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# Quantised coordinate rings of semisimple groups are unique factorisation domains

S Launois and T H Lenagan \*

## Abstract

We show that the quantum coordinate ring of a semisimple group is a unique factorisation domain in the sense of Chatters and Jordan in the case where the deformation parameter  $q$  is a transcendental element.

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*Key words:* Unique factorisation domain, quantum enveloping algebra, quantum coordinate ring.

## Introduction

Throughout this paper,  $\mathbb{C}$  denotes the field of complex numbers,  $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$  and  $q \in \mathbb{C}^*$  is transcendental.

The notion of a noncommutative noetherian unique factorisation domain (UFD for short) has been introduced and studied by Chatters and Jordan in [3, 4]. Recently, the present authors, together with L Rigal, [11], have shown that many quantum algebras are noetherian UFD. In particular, we have shown that the quantum group  $O_q(SL_n)$  is a noetherian UFD.

Let  $G$  be a connected simply connected complex semisimple algebraic group. Since in the classical setting it was shown by Popov, [12], that the ring of regular functions on  $G$  is a unique factorisation domain, one can ask if a similar result holds for the quantisation

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$O_q(G)$  of the coordinate ring of  $G$ . The aim of this note is to provide a positive answer to this question. In order to do this, we use a stratification of the prime spectrum of  $O_q(G)$  that was constructed by Joseph, [8].

# 1 Quantised enveloping algebras and quantum coordinate rings

## 1.1 Quantised enveloping algebras

Let  $\mathfrak{g}$  be a complex semisimple Lie algebra of rank  $n$ . We denote by  $\pi = \{\alpha_1, \dots, \alpha_n\}$  the set of simple roots associated to a triangular decomposition  $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ . Recall that  $\pi$  is a basis of a euclidean vector space  $E$  over  $\mathbb{R}$ , whose inner product is denoted by  $(\cdot, \cdot)$  ( $E$  is usually denoted by  $\mathfrak{h}_{\mathbb{R}}^*$  in Bourbaki). We denote by  $W$  the Weyl group of  $\mathfrak{g}$ ; that is, the subgroup of the orthogonal group of  $E$  generated by the reflections  $s_i := s_{\alpha_i}$ , for  $i \in \{1, \dots, n\}$ , with reflecting hyperplanes  $H_i := \{\beta \in E \mid (\beta, \alpha_i) = 0\}$ , for  $i \in \{1, \dots, n\}$ . If  $w \in W$ , we denote by  $l(w)$  its length. Further, we denote by  $w_0$  the longest element of  $W$ . Throughout this paper, the Coxeter group  $W$  will be endowed with the Bruhat order that we denote by  $\leq$ . We refer the reader to [8, Appendix A1] for the definition and properties of the Bruhat order.

We denote by  $R^+$  the set of positive roots and by  $R$  the set of roots. We set  $Q^+ := \mathbb{N}\alpha_1 \oplus \dots \oplus \mathbb{N}\alpha_n$ . We denote by  $\varpi_1, \dots, \varpi_n$  the fundamental weights, by  $P$  the  $\mathbb{Z}$ -lattice generated by  $\varpi_1, \dots, \varpi_n$ , and by  $P^+$  the set of dominant weights. In the sequel,  $P$  will always be endowed with the following partial order:

$$\lambda \leq \mu \text{ if and only if } \mu - \lambda \in Q^+.$$

Finally, we denote by  $A = (a_{ij}) \in M_n(\mathbb{Z})$  the Cartan matrix associated to these data.

Recall that the scalar product of two roots  $(\alpha, \beta)$  is always an integer. As in [1], we assume that the short roots have length  $\sqrt{2}$ .

