math. +Phy. +Chem.)
A	219	0	0	1	96	123
B	0	0	1	0	0	0
C	0	0	0	1	0	0

4.2.3 DEA models with weighted average preferences

One of the most useful preferences for multi-inputs and multi-outputs is to compare total average. Let $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n)$ be two outputs. Then in this preference, $x \succ y$ if

$$\omega_1 x_1 + \omega_2 x_2 + \dots + \omega_n x_n \geq \omega_1 y_1 + \omega_2 y_2 + \dots + \omega_n y_n,$$

where $\omega_i > 0$ are the assigned weights. Using this preference assumes that the inputs or outputs are ω_i -important and substitutable. For instance, subject examination marks of a student in a university often are considered to be equally important and substitutable. For sake of simplicity, assume that the weights for inputs are all the same. Then the additive DEA model in this preference reads:

$$\begin{aligned}
& \max \quad s^- + s^+ \\
& \text{subject to: } \quad \sum_{j=0}^n x_{1j} \lambda_j + \sum_{j=0}^n x_{2j} \lambda_j + \dots + \sum_{j=0}^n x_{mj} \lambda_j + s^- \leq x_{10} + x_{20} + \dots + x_{m0}, \\
& \quad \quad \quad \omega_1 \sum_{j=0}^n y_{1j} \lambda_j + \omega_2 \sum_{j=0}^n y_{2j} \lambda_j + \dots + \omega_s \sum_{j=0}^n y_{sj} \lambda_j - s^+ = \omega_1 y_{10} + \omega_2 y_{20} + \dots + \omega_s y_{s0}, \\
& \quad \quad \quad \lambda_i \geq 0, \sum \lambda_i = 1.
\end{aligned}$$

One DMU is efficient if its score is zero. Let us have a closer look at this model. Assume that all the DMUs have the same inputs, and then it is clear that:

$$\max s^- + s^+ = \max_{j=1, \dots, n} (\omega_1 y_{1j} + \omega_2 y_{2j} + \dots + \omega_s y_{sj}) - (\omega_1 y_{10} + \omega_2 y_{20} + \dots + \omega_s y_{s0})$$

Thus this DEA model recovers the widely used statistical comprehensive analysis

model with the pre-assigned weights ω_i for the ratio data.

Of course, often adding the inputs or outputs like student numbers and profits together is meaning less. Thus the following preferences are often very useful. Let

$x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n)$ be two positive outputs, and let $\theta_i = \frac{x_i}{y_i}$. One can

express preferences with these ratios. For example, Pareto preference can be described by the condition that $x \succ y$ if $\theta_i \geq 1$ for $i=1,2,\dots,n$. In order to compare the outputs via the total average, we can use the sum of the ratios. We say that $x \succ y$ if

$\sum_{i=1}^n \theta_i \geq 1$ (sometimes it can be replaced by $\sqrt[n]{\theta_1 \theta_2 \cdots \theta_n} \geq 1$). This preference allows

the individual ratios below the unit as long as the total average is not less than one without adding all the inputs or outputs together. Only the inputs or outputs in the same category are compared and this makes a lot of sense. Then with this preference we can have some useful DEA models. For example, in some applications, outputs are equally important and non-substitutable, but inputs are equally important and substitutable. Then the corresponding DEA models are

$$\begin{aligned}
 \text{Min} \quad & \sum_{i=1}^m \frac{\theta_i}{m} \\
 \text{Subject to:} \quad & \sum_{j=1}^n x_{ij} \lambda_j \leq \theta_i x_{i0}, \quad i = 1, \dots, m \\
 & \sum_{j=1}^n y_{rj} \lambda_j \geq y_{r0}, \quad r = 1, \dots, s \\
 & \lambda_j \geq 0, \theta_i \geq 0
 \end{aligned} \tag{4.13}$$

Let us note that these are not just the DEA models with Russell measurement since here the vital individual constraints $\theta_i \leq 1, \phi_r \geq 1$ are dropped so now only the total averages are important, referred to as the DEA models with preference. These models have much more discrimination power than the standard DEA models, and this has been confirmed in Zhu (1996) and Meng et al. (2005).

