Integrable peakon equations with cubic nonlinearity

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Abstract. We present a new integrable partial differential equation found by Vladimir Novikov. Like the Camassa-Holm and Degasperis-Procesi equations, this new equation admits peaked soliton (peakon) solutions, but it has nonlinear terms that are cubic, rather than quadratic. We give a matrix Lax pair for V. Novikov's equation, and show how it is related by a reciprocal transformation to a negative flow in the Sawada-Kotera hierarchy. Infinitely many conserved quantities are found, as well as a bi-Hamiltonian structure. The latter is used to obtain the Hamiltonian form of the finite-dimensional system for the interaction of N peakons, and the two-body dynamics ($N = 2$) is explicitly integrated. Finally, all of this is compared with some analogous results for another cubic peakon derived by Zhijun Qiao.

Submitted to: *J. Phys. A: Math. Gen.*

1. Introduction

The subject of this paper is the partial differential equation (PDE)

$$u_t - u_{xxt} + 4u^2 u_x = 3uu_x u_{xx} + u^2 u_{xxx}, \tag{1}$$

which was discovered very recently by Vladimir Novikov in a symmetry classification of nonlocal PDEs with cubic nonlinearity [18]. The perturbative symmetry approach [17] yields necessary conditions for a PDE to admit infinitely many symmetries. Using this approach, Novikov was able to isolate the equation (1) and find its first few symmetries, and he subsequently found a scalar Lax pair for it, proving that the equation is integrable. Due to the u_{xxt} term on the left hand side of (1), this equation is not an evolutionary PDE for u. However, taking the convolution with the Green's function $g(x) = \exp(-|x|)/2$ for the Helmholtz operator $(1 - \partial_x^2)$ gives the nonlocal (integro-differential) equation $u_t + u^2 u_x + g * [3uu_x u_{xx} + 2(u_x)^3 + 3u^2 u_x] = 0$. It is convenient to define a new dependent variable m to be the Helmholtz operator acting on u, in which case the equation (1) can be more concisely written as

$$m_t + u^2 m_x + 3uu_x m = 0, \qquad m = u - u_{xx}. \tag{2}$$

Henceforth we work with the above form of the equation.

The work of Camassa and Holm [3], who derived the equation

$$m_t + um_x + 2u_x m = 0, \qquad m = u - u_{xx} \tag{3}$$

from an asymptotic approximation to the Hamiltonian for the Green-Naghdi equations in shallow water theory, has attracted a lot of interest in the past fifteen years, for various reasons. To begin with, it is remarkable that the Camassa-Holm equation (3) approximates unidirectional fluid flow in Euler's equations at the next order beyond the KdV equation, and yet preserves the property of being integrable, fitting as it does into the hereditary symmetry framework of Fokas and Fuchssteiner [7], with a bi-Hamiltonian structure and a Lax pair based on a linear spectral problem of second order. Also, while there are smooth soliton solutions of (3) on a non-zero constant background (or equivalently, with the addition of linear dispersion terms), the Camassa-Holm equation has peakon solutions, which are peaked solitons of the form

$$u(x,t) = \sum_{j=1}^{N} p_j(t)\, \exp(-|x - q_j(t)|), \tag{4}$$

where the positions q_j and amplitudes p_j satisfy the system of ODEs

$$\dot{q}_j = \sum_{k=1}^{N} p_k\, e^{-|q_j - q_k|}, \quad \dot{p}_j = p_j \sum_{k=1}^{N} p_k \operatorname{sgn}(q_j - q_k)\, e^{-|q_j - q_k|} \tag{5}$$

for $j = 1, \ldots, N$.

