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Quantised coordinate rings of semisimple groups
are unique factorisation domains

S Launois and T H Lenagan *

Abstract

We show that the quantum coordinate ring of a semisimple group is a unique
factorisation domain in the sense of Chatters and Jordan in the case where the
deformation parameter $q$ is a transcendental element.


Key words: Unique factorisation domain, quantum enveloping algebra, quantum coordinate
ring.

Introduction

Throughout this paper, $\mathbb{C}$ denotes the field of complex numbers, $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$ and $q \in \mathbb{C}^*$
is transcendental.

The notion of a noncommutative noetherian unique factorisation domain (UFD for short) has been introduced and studied by Chatters and Jordan in [3, 4]. Recently, the
present authors, together with L Rigal, [11], have shown that many quantum algebras
are noetherian UFD. In particular, we have shown that the quantum group $O_q(SL_n)$ is a
noetherian UFD.

Let $G$ be a connected simply connected complex semisimple algebraic group. Since in
the classical setting it was shown by Popov, [12], that the ring of regular functions on $G$
is a unique factorisation domain, one can ask if a similar result holds for the quantisation

*This research was supported by a Marie Curie Intra-European Fellowship within the 6th European
Community Framework Programme and by Leverhulme Research Interchange Grant F/00158/X
$O_q(G)$ of the coordinate ring of $G$. The aim of this note is to provide a positive answer to this question. In order to do this, we use a stratification of the prime spectrum of $O_q(G)$ that was constructed by Joseph, [8].

1 Quantised enveloping algebras and quantum coordinate rings

1.1 Quantised enveloping algebras

Let $\mathfrak{g}$ be a complex semisimple Lie algebra of rank $n$. We denote by $\pi = \{\alpha_1, \ldots, \alpha_n\}$ the set of simple roots associated to a triangular decomposition $\mathfrak{g} = n^- \oplus \mathfrak{h} \oplus n^+$. Recall that $\pi$ is a basis of a euclidean vector space $E$ over $\mathbb{R}$, whose inner product is denoted by $(\ , \ )$ ($E$ is usually denoted by $\mathfrak{h}^*_\mathbb{R}$ in Bourbaki). We denote by $W$ the Weyl group of $\mathfrak{g}$; that is, the subgroup of the orthogonal group of $E$ generated by the reflections $s_i := s_{\alpha_i}$, for $i \in \{1, \ldots, n\}$, with reflecting hyperplanes $H_i := \{\beta \in E \mid (\beta, \alpha_i) = 0\}$, for $i \in \{1, \ldots, n\}$. If $w \in W$, we denote by $l(w)$ its length. Further, we denote by $w_0$ the longest element of $W$. Throughout this paper, the Coxeter group $W$ will be endowed with the Bruhat order that we denote by $\leq$. We refer the reader to [8, Appendix A1] for the definition and properties of the Bruhat order.

We denote by $R^+$ the set of positive roots and by $R$ the set of roots. We set $Q^+ := \mathbb{N}\alpha_1 \oplus \cdots \oplus \mathbb{N}\alpha_n$. We denote by $\varpi_1, \ldots, \varpi_n$ the fundamental weights, by $P$ the $\mathbb{Z}$-lattice generated by $\varpi_1, \ldots, \varpi_n$, and by $P^+$ the set of dominant weights. In the sequel, $P$ will always be endowed with the following partial order:

$$\lambda \leq \mu \text{ if and only if } \mu - \lambda \in Q^+.$$ 

Finally, we denote by $A = (a_{ij}) \in M_n(\mathbb{Z})$ the Cartan matrix associated to these data.

Recall that the scalar product of two roots $(\alpha, \beta)$ is always an integer. As in [1], we assume that the short roots have length $\sqrt{2}$.

