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ASSET ALLOCATION TO OPTIMISE LIFE INSURANCE ANNUITY FIRM ECONOMIC CAPITAL AND RISK ADJUSTED PERFORMANCE

BY BRUCE T. PORTEOUS AND PRADIP TAPADAR

ABSTRACT

The impact that asset allocation has on the economic capital and the risk adjusted performance of financial services firms is considered in this article. A stochastic modelling approach is used in conjunction with a life insurance annuity firm illustrative example. It is shown that traditional solvency driven deterministic approaches to financial services firm asset allocation can yield sub optimal results in terms of minimising economic capital or maximising risk adjusted performance. Our results challenge the conventional wisdom that the assets backing life insurance annuities and financial services firm capital should be invested in low risk, bond type, assets. Implications for firms, customers, capital providers and regulators are discussed.

KEYWORDS

Economic Capital; Financial Services Firms; Risk Adjusted Performance; Stochastic Models

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1. INTRODUCTION

With the advent of new risk-based regulations for financial services firms, specifically Basel 2 for banks and Solvency 2 for insurers, there is now a heightened focus on the practical implementation of quantitative risk management techniques for firms operating within the financial services industry. In particular, financial services firms are now expected to self assess and quantify the amount of capital that they need to cover the risks they are running. This self assessed quantum of capital is commonly termed *risk*, or *economic, capital*.

In this article we will use the term economic capital throughout. We will also use the term *actual capital* to mean the total amount of equity and debt capital that the firm has raised to back and support its business.

Porteous and Tapadar (2005) in their recent book, give a very comprehensive discussion of economic capital for financial services firms and conglomerates, and we will build on this work, and their practical examples, wherever possible. See also the articles by Porteous *et al.* (2003), Porteous

(2004), Porteous (2005) and especially Porteous and Tapadar (2008), which provide further detailed examples and discussion of economic capital for financial services firms using the ideas presented in Porteous and Tapadar (2005). The books by Dev (2004) and Matten (2000) also contain relevant material.

1.1 *Business and Actual Capital Assets*

In this article, we distinguish between what we term business assets and actual capital assets as follows.

Business assets are the assets that back a financial services firm's business liabilities. For example, the business assets of a life insurance firm's annuity business liabilities, arising from the annuity premiums paid to the firm by its customers, typically comprise of a range of government and corporate bonds. The business assets of a retail mortgage bank, self evidently, are retail mortgages

Actual capital assets are the assets in which the firm has invested that part of its actual capital that is not tied up in financing the firm's businesses. Such actual capital is used to cover the firm's regulatory and ratings agency capital requirements and, typically, is invested in low risk assets, such as cash and short term government and corporate bonds.

1.2 *Why Economic Capital is of Interest?*

Firms, customers and capital providers have a keen interest in a firm's economic capital, for the following main reasons:

1.2.1 *Economic capital as a risk measure*

Economic capital allows firms, capital providers and regulators to measure explicitly how much risk a firm is taking, holistically, across the entire spectrum of risks the firm accepts.

As a firm's economic capital amount depends on both its business assets and its actual capital assets, economic capital allows firms and capital providers to assess how much risk the firm is taking in aggregate, across all of its risks and, in particular, including the asset risks in both its business and actual capital assets.

1.2.2 *To help risk adjust a firm's business*

A capital constrained firm may not have enough actual capital to cover the risks that it is running. In other words, its actual capital may be less than its economic capital. A firm in this position either needs to raise fresh capital, or de-risk.

In order to reduce the risks that it is running, the firm can examine how its economic capital amount reduces as it accepts less risk, in both the type of business that it writes, and also in the asset risks associated with its business and actual capital assets. By adjusting and reducing its business and asset

risks, the firm can then ensure that it has enough actual capital to cover economic capital, and so help to protect its customers and capital providers from extreme adverse events that might otherwise jeopardise their security.

1.2.3 *Risk adjusted performance measurement*

In order to assess the performance of a firm's management, capital providers will wish to measure the returns that the management is likely to earn on the capital provided to it. In doing this, capital providers should allow explicitly for the risks that the firm is running.

In other words, if a firm is earning high returns, but is taking high risks, then firm performance should be measured after having allowed for its high risk strategy. This will allow capital providers to compare the performance of the firm's management against the management of another firm, which may also be earning good returns, but with a lower risk strategy.

Measuring firm performance using a type of risk adjusted measure based on economic capital, as described in Section 4, allows capital providers to assess firm performance on a risk adjusted, or risk consistent, basis.

1.2.4 *Customer interests*

As described in Section 3, economic capital is usually defined as the amount of capital required to keep a firm solvent with a prescribed probability over a certain time horizon. Economic capital is therefore of great interest to customers as it quantifies their level of security when doing business with a particular firm.

All prudential regulatory regimes are calibrated to achieve a very low, but still non zero, probability of firm insolvency. Zero failure regimes are not practical because they would require such high levels of capital that financial services products would then become prohibitively expensive.

Neither Basel 2 nor Solvency 2, for example, are zero failure regimes and, as a consequence, there will always be a risk of firm insolvency, however small this might be. In the event of insolvency, customers may therefore not receive back all that is owed to them, although guarantee schemes, if available, provide some additional protection.

The position that we take in this article is that firms should, at all times, hold enough actual capital to cover their economic capital requirements, calibrated to provide customers with a certain target level of security. Provided that this target level is achieved, firms should then be free to manage their businesses as they see fit, which may be to maximise the risk adjusted returns earned by their capital providers. This ensures that customer security is never compromised by the way in which the firm is being managed.

1.3 *Key Questions Considered in this Article*

This article is concerned with two important questions regarding how

firms allocate their business and actual capital assets. The first important question that we consider is as follows:

Question 1: How should a capital constrained firm allocate its business and actual capital assets to minimise its economic capital requirement?

As is described in Section 2, published firm solvency has traditionally been the main driver for much of the asset allocation work carried out by financial services firms. Moreover such work has often not had a risk, or economic capital, focus, nor a well defined objective, other than to help ensure that the firm is likely to achieve a target level of solvency, from a regulatory point of view. As a consequence, and as we will demonstrate in this article, firms' traditional approaches to solvency driven asset allocation have tended to be sub optimal.

The second important question that we consider is:

Question 2: How should a firm allocate its business and actual capital assets to optimise its risk adjusted performance?

Similarly to Question 1, when firms, or capital providers, have traditionally considered the problem of how to allocate assets to improve performance, explicit risk, or economic capital, approaches may not always have been used. Instead, the effects that asset allocation choices have on business plan expected and scenario earnings may often have been the main focus. However, expected and scenario earnings effects on their own cannot tell a firm what impact asset allocation decisions have on risk adjusted performance because they do not allow risk to be measured explicitly. As a consequence, and as we will show in this article, traditional approaches to allocating assets to improve firm performance may not always be optimal.

In the Sections that follow, we study and investigate these important questions via the construction of illustrative examples.

1.4 *Other Research*

Although there are many published articles and books that discuss the relationship between asset allocation and risk-reward, this work does not generally approach the problem from the perspective of economic capital. It is this economic capital perspective, especially in the new Basel 2-Solvency 2 world, that makes the work presented here, we feel, of relevance.

The two fundamental questions considered here, which are expressed in economic capital terms, are of great practical importance in this new Basel 2-Solvency 2 world as firms seek to optimise their business performances. We hope, therefore that, not only will the results presented here be of some practical use to firms, but the article will also encourage more work to be carried out in this important area.