The peakons are smooth solutions of (3) except at the peak positions $x = q_j$, where the derivative of u is discontinuous. The equations (5) form an integrable finite-dimensional Hamiltonian system, corresponding to geodesic flow on an N-dimensional

manifold with inverse metric $g^{jk} = \exp(-|q_j - q_k|)$. The positions q_j and momenta p_j satisfy the canonical Poisson bracket $\{q_j, p_k\} = \delta_{jk}$. The dynamics of two peakons ($N = 2$) was solved explicitly in the original paper by Camassa and Holm [3], while the explicit solution for arbitrary N was found by Beals, Sattinger and Szmigielski [2]. Fuchssteiner also showed that the equation (3) is related via a reciprocal transformation to the first negative flow in the hierarchy of the Korteweg–de Vries equation.

One might wonder whether the Camassa-Holm equation is the only integrable PDE of its kind, being a shallow water equation whose dispersionless version has weak soliton solutions. This turns out not to be the case. Degasperis and Procesi used an asymptotic integrability approach to isolate integrable third order equations, and discovered a new equation with the dispersionless form

$$m_t + um_x + 3u_x m = 0, \qquad m = u - u_{xx}. \tag{6}$$

The Degasperis-Procesi equation turns out to be integrable, with a bi-Hamiltonian structure and a Lax pair based on a third order spectral problem [4], and it also arises in shallow water theory [6]. The equation (6) is related by a reciprocal transformation to a negative flow in the hierarchy of the Kaup-Kupershmidt equation [4, 13], and it also has peakon solutions of the form (4) whose dynamics is described by an integrable finite-dimensional Hamiltonian system with a non-canonical Poisson bracket (see [14], and section 4 below). The explicit solution of the N-peakon dynamics was derived by Lundmark and Szmigielski [16]. There are at least two distinct integrable analogues of the Camassa-Holm equation in 2+1 dimensions [12, 15], while the Euler-Poincaré equation on the diffeomorphism group (EPDiff) provides a geometrical generalisation of the Camassa-Holm equation in arbitrary dimension [11], and admits weak solutions with support on lower-dimensional submanifolds. Rosenau also found various PDEs with nonlinear dispersion, which have solutions with compact support [21], some of which are relatives of the Camassa-Holm equation [19].

In what follows we present a bi-Hamiltonian structure for the integrable hierarchy of PDEs of which (2) is a member, present a matrix Lax pair corresponding to a zero curvature representation for this equation, and show how it is related via a reciprocal transformation to a negative flow in the Sawada-Kotera hierarchy. We also present a system of Hamiltonian ODEs for the dynamics of peakon solutions of (2), and explicitly integrate the equations for the interaction of two peakons. Finally, we compare our results with analogous properties of the integrable PDE

$$m_t + \Big(m(u^2 - u_x^2)\Big)_x = 0, \qquad m = u - u_{xx}, \tag{7}$$

which was recently obtained by Zhijun Qiao [20]. Qiao's equation was the original starting point for our study, since it has cubic (rather than quadratic) nonlinear terms, and this is what led us to ask Vladimir Novikov to seek other integrable equations of this kind.

2. Lax pair and reciprocal transformation

The equation (2) arises as a zero curvature equation $\mathbf{F}_t - \mathbf{G}_x + [\mathbf{F}, \mathbf{G}] = 0$, this being the compatibility condition for the linear system

$$\begin{aligned} \mathbf{\Psi}_x &= \mathbf{F}\mathbf{\Psi}, \\ \mathbf{\Psi}_t &= \mathbf{G}\mathbf{\Psi}, \end{aligned} \tag{8}$$

where

$$\mathbf{F} = \begin{pmatrix} 0 & m\lambda & 1 \\ 0 & 0 & m\lambda \\ 1 & 0 & 0 \end{pmatrix}, \quad \mathbf{G} = \begin{pmatrix} \frac{1}{3\lambda^2} - uu_x & \frac{u_x}{\lambda} - u^2 m\lambda & u_x^2 \\ \frac{u}{\lambda} & -\frac{2}{3\lambda^2} & -\frac{u_x}{\lambda} - u^2 m\lambda \\ -u^2 & \frac{u}{\lambda} & \frac{1}{3\lambda^2} + uu_x \end{pmatrix}. \tag{9}$$

We found the linear system (8) directly by applying the prolongation algebra method of Wahlquist and Estabrook (see [22], and also [8]), but the details of this derivation will not be given here‡. In any case, once a Lax pair is given one can use it to derive most of the important properties of an integrable PDE.