For each $i \in \{1, \ldots, n\}$, set $q_i := q^{(\alpha_i, \alpha_i)/2}$ and

$$\begin{bmatrix} m \\ k \\ \vdots \\ 1 \end{bmatrix}_{i} := \frac{(q_i - q_i^{-1}) \cdots (q_i^m - q_i^{-m}) (q_i^m - q_i^{-m})}{(q_i - q_i^{-1}) \cdots (q_i^k - q_i^{-k}) (q_i - q_i^{-1}) \cdots (q_i^m - q_i^{-k}) (q_i - q_i^{-k}) \cdots (q_i^m - q_i^{-m})}$$

for all integers $0 \leq k \leq m$. By convention, we have

$$\begin{bmatrix} m \\ 0 \end{bmatrix}_{i} := 1.$$
We will use the definition of the quantised enveloping algebra given in [1, I.6.3, I.6.4]. The quantised enveloping algebra $U_q(g)$ of $g$ over $\mathbb{C}$ associated to the previous data is the $\mathbb{C}$-algebra generated by indeterminates $E_1, \ldots, E_n, F_1, \ldots, F_n, K_1^{\pm 1}, \ldots, K_n^{\pm 1}$ subject to the following relations:

$$K_i K_j = K_j K_i, \quad K_i K_i^{-1} = 1$$

$$K_i E_j K_i^{-1} = q_i^{a_{ij}} E_j, \quad K_i F_j K_i^{-1} = q_i^{-a_{ij}} F_j$$

$$E_i F_j - F_j E_i = \delta_{ij} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}}$$

and the quantum Serre relations:

$$\sum_{k=0}^{1-a_{ij}} (-1)^k \binom{1-a_{ij}}{k} E_i^{1-a_{ij}-k} E_j E_i^k = 0 \text{ (} i \neq j \text{)}$$

$$\sum_{k=0}^{1-a_{ij}} (-1)^k \binom{1-a_{ij}}{k} F_i^{1-a_{ij}-k} F_j F_i^k = 0 \text{ (} i \neq j \text{)}.$$

Note that $U_q(g)$ is a Hopf algebra; its comultiplication is defined by

$$\Delta(K_i) = K_i \otimes K_i \quad \Delta(E_i) = E_i \otimes 1 + K_i \otimes E_i \quad \Delta(F_i) = F_i \otimes K_i^{-1} + 1 \otimes F_i,$$

its counit by

$$\varepsilon(K_i) = 1 \quad \varepsilon(E_i) = \varepsilon(F_i) = 0,$$

and its antipode by

$$S(K_i) = K_i^{-1} \quad S(E_i) = -K_i^{-1} E_i \quad S(F_i) = -F_i K_i.$$

We refer the reader to [1, 7, 8] for more details on this algebra. Further, as usual, we denote by $U_q^+(g)$ the subalgebra of $U_q(g)$ generated by $E_1, \ldots, E_n$ and by $U_q(b^+)$ the subalgebra of $U_q(g)$ generated by $E_1, \ldots, E_n, K_1^{\pm 1}, \ldots, K_n^{\pm 1}$. In a similar manner, $U_q^-(g)$ is the subalgebra of $U_q(g)$ generated by $F_1, \ldots, F_n$ and $U_q(b^-)$ is the subalgebra of $U_q(g)$ generated by $F_1, \ldots, F_n, K_1^{\pm 1}, \ldots, K_n^{\pm 1}$.

### 1.2 Representation theory of quantised enveloping algebras

It is well-known that the representation theory of the quantised enveloping algebra $U_q(g)$ is analogous to the representation theory of the classical enveloping algebra $U(g)$. In this section, we collect the properties that will be needed in the rest of the paper.
As usual, if $M$ is a left $U_q(\mathfrak{g})$-module, we denote its dual by $M^*$. Observe that $M^*$ is a right $U_q(\mathfrak{g})$-module in a natural way. However, by using the antipode of $U_q(\mathfrak{g})$, this right action of $U_q(\mathfrak{g})$ on $M^*$ can be twisted to a left action, so that $M^*$ can be viewed as a left $U_q(\mathfrak{g})$-module.

