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Quantised coordinate rings of semisimple groups 
are unique factorisation domains

S Launois and T H Lenagan *

Abstract

We show that the quantum coordinate ring of a semisimple group is a unique 
factorisation domain in the sense of Chatters and Jordan in the case where the 
deformation parameter $q$ is a transcendental element.


Key words: Unique factorisation domain, quantum enveloping algebra, quantum coordinate 
ing.

Introduction

Throughout this paper, $\mathbb{C}$ denotes the field of complex numbers, $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$ and $q \in \mathbb{C}^*$ is transcendental.

The notion of a noncommutative noetherian unique factorisation domain (UFD for short) has been introduced and studied by Chatters and Jordan in [3, 4]. Recently, the present authors, together with L Rigal, [11], have shown that many quantum algebras are noetherian UFD. In particular, we have shown that the quantum group $O_q(SL_n)$ is a noetherian UFD.

Let $G$ be a connected simply connected complex semisimple algebraic group. Since in 
the classical setting it was shown by Popov, [12], that the ring of regular functions on $G$ 
is a unique factorisation domain, one can ask if a similar result holds for the quantisation

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$O_q(G)$ of the coordinate ring of $G$. The aim of this note is to provide a positive answer to this question. In order to do this, we use a stratification of the prime spectrum of $O_q(G)$ that was constructed by Joseph, [8].

1 Quantised enveloping algebras and quantum coordinate rings

1.1 Quantised enveloping algebras

Let $\mathfrak{g}$ be a complex semisimple Lie algebra of rank $n$. We denote by $\pi = \{\alpha_1, \ldots, \alpha_n\}$ the set of simple roots associated to a triangular decomposition $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$. Recall that $\pi$ is a basis of a euclidean vector space $E$ over $\mathbb{R}$, whose inner product is denoted by $(,)$ ($E$ is usually denoted by $\mathfrak{h}^* \otimes \mathbb{R}$ in Bourbaki). We denote by $W$ the Weyl group of $\mathfrak{g}$; that is, the subgroup of the orthogonal group of $E$ generated by the reflections $s_i := s_{\alpha_i}$, for $i \in \{1, \ldots, n\}$, with reflecting hyperplanes $H_i := \{\beta \in E \mid (\beta, \alpha_i) = 0\}$, for $i \in \{1, \ldots, n\}$. If $w \in W$, we denote by $l(w)$ its length. Further, we denote by $w_0$ the longest element of $W$. Throughout this paper, the Coxeter group $W$ will be endowed with the Bruhat order that we denote by $\leq$. We refer the reader to [8, Appendix A1] for the definition and properties of the Bruhat order.

We denote by $R^+$ the set of positive roots and by $R$ the set of roots. We set $Q^+ := \mathbb{N}\alpha_1 \oplus \cdots \oplus \mathbb{N}\alpha_n$. We denote by $\varpi_1, \ldots, \varpi_n$ the fundamental weights, by $P$ the $\mathbb{Z}$-lattice generated by $\varpi_1, \ldots, \varpi_n$, and by $P^+$ the set of dominant weights. In the sequel, $P$ will always be endowed with the following partial order:

$$\lambda \leq \mu \text{ if and only if } \mu - \lambda \in Q^+.$$ 

Finally, we denote by $A = (a_{ij}) \in M_n(\mathbb{Z})$ the Cartan matrix associated to these data.

Recall that the scalar product of two roots $(\alpha, \beta)$ is always an integer. As in [1], we assume that the short roots have length $\sqrt{2}$.