For each  $i \in \{1, \dots, n\}$ , set  $q_i := q^{\frac{(\alpha_i, \alpha_i)}{2}}$  and

$$\left[ \begin{matrix} m \\ k \end{matrix} \right]_i := \frac{(q_i - q_i^{-1}) \dots (q_i^{m-1} - q_i^{1-m})(q_i^m - q_i^{-m})}{(q_i - q_i^{-1}) \dots (q_i^k - q_i^{-k})(q_i - q_i^{-1}) \dots (q_i^{m-k} - q_i^{k-m})}$$

for all integers  $0 \leq k \leq m$ . By convention, we have

$$\left[ \begin{matrix} m \\ 0 \end{matrix} \right]_i := 1.$$

We will use the definition of the quantised enveloping algebra given in [1, I.6.3, I.6.4]. The quantised enveloping algebra  $U_q(\mathfrak{g})$  of  $\mathfrak{g}$  over  $\mathbb{C}$  associated to the previous data is the  $\mathbb{C}$ -algebra generated by indeterminates  $E_1, \dots, E_n, F_1, \dots, F_n, K_1^{\pm 1}, \dots, K_n^{\pm 1}$  subject to the following relations:

$$\begin{aligned} K_i K_j &= K_j K_i & K_i K_i^{-1} &= 1 \\ K_i E_j K_i^{-1} &= q_i^{a_{ij}} E_j & K_i F_j K_i^{-1} &= q_i^{-a_{ij}} F_j \\ E_i F_j - F_j E_i &= \delta_{ij} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}} \end{aligned}$$

and the quantum Serre relations:

$$\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1 - a_{ij} \\ k \end{bmatrix}_i E_i^{1-a_{ij}-k} E_j E_i^k = 0 \quad (i \neq j)$$

and

$$\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1 - a_{ij} \\ k \end{bmatrix}_i F_i^{1-a_{ij}-k} F_j F_i^k = 0 \quad (i \neq j).$$

Note that  $U_q(\mathfrak{g})$  is a Hopf algebra; its comultiplication is defined by

$$\Delta(K_i) = K_i \otimes K_i \quad \Delta(E_i) = E_i \otimes 1 + K_i \otimes E_i \quad \Delta(F_i) = F_i \otimes K_i^{-1} + 1 \otimes F_i,$$

its counit by

$$\varepsilon(K_i) = 1 \quad \varepsilon(E_i) = \varepsilon(F_i) = 0,$$

and its antipode by

$$S(K_i) = K_i^{-1} \quad S(E_i) = -K_i^{-1} E_i \quad S(F_i) = -F_i K_i.$$

We refer the reader to [1, 7, 8] for more details on this algebra. Further, as usual, we denote by  $U_q^+(\mathfrak{g})$  the subalgebra of  $U_q(\mathfrak{g})$  generated by  $E_1, \dots, E_n$  and by  $U_q(\mathfrak{b}^+)$  the subalgebra of  $U_q(\mathfrak{g})$  generated by  $E_1, \dots, E_n, K_1^{\pm 1}, \dots, K_n^{\pm 1}$ . In a similar manner,  $U_q^-(\mathfrak{g})$  is the subalgebra of  $U_q(\mathfrak{g})$  generated by  $F_1, \dots, F_n$  and  $U_q(\mathfrak{b}^-)$  is the subalgebra of  $U_q(\mathfrak{g})$  generated by  $F_1, \dots, F_n, K_1^{\pm 1}, \dots, K_n^{\pm 1}$ .

## 1.2 Representation theory of quantised enveloping algebras

It is well-known that the representation theory of the quantised enveloping algebra  $U_q(\mathfrak{g})$  is analogous to the representation theory of the classical enveloping algebra  $U(\mathfrak{g})$ . In this section, we collect the properties that will be needed in the rest of the paper.

As usual, if  $M$  is a left  $U_q(\mathfrak{g})$ -module, we denote its dual by  $M^*$ . Observe that  $M^*$  is a right  $U_q(\mathfrak{g})$ -module in a natural way. However, by using the antipode of  $U_q(\mathfrak{g})$ , this right action of  $U_q(\mathfrak{g})$  on  $M^*$  can be twisted to a left action, so that  $M^*$  can be viewed as a left  $U_q(\mathfrak{g})$ -module.

Let  $M$  be a  $U_q(\mathfrak{g})$ -module and  $m \in M$ . The element  $m$  is said to have weight  $\lambda \in P$  if  $K_i.m = q^{(\lambda, \alpha_i)}m$  for all  $i \in \{1, \dots, n\}$ . For each  $\lambda \in P$ , set

$$M_\lambda := \{m \in M \mid K_i.m = q^{(\lambda, \alpha_i)}m \text{ for all } i \in \{1, \dots, n\}\}.$$

If  $M_\lambda \neq 0$  then  $M_\lambda$  is said to be a weight space of  $M$  and  $\lambda$  is a weight of  $M$ .