1.5 *Structure of the Article*

The remainder of the article is structured as follows. In Section 2, we discuss traditional asset allocation approaches in financial services firms. Section 3 defines and discusses economic capital, with Section 4 describing risk adjusted performance measurement. Section 5 sets out the main assumptions that underpin our illustrative example and Section 6 presents our numerical results where we demonstrate that asset allocation is a key management tool that can be used to reduce firm economic capital and optimise risk adjusted performance. Finally, in Section 7, we draw together the main conclusions of the article and discuss optimal asset allocation processes for financial services firms.

2. TRADITIONAL APPROACHES TO ASSET ALLOCATION

In this section, we briefly discuss some of the approaches that financial services firms have traditionally followed in allocating their assets and the motivations and rationales that have driven these approaches. To illustrate the discussion, we consider life insurance and banking firms separately as they have tended to follow different approaches.

2.1 *Life Insurance Firms*

We are interested in life insurance firms writing non profit life insurance products, such as protection business or annuities, and with profit or variable annuity life insurance products which, ostensibly, are long term savings products with investment guarantees. For such types of business, it can be challenging to match assets and liabilities and asset allocation has traditionally been used by firms mainly to reduce regulatory balance sheet volatility.

Unit linked life insurance firms, where assets and liabilities are almost perfectly matched, and business asset allocation is at the discretion of the customer, or their financial adviser, are of less interest to the questions that this article addresses. We therefore do not consider these firms, or other similar types of firm, further.

Traditionally, life insurance firms have tended to manage the asset allocation of their businesses at the aggregate, or firm-wide, level across all business lines. Managing published solvency, that is managing the ratio of assets to liabilities under the regulatory rules applying to the firm, has usually been a key consideration in firms' asset allocation analyses and decision making. This is because life insurance customers, and their financial advisers, usually prefer to place business with 'strong' firms that they believe are more likely to survive the very long periods of time over which life insurance firm products are written.

As is now acknowledged, life insurance firm regulations can be somewhat ad hoc and arbitrary and, in particular, regulatory capital requirements may

not be well aligned with economic capital. As a consequence, allocating assets to achieve certain published solvency targets will usually not have a genuine risk rationale. To compound matters, true risk adjusted performance measurement, where firm performance is considered after having allowed for the risks in the firm's business, may often not have been carried out. Consider, for example, the very high guarantee, unhedged with profit business that, until recently, was commonly written in the UK market.

Traditional life insurance firm asset allocation may therefore often be sub optimal, in terms of allocating assets to manage risk and to maximise risk adjusted performance.

2.2 *Banks*

Most banks raise funding from a diverse range of sources such as retail, wholesale or the securitisation markets and then lend these funds on to borrowers who are charged an interest rate higher than the bank's cost of funding. The margin between the rate charged by the bank to its borrowers, and that paid to its funders, is used to pay for the bank's credit losses, expenses, taxes and equity and debt capital costs, with any residual amount representing the bank's profit.

Most banks tend to carry out asset allocation at the funding line level. For example, asset allocation for a bank's retail deposit funding source will be managed separately from the bank's wholesale funding source. Even if these lines of funding are allocated to the same, or similar, books of assets, the risk management and hedging of these different books will tend to be carried out separately.

In managing the asset allocation of these books, most banks tend to focus on margin management and cashflow matching. For example, banks may want to minimise margin volatility and down side margin risk and will usually hold excess assets to eliminate the liquidity risks that cannot be mitigated by cashflow matching, or by their contingency funding plans.

This is all fine but, as for life insurance firms, there may not be a genuine risk, or economic capital focus. Eliminating interest rate margin volatility clearly results in more stable earnings, but the impact that this has on a firm's economic capital, and risk adjusted performance, may not be measured. It seems unlikely, therefore, that traditional bank approaches to asset allocation will be optimal in any genuine risk, or risk adjusted performance, sense.

3. ECONOMIC CAPITAL

Although economic capital is a ubiquitous term and is widely used and discussed within the financial services industry, surprisingly, there is no commonly accepted standard definition. In order to move forward and

answer the two questions posed in Section 1, we therefore need a definition of economic capital that we can work with. Let us define economic capital as:

Economic capital is the amount of capital, or excess assets, required to ensure that the market value balance sheet of the firm remains solvent, over a specified time horizon, with a prescribed high probability.

The main rationale and reasoning behind this definition is broadly that: Economic capital should be sufficient to keep the firm's business market value balance sheet solvent. This ensures that economic capital protects the interests of the firm's business customers, over the specified time horizon, at the prescribed confidence level.

Moreover, if the firm decides to invest its actual capital in high risk assets, then economic capital should, additionally, be adequate to cover any losses that may be sustained on these assets as a result of this strategy, again over the specified time horizon and at the prescribed confidence level. This ensures that, if the firm wishes to take risks with its actual capital assets, then these risks are reflected in an increased economic capital amount.

3.1 *Economic Capital Time Horizon*

The proposed new Solvency 2 risk-based regulations for European insurers stipulate a time horizon of 1 year. In other words, Solvency 2 proposes that a firm's regulatory capital requirement should be set broadly equal to the amount of economic capital that a firm needs to stay solvent, at the prescribed probability level, over the next year. This is broadly the same approach used with the Individual Capital Assessment ('ICA') regime that is currently in force in the UK. Under Basel 2, we believe that most banks also use a time horizon of 1 year in their Internal Capital Adequacy Assessment Process ('ICAAP') work.

In this article, we take an alternative view that the time horizon used should equal the entire run off period of the firm's in force business. Given the very long term nature of most life insurance contracts, for example, it seems reasonable that customers should be protected, at the prescribed probability level, over the entire lifetime of their policies, rather than just 1 year.

Many insurance risks, longevity risk for example, can take years to crystallise and so a time horizon of 1 year is, in our view, too short to provide customers with adequate levels of comfort and security. We are aware of at least one major UK life insurance firm that uses a full run off time horizon in its ICA, although this is at a somewhat lower probability level than for those firms using a 1 year time horizon.

For other types of businesses where the associated risks are much shorter, hedge funds or investment bank trading books, for example, it is more

appropriate to use shorter time horizons. Time horizons measured in days or weeks are used for these types of businesses, rather than the periods of tens of years that life insurers should, in our opinion, use.

3.2 Economic Capital Prescribed Probability

The proposed new Solvency 2 risk-based regulations for European insurers and the UK's ICA regime use a probability level of 0.995 over a 1 year time horizon. Under Basel 2, however, the probability level is not prescribed, although we believe most banks also use probabilities of 0.995, or higher.

In the UK, some life insurance firms reduce the prescribed probability level that they use in their ICA with the length of time horizon used. This is to ensure that the ICA capital they calculate with a 'long' time horizon is broadly equivalent to that determined using a 1 year time horizon and a 0.995 prescribed probability. Otherwise such firms would have to hold more ICA capital than 1 year time horizon firms.

As stated earlier, we take the view that a time horizon equal to the full run off period of the firm's in force business is most appropriate and that this should be used in conjunction with the prescribed probability level that gives the firm's stakeholders their desired level of security. For the purposes of this article, we assume a prescribed probability of 0.995, as this is the market standard, but our results could just as easily be presented using any other appropriate probability level.

3.3 Economic Capital Measures

The proposed new Solvency 2 risk-based regulations for European insurers stipulate a Value at Risk ('VAR') risk capital measure, which is also broadly used under the current UK ICA regime. Likewise, it is most common practice for banks to use VAR type measures in their ICAAP work.