For each $i \in \{1, \ldots, n\}$, set $q_i := q^{(\alpha_i, \alpha_i)/2}$ and

$$\begin{bmatrix} m \\ k \end{bmatrix}_i := \frac{(q_i - q_i^{-1})(q_i^{m-1} - q_i^{1-m})(q_i^{m} - q_i^{-m})}{(q_i - q_i^{-1})(q_i^{k} - q_i^{-k})(q_i - q_i^{-1})(q_i^{m-k} - q_i^{-m})}$$

for all integers $0 \leq k \leq m$. By convention, we have

$$\begin{bmatrix} m \\ 0 \end{bmatrix}_i := 1.$$
We will use the definition of the quantised enveloping algebra given in [1, I.6.3, I.6.4]. The quantised enveloping algebra $U_q(\mathfrak{g})$ of $\mathfrak{g}$ over $\mathbb{C}$ associated to the previous data is the $\mathbb{C}$-algebra generated by indeterminates $E_1, \ldots, E_n, F_1, \ldots, F_n, K_1^{\pm 1}, \ldots, K_n^{\pm 1}$ subject to the following relations:

$$K_i K_j = K_j K_i \quad K_i K_i^{-1} = 1$$

$$K_i E_j K_i^{-1} = q_i^{a_{ij}} E_j \quad K_i F_j K_i^{-1} = q_i^{-a_{ij}} F_j$$

$$E_i F_j - F_j E_i = \delta_{ij} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}}$$

and the quantum Serre relations:

$$\sum_{k=0}^{1-a_{ij}} (-1)^k \left[ \frac{1 - a_{ij}}{k} \right] E_i^{1-a_{ij}-k} E_j E_i^k = 0 \quad (i \neq j)$$

and

$$\sum_{k=0}^{1-a_{ij}} (-1)^k \left[ \frac{1 - a_{ij}}{k} \right] F_i^{1-a_{ij}-k} F_j F_i^k = 0 \quad (i \neq j).$$

Note that $U_q(\mathfrak{g})$ is a Hopf algebra; its comultiplication is defined by

$$\Delta(K_i) = K_i \otimes K_i \quad \Delta(E_i) = E_i \otimes 1 + K_i \otimes E_i \quad \Delta(F_i) = F_i \otimes K_i^{-1} + 1 \otimes F_i,$$

its counit by

$$\varepsilon(K_i) = 1 \quad \varepsilon(E_i) = \varepsilon(F_i) = 0,$$

and its antipode by

$$S(K_i) = K_i^{-1} \quad S(E_i) = -K_i^{-1} E_i \quad S(F_i) = -F_i K_i.$$

We refer the reader to [1, 7, 8] for more details on this algebra. Further, as usual, we denote by $U_q^+(\mathfrak{g})$ the subalgebra of $U_q(\mathfrak{g})$ generated by $E_1, \ldots, E_n$ and by $U_q(\mathfrak{b}^+)$ the subalgebra of $U_q(\mathfrak{g})$ generated by $E_1, \ldots, E_n, K_1^{\pm 1}, \ldots, K_n^{\pm 1}$. In a similar manner, $U_q^-(\mathfrak{g})$ is the subalgebra of $U_q(\mathfrak{g})$ generated by $F_1, \ldots, F_n$ and $U_q(\mathfrak{b}^-)$ is the subalgebra of $U_q(\mathfrak{g})$ generated by $F_1, \ldots, F_n, K_1^{\pm 1}, \ldots, K_n^{\pm 1}$.

1.2 Representations of quantised enveloping algebras

It is well-known that the representation theory of the quantised enveloping algebra $U_q(\mathfrak{g})$ is analogous to the representation theory of the classical enveloping algebra $U(\mathfrak{g})$. In this section, we collect the properties that will be needed in the rest of the paper.
As usual, if $M$ is a left $U_q(\mathfrak{g})$-module, we denote its dual by $M^*$. Observe that $M^*$ is a right $U_q(\mathfrak{g})$-module in a natural way. However, by using the antipode of $U_q(\mathfrak{g})$, this right action of $U_q(\mathfrak{g})$ on $M^*$ can be twisted to a left action, so that $M^*$ can be viewed as a left $U_q(\mathfrak{g})$-module.