It is well-known, see, for example [1, 7], that, for each dominant weight  $\lambda \in P^+$ , there exists a unique (up to isomorphism) simple finite dimensional  $U_q(\mathfrak{g})$ -module of highest weight  $\lambda$  that we denote by  $V(\lambda)$ . In the following proposition, we collect some well-known properties of the  $V(\lambda)$ , for  $\lambda \in P^+$ . We refer the reader to [1, especially I.6.12], [6] and [7] for details and proofs.

**Proposition 1.1** *Denote by  $\Omega(\lambda)$  the set of those weights  $\mu \in P$  such that  $V(\lambda)_\mu \neq 0$ .*

1.  $V(\lambda) = \bigoplus_{\mu \in \Omega(\lambda)} V(\lambda)_\mu$
2. *The weights of  $V(\lambda)$  are given by Weyl's character formula. In particular, if  $\mu \in \Omega(\lambda)$ , then  $w\mu \in \Omega(\lambda)$  for all  $w \in W$ .*
3. *For all  $w \in W$ , one has  $\dim_{\mathbb{C}} V(\lambda)_{w\lambda} = 1$ .*
4.  $V(\lambda)^* \simeq V(-w_0\lambda)$ .
5. *The weight  $w_0\lambda$  is the unique lowest weight of  $V(\lambda)$ . In particular, for all  $\mu \in \Omega(\lambda)$ , one has  $w_0\lambda \leq \mu \leq \lambda$ .*
6.  $\Omega(\lambda) = \{\lambda - w\mu \mid w \in W \text{ and } \mu \in P^+ \text{ such that } \mu \leq \lambda\}$ .

For all  $w \in W$  and  $\lambda \in P^+$ , let  $u_{w\lambda}$  denote a nonzero vector of weight  $w\lambda$  in  $V(\lambda)$ . Then we denote by  $V_w^+(\lambda)$  the Demazure module associated to the pair  $\lambda, w$ , that is:

$$V_w^+(\lambda) := U_q^+(\mathfrak{g})u_{w\lambda} = U_q(\mathfrak{b}^+)u_{w\lambda}.$$

We also set

$$V_w^-(\lambda) := U_q^-(\mathfrak{g})u_{w\lambda} = U_q(\mathfrak{b}^-)u_{w\lambda}.$$

(Observe that these definitions are independent of the choice of  $u_{w\lambda}$  because of Proposition 1.1 (3).)

The following result may be well-known; however, we have been unable to locate a precise statement.

**Proposition 1.2** 1.  $V_{w_0}^+(\lambda) = V(\lambda) = V_{id}^-(\lambda)$ .

2. For all  $i, j \in \{1, \dots, n\}$ , one has

$$V_{w_0 s_i}^+(\varpi_j) = \begin{cases} \bigoplus_{\mu \in \Omega(\varpi_j) \setminus \{w_0 \varpi_j\}} V(\varpi_j)_\mu & \text{if } i = j \\ V(\varpi_j) & \text{otherwise,} \end{cases}$$

and

$$V_{s_i}^-(\varpi_j) = \begin{cases} \bigoplus_{\mu \in \Omega(\varpi_j) \setminus \{\varpi_j\}} V(\varpi_j)_\mu & \text{if } i = j \\ V(\varpi_j) & \text{otherwise.} \end{cases}$$

*Proof.* We only prove the assertions corresponding to “positive” Demazure modules, the proof for “negative” Demazure modules is similar.

Since  $w_0\lambda$  is the lowest weight of  $V(\lambda)$ , we have  $U_q^+(\mathfrak{g})u_{w_0\lambda} = V(\lambda)$ ; that is,  $V_{w_0}^+(\lambda) = V(\lambda)$ . This proves the first assertion.

In order to prove the second claim, we distinguish between two cases.

First, let  $i, j \in \{1, \dots, n\}$  with  $i \neq j$ . Then  $s_i(\varpi_j) = \varpi_j$ . Hence, in this case, one has:  $V_{w_0 s_i}^+(\varpi_j) = U_q^+(\mathfrak{g})u_{w_0 s_i \varpi_j} = U_q^+(\mathfrak{g})u_{w_0 \varpi_j} = V_{w_0}^+(\varpi_j) = V(\varpi_j)$ .

