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Quantised coordinate rings of semisimple groups are unique factorisation domains

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Abstract

We show that the quantum coordinate ring of a semisimple group is a unique factorisation domain in the sense of Chatters and Jordan in the case where the deformation parameter $q$ is a transcendental element.


Key words: Unique factorisation domain, quantum enveloping algebra, quantum coordinate ring.

Introduction

Throughout this paper, $\mathbb{C}$ denotes the field of complex numbers, $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$ and $q \in \mathbb{C}^*$ is transcendental.

The notion of a noncommutative noetherian unique factorisation domain (UFD for short) has been introduced and studied by Chatters and Jordan in [3, 4]. Recently, the present authors, together with L Rigal, [11], have shown that many quantum algebras are noetherian UFD. In particular, we have shown that the quantum group $O_q(SL_n)$ is a noetherian UFD.

Let $G$ be a connected simply connected complex semisimple algebraic group. Since in the classical setting it was shown by Popov, [12], that the ring of regular functions on $G$ is a unique factorisation domain, one can ask if a similar result holds for the quantisation

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$O_q(G)$ of the coordinate ring of $G$. The aim of this note is to provide a positive answer to this question. In order to do this, we use a stratification of the prime spectrum of $O_q(G)$ that was constructed by Joseph, [8].

1 Quantised enveloping algebras and quantum coordinate rings

1.1 Quantised enveloping algebras

Let $\mathfrak{g}$ be a complex semisimple Lie algebra of rank $n$. We denote by $\pi = \{\alpha_1, \ldots, \alpha_n\}$ the set of simple roots associated to a triangular decomposition $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$. Recall that $\pi$ is a basis of a euclidean vector space $E$ over $\mathbb{R}$, whose inner product is denoted by $(\ , \ )$ ($E$ is usually denoted by $\mathfrak{g}^* \otimes \mathbb{R}$ in Bourbaki). We denote by $W$ the Weyl group of $\mathfrak{g}$; that is, the subgroup of the orthogonal group of $E$ generated by the reflections $s_i := s_{\alpha_i}$, for $i \in \{1, \ldots, n\}$, with reflecting hyperplanes $H_i := \{\beta \in E \mid (\beta, \alpha_i) = 0\}$, for $i \in \{1, \ldots, n\}$. If $w \in W$, we denote by $l(w)$ its length. Further, we denote by $w_0$ the longest element of $W$. Throughout this paper, the Coxeter group $W$ will be endowed with the Bruhat order that we denote by $\leq$. We refer the reader to [8, Appendix A1] for the definition and properties of the Bruhat order.

We denote by $R^+$ the set of positive roots and by $R$ the set of roots. We set $Q^+ := \mathbb{N}\alpha_1 \oplus \cdots \oplus \mathbb{N}\alpha_n$. We denote by $\varpi_1, \ldots, \varpi_n$ the fundamental weights, by $P$ the $\mathbb{Z}$-lattice generated by $\varpi_1, \ldots, \varpi_n$, and by $P^+$ the set of dominant weights. In the sequel, $P$ will always be endowed with the following partial order:

$$\lambda \leq \mu \text{ if and only if } \mu - \lambda \in Q^+.$$ 

Finally, we denote by $A = (a_{ij}) \in M_n(\mathbb{Z})$ the Cartan matrix associated to these data.

Recall that the scalar product of two roots $(\alpha, \beta)$ is always an integer. As in [1], we assume that the short roots have length $\sqrt{2}$.