The first important observation we wish to make about Vladimir Novikov's equation is that it is connected to a negative flow in the Sawada-Kotera hierarchy via a reciprocal transformation. Upon rewriting the PDE (2) in the form

$$(m^{2/3})_t + (m^{2/3}u^2)_x = 0, \tag{10}$$

it is immediately clear that $m^{2/3}$ is a conserved density. Since each of the equations (3) and (6) has a conserved density of the form $m^{1/b}$, for $b = 2, 3$ respectively, and these densities yield reciprocal transformations to negative flows in more familiar hierarchies, this suggests that we should define the new independent variables X and T by

$$dX = m^{2/3}\,dx - m^{2/3}u^2\,dt, \qquad dT = dt. \tag{11}$$

The closure condition $d^2X = 0$ for the exact one-form dX in the reciprocal transformation (11) is just the conservation law (10). Transforming the time evolution of m in (2), together with the definition $m = u - u_{xx}$, leads to the equations

$$\Big(\frac{1}{V}\Big)_T = \Big(\frac{W^2}{V}\Big)_X, \qquad W_{XX} - \left(\frac{V_{XX}}{2V} - \frac{(V_X)^2}{4V^2} + \frac{1}{V^2}\right) W + 1 = 0, \tag{12}$$

where $V = m^{2/3}$ and $W = um^{1/3}$. The evolution equation for $1/V$ in the new independent variables X, T is the reciprocal transformation of the equation (2). However, in order to recognise (12) as a member of the Sawada-Kotera hierarchy we need to apply the reciprocal transformation to the Lax pair. (For details of the Sawada-Kotera hierarchy and its extensions we refer the reader to [10].)

By writing the column vector $\mathbf{\Psi}$ in components as $\mathbf{\Psi} = (\psi_1, \psi_2, \psi_3)^T$, we can eliminate ψ_1 and ψ_3 from $\mathbf{\Psi}_x = \mathbf{F}\mathbf{\Psi}$ to get a single scalar equation for $\psi = \psi_2$, namely

$$\psi_{xxx} - 2m_x m^{-1}\psi_{xx} - (m_{xx}m^{-1} - 2(m_x)^2 m^{-2} + 1)\psi_x = m^2\lambda^2\psi. \tag{13}$$

‡ V. Novikov told us that he earlier found a scalar Lax pair for the PDE (2) based on a third order spectral problem, by applying a reciprocal transformation to a symmetry of fifth order. Any scalar linear problem for (2) should be equivalent to the matrix system (8), possibly after a gauge transformation.

When the reciprocal transformation (11) is used to transform the x derivatives as $\partial_x = V\partial_X$, the equation (13) becomes

$$\psi_{XXX} + U\psi_X = \lambda^2\psi, \qquad \text{with} \quad U = -\frac{V_{XX}}{2V} + \frac{(V_X)^2}{4V^2} - \frac{1}{V^2}, \tag{14}$$

so that the second equation in (12) has the form $W_{XX} + UW + 1 = 0$ for the same potential U. The third order operator $\partial_X^3 + U\partial_X$ in (14) is the standard Lax operator for the Sawada-Kotera hierarchy, and by transforming the t derivatives in $\mathbf{\Psi}_t = \mathbf{G\Psi}$ according to $\partial_t = \partial_T - W^2\partial_X$ we find that the T evolution of ψ is given by

$$\psi_T = \frac{1}{\lambda^2}\Big(W\psi_{XX} - W_X\psi_X\Big) - \frac{2}{3\lambda^2}\,\psi. \tag{15}$$

After gauging ψ by a factor of $e^{2T/(3\lambda^2)}$ to remove the final term above, and then replacing λ^2 by λ and setting $\phi = -3W$, we see that (14) and (15) are respectively equivalent to equations (2.25) and (2.26) in [13], and the compatibility requirement $\psi_{TXXX} = \psi_{XXXT}$ for this pair of scalar equations gives two conditions, namely that $W_{XX} + UW$ is independent of X, and $U_T + 3W_X = 0$. The latter two conditions follow from (12) provided that U is given in terms of V as in (14).