Next, let  $j \in \{1, \dots, n\}$ . Then  $s_j(\varpi_j) = \varpi_j - \alpha_j$ . Let  $\mu \in \Omega(\varpi_j)$  with  $\mu \neq w_0 \varpi_j$ , and let  $m \in V(\varpi_j)_\mu$  be any nonzero element. It follows from the first assertion that there exists  $x \in U_q^+(\mathfrak{g})$  such that  $m = x.u_{w_0 \varpi_j}$ . The element  $x$  can be written as a linear combination of products  $E_{i_1} \dots E_{i_k}$ , with  $k \in \mathbb{N}^*$  and  $i_1, \dots, i_k \in \{1, \dots, n\}$ . Naturally, one can assume that  $E_{i_1} \dots E_{i_k}.u_{w_0 \varpi_j} \neq 0$  for each such product. Let  $E_{i_1} \dots E_{i_k}$  be one of these products. Since  $w_0\pi = -\pi$ , there exists  $l \in \{1, \dots, n\}$  such that  $w_0\alpha_{i_k} = -\alpha_l$ . We will prove that  $l = j$ . Indeed, assume that  $l \neq j$ . Since  $E_{i_k}.u_{w_0 \varpi_j}$  is a nonzero vector of  $V(\varpi_j)$  of weight  $w_0 \varpi_j + \alpha_{i_k}$ , we get that

$$w_0 \varpi_j + \alpha_{i_k} \in \Omega(\varpi_j).$$

Then, we deduce from Proposition 1.1 that

$$s_l w_0 (w_0 \varpi_j + \alpha_{i_k}) \in \Omega(\varpi_j),$$

that is,

$$s_l \varpi_j + \alpha_l \in \Omega(\varpi_j).$$

Further, since we have assumed that  $l \neq j$ , we get  $s_l \varpi_j = \varpi_j$ , so that

$$\varpi_j + \alpha_l \in \Omega(\varpi_j).$$

This contradicts the fact that  $\varpi_j$  is the highest weight of  $V(\varpi_j)$ .

Thus, we have just proved that  $w_0\alpha_{i_k} = -\alpha_j$  for all products  $E_{i_1} \dots E_{i_k}$  that appear in  $x$ . Now, observe that  $E_{i_k} \cdot u_{w_0\varpi_j}$  is a nonzero vector of  $V(\varpi_j)$  of weight  $w_0\varpi_j + \alpha_{i_k} = w_0(\varpi_j + w_0\alpha_{i_k}) = w_0(\varpi_j - \alpha_j) = w_0s_j\varpi_j$ . Since  $\dim_{\mathbb{C}} V(\varpi_j)_{w_0s_j\varpi_j} = 1$ , we get that  $E_{i_k} \cdot u_{w_0\varpi_j} = au_{w_0s_j\varpi_j}$  for a certain nonzero complex number  $a$ . Hence we get that

$$m = x \cdot u_{w_0\varpi_j} = \sum \bullet E_{i_1} \dots E_{i_k} \cdot u_{w_0\varpi_j} = y \cdot u_{w_0s_j\varpi_j},$$

where  $\bullet$  denote some nonzero complex numbers and  $y \in U_q^+(\mathfrak{g})$ . Thus  $m \in V_{w_0s_j}^+(\varpi_j)$ . This shows that

$$\bigoplus_{\mu \in \Omega(\varpi_j) \setminus \{w_0\varpi_j\}} V(\varpi_j)_{\mu} \subseteq V_{w_0s_j}^+(\varpi_j).$$

As the reverse inclusion is trivial, this finishes the proof.  $\square$

### 1.3 Quantised coordinate rings of semisimple groups and their prime spectra.

Let  $G$  be a connected, simply connected, semisimple algebraic group over  $\mathbb{C}$  with Lie algebra  $\text{Lie}(G) = \mathfrak{g}$ . Since  $U_q(\mathfrak{g})$  is a Hopf algebra, one can define its Hopf dual  $U_q(\mathfrak{g})^*$  (see [8, 1.4]) via

$$U_q(\mathfrak{g})^* := \{f \in \text{Hom}_{\mathbb{C}}(U_q(\mathfrak{g}), \mathbb{C}) \mid f = 0 \text{ on some ideal of finite codimension}\}.$$