For each $i \in \{1, \ldots, n\}$, set $q_i := q_{\frac{\alpha_i, \alpha_i}{2}}$ and

$$\begin{bmatrix} m \\ k \end{bmatrix}_i := \frac{(q_i - q_i^{-1}) \cdots (q_i^{m-1} - q_i^{-1})(q_i^m - q_i^{-m})}{(q_i - q_i^{-1}) \cdots (q_i^k - q_i^{-k})(q_i - q_i^{-1})(q_i^{m-k} - q_i^{-m})}$$

for all integers $0 \leq k \leq m$. By convention, we have

$$\begin{bmatrix} m \\ 0 \end{bmatrix}_i := 1.$$
We will use the definition of the quantised enveloping algebra given in \[I\text{.6.3, I.6.4}\]. The quantised enveloping algebra $U_q(\mathfrak{g})$ of $\mathfrak{g}$ over $\mathbb{C}$ associated to the previous data is the $\mathbb{C}$-algebra generated by indeterminates $E_1, \ldots, E_n, F_1, \ldots, F_n, K_1^{\pm 1}, \ldots, K_n^{\pm 1}$ subject to the following relations:

\[
K_iK_j = K_jK_i \quad K_iK_i^{-1} = 1 \quad K_iE_jK_i^{-1} = q_i^{a_{ij}}E_j \quad K_iF_jK_i^{-1} = q_i^{-a_{ij}}F_j \\
E_iF_j - F_jE_i = \delta_{ij}\frac{K_i - K_i^{-1}}{q_i - q_i^{-1}}
\]

and the quantum Serre relations:

\[
\sum_{k=0}^{1-a_{ij}} (-1)^k \left[ \begin{array}{c} 1 - a_{ij} \\ k \end{array} \right] E_i^{1-a_{ij}-k}E_jE_i^k = 0 \hspace{1em} (i \neq j)
\]

\[
\sum_{k=0}^{1-a_{ij}} (-1)^k \left[ \begin{array}{c} 1 - a_{ij} \\ k \end{array} \right] F_i^{1-a_{ij}-k}F_jF_i^k = 0 \hspace{1em} (i \neq j).
\]

Note that $U_q(\mathfrak{g})$ is a Hopf algebra; its comultiplication is defined by

\[
\Delta(K_i) = K_i \otimes K_i \quad \Delta(E_i) = E_i \otimes 1 + K_i \otimes E_i \quad \Delta(F_i) = F_i \otimes K_i^{-1} + 1 \otimes F_i,
\]

its counit by

\[
\varepsilon(K_i) = 1 \quad \varepsilon(E_i) = \varepsilon(F_i) = 0,
\]

and its antipode by

\[
S(K_i) = K_i^{-1} \quad S(E_i) = -K_i^{-1}E_i \quad S(F_i) = -F_iK_i.
\]

We refer the reader to \[I\text{, 7, 8}\] for more details on this algebra. Further, as usual, we denote by $U_q^+(\mathfrak{g})$ the subalgebra of $U_q(\mathfrak{g})$ generated by $E_1, \ldots, E_n$ and by $U_q(\mathfrak{b}^+)$ the subalgebra of $U_q(\mathfrak{g})$ generated by $E_1, \ldots, E_n, K_1^{\pm 1}, \ldots, K_n^{\pm 1}$. In a similar manner, $U_q^-(\mathfrak{g})$ is the subalgebra of $U_q(\mathfrak{g})$ generated by $F_1, \ldots, F_n$ and $U_q(\mathfrak{b}^-)$ is the subalgebra of $U_q(\mathfrak{g})$ generated by $F_1, \ldots, F_n, K_1^{\pm 1}, \ldots, K_n^{\pm 1}$.

### 1.2 Representation theory of quantised enveloping algebras

It is well-known that the representation theory of the quantised enveloping algebra $U_q(\mathfrak{g})$ is analogous to the representation theory of the classical enveloping algebra $U(\mathfrak{g})$. In this section, we collect the properties that will be needed in the rest of the paper.
As usual, if $M$ is a left $U_q(\mathfrak{g})$-module, we denote its dual by $M^*$. Observe that $M^*$ is a right $U_q(\mathfrak{g})$-module in a natural way. However, by using the antipode of $U_q(\mathfrak{g})$, this right action of $U_q(\mathfrak{g})$ on $M^*$ can be twisted to a left action, so that $M^*$ can be viewed as a left $U_q(\mathfrak{g})$-module.