3. Conserved densities and bi-Hamiltonian structure

The Lax pair (8) can be used to find infinitely many conserved densities for (2). Upon setting $\rho = (\log\psi)_x$ in (13) it is clear that ρ satisfies the equation

$$\rho_{xx} + 3\rho\rho_x + \rho^3 - 2m_xm^{-1}(\rho_x + \rho^2) + \Big(m(m^{-1})_{xx} - 1\Big)\rho = m^2\lambda^2. \tag{16}$$

The corresponding t evolution of ψ implies that $\rho_t = F_x$ for some flux F, and so by expanding ρ in powers of λ one finds coefficients that are conserved densities. The asymptotic expansion for $\lambda \to \infty$ has $\rho^3 \sim m^2\lambda^2$, so $\rho \sim m^{2/3}\lambda^{2/3}$, which extends to an infinite series $\rho \sim m^{2/3}\lambda^{2/3} + \sum_{j=1}^{\infty}\mu_j\lambda^{-2j/3}$. The densities μ_j are all determined recursively from (16) as local functions of m; for example $\mu_1 = m^{-5/3}m_{xx} - \frac{4}{3}m^{-8/3}m_x^2 + 3m^{-2/3}$. An expansion in positive powers of λ for $\lambda \to 0$ can consistently begin with $\rho \sim -mu\lambda^2$, but one must solve a second order differential equation to obtain each subsequent term, which leads to increasingly nonlocal expressions in m and u. Since we know that (2) is reciprocally related to a negative Sawada-Kotera flow, it is natural to regard the μ_j as densities for Hamiltonians that generate a positive hierarchy of flows, with the expansion around $\lambda = 0$ producing Hamiltonian densities for negative flows.

Having found these conserved densities, we require a pair of Hamiltonian operators $\mathcal{B}_1$, $\mathcal{B}_2$ which are compatible (in the sense that $\mathcal{B}_1 + \mathcal{B}_2$, or any linear combination of them, is Hamiltonian) and can be used to generate the hierarchy of flows that commute with (2). From earlier studies on the Camassa-Holm and Degasperis-Procesi equations [13, 14], we know that all nonlocal operators of the form

$$\mathcal{B} = m^{1-1/b}D_xm^{1/b}\hat{G}m^{1/b}D_xm^{1-1/b}, \tag{17}$$

with $\hat{G} = (c_1 D_x + c_2 D_x^3)^{-1}$ for constants b, c_1, c_2, are Hamiltonian, and have Casimir $\int m^{1/b} dx$. In fact, the case $b = 2$ gives the third Hamiltonian structure for the Camassa-Holm equation, and $b = 3$ gives the second Hamiltonian structure for the Degasperis-Procesi equation. Since $\int m^{2/3} dx$ is a conserved quantity for (2), this suggests we should consider the operator (17) with $b = 3/2$, and indeed we find that the equation can be written in Hamiltonian form as

$$m_t = \mathcal{B}_1 \frac{\delta \tilde{H}}{\delta m}, \qquad \tilde{H} = \frac{1}{4} \int mu \, dx, \tag{18}$$

for $\mathcal{B}_1 = -18\mathcal{B}|_{b=3/2}$ in the case $c_1 = 4$, $c_2 = -1$. Some other conserved quantities are $H_1 = \int \frac{1}{8} \left(u^4 + 2u^2 u_x^2 - \frac{u_x^4}{3} \right) dx$, $H_5 = \int m^{2/3} dx$, $H_7 = \int \frac{1}{3} (m^{-8/3} m_x^2 + 9m^{-2/3}) dx = \int \mu_1 dx$, and the next one has leading term $H_{11} = \int (m^{-16/3} m_{xxx}^2 + \ldots) dx$ (up to rescaling). These are the first few in the sequence of Hamiltonians that generate local symmetries of weight $k \equiv \pm 1 \bmod 6$ according to