Let $M$ be a $U_q(\mathfrak{g})$-module and $m \in M$. The element $m$ is said to have weight $\lambda \in P$ if $K_i.m = q^{(\lambda, \alpha_i)}m$ for all $i \in \{1, \ldots, n\}$. For each $\lambda \in P$, set

$$M_\lambda := \{m \in M \mid K_i.m = q^{(\lambda, \alpha_i)}m \text{ for all } i \in \{1, \ldots, n\}\}.$$ 

If $M_\lambda \neq 0$ then $M_\lambda$ is said to be a weight space of $M$ and $\lambda$ is a weight of $M$.

It is well-known, see, for example [1, 7], that, for each dominant weight $\lambda \in P^+$, there exists a unique (up to isomorphism) simple finite dimensional $U_q(\mathfrak{g})$-module of highest weight $\lambda$ that we denote by $V(\lambda)$. In the following proposition, we collect some well-known properties of the $V(\lambda)$, for $\lambda \in P^+$. We refer the reader to [1, especially I.6.12], [6] and [7] for details and proofs.

**Proposition 1.1** Denote by $\Omega(\lambda)$ the set of those weights $\mu \in P$ such that $V(\lambda)_\mu \neq 0$.

1. $V(\lambda) = \bigoplus_{\mu \in \Omega(\lambda)} V(\lambda)_\mu$

2. The weights of $V(\lambda)$ are given by Weyl’s character formula. In particular, if $\mu \in \Omega(\lambda)$, then $w\mu \in \Omega(\lambda)$ for all $w \in W$.

3. For all $w \in W$, one has $\dim \mathbb{C} V(\lambda)_{w\lambda} = 1$.

4. $V(\lambda)^* \simeq V(-w_0\lambda)$.

5. The weight $w_0\lambda$ is the unique lowest weight of $V(\lambda)$.

   In particular, for all $\mu \in \Omega(\lambda)$, one has $w_0\lambda \leq \mu \leq \lambda$.

6. $\Omega(\lambda) = \{\lambda - w\mu \mid w \in W \text{ and } \mu \in P^+ \text{ such that } \mu \leq \lambda\}$.

For all $w \in W$ and $\lambda \in P^+$, let $u_{w\lambda}$ denote a nonzero vector of weight $w\lambda$ in $V(\lambda)$. Then we denote by $V^+_w(\lambda)$ the Demazure module associated to the pair $\lambda, w$, that is:

$$V^+_w(\lambda) := U_q^+(\mathfrak{g})u_{w\lambda} = U_q(b^+)u_{w\lambda}.$$ 

We also set

$$V^-_w(\lambda) := U_q^-(\mathfrak{g})u_{w\lambda} = U_q(b^-)u_{w\lambda}.$$ 

(Observe that these definitions are independent of the choice of $u_{w\lambda}$ because of Proposition [1,1 (3).]
The following result may be well-known; however, we have been unable to locate a precise statement.

**Proposition 1.2**

1. \( V^{+}_{w_0}(\lambda) = V(\lambda) = V^{-}_{id}(\lambda) \).

2. For all \( i, j \in \{1, \ldots, n\} \), one has

\[
V^{+}_{w_0 s_i}(\varpi_j) = \begin{cases} 
\bigoplus_{\mu \in \Omega(\varpi_j) \setminus \{w_0 \varpi_j\}} V(\varpi_j)_{\mu} & \text{if } i = j \\
V(\varpi_j) & \text{otherwise,}
\end{cases}
\]

and

\[
V^{-}_{s_i}(\varpi_j) = \begin{cases} 
\bigoplus_{\mu \in \Omega(\varpi_j) \setminus \{\varpi_j\}} V(\varpi_j)_{\mu} & \text{if } i = j \\
V(\varpi_j) & \text{otherwise.}
\end{cases}
\]

**Proof.** We only prove the assertions corresponding to “positive” Demazure modules, the proof for “negative” Demazure modules is similar.

Since \( w_0 \lambda \) is the lowest weight of \( V(\lambda) \), we have \( U^+_{q}(g) u_{w_0 \lambda} = V(\lambda) \); that is, \( V^+_{w_0}(\lambda) = V(\lambda) \). This proves the first assertion.

In order to prove the second claim, we distinguish between two cases.