Let $M$ be a $U_q(\mathfrak{g})$-module and $m \in M$. The element $m$ is said to have weight $\lambda \in P$ if $K_i.m = q^{(\lambda,\alpha_i)}m$ for all $i \in \{1, \ldots, n\}$. For each $\lambda \in P$, set

$$M_\lambda := \{m \in M \mid K_i.m = q^{(\lambda,\alpha_i)}m \text{ for all } i \in \{1, \ldots, n\}\}.$$

If $M_\lambda \neq 0$ then $M_\lambda$ is said to be a weight space of $M$ and $\lambda$ is a weight of $M$.

It is well-known, see, for example [1, 7], that, for each dominant weight $\lambda \in P^+$, there exists a unique (up to isomorphism) simple finite dimensional $U_q(\mathfrak{g})$-module of highest weight $\lambda$ that we denote by $V(\lambda)$. In the following proposition, we collect some well-known properties of the $V(\lambda)$, for $\lambda \in P^+$. We refer the reader to [1 especially I.6.12], [6] and [7] for details and proofs.

**Proposition 1.1** Denote by $\Omega(\lambda)$ the set of those weights $\mu \in P$ such that $V(\lambda)_\mu \neq 0$.

1. $V(\lambda) = \bigoplus_{\mu \in \Omega(\lambda)} V(\lambda)_\mu$

2. The weights of $V(\lambda)$ are given by Weyl’s character formula. In particular, if $\mu \in \Omega(\lambda)$, then $w\mu \in \Omega(\lambda)$ for all $w \in W$.

3. For all $w \in W$, one has $\dim \mathbb{C} V(\lambda)_w = 1$.

4. $V(\lambda)^* \simeq V(-w_0 \lambda)$.

5. The weight $w_0 \lambda$ is the unique lowest weight of $V(\lambda)$.

In particular, for all $\mu \in \Omega(\lambda)$, one has $w_0 \lambda \leq \mu \leq \lambda$.

6. $\Omega(\lambda) = \{\lambda - w\mu \mid w \in W \text{ and } \mu \in P^+ \text{ such that } \mu \leq \lambda\}$.

For all $w \in W$ and $\lambda \in P^+$, let $u_{w\lambda}$ denote a nonzero vector of weight $w\lambda$ in $V(\lambda)$. Then we denote by $V_w^+(\lambda)$ the Demazure module associated to the pair $\lambda, w$, that is:

$$V_w^+(\lambda) := U_q^+(\mathfrak{g})u_{w\lambda} = U_q(b^+)u_{w\lambda}.$$

We also set

$$V_w^-(\lambda) := U_q^-(\mathfrak{g})u_{w\lambda} = U_q(b^-)u_{w\lambda}.$$

(Observe that these definitions are independent of the choice of $u_{w\lambda}$ because of Proposition 1.1 (3).)
The following result may be well-known; however, we have been unable to locate a precise statement.

**Proposition 1.2**

1. \( V^+_{w_0}(\lambda) = V(\lambda) = V_{id}(\lambda) \).

2. For all \( i, j \in \{1, \ldots, n\} \), one has

\[
V^+_{w_0s_i}(\varpi_j) = \begin{cases} \bigoplus_{\mu \in \Omega(\varpi_j) \setminus \{w_0 \varpi_j\}} V(\varpi_j)_{\mu} & \text{if } i = j \\ V(\varpi_j) & \text{otherwise}, \end{cases}
\]

and

\[
V^-_{s_i}(\varpi_j) = \begin{cases} \bigoplus_{\mu \in \Omega(\varpi_j) \setminus \{\varpi_j\}} V(\varpi_j)_{\mu} & \text{if } i = j \\ V(\varpi_j) & \text{otherwise}. \end{cases}
\]

**Proof.** We only prove the assertions corresponding to “positive” Demazure modules, the proof for “negative” Demazure modules is similar.

Since \( w_0 \lambda \) is the lowest weight of \( V(\lambda) \), we have \( U^+_q(g) u_{w_0 \lambda} = V(\lambda) \); that is, \( V^+_{w_0}(\lambda) = V(\lambda) \). This proves the first assertion.