Although VAR is the risk capital measure that is most often used in practice, other measures, such as tail VAR, conditional tail expectations in other words, could also be used.

Provided that firms can estimate the distribution of their solvency capital requirements, as described in Section 6.1.1, pretty much any economic capital measure can be calculated and used in practice and we are relatively agnostic as to which measure should be used. In this article, however, we use traditional VAR, simply because it is the standard measure used by most firms and regulators.

3.4 Further Information

Economic capital is discussed very fully in Porteous and Tapadar (2005) and Porteous and Tapadar (2008), where a large number of worked examples are developed to illustrate the use of economic capital in practice. We refer the reader to these references for more information.

4. RISK ADJUSTED PERFORMANCE

The textbook Matten (2000) gives an excellent introduction to the problem of measuring bank performance, after having allowed for the risks that the bank is running. Clearly, a financial services firm can generate higher expected returns and earnings in a number of ways.

For example, it could write more risky business, it could gear up its Tier 1, or equity, capital, or it could take more risk in its actual capital assets. The problem that the providers of a firm's Tier 1, or equity, capital have is in understanding if they are better off with these higher, but more risky returns, as compared to lower, more certain, returns.

In this article, we tackle this problem by measuring financial services firm 'riskiness' using economic capital and risk adjusted performance by the returns earned on this economic capital. Specifically, we measure risk adjusted, or risk consistent, performance as the firm's earned rate of return on the Tier 1, or equity, capital backing economic capital, over the projected lifetime of the firm. This is our key measure of risk adjusted, or risk consistent, financial performance and which we denote by ROEC.

ROEC is, therefore, a conventional IRR calculated using the firm's projected cashflows over the entire run off period of the in force book of business. The firm's actual capital is assumed to equal economic capital, at all durations, with this actual capital backed entirely by Tier 1 capital and invested in a range of assets, as described in the article. So, ROEC measures the amount of shareholder value that is created by the firm after having allowed for the amount of risk that the firm is taking, as measured by economic capital.

Therefore, if a highly profitable firm is generating good returns, but only by taking excessive risks, this might be reflected in a low ROEC value. It is then up to the providers of the firm's Tier 1, or equity, capital to decide if they are being adequately compensated for the risks that the firm is running.

4.1 *ROEC as a Random Variable*

We can also think of ROEC as a random variable having a well defined probability distribution. Obviously a ROEC distribution with a 'high' mean value is desirable, but other characteristics of the ROEC distribution are arguably even more important.

For example, if the ROEC variable has high volatility, or standard deviation, then certainty of returns is reduced, which is less attractive to Tier 1, or equity, capital providers than a low volatility ROEC variable.

Likewise, a positively skewed ROEC distribution, where high extreme values are possible, but low extreme values are generally not, is preferable to a negatively skewed distribution, where the converse applies.

5. THE EXAMPLE AND KEY ASSUMPTIONS

In this section, we describe the broad details of the example that we use to illustrate our results. The example is very closely related to an example that we have used elsewhere, specifically the life insurance annuity firm example described in Porteous and Tapadar (2005, Section 9.2.2) and Porteous and Tapadar (2008). In the interests of brevity, and to avoid repetition, we will therefore provide enough high level detail for this article to be self contained, whilst not repeating the very full details that are available in the aforementioned references.

5.1 *Definition of Economic Capital Used*

In our definition of economic capital, we use a VAR measure with a prescribed probability of 0.995 over a time horizon equal to the run off period of the life insurance annuity firm's in force business. So, the time horizon is determined by the time it takes for the firm's very last annuitant to have died.

5.2 *The Stochastic Model*

The stochastic model that we use to drive the example is a high dimensional multivariate time series model. The *within variable* dependency structure is modelled using first order autoregressive time series models and the *between variable* dependency structure is modelled using graphical models. Please see Lauritzen (1996) for a comprehensive treatment of graphical models, which have useful dimension reduction characteristics.

The model allows the expected values, standard deviations and correlations of the modelled response variables to be user specified. Stochastic standard deviations and correlations can also be accommodated, but we concentrate on stationary values in this example.

Based on our experience of using this model in practice, as evidenced in Porteous (2004, 2005) and Porteous and Tapadar (2005, 2008), we have found it to be an excellent first order approximation to the markets that it models. Please see Porteous and Tapadar (2008) and Appendix 1 for full details of the specific parameterisation of the model used in this example. The key modelled UK investment variables that we use in the example are set out below.

- Retail price inflation ("RPI").
- Equity earnings growth.
- Equity dividend yield.
- Short term cash yield.
- Long term corporate bond yield.

5.3 *Demographic Variables*

The key demographic variable used in the example is the rate at which future mortality improves. As is described in Porteous and Tapadar

(2005), actuaries and demographers have a track record of consistently underestimating rates of mortality improvement.

Improvement rates depend on a number of factors including sex, age and cohort. So, for example, we now have evidence to conclude that improvement rates are accelerating for older groups and decelerating for younger groups. The article by Willets *et al.* (2004) is an excellent recent investigation into these and other effects.

Our approach to modelling mortality improvement risk is to start with a base mortality table that represents a good estimate of current mortality. Expected improvement factors, as set out in Porteous and Tapadar (2008), are postulated and these are then perturbed by the addition of a stochastic error term in order to model the uncertainty that is prevalent in estimating future mortality improvements. These stochastically generated improvement factors are finally used to project the base mortality table forward in time.

Again, please see Porteous and Tapadar (2008) for full details of the specific parameterisation of the model used in this example.

5.4 *Actual Capital*

In our example, we assume that the firm's economic capital amount is backed exactly by actual capital comprising of Tier 1 capital only. Tier 1 capital often takes the form of fully paid up ordinary share capital and, for this reason, we refer to Tier 1 capital as equity capital. We assume that actual capital is invested in a range of long term corporate bonds and equities as described in the article.

By Tier 2 capital, we mean subordinated debt, either perpetual, or term. Porteous and Tapadar (2008) describe how economic capital and risk adjusted performance varies when a combination of Tier 1 and Tier 2 capital is used to back economic capital.

5.5 *Annuity Example*

We model a joint life last survivor annuity for a male aged 65 and a female aged 65, for a single premium of £250,000. The annuity income, £1,500 per month, is assumed to be level and is paid until both lives have died. The very detailed assumptions that underpin this example are set out in Porteous and Tapadar (2008). We assume that the life insurance annuity business assets are allocated to a range of mixes of long term corporate bonds and equities.

Based on the stochastic elements of the example, the economic capital requirement of the firm's annuity business, as generated by the stochastic model, is the amount of capital that is needed to cover aggregate, or holistic:

- Market risks as a result of corporate bonds or equities generating inadequate returns;
- Credit risks as a result of corporate bonds generating excessive credit losses;

- Expense risks as a consequence of RPI-linked unit costs increasing in line with high levels of RPI;
- Mortality risks as a consequence of rapidly improving mortality experience; and

all beyond the expected amounts loaded into the life insurance annuity product pricing to cover these risks.

6. RESULTS

Our results are presented in the form of a series of graphs and a table, but we first introduce some notation that helps us to describe our results, as follows: let $EC(x, y)$ denote the economic capital curve when x per cent of business assets and y per cent of actual capital assets, respectively, are invested in long term corporate bonds. The balance of both business and actual capital assets are then assumed to be invested in equities.

Similarly, $ROEC(x, y)$ denotes the corresponding earned ROEC when x per cent of business assets and y per cent of actual capital assets are invested in long term corporate bonds.