$$m_{t_k} = \mathcal{B}_2 \frac{\delta H_k}{\delta m} = \mathcal{B}_1 \frac{\delta H_{k+6}}{\delta m}, \tag{19}$$

where $\mathcal{B}_2 = (1 - D_x^2) m^{-1} D_x m^{-1} (1 - D_x^2)$. The recursion operator is $\mathcal{R} = \mathcal{B}_2 \mathcal{B}_1^{-1}$, and it generates the flows $\mathcal{R}^n m_x$ of weight $6n + 1$ and the flows $\mathcal{R}^n m_{t_5}$ of weight $6n + 5$. However, when $k = 5$ or 7 the rightmost part of the identity (19) fails, since both H_5 and H_7 are Casimirs for $\mathcal{B}_1$; and the Hamiltonian $\tilde{H}$ is a Casimir for $\mathcal{B}_2$. The proof of the following theorem will be presented in a forthcoming article.

Theorem 1 *The operators* $\mathcal{B}_1 = -2(3mD_x + 2m_x)(4D_x - D_x^3)^{-1}(3mD_x + m_x)$ *and* $\mathcal{B}_2 = (1 - D_x^2) m^{-1} D_x m^{-1} (1 - D_x^2)$ *provide a bi-Hamiltonian structure for the hierarchy of symmetries of the equation (2).*

4. Peakon solutions

From (10) the travelling waves $u = u(z)$, $z = x - ct$ of (2) satisfy $(u^2 - c)m^{2/3} = \text{const}$. In the general case this gives $m = \frac{1}{2} c^2 D (u^2 - c)^{-3/2}$ for constant $D \neq 0$, which integrates further to $(u')^2 = u^2 + cDu(u^2 - c)^{-1/2} + cE$, for another constant E. This can be reduced to a quadrature which is the sum of elliptic integrals of the third kind, namely

$$dz = \frac{(\frac{1}{w-1} - \frac{1}{w+1}) dw}{2\sqrt{(Dw + E)(w^2 - 1) + w^2}}, \quad w = u(u^2 - c)^{-1/2}. \tag{20}$$

However, if we require waves that vanish at spatial infinity, then $D = 0$, which implies that $m = 0$ whenever $u^2 \neq c$. No smooth solution can satisfy the latter requirement, but this observation suggests that there should be a weak solution of the form

$$u(x,t) = \pm \sqrt{c} \, e^{-|x - ct - x_0|}, \quad c > 0, \quad x_0 \text{ constant}, \tag{21}$$

which has the same form as the peakon for the Camassa-Holm and Degasperis-Procesi equations, except that the amplitude is the square root of the speed rather than being equal to the speed, as is the case for the peakon solutions of (3) and (6). The expression (21) has $m = 0$ away from the peak, and $u^2 = c$ at the peak, but to regard it as a weak

solution of (2) it is necessary to substitute it into the equation and integrate against suitable test functions with support around the peak. For the single peakon (21) we have $m = \pm 2\sqrt{c}\,\delta(x - ct - x_0)$, but there is some subtlety in interpreting this as a solution, because $u_x = \mp\sqrt{c}\,\mathrm{sgn}(x - ct - x_0)e^{-|x-ct-x_0|}$ and m are distributions, while the equation (2) includes the product $u_x m$. The integrals can be regularised by taking the convention $\mathrm{sgn}(0) = 0$, but a more rigorous alternative is to construct the peakon distribution as a limit of smooth solutions of the PDE. For the Camassa-Holm equation it is known that the single peakon arises as a weak solution in this way (see [19] for a very detailed treatment), and multi-peakons arise similarly as a degenerate limit of algebro-geometric solutions [1].

If we take u to be a linear superposition of N peakons, as in (4), so that $m = 2\sum_{j=1}^{N} p_j(t)\delta(x - q_j(t))$, then substituting into the equation (2) and integrating against test functions supported at $x = q_j$ gives the equations of motion for the peak positions and amplitudes.