In order to prove the second claim, we distinguish between two cases.

First, let \( i, j \in \{1, \ldots, n\} \) with \( i \neq j \). Then \( s_i(\varpi_j) = \varpi_j \). Hence, in this case, one has:

\[
V^+_{w_0s_i}(\varpi_j) = U^+_q(g) u_{w_0s_i \varpi_j} = U^+_q(g) u_{w_0 \varpi_j} = V^+_{w_0}(\varpi_j) = V(\varpi_j).
\]

Next, let \( j \in \{1, \ldots, n\} \). Then \( s_j(\varpi_j) = \varpi_j - \alpha_j \). Let \( \mu \in \Omega(\varpi_j) \) with \( \mu \neq w_0 \varpi_j \), and let \( m \in V(\varpi_j)_{\mu} \) be any nonzero element. It follows from the first assertion that there exists \( x \in U^+_q(g) \) such that \( m = x. u_{w_0 \varpi_j} \). The element \( x \) can be written as a linear combination of products \( E_{i_1} \ldots E_{i_k} \), with \( k \in \mathbb{N}^+ \) and \( i_1, \ldots, i_k \in \{1, \ldots, n\} \). Naturally, one can assume that \( E_{i_1} \ldots E_{i_k} u_{w_0 \varpi_j} \neq 0 \) for each such product. Let \( E_{i_1} \ldots E_{i_k} \) be one of these products. Since \( w_0 \pi = -\pi \), there exists \( l \in \{1, \ldots, n\} \) such that \( w_0 \alpha_{i_k} = -\alpha_l \). We will prove that \( l = j \). Indeed, assume that \( l \neq j \). Since \( E_{i_k} u_{w_0 \varpi_j} \) is a nonzero vector of \( V(\varpi_j) \) of weight \( w_0 \varpi_j + \alpha_{i_k} \), we get that

\[
w_0 \varpi_j + \alpha_{i_k} \in \Omega(\varpi_j).
\]

Then, we deduce from Proposition \[\Box\] that

\[
s_l w_0 (w_0 \varpi_j + \alpha_{i_k}) \in \Omega(\varpi_j),
\]

that is,

\[
s_l \varpi_j + \alpha_l \in \Omega(\varpi_j).
\]

Further, since we have assumed that \( l \neq j \), we get \( s_l \varpi_j = \varpi_j \), so that

\[
\varpi_j + \alpha_l \in \Omega(\varpi_j).
\]
This contradicts the fact that $\varpi_j$ is the highest weight of $V(\varpi_j)$.

Thus, we have just proved that $w_0\alpha_{i_k} = -\alpha_j$ for all products $E_{i_1} \ldots E_{i_k}$ that appear in $x$. Now, observe that $E_{i_k}.u_{w_0\varpi_j}$ is a nonzero vector of $V(\varpi_j)$ of weight $w_0\varpi_j + \alpha_{i_k} = w_0(\varpi_j + w_0\alpha_{i_k}) = w_0(\varpi_j - \alpha_j) = w_0s_j\varpi_j$. Since $\dim \mathbb{C}V(\varpi_j)_{w_0s_j}\varpi_j = 1$, we get that $E_{i_k}.u_{w_0\varpi_j} = au_{w_0s_j}\varpi_j$ for a certain nonzero complex number $a$. Hence we get that

$$m = x.u_{w_0\varpi_j} = \sum \bullet E_{i_1} \ldots E_{i_k}.u_{w_0\varpi_j} = y.u_{w_0s_j}\varpi_j,$$

where $\bullet$ denote some nonzero complex numbers and $y \in U_q^+(\mathfrak{g})$. Thus $m \in V_{w_0s_j}^+(\varpi_j)$. This shows that

$$\bigoplus_{\mu \in \Omega(\varpi_j) \setminus \{w_0\varpi_j\}} V(\varpi_j)_\mu \subseteq V_{w_0s_j}^+(\varpi_j).$$