The quantised coordinate ring  $O_q(G)$  of  $G$  is the subalgebra of  $U_q(\mathfrak{g})^*$  generated by the coordinate functions  $c_{\xi, v}^{\lambda}$  for all  $\lambda \in P^+$ ,  $\xi \in V(\lambda)^*$  and  $v \in V(\lambda)$ , where  $c_{\xi, v}^{\lambda}$  is the element of  $U_q(\mathfrak{g})^*$  defined by

$$c_{\xi, v}^{\lambda}(u) := \xi(uv) \text{ for all } u \in U_q(\mathfrak{g}),$$

see, for example, [8, Chapter 9]. As usual, if  $\xi \in V(\lambda)_{\eta}^*$  and  $v \in V(\lambda)_{\mu}$ , we write  $c_{\eta, \mu}^{\lambda}$  instead of  $c_{\xi, v}^{\lambda}$ . Naturally, this leads to some ambiguity. However, when  $\mu \in W\lambda$  and  $\eta \in W(-w_0\lambda)$ , then  $\dim(V(\lambda)_{\mu}) = 1 = \dim(V(\lambda)_{\eta}^*)$ , so that this ambiguity is very minor.

It is well-known that  $O_q(G)$  is a noetherian domain and a Hopf-subalgebra of  $U_q(\mathfrak{g})^*$ , see [1, 8]. This latter structure allows us to define the so-called left and right winding automorphisms (see, for instance, [1, 1.9.25] or [8, 1.3.5]), and then to obtain an action of the torus  $\mathcal{H} := (\mathbb{C}^*)^{2n}$  on  $O_q(G)$  (see [2, 5.2]). More precisely, observe that the torus  $H := (\mathbb{C}^*)^n$  can be identified with  $\text{Hom}(P, \mathbb{C}^*)$  via:

$$h(\lambda) = h_1^{\lambda_1} \dots h_n^{\lambda_n},$$

where  $h = (h_1, \dots, h_n) \in H$  and  $\lambda = \lambda_1 \varpi_1 + \dots + \lambda_n \varpi_n$  with  $\lambda_1, \dots, \lambda_n \in \mathbb{Z}$ . Then, it is known (see [5, 3.3] or [1, I.1.18]) that the torus  $\mathcal{H}$  acts rationally by  $\mathbb{C}$ -algebra automorphisms on  $O_q(G)$  via:

$$g.c_{\xi,v}^\lambda = g_1(\mu)g_2(\eta)c_{\xi,v}^\lambda,$$

for all  $g = (g_1, g_2) \in \mathcal{H} = H \times H$ ,  $\lambda \in P^+$ ,  $\xi \in V(\lambda)_\mu^*$  and  $v \in V(\lambda)_\eta$ .

(We refer the reader to [1, II.2.6] for the definition of a rational action.)

As usual, we denote by  $\text{Spec}(O_q(G))$  the set of prime ideals in  $O_q(G)$ . Recall that Joseph has proved [9] that every prime in  $O_q(G)$  is completely prime.

Since  $\mathcal{H}$  acts by automorphisms on  $O_q(G)$ , this induces an action of  $\mathcal{H}$  on the prime spectrum of  $O_q(G)$ . As usual, we denote by  $\mathcal{H}\text{-Spec}(O_q(G))$  the set of those primes ideals of  $O_q(G)$  that are  $\mathcal{H}$ -invariant. This is a finite set since Brown and Goodearl [2, Section 5] (see also [1, II.4]) have shown using previous results of Joseph that

$$\mathcal{H}\text{-Spec}(O_q(G)) = \{Q_{w_+,w_-} \mid (w_+, w_-) \in W \times W\},$$

where

$$Q_{w_+}^+ := \langle c_{\xi,v}^\lambda \mid \lambda \in P^+, v \in V(\lambda)_{w_+} \text{ and } \xi \in (V_{w_+}^+(\lambda))^\perp \subseteq V(\lambda)^* \rangle,$$

$$Q_{w_-}^- := \langle c_{\xi,v}^\lambda \mid \lambda \in P^+, v \in V(\lambda)_{w_0 w_-} \text{ and } \xi \in (V_{w_- w_0}^-(\lambda))^\perp \subseteq V(\lambda)^* \rangle,$$

and

$$Q_{w_+,w_-} := Q_{w_+}^+ + Q_{w_-}^-.$$

Since  $q$  is transcendental, it follows from [10, Théorème 3] that it is enough to consider the fundamental weights in the definition of  $Q_{w_+}^+$  and  $Q_{w_-}^-$ . More precisely, we deduce from [10, Théorème 3] the following result.