First, let \( i, j \in \{1, \ldots, n\} \) with \( i \neq j \). Then \( s_i(\varpi_j) = \varpi_j \). Hence, in this case, one has:

\[
V^{+}_{w_0 s_i}(\varpi_j) = U^+_{q}(g) u_{w_0 s_i, \varpi_j} = U^+_{q}(g) u_{w_0 \varpi_j} = V^+_{w_0}(\varpi_j) = V(\varpi_j).
\]

Next, let \( j \in \{1, \ldots, n\} \). Then \( s_j(\varpi_j) = \varpi_j - \alpha_j \). Let \( \mu \in \Omega(\varpi_j) \) with \( \mu \neq w_0 \varpi_j \), and let \( m \in V(\varpi_j)_{\mu} \) be any nonzero element. It follows from the first assertion that there exists \( x \in U^+_{q}(g) \) such that \( m = x \cdot u_{w_0 \varpi_j} \). The element \( x \) can be written as a linear combination of products \( E_{i_1} \ldots E_{i_k} \), with \( k \in \mathbb{N}^+ \) and \( i_1, \ldots, i_k \in \{1, \ldots, n\} \). Naturally, one can assume that \( E_{i_1} \ldots E_{i_k} u_{w_0 \varpi_j} \neq 0 \) for each such product. Let \( E_{i_1} \ldots E_{i_k} \) be one of these products. Since \( w_0 \pi = -\pi \), there exists \( l \in \{1, \ldots, n\} \) such that \( w_0 \alpha_{i_k} = -\alpha_l \). We will prove that \( l = j \). Indeed, assume that \( l \neq j \). Since \( E_{i_k} u_{w_0 \varpi_j} \) is a nonzero vector of \( V(\varpi_j) \) of weight \( w_0 \varpi_j + \alpha_{i_k} \), we get that

\[
w_0 \varpi_j + \alpha_{i_k} \in \Omega(\varpi_j).
\]

Then, we deduce from Proposition\ref{equiv} that

\[
s_l w_0 (w_0 \varpi_j + \alpha_{i_k}) \in \Omega(\varpi_j),
\]

that is,

\[
s_l \varpi_j + \alpha_l \in \Omega(\varpi_j).
\]

Further, since we have assumed that \( l \neq j \), we get \( s_l \varpi_j = \varpi_j \), so that

\[
\varpi_j + \alpha_l \in \Omega(\varpi_j).
\]
This contradicts the fact that \( w_j \) is the highest weight of \( V(w_j) \).

Thus, we have just proved that \( w_0\alpha_{ik} = -\alpha_j \) for all products \( E_{i_1} \cdots E_{i_k} \) that appear in \( x \). Now, observe that \( E_{i_k}.u_{w_0w_j} \) is a nonzero vector of \( V(w_j) \) of weight \( w_0(w_j + \alpha_{ik}) = w_0(w_j - \alpha_j) = w_0s_jw_j \). Since \( \dim \mathbb{C} V(w_j)w_0s_jw_j = 1 \), we get that \( E_{i_k}.u_{w_0w_j} = au_{w_0s_jw_j} \) for a certain nonzero complex number \( a \). Hence we get that

\[
m = x.u_{w_0w_j} = \sum \bullet E_{i_1} \cdots E_{i_k}.u_{w_0w_j} = y.u_{w_0s_jw_j},
\]

where \( \bullet \) denote some nonzero complex numbers and \( y \in U_q^+(g) \). Thus \( m \in V_{w_0s_j}^+(w_j) \). This shows that

\[
\bigoplus_{\mu \in \Omega(w_j) \setminus \{w_0w_j\}} V(w_j)_\mu \subseteq V_{w_0s_j}^+(w_j).
\]

As the reverse inclusion is trivial, this finishes the proof. \( \square \)