6.1 *Base Case Results When Actual Capital Assets Are Invested 100 Per Cent in Bonds*

The first graph in Figure 1 shows 99.5th percentile economic capital curves for a range of business asset mixes, when actual capital assets are invested 100 per cent in long term corporate bonds. In particular, economic capital, per unit of in force business at time zero, is shown on the vertical axis of Figure 1, with the duration to which the firm's balance sheet has been projected shown on the horizontal axis. All economic capital figures provided in this article follow this format.

The time horizon that we use throughout, in accordance with our definition of economic capital set out in Section 5.1, is the run off period of the life insurance annuity firm's in force business, at each projected duration. In other words, the outstanding time it takes for the last annuitant to die, from the point of each projected firm balance sheet.

6.1.1 *Economic capital curve construction*

The economic capital curves are constructed from 10,000 runs of the stochastic model in a four stage process as follows:

Stage 1 — We first define the firm's *capital requirement* at projected duration t as the amount of capital that is required to ensure that the duration t firm balance sheet is solvent, under a specific run of the stochastic model.

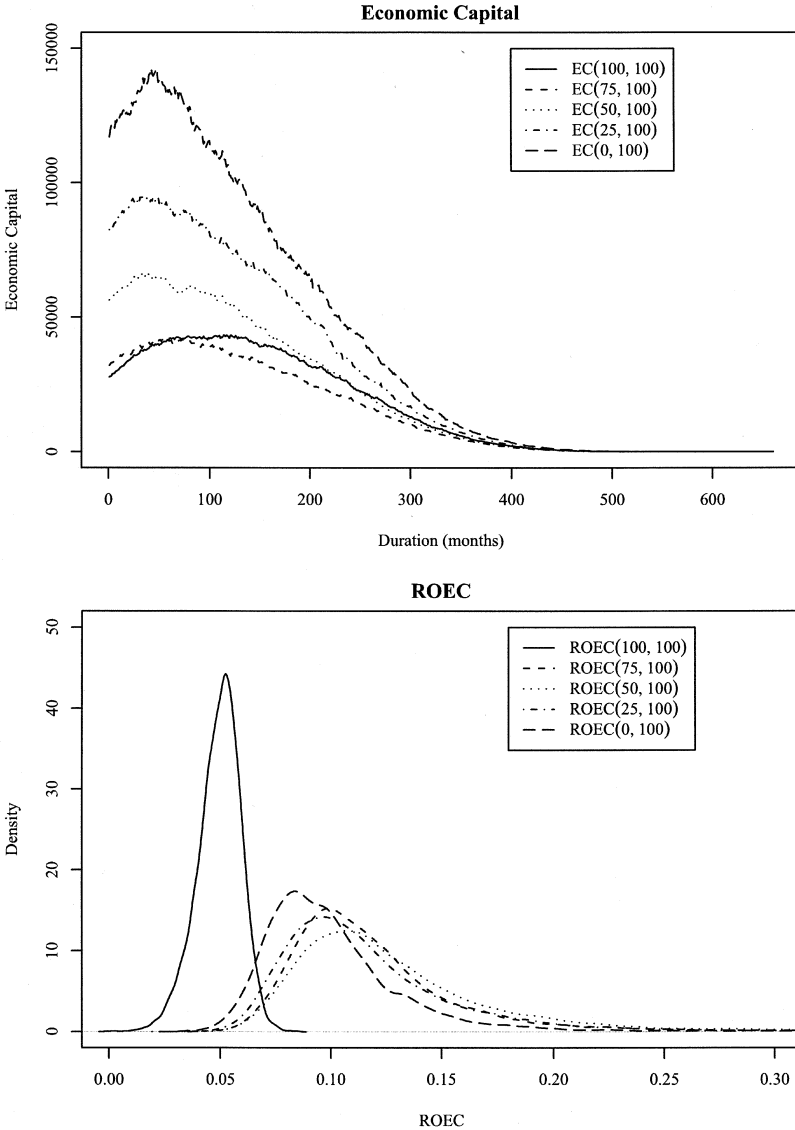


Figure 1. 99.5th percentile economic capital and the associated ROEC probability density functions, when actual capital assets are allocated 100 per cent to long term corporate bonds and business assets to a range of long term corporate bonds and equities, as indicated in the graph keys

- Stage 2** — We next define the firm's *discounted capital requirement* at duration $s \leq t$, in respect of the firm's projected duration t balance sheet. We discount the projected duration t capital requirement from t to s using the rates of return earned on the assets in which the firm's actual capital is assumed to be invested, again under one specific run of the stochastic model. This discounted capital requirement at time s , plus the returns earned on the assets in which the firm's actual capital is assumed to be invested, over the period from s to t , is then sufficient to ensure that the firm's projected duration t balance sheet is solvent, under this one run of the stochastic model.
- Stage 3** — The firm's *discounted solvency capital requirement*, at duration s , is defined as the maximum of the discounted capital requirements over all durations $t \geq s$. As a consequence, discounted solvency capital at duration s , plus the returns earned on the assets in which the firm's actual capital is assumed to be invested over the period from s to t , ensures that all projected balance sheets for durations $t \geq s$ remain solvent, again under this one run of the stochastic model.
- Stage 4** — Finally, the firm's *economic capital requirement* at duration s is defined as the amount of capital that is required to ensure that all projected duration t firm balance sheets remain solvent, for $t \geq s$, with the prescribed probability level, under multiple runs of the stochastic model. We estimate economic capital using the percentiles of the firm's corresponding discounted solvency capital requirements, as described in Stage 3 above, under 10,000 runs of the stochastic model.

6.1.2 *Basic shape of economic capital curves*

The initial rise in the economic capital curves is due to the fact that, as the outstanding duration of the firm's in force book of annuity business declines, discounted solvency capital increases. This is because there is less time available for the firm's market value balance sheet to recover from the extreme events that drive discounted solvency capital.

However, as explained in Section 6.1.1, the time zero actual capital amounts are adequate to cover the initial economic capital increases and no additional injections of actual capital are required. This is because the returns earned on the assets in which the firm's actual capital is assumed to be invested are sufficient to fund the economic capital increases exactly, as a consequence of the way in which economic capital is defined.

After these initial economic capital increases, economic capital generally declines as the firm's in force book of annuity business begins to run off faster, as annuitants age and die. The surplus actual capital backing economic capital is then released back to the capital providers.

6.1.3 ROEC probability density function curves construction

The second graph in Figure 1 shows the ROEC probability density function estimates, where the ROECs are described as previously in Section 4. In Table 1 we show corresponding summary statistics for the ROEC probability density functions.

In order to obtain the 'smooth' density functions shown in Figure 1, standard statistical kernel density estimation techniques were used to smooth the ROEC probability density histograms calculated from the 10,000 runs of the stochastic model used to determine economic capital.

6.1.4 Base case economic capital results

We first note that, as business assets are switched out of bonds and into equities, two counteracting first order effects are anticipated to occur.

Firstly, because equity investment returns are more volatile than bond returns, the volatility of the firm's market value balance sheet, which it is important to remember is calculated over the entire run off period of the firm's in force book of business at each duration, will also increase. Increased balance sheet volatility increases firm discounted solvency capital and so, therefore, economic capital. It is important to note that this effect will be seen irrespective of the time horizon over which economic capital is determined although, the longer the time horizon used, the more marked the effect will be.

The second anticipated effect is that business asset returns will increase on average because equity investment expected returns exceed those of bonds. All else being equal, this reduces firm discounted solvency capital and so economic capital. Again, this effect will be seen regardless of the time horizon over which economic capital is calculated.