Let $M$ be a $U_q(\mathfrak{g})$-module and $m \in M$. The element $m$ is said to have weight $\lambda \in P$ if $K_i.m = q^{(\lambda, \alpha_i)}m$ for all $i \in \{1, \ldots, n\}$. For each $\lambda \in P$, set

$$M_\lambda := \{m \in M \mid K_i.m = q^{(\lambda, \alpha_i)}m \text{ for all } i \in \{1, \ldots, n\}\}.$$ 

If $M_\lambda \neq 0$ then $M_\lambda$ is said to be a weight space of $M$ and $\lambda$ is a weight of $M$.

It is well-known, see, for example [1, 7], that, for each dominant weight $\lambda \in P^+$, there exists a unique (up to isomorphism) simple finite dimensional $U_q(\mathfrak{g})$-module of highest weight $\lambda$ that we denote by $V(\lambda)$. In the following proposition, we collect some well-known properties of the $V(\lambda)$, for $\lambda \in P^+$. We refer the reader to [1 especially I.6.12], [6] and [7] for details and proofs.

**Proposition 1.1** Denote by $\Omega(\lambda)$ the set of those weights $\mu \in P$ such that $V(\lambda)_\mu \neq 0$.

1. $V(\lambda) = \bigoplus_{\mu \in \Omega(\lambda)} V(\lambda)_\mu$
2. The weights of $V(\lambda)$ are given by Weyl’s character formula. In particular, if $\mu \in \Omega(\lambda)$, then $w\mu \in \Omega(\lambda)$ for all $w \in W$.
3. For all $w \in W$, one has $\dim_C V(\lambda)_{w\lambda} = 1$.
4. $V(\lambda)^* \simeq V(-w_0\lambda)$.
5. The weight $w_0\lambda$ is the unique lowest weight of $V(\lambda)$.
   In particular, for all $\mu \in \Omega(\lambda)$, one has $w_0\lambda \leq \mu \leq \lambda$.
6. $\Omega(\lambda) = \{\lambda - w\mu \mid w \in W \text{ and } \mu \in P^+ \text{ such that } \mu \leq \lambda\}$.

For all $w \in W$ and $\lambda \in P^+$, let $u_{w\lambda}$ denote a nonzero vector of weight $w\lambda$ in $V(\lambda)$. Then we denote by $V^+_w(\lambda)$ the Demazure module associated to the pair $\lambda, w$, that is:

$$V^+_w(\lambda) := U_q^+(\mathfrak{g})u_{w\lambda} = U_q(\mathfrak{b}^+)u_{w\lambda}.$$ 

We also set

$$V^-_w(\lambda) := U_q^-\mathfrak{g}u_{w\lambda} = U_q(\mathfrak{b}^-)u_{w\lambda}.$$ 

(Observe that these definitions are independent of the choice of $u_{w\lambda}$ because of Proposition 1.1 (3).)
The following result may be well-known; however, we have been unable to locate a precise statement.

**Proposition 1.2**

1. \( V_{w_0}^+(\lambda) = V(\lambda) = V_{id}^-(\lambda) \).

2. For all \( i, j \in \{1, \ldots, n\} \), one has

\[
V_{w_0 s_i}^+ (\varpi_j) = \begin{cases} \bigoplus_{\mu \in \Omega(\varpi_j) \setminus \{ w_0 \varpi_j \}} V(\varpi_j)_\mu & \text{if } i = j \\ V(\varpi_j) & \text{otherwise}, \end{cases}
\]

and

\[
V_{s_i}^- (\varpi_j) = \begin{cases} \bigoplus_{\mu \in \Omega(\varpi_j) \setminus \{ \varpi_j \}} V(\varpi_j)_\mu & \text{if } i = j \\ V(\varpi_j) & \text{otherwise}. \end{cases}
\]

**Proof.** We only prove the assertions corresponding to “positive” Demazure modules, the proof for “negative” Demazure modules is similar.

Since \( w_0 \lambda \) is the lowest weight of \( V(\lambda) \), we have \( U_q^+ (g) u_{w_0 \lambda} = V(\lambda) \); that is, \( V_{w_0}^+(\lambda) = V(\lambda) \). This proves the first assertion.