### 1.3 Quantised coordinate rings of semisimple groups and their prime spectra.

Let \( G \) be a connected, simply connected, semisimple algebraic group over \( \mathbb{C} \) with Lie algebra \( \text{Lie}(G) = g \). Since \( U_q(g) \) is a Hopf algebra, one can define its Hopf dual \( U_q(g)^* \) (see [3 1.4]) via

\[
U_q(g)^* := \left\{ f \in \text{Hom}_{\mathbb{C}}(U_q(g), \mathbb{C}) \mid f = 0 \text{ on some ideal of finite codimension} \right\}.
\]

The quantised coordinate ring \( O_q(G) \) of \( G \) is the subalgebra of \( U_q(g)^* \) generated by the coordinate functions \( c_{\xi,v}^\lambda \) for all \( \lambda \in \mathbb{P}^+, \xi \in V(\lambda)^* \) and \( v \in V(\lambda) \), where \( c_{\xi,v}^\lambda \) is the element of \( U_q(g)^* \) defined by

\[
c_{\xi,v}^\lambda(u) := \xi(uv) \text{ for all } u \in U_q(g),
\]

see, for example, [3 Chapter 9]. As usual, if \( \xi \in V(\lambda)^*_\eta \) and \( v \in V(\lambda)_\mu \), we write \( c_{\eta,\mu}^\lambda \) instead of \( c_{\xi,v}^\lambda \). Naturally, this leads to some ambiguity. However, when \( \mu \in W.\lambda \) and \( \eta \in W.(-w_0\lambda) \), then \( \dim(V(\lambda)_\mu) = 1 = \dim(V(\lambda)^*_\eta) \), so that this ambiguity is very minor.

It is well-known that \( O_q(G) \) is a noetherian domain and a Hopf-subalgebra of \( U_q(g)^* \), see [1 8]. This latter structure allows us to define the so-called left and right winding automorphisms (see, for instance, [1 1.9.25] or [3 1.3.5]), and then to obtain an action of the torus \( H := (\mathbb{C}^*)^{2n} \) on \( O_q(G) \) (see [2 5.2]). More precisely, observe that the torus \( H := (\mathbb{C}^*)^n \) can be identified with \( \text{Hom}(P, \mathbb{C}^*) \) via:

\[
h(\lambda) = h_1^\lambda \cdots h_n^\lambda,
\]

\[6\]
where \( h = (h_1, \ldots, h_n) \in H \) and \( \lambda = \lambda_1 \omega_1 + \cdots + \lambda_n \omega_n \) with \( \lambda_1, \ldots, \lambda_n \in \mathbb{Z} \). Then, it is known (see [3, 3.3] or [1, I.1.18]) that the torus \( \mathcal{H} \) acts rationally by \( \mathbb{C} \)-algebra automorphisms on \( O_q(G) \) via:

\[
g.c^\lambda_{\xi, v} = g_1(\mu)g_2(\eta)c^\lambda_{\xi, v},
\]

for all \( g = (g_1, g_2) \in \mathcal{H} = H \times H, \lambda \in P^+, \xi \in V(\lambda)^+ \) and \( v \in V(\lambda) \).

(We refer the reader to [1, II.2.6] for the definition of a rational action.)

As usual, we denote by \( \text{Spec}(O_q(G)) \) the set of prime ideals in \( O_q(G) \). Recall that Joseph has proved [9] that every prime in \( O_q(G) \) is completely prime.

Since \( \mathcal{H} \) acts by automorphisms on \( O_q(G) \), this induces an action of \( \mathcal{H} \) on the prime spectrum of \( O_q(G) \). As usual, we denote by \( \mathcal{H}\text{-Spec}(O_q(G)) \) the set of those primes ideals of \( O_q(G) \) that are \( \mathcal{H} \)-invariant. This is a finite set since Brown and Goodearl [2, Section 5] (see also [11, II.4]) have shown using previous results of Joseph that

\[
\mathcal{H}\text{-Spec}(O_q(G)) = \{ Q_{w_+, w_-} \mid (w_+, w_-) \in W \times W \},
\]

where

\[
Q_{w_+}^+ := \langle c^\lambda_{\xi, v} \mid \lambda \in P^+, v \in V(\lambda)_+ \text{ and } \xi \in (V_{w_+}(\lambda))^+ \subseteq V(\lambda)^+ \rangle,
\]

\[
Q_{w_-}^- := \langle c^\lambda_{\xi, v} \mid \lambda \in P^+, v \in V(\lambda)_{w_0}^+ \text{ and } \xi \in (V_{w_-}(\lambda))^+ \subseteq V(\lambda)^+ \rangle,
\]

and

\[
Q_{w_+, w_-} := Q_{w_+}^+ + Q_{w_-}^-.
\]