As the reverse inclusion is trivial, this finishes the proof. \hfill \Box

### 1.3 Quantised coordinate rings of semisimple groups and their prime spectra.

Let $G$ be a connected, simply connected, semisimple algebraic group over $\mathbb{C}$ with Lie algebra $\text{Lie}(G) = \mathfrak{g}$. Since $U_q(\mathfrak{g})$ is a Hopf algebra, one can define its Hopf dual $U_q(\mathfrak{g})^*$ (see \cite{8} 1.4) via

$$U_q(\mathfrak{g})^* := \{ f \in \text{Hom}_\mathbb{C}(U_q(\mathfrak{g}), \mathbb{C}) \mid f = 0 \text{ on some ideal of finite codimension} \}.$$ 

The quantised coordinate ring $O_q(G)$ of $G$ is the subalgebra of $U_q(\mathfrak{g})^*$ generated by the coordinate functions $c_{\xi,v}^\lambda$ for all $\lambda \in P^+$, $\xi \in V(\lambda)^*$ and $v \in V(\lambda)$, where $c_{\xi,v}^\lambda$ is the element of $U_q(\mathfrak{g})^*$ defined by

$$c_{\xi,v}^\lambda(u) := \xi(uv) \text{ for all } u \in U_q(\mathfrak{g}),$$

see, for example, \cite{8} Chapter 9]. As usual, if $\xi \in V(\lambda)_\eta^*$ and $v \in V(\lambda)_\mu$, we write $c_{\eta,\mu}^\lambda$ instead of $c_{\xi,v}^\lambda$. Naturally, this leads to some ambiguity. However, when $\mu \in W.\lambda$ and $\eta \in W.(-w_0\lambda)$, then $\dim(V(\lambda)_\mu) = 1 = \dim(V(\lambda)_\eta^*)$, so that this ambiguity is very minor.

It is well-known that $O_q(G)$ is a noetherian domain and a Hopf-subalgebra of $U_q(\mathfrak{g})^*$, see \cite{11} \cite{8}. This latter structure allows us to define the so-called left and right winding automorphisms (see, for instance, \cite{11} 1.9.25 or \cite{8} 1.3.5), and then to obtain an action of the torus $\mathcal{H} := (\mathbb{C}^*)^{2n}$ on $O_q(G)$ (see \cite{2} 5.2). More precisely, observe that the torus $H := (\mathbb{C}^*)^n$ can be identified with $\text{Hom}(P, \mathbb{C}^*)$ via:

$$h(\lambda) = h_{1}^{\lambda_1} \ldots h_{n}^{\lambda_n},$$
where $h = (h_1, \ldots, h_n) \in H$ and $\lambda = \lambda_1 \varpi_1 + \cdots + \lambda_n \varpi_n$ with $\lambda_1, \ldots, \lambda_n \in \mathbb{Z}$. Then, it is known (see [3, 3.3] or [1, I.1.18]) that the torus $\mathcal{H}$ acts rationally by $\mathbb{C}$-algebra automorphisms on $O_q(G)$ via:

$$g. c^\lambda_{\xi, v} = g_1(\mu)g_2(\eta)c^\lambda_{\xi, v},$$

for all $g = (g_1, g_2) \in \mathcal{H} = H \times H$, $\lambda \in P^+$, $\xi \in V(\lambda)^*$ and $v \in V(\lambda)_\eta$. (We refer the reader to [1, II.2.6] for the definition of a rational action.)

As usual, we denote by $\text{Spec}(O_q(G))$ the set of prime ideals in $O_q(G)$. Recall that Joseph has proved [9] that every prime in $O_q(G)$ is completely prime.