In order to prove the second claim, we distinguish between two cases.

First, let \( i, j \in \{1, \ldots, n\} \) with \( i \neq j \). Then \( s_i (\varpi_j) = \varpi_j \). Hence, in this case, one has:

\[
V_{w_0 s_i}^+ (\varpi_j) = U_q^+ (g) u_{w_0 s_i \varpi_j} = U_q^+ (g) u_{w_0 \varpi_j} = V_{w_0}^+ (\varpi_j) = V(\varpi_j).
\]

Next, let \( j \in \{1, \ldots, n\} \). Then \( s_j (\varpi_j) = \varpi_j - \alpha_j \). Let \( \mu \in \Omega(\varpi_j) \) with \( \mu \neq w_0 \varpi_j \), and let \( m \in V(\varpi_j)_\mu \) be any nonzero element. It follows from the first assertion that there exists \( x \in U_q^+ (g) \) such that \( m = x, u_{w_0 \varpi_j} \). The element \( x \) can be written as a linear combination of products \( E_{i_1} \cdots E_{i_k} \), with \( k \in \mathbb{N}^+ \) and \( i_1, \ldots, i_k \in \{1, \ldots, n\} \). Naturally, one can assume that \( E_{i_1} \cdots E_{i_k} u_{w_0 \varpi_j} \neq 0 \) for each such product. Let \( E_{i_1} \cdots E_{i_k} \) be one of these products. Since \( w_0 \pi = -\pi \), there exists \( l \in \{1, \ldots, n\} \) such that \( w_0 \alpha_{i_k} = -\alpha_l \). We will prove that \( l = j \). Indeed, assume that \( l \neq j \). Since \( E_{i_k} u_{w_0 \varpi_j} \) is a nonzero vector of \( V(\varpi_j) \) of weight \( w_0 \varpi_j + \alpha_{i_k} \), we get that

\[
w_0 \varpi_j + \alpha_{i_k} \in \Omega(\varpi_j).
\]

Then, we deduce from Proposition 1.1 that

\[
s_l w_0 (w_0 \varpi_j + \alpha_{i_k}) \in \Omega(\varpi_j),
\]

that is,

\[
s_l \varpi_j + \alpha_l \in \Omega(\varpi_j).
\]

Further, since we have assumed that \( l \neq j \), we get \( s_l \varpi_j = \varpi_j \), so that

\[
\varpi_j + \alpha_l \in \Omega(\varpi_j).
\]

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This contradicts the fact that \( w_j \) is the highest weight of \( V(w_j) \).

Thus, we have just proved that \( w_0 \alpha_i = -\alpha_j \) for all products \( E_{i_1} \ldots E_{i_k} \) that appear in \( x \). Now, observe that \( E_{i_k}.u_{w_0w_j} \) is a nonzero vector of \( V(w_j) \) of weight \( w_0(w_j + \alpha_i) = w_0(w_j - \alpha_j) = w_0s_jw_j \). Since \( \text{dim}_\mathbb{C} V(w_j)_{w_0s_jw_j} = 1 \), we get that \( E_{i_k}.u_{w_0w_j} = au_{w_0s_jw_j} \) for a certain nonzero complex number \( a \). Hence we get that

\[
m = x.u_{w_0w_j} = \sum \bullet E_{i_1} \ldots E_{i_k}.u_{w_0w_j} = y.u_{w_0s_jw_j},
\]

where \( \bullet \) denote some nonzero complex numbers and \( y \in U_q^+(g) \). Thus \( m \in V^+_{w_0s_j}(w_j) \). This shows that

\[
\bigoplus_{\mu \in \Omega(w_j) \setminus \{w_0w_j\}} V(w_j)_\mu \subseteq V^+_{w_0s_j}(w_j).
\]

As the reverse inclusion is trivial, this finishes the proof. \[\square\]