Since \( q \) is transcendental, it follows from [10, Théorème 3] that it is enough to consider the fundamental weights in the definition of \( Q_{w_+}^+ \) and \( Q_{w_-}^- \). More precisely, we deduce from [10, Théorème 3] the following result.

**Theorem 1.3 (Joseph)**

\[
\mathcal{H}\text{-Spec}(O_q(G)) = \{ Q_{w_+, w_-} \mid (w_+, w_-) \in W \times W \},
\]

where

\[
Q_{w_+}^+ := \langle c^\lambda_{\xi, v} \mid j \in \{1, \ldots, n\}, v \in V(\omega_j)_{\omega_j}^+ \text{ and } \xi \in (V_{w_+}^+(\omega_j))^+ \subseteq V(\omega_j)^+ \rangle,
\]

\[
Q_{w_-}^- := \langle c^\lambda_{\xi, v} \mid j \in \{1, \ldots, n\}, v \in V(\omega_j)_{w_0\omega_j}^+ \text{ and } \xi \in (V_{w_-}(\omega_j))^+ \subseteq V(\omega_j)^+ \rangle,
\]

and

\[
Q_{w_+, w_-} := Q_{w_+}^+ + Q_{w_-}^-.
\]

Moreover the prime ideals \( Q_{w_+, w_-} \), for \( (w_+, w_-) \in W \times W \), are pairwise distinct.
2 \(O_q(G)\) is a noetherian UFD.

In this section, we prove that \(O_q(G)\) is a noetherian UFD (We refer the reader to Section 1 for the definition of a noetherian UFD; the key point is that each height one prime ideal should be generated by a normal element.) In order to do this, we proceed in three steps.

1. First, by using results of Joseph, we show that there exist a finite number of nonzero normal \(H\)-eigenvectors \(r_1, \ldots, r_k\) of \(O_q(G)\) such that each \(\langle r_i \rangle\) is (completely) prime, and that each nonzero \(H\)-invariant prime ideal of \(O_q(G)\) contains one of the \(r_i\). This property may be thought of as a “weak factoriality” result: \(O_q(G)\) is an \(H\)-UFD in the terminology of [11].

2. Secondly, by using the \(H\)-stratification theory of Goodearl and Letzter (see II), we show that the localisation of \(O_q(G)\) with respect to the multiplicative system generated by the \(r_i\) is a noetherian UFD.

3. Finally, we use a noncommutative analogue of Nagata’s Lemma (see Proposition 1.6]) to prove that \(O_q(G)\) itself is a noetherian UFD.

2.1 \(O_q(G)\) is an \(H\)-UFD

This aim of this section is two-fold. First, we show that for each \(i \in \{1, \ldots, n\}\), the ideal generated by the normal element \(c_{-w_0}^{\omega_i}w_0\omega_i\) or \(c_{-w_0}^{\omega_i}w_0\omega_i\) is (completely) prime and then we prove that every nonzero \(H\)-invariant prime ideal of \(O_q(G)\) contains either one of the \(c_{-w_0}^{\omega_i}w_0\omega_i\) or one of the \(c_{-w_0}^{\omega_i}w_0\omega_i\).

**Lemma 2.1** Let \(i \in \{1, \ldots, n\}\). Then \(Q_{w_0s_iw_0} = \langle c_{-w_0}^{\omega_i}w_0\omega_i\rangle\) and \(Q_{w_0s_iw_0} = \langle c_{-w_0}^{\omega_i}w_0\omega_i\rangle\).