Since $\mathcal{H}$ acts by automorphisms on $O_q(G)$, this induces an action of $\mathcal{H}$ on the prime spectrum of $O_q(G)$. As usual, we denote by $\mathcal{H}$-$\text{Spec}(O_q(G))$ the set of those prime ideals of $O_q(G)$ that are $\mathcal{H}$-invariant. This is a finite set since Brown and Goodearl [2, Section 5] (see also [11 II.4]) have shown using previous results of Joseph that

$$\mathcal{H}$-$\text{Spec}(O_q(G)) = \{ Q_{w_+, w_-} \mid (w_+, w_-) \in W \times W \},$$

where

$$Q^+_{w_+} := \langle c^\lambda_{\xi, v} \mid \lambda \in P^+, v \in V(\lambda)_\lambda \text{ and } \xi \in (V^+_{w_+}(\lambda))^\perp \subseteq V(\lambda)^* \rangle,$$

$$Q^-_{w_-} := \langle c^\lambda_{\xi, v} \mid \lambda \in P^+, v \in V(\lambda)_w \text{ and } \xi \in (V^-_{w_-}(\lambda))^\perp \subseteq V(\lambda)^* \rangle,$$

and

$$Q_{w_+, w_-} := Q^+_{w_+} + Q^-_{w_-}.$$ 

Since $q$ is transcendental, it follows from [11 Théorème 3] that it is enough to consider the fundamental weights in the definition of $Q^+_{w_+}$ and $Q^-_{w_-}$. More precisely, we deduce from [11 Théorème 3] the following result.

**Theorem 1.3 (Joseph)**

$$\mathcal{H}$-$\text{Spec}(O_q(G)) = \{ Q_{w_+, w_-} \mid (w_+, w_-) \in W \times W \},$$

where

$$Q^+_{w_+} := \langle c^\lambda_{\xi, v} \mid j \in \{1, \ldots, n\}, v \in V(\varpi_j)_{\varpi_j} \text{ and } \xi \in (V^+_{w_+}(\varpi_j))^\perp \subseteq V(\varpi_j)^* \rangle,$$

$$Q^-_{w_-} := \langle c^\lambda_{\xi, v} \mid j \in \{1, \ldots, n\}, v \in V(\varpi_j)_w \varpi_j \text{ and } \xi \in (V^-_{w_-}(\varpi_j))^\perp \subseteq V(\varpi_j)^* \rangle,$$

and

$$Q_{w_+, w_-} := Q^+_{w_+} + Q^-_{w_-}.$$ 

Moreover the prime ideals $Q_{w_+, w_-}$, for $(w_+, w_-) \in W \times W$, are pairwise distinct.
2 $O_q(G)$ is a noetherian UFD.

In this section, we prove that $O_q(G)$ is a noetherian UFD (We refer the reader to [I] Section 1 for the definition of a noetherian UFD; the key point is that each height one prime ideal should be generated by a normal element.) In order to do this, we proceed in three steps.

1. First, by using results of Joseph, we show that there exist a finite number of nonzero normal $H$-eigenvectors $r_1, \ldots, r_k$ of $O_q(G)$ such that each $\langle r_i \rangle$ is (completely) prime, and that each nonzero $H$-invariant prime ideal of $O_q(G)$ contains one of the $r_i$. This property may be thought of as a “weak factoriality” result: $O_q(G)$ is an $H$-UFD in the terminology of [II].

2. Secondly, by using the $H$-stratification theory of Goodearl and Letzter (see [I, II]), we show that the localisation of $O_q(G)$ with respect to the multiplicative system generated by the $r_i$ is a noetherian UFD.

3. Finally, we use a noncommutative analogue of Nagata’s Lemma (see [II Proposition 1.6]) to prove that $O_q(G)$ itself is a noetherian UFD.

2.1 $O_q(G)$ is an $H$-UFD

This aim of this section is two-fold. First, we show that for each $i \in \{1, \ldots, n\}$, the ideal generated by the normal element $c_{i, w_0, \omega_i}$ is (completely) prime and then we prove that every nonzero $H$-invariant prime ideal of $O_q(G)$ contains either one of the $c_{i, w_0, \omega_i}$ or one of the $c_{\omega_i, w_0, \omega_i}$.