### 1.3 Quantised coordinate rings of semisimple groups and their prime spectra.

Let \( G \) be a connected, simply connected, semisimple algebraic group over \( \mathbb{C} \) with Lie algebra \( \text{Lie}(G) = g \). Since \( U_q(g) \) is a Hopf algebra, one can define its Hopf dual \( U_q(g)^* \) (see §1.4) via

\[
U_q(g)^* := \{ f \in \text{Hom}_\mathbb{C}(U_q(g), \mathbb{C}) \mid f = 0 \text{ on some ideal of finite codimension} \}.
\]

The quantised coordinate ring \( O_q(G) \) of \( G \) is the subalgebra of \( U_q(g)^* \) generated by the coordinate functions \( c^\lambda_{\xi,v} \) for all \( \lambda \in P^+, \xi \in V(\lambda)^* \) and \( v \in V(\lambda) \), where \( c^\lambda_{\xi,v} \) is the element of \( U_q(g)^* \) defined by

\[
c^\lambda_{\xi,v}(u) := \xi(uv) \text{ for all } u \in U_q(g),
\]

see, for example, [§ Chapter 9]. As usual, if \( \xi \in V(\lambda)^* \) and \( v \in V(\lambda)_\mu \), we write \( c^\lambda_{\eta,\mu} \) instead of \( c^\lambda_{\xi,v} \). Naturally, this leads to some ambiguity. However, when \( \mu \in W\lambda \) and \( \eta \in W.(-w_0\lambda) \), then \( \text{dim}(V(\lambda)_\mu) = 1 = \text{dim}(V(\lambda)^*_\eta) \), so that this ambiguity is very minor.

It is well-known that \( O_q(G) \) is a noetherian domain and a Hopf-subalgebra of \( U_q(g)^* \), see [¶ § 1.8]. This latter structure allows us to define the so-called left and right winding automorphisms (see, for instance, [¶ 1.9.25] or [§ 1.3.5]), and then to obtain an action of the torus \( H := (\mathbb{C}^*)^{2n} \) on \( O_q(G) \) (see [¶ 5.2]). More precisely, observe that the torus \( H := (\mathbb{C}^*)^n \) can be identified with \( \text{Hom}(P, \mathbb{C}^*) \) via:

\[
h(\lambda) = h_1^{\lambda_1} \ldots h_n^{\lambda_n},
\]

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where $h = (h_1, \ldots, h_n) \in H$ and $\lambda = \lambda_1 \varpi_1 + \cdots + \lambda_n \varpi_n$ with $\lambda_1, \ldots, \lambda_n \in \mathbb{Z}$. Then, it is known (see \[5, 3.3\] or \[1, I.1.18\]) that the torus $\mathcal{H}$ acts rationally by $\mathbb{C}$-algebra automorphisms on $O_q(G)$ via:
\[
g \cdot c^\lambda_{\xi,v} = g_1(\mu)g_2(\eta)c^\lambda_{\xi,v},
\]
for all $g = (g_1, g_2) \in \mathcal{H} = H \times H$, $\lambda \in P^+$, $\xi \in V(\lambda)^*$, and $v \in V(\lambda)_\eta$.
(We refer the reader to \[1, II.2.6\] for the definition of a rational action.)

As usual, we denote by $\text{Spec}(O_q(G))$ the set of prime ideals in $O_q(G)$. Recall that Joseph has proved \[9\] that every prime in $O_q(G)$ is completely prime.