**Proof.** Recall that

\[
Q_{w_0s_iw_0} = Q_{w_0s_iw_0}^+ + Q_{w_0s_iw_0}^-,
\]

where

\[
Q_{w_0}^+ = \langle c_{\xi, v}^{\omega_j} | j \in \{1, \ldots, n\}, v \in V(\omega_j)_{\omega_j} \text{ and } \xi \in (V_{w_0}^{\omega_j})^* \subseteq V(\omega_j)^* \rangle,
\]

\[
Q_{s_iw_0}^- = \langle c_{\xi, v}^{\omega_j} | j \in \{1, \ldots, n\}, v \in V(\omega_j)_{w_0\omega_j} \text{ and } \xi \in (V_{s_i}^{\omega_j})^* \subseteq V(\omega_j)^* \rangle.
\]

Next, it follows from Proposition 1.2(1) that \(V_{w_0}^{\omega_j} = V(\omega_j)\) for all \(j\), so that \(Q_{w_0}^+ = (0)\). Also, we deduce from Proposition 1.2(2) that \(V_{s_i}^{\omega_j} = V(\omega_j)\) if \(j \neq i\), and \(V_{s_i}^{\omega_i} = \oplus_{\mu \in \Omega(\omega_i)\setminus\{\omega_i\}} V(\omega_i)^\mu\). Hence,

\[
Q_{s_iw_0}^- = \langle c_{\xi, v}^{\omega_i} | v \in V(\omega_i)_{w_0\omega_i} \text{ and } \xi \in V(\omega_i)^*_{\omega_i} \rangle.
\]
that is, $Q_{s_iw_0}^{-} = \langle c_{-\varpi_i, w_0\varpi_i}^{-} \rangle$. Therefore $Q_{w_0s_iw_0} = Q_{s_iw_0}^{+} + Q_{s_iw_0}^{-} = \langle c_{-\varpi_i, w_0\varpi_i}^{-} \rangle$, as desired.

The second claim of the lemma is obtained in the same way. $\square$

Now observe that, in [8], Joseph uses slightly different conventions for the dual $M^*$ of a left $U_q(\mathfrak{g})$-module. Indeed, it is mentioned in [8, Corollary 9.1.4] that the dual $M^*$ is viewed with its natural right $U_q(\mathfrak{g})$-module structure. As a consequence, Joseph’s convention for the weights of the dual $L(\lambda)^*$ of $L(\lambda)$, for $\lambda \in P^+$, is not exactly the same as our convention. In particular, the elements $c_{\varpi_i, w_0\varpi_i}^{\varpi_i}$ and $c_{\varpi_i, w_0\varpi_i}$, $i \in \{1, \ldots, n\}$, that appear in [8, Corollary 9.1.4], correspond to the elements $c_{-\varpi_i, w_0\varpi_i}$ and $c_{-\varpi_i, w_0\varpi_i}$ in our notation. With this in mind, it follows from [8, Corollary 9.1.4] that the elements $c_{\varpi_i, w_0\varpi_i}$ and $c_{\varpi_i, w_0\varpi_i}$, for $i \in \{1, \ldots, n\}$, are normal in $O_q(G)$. Thus we deduce from Lemma 2.1 the following result which will allow us later to use a noncommutative analogue of Nagata’s Lemma in order to prove that $O_q(G)$ is a noetherian UFD.

**Corollary 2.2** The $2n$ elements $c_{\varpi_i, w_0\varpi_i}$ and $c_{\varpi_i, w_0\varpi_i}$, for $i \in \{1, \ldots, n\}$, are nonzero normal elements of $O_q(G)$ and they generate pairwise distinct completely prime ideals of $O_q(G)$.

Since the $c_{\varpi_i, w_0\varpi_i}$ and $c_{\varpi_i, w_0\varpi_i}$, for $i \in \{1, \ldots, n\}$, are $\mathcal{H}$-eigenvectors of $O_q(G)$, in order to prove that $O_q(G)$ is an $\mathcal{H}$-UFD in the sense of [11, Definition 2.7], it only remains to prove that every nonzero $\mathcal{H}$-invariant prime ideal of $O_q(G)$ contains either one of the $c_{\varpi_i, w_0\varpi_i}$ or one of the $c_{\varpi_i, w_0\varpi_i}$. This is what we do next.