Proposition 1 *The equation (2) has peakon solutions of the form (4), whose positions $q_j(t)$ and amplitudes $p_j(t)$ evolve according to the dynamical system*

$$\begin{aligned} \dot{q}_j &= \textstyle\sum_{k,\ell=1}^{N} p_k p_\ell\, e^{-|q_j-q_k|-|q_j-q_\ell|}, \\ \dot{p}_j &= p_j \textstyle\sum_{k,\ell=1}^{N} p_k p_\ell\, \mathrm{sgn}(q_j - q_k)\, e^{-|q_j-q_k|-|q_j-q_\ell|}. \end{aligned} \tag{22}$$

The above equations are not in canonical Hamiltonian form. However, in [14] one of us showed how Hamiltonian operators of the form (17) are reduced to Poisson structures on the finite-dimensional submanifold of N peaks or pulses, resulting in the Poisson bracket

$$\begin{aligned} \{q_j, q_k\} = G(q_j - q_k), \qquad \{q_j, p_k\} &= (b-1)G'(q_j - q_k)p_k, \\ \{p_j, p_k\} &= -(b-1)^2 G''(q_j - q_k)p_j p_k, \end{aligned} \tag{23}$$

where G is the skew-symmetric Green's function for the operator $\hat{G}$. For $N > 2$, the Jacobi identity holds for this bracket if and only if G satisfies the functional equation

$$G'(\alpha)(G(\beta) + G(\gamma)) + \text{cyclic} = 0 \quad \text{for} \quad \alpha + \beta + \gamma = 0. \tag{24}$$

This functional equation is also a sufficient condition for the operator (17) to be Hamiltonian, and Braden and Byatt-Smith proved in the appendix to [14] that the unique continuously differentiable, odd solution of equation (24) is $G(x) = A\,\mathrm{sgn}(x)(1 - e^{-B|x|})$ for arbitrary constants A, B. Up to rescaling x, this is the Green's function for the operator $\hat{G} = (D_x - D_x^3)^{-1}$ (or $\hat{G} = D_x^{-1}$ in the degenerate case $B \to \infty$). In the case at hand, the operator $\mathcal{B}$ in Theorem 1 has $\hat{G} = (4D_x - D_x^3)^{-1}$, and the Hamiltonian $\tilde{H}$ reduces to a conserved quantity h for the equations of motion (22), which is quadratic in the amplitudes p_j.

Theorem 2 *The equations (22) for the motion of N peakons in the PDE (2) are an Hamiltonian vector field*

$$\dot{q}_j = \{q_j, h\}, \qquad \dot{p}_j = \{p_j, h\}$$

for the Hamiltonian $h = \frac{1}{2}\sum_{j,k=1}^{N} p_j p_k \exp(-|q_j - q_k|)$*, with the Poisson bracket specified by*

$$\begin{aligned} \{q_j, q_k\} = \mathrm{sgn}(q_j - q_k)(1 - e^{-2|q_j-q_k|}), \qquad & \{q_j, p_k\} = e^{-2|q_j-q_k|} p_k, \\ \{p_j, p_k\} \;=\; & \mathrm{sgn}(q_j - q_k) e^{-2|q_j-q_k|} p_j p_k. \end{aligned} \tag{25}$$

We conjecture that the equations (22) constitute a Liouville integrable Hamiltonian system with N degrees of freedom. For $N = 1$ this is trivial, and for $N = 2$ the result follows from the existence of a second independent integral in involution with h, namely

$$k = p_1^2 p_2^2 (1 - e^{-2|q_1-q_2|}), \qquad \{k, h\} = 0. \tag{26}$$

The invariant k is degree four in the amplitudes, and for all N there is an analogous integral, quartic in p_j, obtained by restricting the Hamiltonian H_1 to the peakon submanifold. Indeed, the conserved densities for the negative flows in the hierarchy of the PDE (2) should all reduce to integrals for the N-peakon dynamics, but the explicit construction of N independent Poisson-commuting integrals for (22) is still in progress. It is also worth mentioning that the Lax pair (8) can be used to obtain an $N \times N$ Lax matrix for the finite-dimensional system, satisfying