Since $\mathcal{H}$ acts by automorphisms on $O_q(G)$, this induces an action of $\mathcal{H}$ on the prime spectrum of $O_q(G)$. As usual, we denote by $\mathcal{H}\text{-Spec}(O_q(G))$ the set of those prime ideals of $O_q(G)$ that are $\mathcal{H}$-invariant. This is a finite set since Brown and Goodearl \[2, Section 5\] (see also \[11 II.4\]) have shown using previous results of Joseph that
\[
\mathcal{H}\text{-Spec}(O_q(G)) = \{ Q_{w_+, w_-} \mid (w_+, w_-) \in W \times W \},
\]
where
\[
Q^+_{w_+} := \langle c^\lambda_{\xi,v} \mid \lambda \in P^+, v \in V(\lambda)_\lambda \text{ and } \xi \in (V^+_{w_+}(\lambda))^0 \subseteq V(\lambda)^* \rangle,
\]
\[
Q^-_{w_-} := \langle c^\lambda_{\xi,v} \mid \lambda \in P^+, v \in V(\lambda)_{w_0\lambda} \text{ and } \xi \in (V^-_{w_-w_0}(\lambda))^0 \subseteq V(\lambda)^* \rangle,
\]
and
\[
Q_{w_+, w_-} := Q^+_{w_+} + Q^-_{w_-}.
\]
Since $q$ is transcendental, it follows from \[10, Théorème 3\] that it is enough to consider the fundamental weights in the definition of $Q^+_{w_+}$ and $Q^-_{w_-}$. More precisely, we deduce from \[10, Théorème 3\] the following result.

**Theorem 1.3 (Joseph)**

\[
\mathcal{H}\text{-Spec}(O_q(G)) = \{ Q_{w_+, w_-} \mid (w_+, w_-) \in W \times W \},
\]
where
\[
Q^+_{w_+} := \langle c^j_{\xi,v} \mid j \in \{1, \ldots, n\}, v \in V(\omega_j)_{\omega_j} \text{ and } \xi \in (V^+_{w_+}(\omega_j))^0 \subseteq V(\omega_j)^* \rangle,
\]
\[
Q^-_{w_-} := \langle c^j_{\xi,v} \mid j \in \{1, \ldots, n\}, v \in V(\omega_j)_{w_0\omega_j} \text{ and } \xi \in (V^-_{w_-w_0}(\omega_j))^0 \subseteq V(\omega_j)^* \rangle,
\]
and
\[
Q_{w_+, w_-} := Q^+_{w_+} + Q^-_{w_-}.
\]
Moreover the prime ideals $Q_{w_+, w_-}$, for $(w_+, w_-) \in W \times W$, are pairwise distinct.
2 \(O_q(G)\) is a noetherian UFD.

In this section, we prove that \(O_q(G)\) is a noetherian UFD (We refer the reader to [11, Section 1] for the definition of a noetherian UFD; the key point is that each height one prime ideal should be generated by a normal element.) In order to do this, we proceed in three steps.

1. First, by using results of Joseph, we show that there exist a finite number of nonzero normal \(\mathcal{H}\)-eigenvectors \(r_1, \ldots, r_k\) of \(O_q(G)\) such that each \(\langle r_i \rangle\) is (completely) prime, and that each nonzero \(\mathcal{H}\)-invariant prime ideal of \(O_q(G)\) contains one of the \(r_i\). This property may be thought of as a “weak factoriality” result: \(O_q(G)\) is an \(\mathcal{H}\)-UFD in the terminology of [11].

2. Secondly, by using the \(H\)-stratification theory of Goodearl and Letzter (see [1, II]), we show that the localisation of \(O_q(G)\) with respect to the multiplicative system generated by the \(r_i\) is a noetherian UFD.

3. Finally, we use a noncommutative analogue of Nagata’s Lemma (see [11, Proposition 1.6]) to prove that \(O_q(G)\) itself is a noetherian UFD.

2.1 \(O_q(G)\) is an \(\mathcal{H}\)-UFD

This aim of this section is two-fold. First, we show that for each \(i \in \{1, \ldots, n\}\), the ideal generated by the normal element \(c_{w_i,0,w_0}^{-1}\) or \(c_{w_i,0,w_0}^{-1}\) is (completely) prime and then we prove that every nonzero \(\mathcal{H}\)-invariant prime ideal of \(O_q(G)\) contains either one of the \(c_{w_i,0,w_0}^{-1}\) or one of the \(c_{w_i,0,w_0}^{-1}\).

Lemma 2.1 Let \(i \in \{1, \ldots, n\}\). Then \(Q_{w_0,0,w_0}^{-1} = \langle c_{w_i,0,w_0}^{-1} \rangle\) and \(Q_{w_0,0,w_0}^{-1} = \langle c_{w_i,0,w_0}^{-1} \rangle\).