**Theorem 1.3 (Joseph)**

$$\mathcal{H}\text{-Spec}(O_q(G)) = \{Q_{w_+,w_-} \mid (w_+, w_-) \in W \times W\},$$

where

$$Q_{w_+}^+ := \langle c_{\xi,v}^{\varpi_j} \mid j \in \{1, \dots, n\}, v \in V(\varpi_j)_{\varpi_j} \text{ and } \xi \in (V_{w_+}^+(\varpi_j))^\perp \subseteq V(\varpi_j)^* \rangle,$$

$$Q_{w_-}^- := \langle c_{\xi,v}^{\varpi_j} \mid j \in \{1, \dots, n\}, v \in V(\varpi_j)_{w_0 \varpi_j} \text{ and } \xi \in (V_{w_- w_0}^-(\varpi_j))^\perp \subseteq V(\varpi_j)^* \rangle,$$

and

$$Q_{w_+,w_-} := Q_{w_+}^+ + Q_{w_-}^-.$$

Moreover the prime ideals  $Q_{w_+,w_-}$ , for  $(w_+, w_-) \in W \times W$ , are pairwise distinct.



## 2 $O_q(G)$ is a noetherian UFD.

In this section, we prove that  $O_q(G)$  is a noetherian UFD (We refer the reader to [11, Section 1] for the definition of a noetherian UFD; the key point is that each height one prime ideal should be generated by a normal element.) In order to do this, we proceed in three steps.

1. First, by using results of Joseph, we show that there exist a finite number of nonzero normal  $\mathcal{H}$ -eigenvectors  $r_1, \dots, r_k$  of  $O_q(G)$  such that each  $\langle r_i \rangle$  is (completely) prime, and that each nonzero  $\mathcal{H}$ -invariant prime ideal of  $O_q(G)$  contains one of the  $r_i$ . This property may be thought of as a “weak factoriality” result:  $O_q(G)$  is an  $\mathcal{H}$ -UFD in the terminology of [11].
2. Secondly, by using the  $H$ -stratification theory of Goodearl and Letzter (see [1, II]), we show that the localisation of  $O_q(G)$  with respect to the multiplicative system generated by the  $r_i$  is a noetherian UFD.
3. Finally, we use a noncommutative analogue of Nagata’s Lemma (see [11, Proposition 1.6]) to prove that  $O_q(G)$  itself is a noetherian UFD.

### 2.1 $O_q(G)$ is an $\mathcal{H}$ -UFD

This aim of this section is two-fold. First, we show that for each  $i \in \{1, \dots, n\}$ , the ideal generated by the normal element  $c_{-\varpi_i, w_0 \varpi_i}^{\varpi_i}$  or  $c_{-w_0 \varpi_i, \varpi_i}^{\varpi_i}$  is (completely) prime and then we prove that every nonzero  $\mathcal{H}$ -invariant prime ideal of  $O_q(G)$  contains either one of the  $c_{-\varpi_i, w_0 \varpi_i}^{\varpi_i}$  or one of the  $c_{-w_0 \varpi_i, \varpi_i}^{\varpi_i}$ .

**Lemma 2.1** *Let  $i \in \{1, \dots, n\}$ . Then  $Q_{w_0, s_i w_0} = \langle c_{-\varpi_i, w_0 \varpi_i}^{\varpi_i} \rangle$  and  $Q_{w_0 s_i, w_0} = \langle c_{-w_0 \varpi_i, \varpi_i}^{\varpi_i} \rangle$ .*