4.2.4. DEA models with lexicographic preference

In this section we present some DEA models with lexicographic preference via examining a typical application. Olympic ranking is a typical example of applying lexicographic preference, where countries are ranked in accordance with the number of gold, silver and bronze medals that their athletes have won. Gold medals are of

dominant importance, as the countries that won a number of silver and bronze medals but none of gold are often ranked below the countries that have just won a single gold medal. When the gold medals are the same, then the silver and then bronze medals are used to rank. Thus this procedure typically uses lexicographic preference.

Since such Olympic ranking only focuses on outputs (medals), in other words, it is a typical effectiveness evaluation that regardless resources utilisation. It arises an interesting issue on examining the participant countries' efficiency. In fact there are already several papers related with efficiency evaluation of Olympic Games by using DEA approach. Lozano et al. (2002) and Lins et al. (2003) analysed the relative efficiency of countries that win at least one medal in Olympic Games in relation to its available resources, where inputs were a country's population and GDP, outputs were the number of gold, silver and bronze medals. These two papers used weights restrictions to incorporate the value judgments that gold medals were worth more than silver ones and that the latter were worth more than bronze medals. Let x^1 , x^2 represent the population and the GDP, and y^1 , y^2 , y^3 represent the number of gold, silver, and bronze medals respectively. For example, Lozabo et al. (2002) incorporated this value judgment in the multiplier DEA model by restricting

$$\frac{u_1}{u_2} \geq \alpha, \frac{u_2}{u_3} \geq \beta, \text{ where } \alpha, \beta \text{ were numerical scalar.}$$

Churilov and Flitman instead aggregated four outputs according to four possible preferences by following formula $y_j = \sum_{r=1}^3 u_r y_{rj}$, where y_{rj} were the corresponding number of gold, silver, and bronze medals won by DMU_j, u_r were the weights. For example, the first output could be accounted by assigning weights (10, 7, 2) to gold, silver and bronze medal. This implies that winning a gold medal is five times as important as winning a bronze medal. With selected inputs and aggregated four outputs, standard DEA models were used to examine different countries' technical efficiency and scale efficiency.

However it is clear that these DEA models on evaluation of Olympic Games did not use lexicographic preference, but some kinds of matrix preferences. Here we indicate how to incorporate lexicographic preference in DEA in this particular application, in order to illustrate the general idea. Output oriented model is used, and Pareto

preference is adopted for the inputs, as we regard the population and GDP are equally important and non-substitutable. Let us select lexicographic preference for the outputs with the radial measurement. Then the BCC model with lexicographic preference can be presented via a multi-phase procedure as follows: (for simplicity, we omit slacks)

Step 1: First consider the number of gold medals as the output

$$\begin{aligned}
& \max && \theta^1 \\
& \text{subject to} && \sum_{j=1}^n \lambda_j x_j^1 \leq x_0^1, \quad \sum_{j=1}^n \lambda_j x_j^2 \leq x_0^2 \\
& && \sum_{j=1}^n \lambda_j y_j^1 \leq \theta^1 y_0^1 \\
& && \sum_{j=1}^n \lambda_j = 1, \lambda_j \geq 0, j = 1, \dots, n
\end{aligned} \tag{4.14}$$

If all countries are assigned different efficiency scores in this step, then we have completed the ranking. Otherwise, those DMUs have the same efficiency scores need to be further ranked in Step 2 as below. This means that these countries have same efficiency scores from the point of view of obtaining the gold medals, and need to be further compared using their silver medals.

Step 2: Substitute the optimal value θ^{1*} obtained from Step 1 into the following model. Then rank these countries by comparing their relative efficiency from a point of view of winning silver medals:

$$\begin{aligned}
& \max && \theta^2 \\
& \text{subject to} && \sum_{j=1}^n \lambda_j x_j^1 \leq x_0^1, \quad \sum_{j=1}^n \lambda_j x_j^2 \leq x_0^2 \\
& && \sum_{j=1}^n \lambda_j y_j^1 = \theta^{1*} y_0^1, \quad \sum_{j=1}^n \lambda_j y_j^2 \leq \theta^2 y_0^2 \\
& && \sum_{j=1}^n \lambda_j = 1, \lambda_j \geq 0, j = 1, \dots, n
\end{aligned} \tag{4.15}$$

Then if some DMUs still have the same efficiency scores in Step 2, we need to move to Step 3 as below:

Step 3: Substitute the optimal values of θ^{1*} , and θ^{2*} , which was obtained from Step-2, into the following model to measure the relative efficiency to win bronze

medals for these countries.

$$\begin{aligned}
& \max && \theta^3 \\
& \text{subject to} && \sum_{j=1}^n \lambda_j x_j^1 \leq x_0^1, \sum_{j=1}^n \lambda_j x_j^2 \leq x_0^2 \\
& && \sum_{j=1}^n \lambda_j y_j^2 = \theta^{2*} y_0^2, \sum_{j=1}^n \lambda_j y_j^1 = \theta^{1*} y_0^1 \quad (4.16) \\
& && \sum_{j=1}^n \lambda_j y_j^3 \leq \theta^3 y_0^3 \\
& && \sum_{j=1}^n \lambda_j = 1, \lambda_j \geq 0, j = 1, \dots, n
\end{aligned}$$

This approach clearly reflects the value judgment that the number of gold medals is of the dominant importance, and so on. Then interpretation of the results is easier and clearer. For any two countries A and B, $DMU_A > DMU_B$ (i.e., A is ranked higher than B) if and only if $\theta_A^1 > \theta_B^1$. If $\theta_A^1 = \theta_B^1$, then we move to Step 2. If $\theta_A^2 > \theta_B^2$, then $DMU_A > DMU_B$. Otherwise, we further move to Step 3. If the two countries still have the same efficiency score at Step 3, we give them the same rank.

4. 3. Discussions

The core of the preferences changes approach is to quantify value judgments into suitable preferences and then incorporate them into DEA models. This approach has the advantage of incorporating value judgments in DEA more directly. The most difficult part is to quantify value judgments into suitable preferences, and this is not always possible. Sometimes, only a part of the value judgments can be reflected. Often the issue of preferences is silently hid inside mission statements of DMUs and needs to be clarified. For instance, when stakeholders in an application tell an evaluator that one particular output is more important than the others, this statement can be quantified via a) the lexicographic preference – then it is of dominant importance; or b) via the preference used in the chemistry department in Section 4.2.2 – then the marks of chemistry are not substitutable although the others are with the equal weights; or c) via the weighted total preference with a high assigned weight as in Section 4.2.3 – then all the marks are substitutable although the chemistry marks have a higher weight; or d) something more suitable for this situation. Agreement needs to be made through close communications with the DMs and the stakeholders from all sides. A useful approach is that the evaluator first listens to and understands

the requirements from all stakeholders, and present several possible alternatives (as the above) with clear explanations of their meanings and differences in daily languages for discussions. This often can lead to an agreement on the preference to be used. For instance when evaluating a group research institutes as requested by a funding body, if the evaluator is not sure what preference to be used, then he or she should present several possible alternative for discussion and to communicate with the researchers, administrators, and the funding body in order to have an agreed solution. This is of course quite demanding for the evaluator. In this sense weights restrictions are easier to use. However in fact it is much more difficult to know whether the resulting DEA models with weights restrictions correctly reflect the value judgments requested in a particular application, as to be seen in the next section.

More discussions and useful preferences can be found in Liu et al. (2006). For instance, there the approach of preferences changes was used to handle the cases where parts of the inputs or outputs are undesirable, and also economics preference is defined to handle **allocative** efficiency. Furthermore it is important to realize that the performance measurements may affect incorporation of value judgments as well with selecting the preferences, see the above reference again for the details.

5. Interpretation and relationship of different approaches

Interpreting results of DEA models with value judgments may not be a trivial task. Among the several approaches discussed in this chapter, interpretation is generally more straightforward for the approach of preferences changes, as an important part of interpretation has been included in the procedure of quantifying value judgments into suitable preferences. For example, when incorporating the value judgment that the marks of chemistry are more important than the others in DEA with this approach, the evaluator should have already discussed with the stakeholders which of the precise statements a)-d) in the section above most accurately reflects the real meanings of “more important” for the department. Thus an important part of the interpretation task has been transferred to the DMs and stakeholders. The rest of the task follows the standard interpretations of dual DEA models, as illustrated in Liu et al. (2006).