Whether economic capital increases, or decreases, as business assets are switched out of bonds and into equities therefore depends on which of these two principal first order effects dominate. If increased volatility over the run off period of the market value balance sheet dominates, then economic capital will increase, and vice versa.

Inspecting the economic capital curves in Figure 1 allows us to see that, perhaps as expected, economic capital is high when the bulk of business assets is allocated to equities, rather than to bonds. This is because, with increasing equity investment, the higher volatility of equity investment returns is dominant.

The one exception to this is the *EC(75, 100)* curve, when business assets are invested 75 per cent in bonds and 25 per cent in equities. In this case, it can be seen that economic capital falls at most durations, beyond the very short durations, relative to 100 per cent bond investment. That is, the higher expected asset return effect is dominant, except at very short durations, and a small proportion of equity investment can actually reduce risk.

At the very short durations, the higher volatility of equities is more dominant as it interacts with the longer, and therefore more volatile, market value balance sheet.

Although not shown explicitly here, we have also re-calculated the economic capital curves shown in Figure 1 using a one year time horizon, rather than a run off horizon. As expected, the one year curves are always lower because, although economic capital is still calculated as the maximum discounted solvency capital requirement, this maximum is over the shorter one year time period. However, qualitatively, the one year curves are identical to the run off curves and again show that investing 25 per cent of business assets in equities reduces risk.

So, we are able to conclude that a capital constrained life insurance annuity firm can reduce risk by switching a small proportion of its business assets out of bonds and into equities. It is interesting to note that this type of asset allocation decision is not at all common in the market.

6.1.5 Base case ROEC results

The second graph in Figure 1 shows the estimated ROEC probability density functions corresponding to the economic capital amounts shown in the first graph.

The first point that can be made is that, as soon as equity investment is introduced into business assets, the densities move significantly to the right and become much more spread, or diverse, and positively skewed. In other words, mean, standard deviation and skewness of ROECs increase. This pattern is also evident in Table 1.

Reading the Figure 1 ROEC probability density function estimates from left to right, we can see that the broad order is as follows: 100 per cent bonds, 0 per cent bonds, 25 per cent bonds, 75 per cent bonds followed, finally, by 50 per cent bonds. As well as the densities shifting to the right in this broad order, the densities also become more spread and positively skewed in this order.

It can be seen therefore that, as business assets are increasingly allocated to equities up to 50 per cent, the $ROEC(\cdot, 100)$ densities shift to the right and become more positively skewed. This is because the higher returns that are expected to be earned by equity investment dominate the corresponding economic capital increases which are due to the higher volatility of equity investment returns.

However, as the equity investment percentage increases beyond 50 per cent, the $ROEC(\cdot, 100)$ densities move back towards the $ROEC(100, 100)$ density. This behaviour is explained by the larger amounts of economic capital required when the equity investment percentage increases beyond 50 per cent, as evidenced in the economic capital curves shown in the first graph of Figure 1. In other words, the higher economic capital amounts pull the ROEC densities back towards the $ROEC(100, 100)$ density, as the increased

economic capital amounts more than compensate for the higher returns that are expected to be earned from equity investment.

By inspecting the way that the ROEC density curves overlap, it can also be observed that the probability of $ROEC(x, 100) < ROEC(100, 100)$ for $x < 100$ is rather small. In other words, the likelihood of a lower ROEC, when a proportion of business assets is allocated to equities, as compared to 100 per cent investment in bonds, is very small.

Perhaps more importantly, the upside potential of equity investment on ROEC can be seen to be very significant. This is because of the positive skew shape of the equity investment ROEC densities. For this same reason, the down side risks are also minimal and, by comparing the shapes of the ROEC probability density curves shown in Figure 1, the merit of allocating business assets to equities is apparent. Again, it is interesting to note that most life insurance annuity firms operating in the market invest their business assets 100 per cent in bond type assets.

In Table 1 we have shown what we call the ‘Sharpe ratio’, defined as mean return, divided by standard deviation return, and which is a potential measure of risk adjusted performance. In statistical terminology, our Sharpe ratio is equivalent to the inverse of a coefficient of variation.

If we based our asset allocation decisions on the Sharpe ratio displayed in Table 1, 100 per cent bond allocation appears to be the optimal business asset allocation choice. The very low estimated standard deviation, or volatility, associated with the $ROEC(100, 100)$ density dominates the measure, implying that highly risk averse investors, who measure risk using standard deviation, or volatility, may forgo the prospect of the higher returns available with equity investment.

We would recommend, however, that better asset allocation decision making is obtained by considering the full shape of the $ROEC(\cdot, 100)$ probability density functions, and this indicates that some element of equity investment for business assets is optimal. The same conclusion is arrived at by considering the quartiles of the $ROEC(\cdot, 100)$ distributions shown in Table 1.

6.1.6 Base case results summary

It can be seen from the results for the example considered here, based on the specific model parameterisation, that allocating all business assets to bonds appears to be sub optimal from the perspective of both minimising economic capital and also optimising risk adjusted performance. A small amount of equity investment appears to be beneficial for a capital constrained life insurance annuity firm as this reduces economic capital.

If a life insurance annuity firm wishes to maximise its risk adjusted performance, and it is not capital constrained, then an even higher amount of equity investment appears optimal. Beyond a certain level, greater equity investment becomes sub optimal on both counts, however.

Table 1. Summary statistics for 99.5th percentile $ROEC(x, y)$ when both business and actual capital assets are allocated to a range of long term corporate bonds and equities, as indicated in the table. (μ = Mean, σ = Standard Deviation, Q_1 = First Quartile, Q_2 = Second Quartile (Median), Q_3 = Third Quartile)

y	x	μ	σ	Sharpe ratio	Q_1	Q_2	Q_3
100	100	0.049	0.010	5.023	0.044	0.050	0.056
	75	0.124	0.081	1.525	0.094	0.110	0.134
	50	0.135	0.082	1.652	0.097	0.117	0.147
	25	0.121	0.065	1.850	0.090	0.108	0.133
	0	0.102	0.042	2.428	0.080	0.094	0.113
75	100	0.057	0.010	5.672	0.051	0.057	0.063
	75	0.160	0.159	1.004	0.105	0.129	0.165
	50	0.191	0.303	0.631	0.110	0.139	0.187
	25	0.147	0.116	1.262	0.097	0.121	0.158
	0	0.131	0.116	1.135	0.089	0.110	0.140
50	100	0.064	0.013	5.019	0.057	0.064	0.071
	75	0.195	0.351	0.555	0.112	0.140	0.186
	50	0.229	0.501	0.456	0.115	0.147	0.204
	25	0.202	0.744	0.271	0.106	0.135	0.183
	0	0.143	0.176	0.815	0.092	0.115	0.150
25	100	0.071	0.016	4.383	0.063	0.071	0.078
	75	0.260	1.140	0.228	0.119	0.152	0.211
	50	0.300	2.214	0.136	0.120	0.155	0.217
	25	0.220	0.623	0.353	0.109	0.140	0.193
	0	0.161	0.247	0.652	0.096	0.121	0.163
0	100	0.079	0.022	3.574	0.068	0.077	0.086
	75	0.232	0.540	0.430	0.119	0.152	0.208
	50	0.353	2.480	0.142	0.119	0.155	0.223
	25	0.221	0.689	0.321	0.109	0.139	0.189
	0	0.170	0.306	0.556	0.099	0.125	0.167

Contrary to these conclusions, most life insurance annuity firms operating in the market allocate 100 per cent of their business assets to bond type assets.