*Proof.* Recall that

$$Q_{w_0, s_i w_0} = Q_{w_0}^+ + Q_{s_i w_0}^-,$$

where

$$Q_{w_0}^+ = \langle c_{\xi, v}^{\varpi_j} \mid j \in \{1, \dots, n\}, v \in V(\varpi_j)_{\varpi_j} \text{ and } \xi \in (V_{w_0}^+(\varpi_j))^\perp \subseteq V(\varpi_j)^* \rangle,$$

$$Q_{s_i w_0}^- = \langle c_{\xi, v}^{\varpi_j} \mid j \in \{1, \dots, n\}, v \in V(\varpi_j)_{w_0 \varpi_j} \text{ and } \xi \in (V_{s_i}^-(\varpi_j))^\perp \subseteq V(\varpi_j)^* \rangle.$$

Next, it follows from Proposition 1.2(1) that  $V_{w_0}^+(\varpi_j) = V(\varpi_j)$  for all  $j$ , so that  $Q_{w_0}^+ = (0)$ . Also, we deduce from Proposition 1.2(2) that  $V_{s_i}^-(\varpi_j) = V(\varpi_j)$  if  $j \neq i$ , and  $V_{s_i}^-(\varpi_i) = \bigoplus_{\mu \in \Omega(\varpi_i) \setminus \{\varpi_i\}} V(\varpi_i)_\mu$ . Hence,

$$Q_{s_i w_0}^- = \langle c_{\xi, v}^{\varpi_i} \mid v \in V(\varpi_i)_{w_0 \varpi_i} \text{ and } \xi \in V(\varpi_i)_{-\varpi_i}^* \rangle,$$

that is,  $Q_{s_i w_0}^- = \langle c_{-\varpi_i, w_0 \varpi_i}^{\varpi_i} \rangle$ . Therefore  $Q_{w_0, s_i w_0} = Q_{w_0}^+ + Q_{s_i w_0}^- = \langle c_{-\varpi_i, w_0 \varpi_i}^{\varpi_i} \rangle$ , as desired.

The second claim of the lemma is obtained in the same way.  $\square$

Now observe that, in [8], Joseph uses slightly different conventions for the dual  $M^*$  of a left  $U_q(\mathfrak{g})$ -module. Indeed, it is mentioned in [8, 9.1] that the dual  $M^*$  is viewed with its natural right  $U_q(\mathfrak{g})$ -module structure. As a consequence, Joseph's convention for the weights of the dual  $L(\lambda)^*$  of  $L(\lambda)$ , for  $\lambda \in P^+$ , is not exactly the same as our convention. In particular, the elements  $c_{\varpi_i, w_0 \varpi_i}^{\varpi_i}$  and  $c_{w_0 \varpi_i, \varpi_i}^{\varpi_i}$ ,  $i \in \{1, \dots, n\}$ , that appear in [8, Corollary 9.1.4], correspond to the elements  $c_{-\varpi_i, w_0 \varpi_i}^{\varpi_i}$  and  $c_{-w_0 \varpi_i, \varpi_i}^{\varpi_i}$  in our notation. With this in mind, it follows from [8, Corollary 9.1.4] that the elements  $c_{-\varpi_i, w_0 \varpi_i}^{\varpi_i}$  and  $c_{-w_0 \varpi_i, \varpi_i}^{\varpi_i}$ , for  $i \in \{1, \dots, n\}$ , are normal in  $O_q(G)$ . Thus we deduce from Lemma 2.1 the following result which will allow us later to use a noncommutative analogue of Nagata's Lemma in order to prove that  $O_q(G)$  is a noetherian UFD.

**Corollary 2.2** *The  $2n$  elements  $c_{-\varpi_i, w_0 \varpi_i}^{\varpi_i}$  and  $c_{-w_0 \varpi_i, \varpi_i}^{\varpi_i}$ , for  $i \in \{1, \dots, n\}$ , are nonzero normal elements of  $O_q(G)$  and they generate pairwise distinct completely prime ideals of  $O_q(G)$ .*

Since the  $c_{-\varpi_i, w_0 \varpi_i}^{\varpi_i}$  and  $c_{-w_0 \varpi_i, \varpi_i}^{\varpi_i}$ , for  $i \in \{1, \dots, n\}$ , are  $\mathcal{H}$ -eigenvectors of  $O_q(G)$ , in order to prove that  $O_q(G)$  is an  $\mathcal{H}$ -UFD in the sense of [11, Definition 2.7], it only remains to prove that every nonzero  $\mathcal{H}$ -invariant prime ideal of  $O_q(G)$  contains either one of the  $c_{-\varpi_i, w_0 \varpi_i}^{\varpi_i}$  or one of the  $c_{-w_0 \varpi_i, \varpi_i}^{\varpi_i}$ . This is what we do next.