Lemma 2.1 Let $i \in \{1, \ldots, n\}$. Then $Q_{w_0, s_i w_0} = \langle c_{\omega_i, w_0, \omega_i} \rangle$ and $Q_{w_0 s_i, w_0} = \langle c_{\omega_i, w_0, \omega_i} \rangle$.

Proof. Recall that

$$Q_{w_0, s_i w_0} = Q_{w_0}^+ + Q_{s_i w_0}^-,$$

where

$$Q_{w_0}^+ = \langle c_{\omega_j} \mid j \in \{1, \ldots, n\}; v \in V(\omega_j)_{\omega_j} \text{ and } \xi \in (V_{w_0}^+(\omega_j))^{\perp} \subseteq V(\omega_j)^* \rangle,$$

$$Q_{s_i w_0}^- = \langle c_{\omega_j} \mid j \in \{1, \ldots, n\}; v \in V(\omega_j)_{w_0, \omega_j} \text{ and } \xi \in (V_{s_i}^- (\omega_j))^{\perp} \subseteq V(\omega_j)^* \rangle.$$

Next, it follows from Proposition 2.2 (1) that $V_{w_0}^+(\omega_j) = V(\omega_j)$ for all $j$, so that $Q_{w_0}^+ = (0)$. Also, we deduce from Proposition 2.2 (2) that $V_{s_i}^- (\omega_j) = V(\omega_j)$ if $j \neq i$, and $V_{s_i}^- (\omega_i) = \bigoplus_{\mu \in R(\omega_i) \setminus \{\omega_i\}} V(\omega_i)_{\mu}$. Hence,

$$Q_{s_i w_0}^- = \langle c_{\omega_i} \mid v \in V(\omega_i)_{w_0, \omega_i} \text{ and } \xi \in V(\omega_i)^*_{\omega_i} \rangle.$$
that is, $Q_{s_i w_0} = \langle c_{s_i, w_0 \omega_i} \rangle$. Therefore $Q_{w_0 s_i w_0} = Q^+_{w_0} + Q^-_{s_i w_0} = \langle c_{s_i, w_0 \omega_i} \rangle$, as desired.

The second claim of the lemma is obtained in the same way. \hfill \square

Now observe that, in [8], Joseph uses slightly different conventions for the dual $M^*$ of a left $U_q(\mathfrak{g})$-module. Indeed, it is mentioned in [8, 9.1] that the dual $M^*$ is viewed with its natural right $U_q(\mathfrak{g})$-module structure. As a consequence, Joseph’s convention for the weights of the dual $L(\lambda)^*$ of $L(\lambda)$, for $\lambda \in P^+$, is not exactly the same as our convention. In particular, the elements $c_{s_i, w_0 \omega_i}$ and $c_{w_0 \omega_i, \omega_i}$, $i \in \{1, \ldots, n\}$, that appear in [8, Corollary 9.1.4], correspond to the elements $c_{-s_i, w_0 \omega_i}$ and $c_{-w_0 \omega_i, \omega_i}$ in our notation. With this in mind, it follows from [8, Corollary 9.1.4] that the elements $c_{-s_i, w_0 \omega_i}$ and $c_{-w_0 \omega_i, \omega_i}$, for $i \in \{1, \ldots, n\}$, are normal in $O_q(G)$. Thus we deduce from Lemma 2.1 the following result which will allow us later to use a noncommutative analogue of Nagata’s Lemma in order to prove that $O_q(G)$ is a noetherian UFD.

**Corollary 2.2** The $2n$ elements $c_{-s_i, w_0 \omega_i}$ and $c_{-w_0 \omega_i, \omega_i}$, for $i \in \{1, \ldots, n\}$, are nonzero normal elements of $O_q(G)$ and they generate pairwise distinct completely prime ideals of $O_q(G)$.