On the other hand, interpreting results of DEA models with weights restrictions often needs careful thinking, although applying this approach in applications seems more straightforward. For example, in the absence of weights restrictions, the CCR efficiency score for a DMU is traditionally interpreted as a measure of the radial

contraction of inputs (or expansion of outputs) necessary for efficient operation. However this interpretation of the CCR efficiency measure no longer holds under weights restrictions. The standard interpretation that a DEA model seeks the virtual DMUs of the best performance in PPS often breaks as well in the presence of WRs. As stated in the general guidelines for interpreting results of DEA models with WRs, there may exist substantial changes to the mix of inputs and outputs of a given DMU, and deteriorations in some observed inputs or output level. There are comprehensive studies on effects of DEA weights restrictions on the efficiency scores, DEA targets, and efficient peers, etc, see, Allen et al. (1997), Thanassoulis et al. (2004).

However here we wish to emphasize that these changes brought by the WRs may be perfectly in line with intuition, and suitable for incorporating value judgments in a particular application, as long as the evaluators are fully aware of, and approve these changes. Take the mathematical department case in Section 4.2.2 for example. Suppose that we wish to incorporate the value judgment that marks of mathematics are more important than these of physics with weights restrictions, say, $u_1 \geq u_2 \geq 0$ in the multiplier-DEA model. Then it can be shown that the dual model reads:

$$\begin{aligned}
 & \text{Max} && \theta \\
 & \text{Subject to:} && \sum_{j=1}^n \lambda_j x_{ij} \leq x_{i0}, \quad i = 1, \dots, m \\
 & && \sum_{j=1}^n \lambda_j y_{1j} \geq \theta y_{10}, \quad \sum_{j=1}^n \lambda_j y_{1j} + \sum_{j=1}^n \lambda_j y_{2j} \geq \theta (y_{10} + y_{20}), \\
 & && \sum_{j=1}^n \lambda_j = 1, \quad \lambda_j \geq 0, \quad j = 1, \dots, n
 \end{aligned} \tag{5.1}$$

Thus in this case, the model with WRs exactly coincides with the PCs model in Section 4.2.2. Therefore implicitly this weights restrictions model assumes the preference in option b) for the outputs as adopted by the mathematical department in Section 4.2.2

$$\begin{aligned}
 y_{1i} & \geq y_{1j} \\
 y_{1i} + y_{2i} & \geq y_{1j} + y_{2j}
 \end{aligned}$$

Of course as long as these implicit features are in line with the DMs value judgments, they are all perfectly acceptable. However it is not very often that implications of WRs can be clearly demonstrated as in this simple case. Until the dual models are clearly presented and analyzed, an evaluator may not easily realize these implicit

features and changes, and therefore the implications and consequences that they will bring to the assessment outcomes.

As illustrated above, the DEA models from weights restrictions have some overlaps with the DEA models from preferences changes. Also sometimes, the PCs approach can be viewed as a kind of data transformations as in the Cone-ratio model. It is easy to see that many modified DEA models with weights restrictions cannot be produced from the PCs approach. On the other hand, it is not always possible to recover a modified DEA model with PCs via WRs. For example, a DEA model of type (4.13) cannot be recovered from any weights restrictions of a standard DEA model, see Zhu (1996). In fact it does not seem to be straightforward to recover any of the models after (4.7) via weights restrictions of a standard DEA model.

6. Conclusions

The issue of incorporating value judgments in DEA models may arise in many real-life applications with several different goals, and may be dealt with via very different approaches. However none of the approaches studied or briefly outlined in this chapter is all-purpose, or free of disadvantages. It is important for evaluators to be aware of their advantages and the pitfalls.

It seems to be more straightforward to interpret mathematical and managerial implications of introducing value judgments in DEA via the approach of preferences changes. This approach often requires the explicit participation of the DMs and stakeholders, although it is now considered to be increasingly important to involve these stakeholders in the evaluation processes to reach agreements on appropriate ways to reflect the value judgments. On the other hand, the approach of weights restrictions seems to be simpler to use and more flexible in incorporating value judgments in DEA, although it is still a current issue how to interpret the efficiency scores from weights restricted DEA models, as seen in the section above.

The most suitable approaches to be used for incorporating value judgments will depend on many factors like objectives of the evaluations and the data available to the evaluators. It is also important for the evaluators to bear in mind that there may be other methods such as selecting suitable sets of input-outputs or data transformation, which can better handle incorporation of value judgments in their situations.

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