**Lemma 2.3** *Let  $\mathbf{w} = (w_+, w_-) \in W \times W$ , with  $\mathbf{w} \neq (w_0, w_0)$ . Then  $Q_{\mathbf{w}}$  contains either one of the  $c_{-\varpi_i, w_0 \varpi_i}^{\varpi_i}$ , or one of the  $c_{-w_0 \varpi_i, \varpi_i}^{\varpi_i}$ .*

*Proof.* Since  $\mathbf{w} \neq (w_0, w_0)$ , either  $w_+ \neq w_0$ , or  $w_- \neq w_0$ . Assume, for instance, that  $w_+ \neq w_0$ , so that there exists  $i \in \{1, \dots, n\}$  such that  $w_+ \leq w_0 s_i$ . One can easily check from the definition of  $Q_{\mathbf{w}}$  that this forces  $c_{-w_0 \varpi_i, \varpi_i}^{\varpi_i} \in Q_{w_+}^+$ , so that

$$c_{-w_0 \varpi_i, \varpi_i}^{\varpi_i} \in Q_{w_+}^+ \subseteq Q_{\mathbf{w}},$$

as required.  $\square$

As a consequence of Corollary 2.2 and Lemma 2.3, we get the following result.

**Corollary 2.4**  *$O_q(G)$  is an  $\mathcal{H}$ -UFD.*

*Proof.* Theorem 1.3 establishes that  $\mathcal{H}\text{-Spec}(O_q(G)) = \{Q_{w_+, w_-} \mid (w_+, w_-) \in W \times W\}$ . Note that  $Q_{w_+, w_-} = 0$  precisely when  $w_+ = w_- = w_0$ . Thus, Corollary 2.2 and Lemma 2.3 show that each nonzero  $\mathcal{H}$ -prime ideal of  $O_q(G)$  contains a nonzero  $\mathcal{H}$ -prime of height one that is generated by a normal  $\mathcal{H}$ -eigenvector. Thus,  $O_q(G)$  is an  $\mathcal{H}$ -UFD.  $\square$

## 2.2 $O_q(G)$ is a noetherian UFD.

Set  $T$  to be the localisation of  $O_q(G)$  with respect to the multiplicatively closed set generated by the normal  $\mathcal{H}$ -eigenvectors  $c_{-\varpi_i, w_0 \varpi_i}^{\varpi_i}$  and  $c_{-w_0 \varpi_i, \varpi_i}^{\varpi_i}$ , for  $i \in \{1, \dots, n\}$ . Then the rational action of  $\mathcal{H}$  on  $O_q(G)$  extends to an action of  $\mathcal{H}$  on the localisation  $T$  by  $\mathbb{C}$ -algebra automorphisms, since we are localising with respect to  $\mathcal{H}$ -eigenvectors, and this action of  $\mathcal{H}$  on  $T$  is also rational, by using [1, II.2.7]. The following result is a consequence of Corollary 2.4 and [11, Proposition 3.5].

**Proposition 2.5** *The ring  $T$  is  $\mathcal{H}$ -simple; that is, the only  $\mathcal{H}$ -ideals of  $T$  are 0 and  $T$ .*

We are now in position to show that  $O_q(G)$  is a noetherian UFD.

**Theorem 2.6**  *$O_q(G)$  is a noetherian UFD.*

*Proof.* By [11, Proposition 1.6], it is enough to prove that the localisation  $T$  is a noetherian UFD. Now, as proved in Proposition 2.5,  $T$  is an  $\mathcal{H}$ -simple ring. Thus, using [1, II.3.9],  $T$  is a noetherian UFD, as required.  $\square$

As a consequence, we deduce from Theorem 2.6 and [4, Theorem 2.4] the following result.

**Corollary 2.7**  *$O_q(G)$  is a maximal order.*

The fact that  $O_q(G)$  is a maximal order can also be proved directly by using a suitable localisation of  $O_q(G)$ , [8, Corollary 9.3.10], which is itself a maximal order.

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