Since the $c_{s_i, w_0 \omega_i}$ and $c_{-w_0 \omega_i, \omega_i}$, for $i \in \{1, \ldots, n\}$, are $\mathcal{H}$-eigenvectors of $O_q(G)$, in order to prove that $O_q(G)$ is an $\mathcal{H}$-UFD in the sense of [11, Definition 2.7], it only remains to prove that every nonzero $\mathcal{H}$-invariant prime ideal of $O_q(G)$ contains either one of the $c_{-s_i, w_0 \omega_i}$ or one of the $c_{-w_0 \omega_i, \omega_i}$. This is what we do next.

**Lemma 2.3** Let $w = (w_+, w_-) \in W \times W$, with $w \neq (w_0, w_0)$. Then $Q_w$ contains either one of the $c_{-s_i, w_0 \omega_i}$, or one of the $c_{-w_0 \omega_i, \omega_i}$.

**Proof.** Since $w \neq (w_0, w_0)$, either $w_+ \neq w_0$, or $w_- \neq w_0$. Assume, for instance, that $w_+ \neq w_0$, so that there exists $i \in \{1, \ldots, n\}$ such that $w_+ \leq w_0 s_i$. One can easily check from the definition of $Q_w$ that this forces $c_{-w_0 \omega_i, \omega_i} \in Q^+_{w_+}$, so that
\[
c_{-w_0 \omega_i, \omega_i} \in Q^+_{w_+} \subseteq Q_w,
\]
as required. \hfill \square

As a consequence of Corollary 2.2 and Lemma 2.3, we get the following result.

**Corollary 2.4** $O_q(G)$ is an $\mathcal{H}$-UFD.

**Proof.** Theorem 1.3 establishes that $\mathcal{H}$-Spec($O_q(G)$) = \{ $Q_{w_+, w_-}$ | $(w_+, w_-) \in W \times W$ \}. Note that $Q_{w_+, w_-} = 0$ precisely when $w_+ = w_- = w_0$. Thus, Corollary 2.2 and Lemma 2.3 show that each nonzero $\mathcal{H}$-prime ideal of $O_q(G)$ contains a nonzero $\mathcal{H}$-prime of height one that is generated by a normal $\mathcal{H}$-eigenvector. Thus, $O_q(G)$ is an $\mathcal{H}$-UFD. \hfill \square
2.2 \( O_q(G) \) is a noetherian UFD.

Set \( T \) to be the localisation of \( O_q(G) \) with respect to the multiplicatively closed set generated by the normal \( \mathcal{H} \)-eigenvectors \( c_{\omega_i^1,\omega_i^2}^{\omega_i^1,\omega_i^2} \) and \( c_{-\omega_i^1,\omega_i^2}^{\omega_i^1,\omega_i^2} \), for \( i \in \{1, \ldots, n\} \). Then the rational action of \( \mathcal{H} \) on \( O_q(G) \) extends to an action of \( \mathcal{H} \) on the localisation \( T \) by \( \mathbb{C} \)-algebra automorphisms, since we are localising with respect to \( \mathcal{H} \)-eigenvectors, and this action of \( \mathcal{H} \) on \( T \) is also rational, by using [11 II.2.7]. The following result is a consequence of Corollary 2.4 and [11 Proposition 3.5].

**Proposition 2.5** The ring \( T \) is \( \mathcal{H} \)-simple; that is, the only \( \mathcal{H} \)-ideals of \( T \) are 0 and \( T \).

We are now in position to show that \( O_q(G) \) is a noetherian UFD.

**Theorem 2.6** \( O_q(G) \) is a noetherian UFD.

**Proof.** By [11 Proposition 1.6], it is enough to prove that the localisation \( T \) is a noetherian UFD. Now, as proved in Proposition 2.5, \( T \) is an \( \mathcal{H} \)-simple ring. Thus, using [11 II.3.9], \( T \) is a noetherian UFD, as required. \( \square \)

As a consequence, we deduce from Theorem 2.6 and [11 Theorem 2.4] the following result.

**Corollary 2.7** \( O_q(G) \) is a maximal order.

The fact that \( O_q(G) \) is a maximal order can also be proved directly by using a suitable localisation of \( O_q(G) \), [8 Corollary 9.3.10], which is itself a maximal order.

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