**Lemma 2.3** Let $\mathbf{w} = (w_+, w_-) \in W \times W$, with $\mathbf{w} \neq (w_0, w_0)$. Then $Q_{\mathbf{w}}$ contains either one of the $c_{-\varpi_i, w_0\varpi_i}$ or one of the $c_{-\varpi_i, w_0\varpi_i}$.

*Proof.* Since $\mathbf{w} \neq (w_0, w_0)$, either $w_+ \neq w_0$, or $w_- \neq w_0$. Assume, for instance, that $w_+ \neq w_0$, so that there exists $i \in \{1, \ldots, n\}$ such that $w_+ \leq w_0s_i$. One can easily check from the definition of $Q_{\mathbf{w}}$ that this forces $c_{-\varpi_i, w_0\varpi_i} \in Q_{w_+}^+$, so that

$$c_{-\varpi_i, w_0\varpi_i} \in Q_{w_+}^+ \subseteq Q_{\mathbf{w}},$$

as required. $\square$

As a consequence of Corollary 2.2 and Lemma 2.3 we get the following result.

**Corollary 2.4** $O_q(G)$ is an $\mathcal{H}$-UFD.

*Proof.* Theorem 1.3 establishes that $\mathcal{H}$-Spec($O_q(G)$) = \{ $Q_{w_+, w_-}$ | $(w_+, w_-) \in W \times W$ \}. Note that $Q_{w_+, w_-} = 0$ precisely when $w_+ = w_- = w_0$. Thus, Corollary 2.2 and Lemma 2.3 show that each nonzero $\mathcal{H}$-prime ideal of $O_q(G)$ contains a nonzero $\mathcal{H}$-prime of height one that is generated by a normal $\mathcal{H}$-eigenvector. Thus, $O_q(G)$ is an $\mathcal{H}$-UFD. $\square$
2.2 $O_q(G)$ is a noetherian UFD.

Set $T$ to be the localisation of $O_q(G)$ with respect to the multiplicatively closed set generated by the normal $\mathcal{H}$-eigenvectors $c_{\omega_i,\omega_j}$ and $c_{-\omega_i,\omega_j}$, for $i \in \{1, \ldots, n\}$. Then the rational action of $\mathcal{H}$ on $O_q(G)$ extends to an action of $\mathcal{H}$ on the localisation $T$ by $\mathbb{C}$-algebra automorphisms, since we are localising with respect to $\mathcal{H}$-eigenvectors, and this action of $\mathcal{H}$ on $T$ is also rational, by using [1 II.2.7]. The following result is a consequence of Corollary 2.4 and [11 Proposition 3.5].

**Proposition 2.5** The ring $T$ is $\mathcal{H}$-simple; that is, the only $\mathcal{H}$-ideals of $T$ are $0$ and $T$.

We are now in position to show that $O_q(G)$ is a noetherian UFD.

**Theorem 2.6** $O_q(G)$ is a noetherian UFD.

**Proof.** By [11 Proposition 1.6], it is enough to prove that the localisation $T$ is a noetherian UFD. Now, as proved in Proposition 2.5, $T$ is an $\mathcal{H}$-simple ring. Thus, using [11 II.3.9], $T$ is a noetherian UFD, as required. \hfill \Box

As a consequence, we deduce from Theorem 2.6 and [1 Proposition 2.4] the following result.

**Corollary 2.7** $O_q(G)$ is a maximal order.

The fact that $O_q(G)$ is a maximal order can also be proved directly by using a suitable localisation of $O_q(G)$, [8 Corollary 9.3.10], which is itself a maximal order.

**Acknowledgment** We thank Laurent Rigal with whom we first discussed this problem. We also thank Christian Ohn for a very helpful conversation concerning the representation theory of $U(\mathfrak{g})$ during a meeting of the Groupe de Travail Inter-universitaire en Algèbre in La Rochelle and thank the organisers for the opportunity to attend this meeting.

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