6.2 General Results

6.2.1 General economic capital results

In Figure 2, for each individual graph, we show 99.5th percentile economic capital for a range of equity and bond allocations of both business and actual capital assets, as indicated in the figure. These graphs are directly comparable with the first graph displayed in Figure 1, which shows 99.5th percentile economic capital, for the same range of asset allocations of

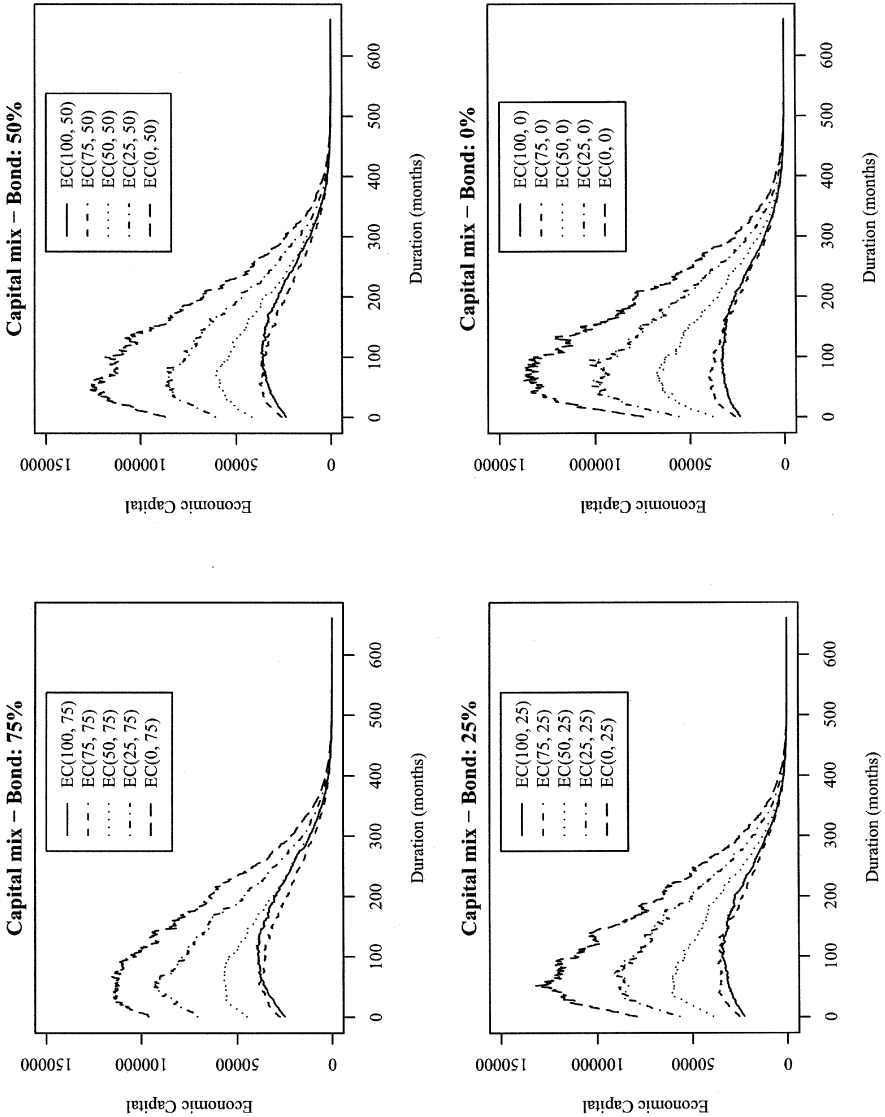


Figure 2. 99.5th percentile economic capital when both business and actual capital assets are allocated to a range of long term corporate bonds and equities, as indicated in the graph keys

business assets, whilst holding actual capital asset allocation fixed at 100 per cent to bonds.

Comparing the graphs in Figures 1 and 2 shows that, as actual capital assets are switched from bonds into equities, consequent economic capital changes are all relatively small. The most discernible change is when actual capital assets are switched from 100 per cent bonds to 75 per cent bonds. In this case, economic capital falls across the board as the additional returns earned on the equity investment more than compensate for the extra risk.

For the other graphs shown in Figure 2, the changes are all smaller. Close inspection of the graphs does reveal, however, a type of ‘diversification’ benefit effect between equity and bond asset allocation that can, for example, cause economic capital to reduce as actual capital assets are allocated to equities, when business assets are predominantly allocated to bonds. This diversification effect is simply due to the fact that, in both the real investment world, and as modelled in this article, equity and bond investment returns are not perfectly dependent.

By way of illustration, when business assets are allocated 100 per cent to bonds, economic capital is lowest when actual capital assets are allocated 25 per cent to bonds. Similarly, when 75 per cent of business assets are allocated to bonds, economic capital again is lowest when actual capital assets are allocated 25 per cent to bonds. In the limit, when business assets are allocated 0 per cent to bonds, economic capital is lowest when actual capital assets are allocated 75 per cent to bonds.

Comparing the economic capital graphs in Figures 1 and 2, shows that economic capital generally increases as actual capital assets are allocated to equities, due to the higher riskiness of equities. But, as the business asset bond proportion increases, this effect is increasingly mitigated by the diversification benefit effect described above. If we superimpose this diversification benefit effect onto the more dominant effect that actual capital asset equity allocation beyond 25 per cent generally increases economic capital, the changes witnessed in Figure 2 can be explained.

The main conclusion that we can draw from this section, therefore, is that, when firms are attempting to reduce economic capital, or reduce risk, they need to be aware, not just of the relative riskiness of the asset categories in which they are investing, but also of the nature of the dependencies between the assets.

Analysis of the underlying economic capital data show that, near optimal business and actual capital asset allocations, in terms of minimising economic capital, correspond to the $EC(100, 25)$ curve at shorter durations and the $EC(75, 25)$ curve at longer durations. It is again worth noting that these types of asset allocations are not at all common in the market.