Proof. Recall that
\[
Q_{w_0,0,w_0}^- = Q_{w_0,0,w_0}^+ + Q_{s_i,w_0}^-,
\]
where
\[
Q_{w_0}^+ = \langle c_{\xi,\nu}^j \mid j \in \{1, \ldots, n\}, \nu \in V(\varpi_j)_{\varpi_j} \text{ and } \xi \in (V_{w_0}^+ (\varpi_j))^\perp \subseteq V(\varpi_j)^*\rangle,
\]
\[
Q_{s_i,w_0}^- = \langle c_{\xi,\nu}^j \mid j \in \{1, \ldots, n\}, \nu \in V(\varpi_j)_{w_0,\varpi_j} \text{ and } \xi \in (V_{s_i}^- (\varpi_j))^\perp \subseteq V(\varpi_j)^*\rangle.
\]
Next, it follows from Proposition 12.2(1) that \(V_{w_0}^+ (\varpi_j) = V(\varpi_j)\) for all \(j\), so that \(Q_{w_0}^+ = (0)\). Also, we deduce from Proposition 12.2(2) that \(V_{s_i}^- (\varpi_j) = V(\varpi_j)\) if \(j \neq i\), and \(V_{s_i}^- (\varpi_i) = \oplus_{\mu \in R(\varpi_i) \setminus \{\varpi_i\}} V(\varpi_i)_\mu\). Hence,
\[
Q_{s_i,w_0}^- = \langle c_{\xi,\nu}^j \mid \nu \in V(\varpi_i)_{w_0,\varpi_i} \text{ and } \xi \in V(\varpi_i)^*_{\varpi_i}\rangle,
\]
that is, \( Q_{s_iw_0} = \langle c_{-w_i,w_0w_i} \rangle \). Therefore \( Q_{w_0s_iw_0} = Q^+_{w_0} + Q^-_{s_iw_0} = \langle c_{-w_i,w_0w_i} \rangle \), as desired.

The second claim of the lemma is obtained in the same way. □

Now observe that, in [8], Joseph uses slightly different conventions for the dual \( M^* \) of a left \( U_q(\mathfrak{g}) \)-module. Indeed, it is mentioned in [8] 9.1] that the dual \( M^* \) is viewed with its natural right \( U_q(\mathfrak{g}) \)-module structure. As a consequence, Joseph’s convention for the weights of the dual \( L(\lambda)^* \) of \( L(\lambda) \), for \( \lambda \in P^+ \), is not exactly the same as our convention. In particular, the elements \( c^i_{-w_i,w_0w_i} \) and \( c^i_{w_0w_i,-w_i} \), \( i \in \{1, \ldots, n\} \), that appear in [8] Corollary 9.1.4], correspond to the elements \( c^i_{-w_i,w_0w_i} \) and \( c^i_{-w_0w_i,w_i} \) in our notation. With this in mind, it follows from [8] Corollary 9.1.4] that the elements \( c^i_{-w_i,w_0w_i} \) and \( c^i_{w_0w_i,-w_i} \), for \( i \in \{1, \ldots, n\} \), are normal in \( O_q(G) \). Thus we deduce from Lemma 2.1 the following result which will allow us later to use a noncommutative analogue of Nagata’s Lemma in order to prove that \( O_q(G) \) is a noetherian UFD.

**Corollary 2.2** The 2n elements \( c^i_{-w_i,w_0w_i} \) and \( c^i_{w_0w_i,-w_i} \), for \( i \in \{1, \ldots, n\} \), are nonzero normal elements of \( O_q(G) \) and they generate pairwise distinct completely prime ideals of \( O_q(G) \).