6.2.2 *General ROEC results*

In this section, we now move on to consider the ROEC probability

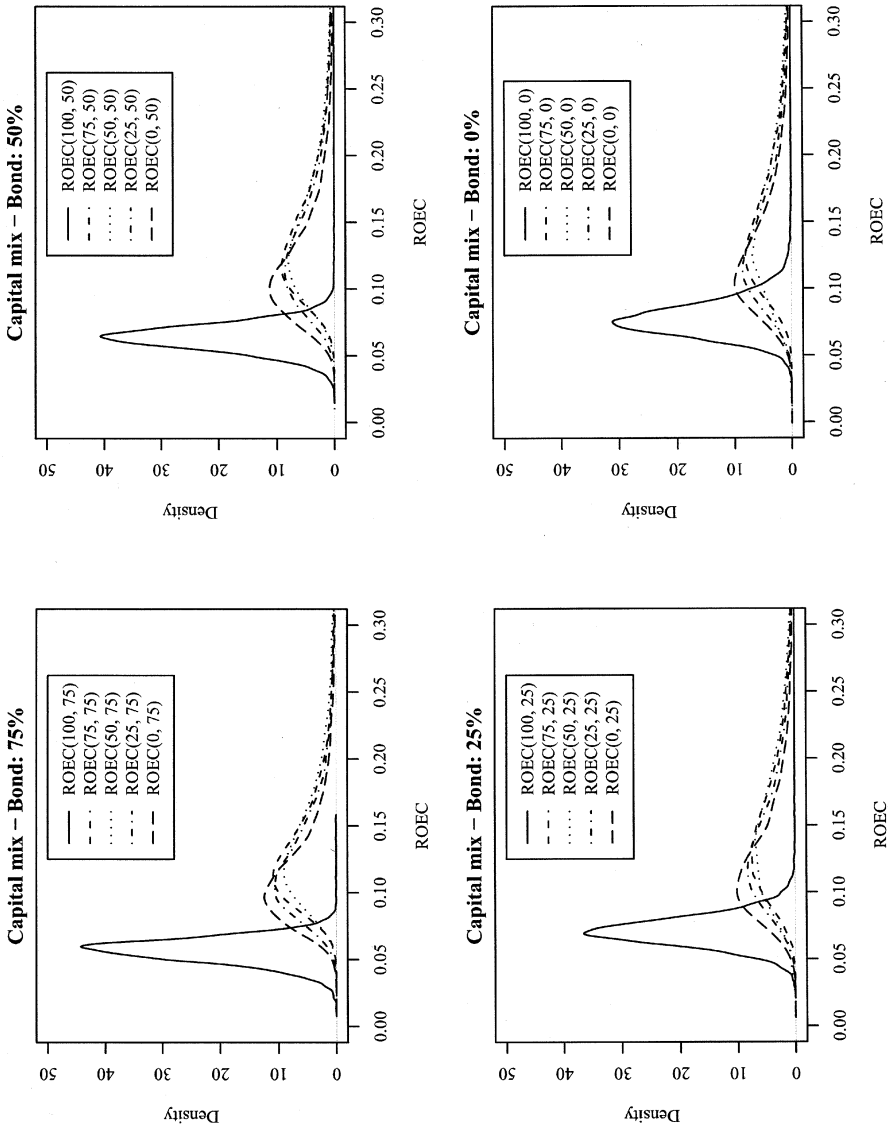


Figure 3. 99.5th percentile ROEC probability density functions when both business and actual assets are allocated to a range of long term corporate bonds and equities, as indicated in the graph keys

density functions and summary statistics corresponding to the economic capital amounts shown in Figure 2. The ROEC density functions are shown in Figure 3 with selected summary statistics again provided in Table 1.

Comparing the graphs displayed within Figures 1 and 3 shows that, as actual capital assets are increasingly allocated to equities, rather than bonds, the density curves shift to the right, and become more diverse and positively skewed. In other words, actual capital equity investment increases ROEC returns, spreads and skewness. The ordering of these density curves by business asset mix, within each individual actual capital graph, remains constant with the *ROEC*(50, ·) density curve being the density furthest right, most spread and skewed.

Similar to the discussion set out in Section 6.1.5, it can be seen from the shape of the density curves in Figures 1 and 3, and the way that they overlap, that there is little down side risk, and plenty of upside potential, to allocating business assets 50 per cent to bonds, as compared to 100 per cent. The summary statistic quartiles shown in Table 1 again confirm this conclusion.

The Sharpe ratio shown in Table 1 is again seen to be relatively unhelpful in the asset allocation decision making process.

Based on the estimated probability density functions shown in Figures 1 and 3, we would suggest that a near optimal asset allocation mix for this firm, in terms of maximising ROEC, is close to the *ROEC*(50, 0) density. This choice gives high mean ROECs, with some down side risks, but very significant up side potential. The summary statistics shown in Table 1 again support this view.

We note that this asset allocation choice would be extremely unusual in the market.

6.2.3 *General results summary*

It can be seen from the results for the example presented here, that allocating all business and actual capital assets to bonds appears to be sub optimal from the perspective of both minimising economic capital and maximising risk adjusted performance.

For example, a substantial amount of equity investment for actual capital assets appears to be beneficial for a capital constrained life insurance annuity firm as this reduces economic capital. Likewise, if a life insurance annuity firm wishes to maximise its risk adjusted performance, and is not capital constrained, then substantial amounts of equity investment for both business and actual capital assets appears to be near optimal.

As a consequence of the fact that equity and bond investment returns are not perfectly dependent, the optimal business asset equity allocation proportion will depend on the proportion of actual capital assets that are allocated to equities, and vice versa.

Contrary to these conclusions, most life insurance annuity firms operating

in the market invest 100 per cent of both their business and actual capital assets in bond type assets.

7. CONCLUSION

In this article, for a simple life insurance annuity firm, we have considered the two key questions set out in Section 1.3 concerning the impact that asset allocation has on firm economic capital and risk adjusted performance. We have not followed the traditional approach to answering these questions, as described in Section 2, but have instead followed an approach based on stochastically generated economic capital.

In our view, the approach adopted here is theoretically more appropriate as the risks involved in the firm's business and assets are modelled explicitly, using a stochastic model, rather than, at best, implicitly using deterministic techniques. Moreover, in determining our optimal asset allocations, we have properly defined our objective, either minimising economic capital, or maximising ROEC, rather than leaving the objective vague or unspecified.

For the simple firm considered here, it is traditional for business assets to be allocated 100 per cent to bonds and actual capital assets to be allocated 100 per cent to low risk assets such as cash or bonds. As can be seen from Section 6, near optimal asset allocation to minimise economic capital, or maximise ROEC, involves allocating actual capital assets, or both business assets and actual capital assets, to equities in substantial proportions.

This goes very much against the mainstream and conventional wisdom. In defence of life insurance annuity firms, it should be noted, however, that life insurance regulations may be a cause of this behaviour as they can penalise equity investment inappropriately. Porteous and Tapadar (2005, Section 9.2.3.6) provide a fuller discussion of this point.

In this article we have allowed the asset allocation of both business and actual capital assets to move in discrete jumps of 25 per cent, which is adequate for our purposes. In practice, firms will want to know to a higher level of granularity what their optimal asset allocation is and this can obviously be achieved by using smaller discrete jumps, especially in the vicinity of the optimal allocation.

Although our results and conclusions are based on one simple example, we believe that they do apply more widely and will report other examples in due course. The implications of the general conclusions presented here are, we feel, of fundamental importance to firms, customers, capital providers and also to regulators for the following main reasons:

- It is incumbent on a firm's management to run the business as optimally and efficiently as possible. This is what they are paid to do. By not allocating assets in a near optimal fashion, directors and managers are not fulfilling their obligations to capital providers

because sub optimal asset allocation deprives capital providers of their full and fair risk adjusted returns.

- Sub optimal asset allocation, in conjunction with, or caused by, risk insensitive regulations, may lead firms to hold more, or less, actual capital than is strictly necessary. The whole of the financial system, and the allocation of capital throughout it, will therefore be inefficient. This is clearly bad news for the financial services industry but, more importantly, for the wider economy more generally.

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APPENDIX 1

STOCHASTIC MODEL INVESTMENT RESPONSE VARIABLE
DETAILS

The stochastic model that we use is a fairly standard 21 dimensional multivariate normal first order autoregressive time series model, where the multivariate dependency structure of the 21 individual response variables is modelled using a graphical model, as explained below.

The 21 individual univariate time series models, described below, model the *within-series* dependency, or serial dependency, structures of each of the individual response variables.

To model the *between-series*, or multivariate, dependency structure of the 21 individual response variables, we use a multivariate normal model where the correlation structure of the 21 dimensions is modelled using a graphical model, which describes the assumed conditional independence properties of the 21 dimensions.

In particular, we model the i th response variable at time t , Z_{it} , as the sum of an unconditional expectation μ_i and a first order autoregressive time series with constant volatilities and correlations, although stochastic volatilities and correlations can also be accommodated if required. In other words, $Z_{it} = \mu_i + Y_{it}$, where:

$$Y_{it} = \beta_i Y_{i(t-1)} + \epsilon_{it}. \quad (1)$$

The error terms $\epsilon_{it} \sim N(0, \sigma_i^2)$ and are assumed to be independently distributed across time. Note that $E[Z_{it}] = \mu_i$ and that the first order autoregressive parameter β_i controls the strength of the within-series dependency for the i th response variable. So, for example, 'large' positive values of β_i will mean that there will be very strong positive serial dependency within the i th response variable.

Table 2 shows the parameterisation of the 21 univariate autoregressive time series models that make up the full 21 dimensional stochastic investment response model. The annual expected values, the μ_i s, and the annual standard deviations, the $\sigma_i/\sqrt{1 - \beta_i^2}$ s, of the individual response variables are also shown.

The correlation structure of the ϵ_i error terms is modelled using a graphical model, as displayed in Figure 4. In this figure, response variable error terms that are directly connected to each other are correlated, with the assumed constant correlation coefficient values ρ_{ij} , as set out in Table 3.

Response variable error terms that are indirectly connected in Figure 4, via other directly connected error terms, are still statistically dependent, and

Table 2. Stochastic model: univariate model parameterisation

Investment response variable	Unconditional expectation μ_i	First order autoregressive parameter β_i	Unconditional standard deviation of error terms $\sigma_i/\sqrt{1-\beta_i^2}$
<i>U.K. economic</i>			
1 Retail Price Inflation (“RPI”)	0.0275	0.975	0.00750
2 Equity earnings/dividend growth	0.0425	0.950	0.02000
3 Equity dividend yield	0.0325	0.975	0.00750
4 Short term cash yield	0.0475	0.975	0.00750
5 Medium term government bond yield	0.0500	0.975	0.01875
6 Medium term corporate bond yield	0.0550	0.975	0.01875
7 Long term government bond yield	0.0525	0.975	0.01875
8 Long term corporate bond yield	0.0575	0.975	0.01875
9 Mortgage yield	0.0575	0.975	0.00750
10 Property rental growth	0.0325	0.950	0.01875
11 Property rental yield	0.0425	0.975	0.00750
12 £ appreciation against \$	0.0200	0.000	0.02500
<i>U.S. economic</i>			
13 Consumer Price Inflation (“CPI”)	0.0200	0.975	0.00750
14 Equity earnings/dividend growth	0.0625	0.950	0.02000
15 Equity dividend yield	0.0200	0.975	0.00750
16 Short term cash yield	0.0200	0.975	0.00750
17 Medium term government bond yield	0.0375	0.975	0.01875
18 Medium term corporate bond yield	0.0425	0.975	0.01875
19 Long term government bond yield	0.0450	0.975	0.01875
20 Long term corporate bond yield	0.0500	0.975	0.01875
21 Mortgage yield	0.0425	0.975	0.00750

so are correlated, but more weakly so. This is a property of graphical models. Such error terms are, however, conditionally independent of each other, given the error terms that connect them, again a property of graphical models. See Porteous and Tapadar (2005) Appendix 7.1 for the full 21 X 21 error term correlation matrix which is completely specified by the graphical model parameterisation described here.

Graphical models are fully described in Lauritzen (1996) and Porteous and Tapadar (2005), for example. They are extremely useful dimension reduction tools that can be used to explain very high dimensional dependency relationships amongst random variables using low dimensional clusters, or cliques, of relationships.

A full justification for the Table 2 and 3 parameterisations of the stochastic model, and the error term multivariate normality assumption, is provided in Porteous and Tapadar (2005) and the references contained therein. It is demonstrated there, and also in Porteous (2004) and Porteous (2005), that this stochastic model represents a very good first order approximation to the

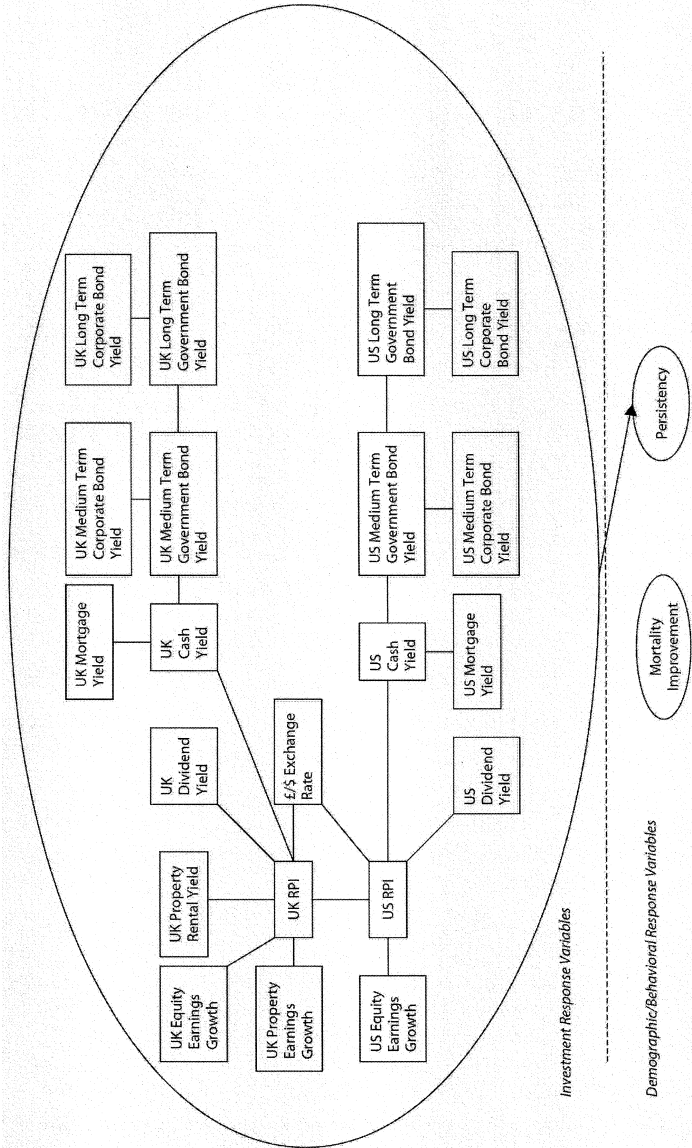


Figure 4. Graphical model of between response variable error term dependency

Table 3. Stochastic model: error term correlation parameterisation

ϵ -pair	Correlation coefficient ρ_{ij}
1, 2	0.1
1, 3	0.3
1, 4	0.6
1, 10	0.1
1, 11	0.3
1, 12	0.6
1, 13	0.3
4, 5	0.6
4, 9	0.6
5, 6	0.6
5, 7	0.6
7, 8	0.6
12, 13	0.6
13, 14	0.1
13, 15	0.3
13, 16	0.6
16, 17	0.6
16, 21	0.6
17, 18	0.6
17, 19	0.6
19, 20	0.6

real investment markets that it is modelling. Summary statistics of model output are provided in Table 11.2 of Porteous and Tapadar (2005).

Two demographic/behavioural response variables, mortality improvement and customer persistency, can also be seen in Figure 4. Whilst the rate at which mortality improves does not directly depend on investment markets, customer persistency behaviour can, in practice, depend on investment conditions.

In the examples that we consider in this article we will, in fact, use only a relatively small subset of the 21 response variables. In other words, we will use only those response variables that are needed to construct the examples, as is explained more fully in Porteous and Tapadar (2008).