Since the \( c^i_{-w_i,w_0w_i} \) and \( c^i_{w_0w_i,-w_i} \), for \( i \in \{1, \ldots, n\} \), are \( \mathcal{H} \)-eigenvectors of \( O_q(G) \), in order to prove that \( O_q(G) \) is an \( \mathcal{H} \)-UFD in the sense of [11] Definition 2.7], it only remains to prove that every nonzero \( \mathcal{H} \)-invariant prime ideal of \( O_q(G) \) contains either one of the \( c^i_{-w_i,w_0w_i} \) or one of the \( c^i_{w_0w_i,-w_i} \). This is what we do next.

**Lemma 2.3** Let \( w = (w_+, w_-) \in W \times W \), with \( w \neq (w_0, w_0) \). Then \( Q_w \) contains either one of the \( c^i_{-w_i,w_0w_i} \) or one of the \( c^i_{w_0w_i,-w_i} \).

**Proof.** Since \( w \neq (w_0, w_0) \), either \( w_+ \neq w_0 \), or \( w_- \neq w_0 \). Assume, for instance, that \( w_+ \neq w_0 \), so that there exists \( i \in \{1, \ldots, n\} \) such that \( w_+ \leq w_0s_i \). One can easily check from the definition of \( Q_w \) that this forces \( c^i_{-w_0w_i,w_i} \in Q^+_{w_+} \), so that

\[
c^i_{-w_0w_i,w_i} \in Q^+_{w_+} \subseteq Q_w,
\]

as required. □

As a consequence of Corollary 2.2 and Lemma 2.3, we get the following result.

**Corollary 2.4** \( O_q(G) \) is an \( \mathcal{H} \)-UFD.

**Proof.** Theorem 1.3 establishes that \( \mathcal{H} \)-Spec\( (O_q(G)) = \{Q_{w_+,w_-} \mid (w_+, w_-) \in W \times W \} \). Note that \( Q_{w_+,w_-} = 0 \) precisely when \( w_+ = w_- = w_0 \). Thus, Corollary 2.2 and Lemma 2.3 show that each nonzero \( \mathcal{H} \)-prime ideal of \( O_q(G) \) contains a nonzero \( \mathcal{H} \)-prime of height one that is generated by a normal \( \mathcal{H} \)-eigenvector. Thus, \( O_q(G) \) is an \( \mathcal{H} \)-UFD. □
2.2 \( O_q(G) \) is a noetherian UFD.

Set \( T \) to be the localisation of \( O_q(G) \) with respect to the multiplicatively closed set generated by the normal \( \mathcal{H} \)-eigenvectors \( e_{c_i}^{\omega_i}, w_0 c_i, w_0 c_i \), for \( i \in \{1, \ldots, n\} \). Then the rational action of \( \mathcal{H} \) on \( O_q(G) \) extends to an action of \( \mathcal{H} \) on the localisation \( T \) by \( \mathbb{C} \)-algebra automorphisms, since we are localising with respect to \( \mathcal{H} \)-eigenvectors, and this action of \( \mathcal{H} \) on \( T \) is also rational, by using [11 II.2.7]. The following result is a consequence of Corollary 2.4 and [11 Proposition 3.5].

**Proposition 2.5** The ring \( T \) is \( \mathcal{H} \)-simple; that is, the only \( \mathcal{H} \)-ideals of \( T \) are \( 0 \) and \( T \).

We are now in position to show that \( O_q(G) \) is a noetherian UFD.

**Theorem 2.6** \( O_q(G) \) is a noetherian UFD.

**Proof.** By [11 Proposition 1.6], it is enough to prove that the localisation \( T \) is a noetherian UFD. Now, as proved in Proposition 2.5, \( T \) is an \( \mathcal{H} \)-simple ring. Thus, using [11 II.3.9], \( T \) is a noetherian UFD, as required. \( \square \)

As a consequence, we deduce from Theorem 2.6 and [11 Theorem 2.4] the following result.

**Corollary 2.7** \( O_q(G) \) is a maximal order.

The fact that \( O_q(G) \) is a maximal order can also be proved directly by using a suitable localisation of \( O_q(G) \), [8 Corollary 9.3.10], which is itself a maximal order.

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