$$\mathbf{L}\mathbf{\Phi} = -\lambda^{-2}\mathbf{\Phi}, \qquad \mathbf{L} = \mathbf{SPEP}, \tag{27}$$

where $\mathbf{S}_{jk} = \mathrm{sgn}(q_j - q_k)$, $\mathbf{P} = \mathrm{diag}(p_1, \ldots, p_N)$, $\mathbf{E}_{jk} = \exp(-|q_j - q_k|)$. The jth component of the vector $\mathbf{\Phi}$ is just $\psi_2(q_j(t), t)$, where $\psi_2(x, t)$ is the second component of $\mathbf{\Psi}$ in (8), and the corresponding time evolution $\dot{\mathbf{\Phi}} = \mathbf{M}\mathbf{\Phi}$ yields the Lax equation $\dot{\mathbf{L}} = [\mathbf{M}, \mathbf{L}]$ for the system (22). However, unfortunately the spectral invariants of $\mathbf{L}$, which are the coefficients of the characteristic polynomial $\det(\mathbf{L} + \lambda^{-2}\mathbf{I})$ (a polynomial in λ^{-2}), do not provide enough integrals. For instance, when $N = 2$ we find that the trace of $\mathbf{L}$ vanishes, while the trace of $\mathbf{L}^2$ gives k, but h does not appear. For higher values of N we have found that the spectral invariants of $\mathbf{L}$ have degrees $4, 8, 12, \ldots$ but the integrals of degrees $2, 6, 10, \ldots$ are missing. This leads us to expect that there should be another Lax representation for this system which would provide the correct number of integrals for Liouville's theorem.

For the two-peakon dynamics, the equations of motion are

$$\begin{aligned} \dot{q}_1 &= (p_1 + p_2 e^{-|q_1-q_2|})^2 \quad \dot{q}_2 = (p_2 + p_1 e^{-|q_1-q_2|})^2 \\ \dot{p}_1 &= \mathrm{sgn}(q_1 - q_2) e^{-|q_1-q_2|}(p_1 + p_2 e^{-|q_1-q_2|}) p_1 p_2, \\ \dot{p}_2 &= -\mathrm{sgn}(q_1 - q_2) e^{-|q_1-q_2|}(p_2 + p_1 e^{-|q_1-q_2|}) p_1 p_2, \end{aligned} \tag{28}$$

and without loss of generality we consider the case where the peaks are initially well separated, so that $q_1 << q_2$ with $q_1 \sim c_1 t$, $q_2 \sim c_2 t$ (for $c_1 > c_2 > 0$), and we assume that both amplitudes are positive, so $p_1 \to \sqrt{c_1}$ and $p_2 \to \sqrt{c_2}$ as $t \to -\infty$. In terms of these asymptotic speeds the Hamiltonian is $h = \frac{1}{2}(c_1 + c_2)$ and the quartic invariant is $k = c_1 c_2$. Upon integrating the equations (28) we find elementary formulae for $p_2^2 - p_1^2$, $p_1 p_2$ and $e^{-|q_1-q_2|}$, leading to the expressions

$$\begin{aligned} p_2^2 - p_1^2 &= (c_1 - c_2)\tanh T \qquad\qquad p_1 p_2 = \sqrt{c_1 c_2 + \tfrac{(c_1-c_2)^4}{16(c_1+c_2)^2}\mathrm{sech}^4 T} \\ q_2 - q_1 &= \tfrac{1}{2}\log\left(1 + \tfrac{16 c_1 c_2 (c_1+c_2)^2}{(c_1-c_2)^4}\cosh^4 T\right), \end{aligned} \tag{29}$$

where $T = (c_1 - c_2)(t - t_0)/2$, with t_0 being an arbitrary constant. The formula for $q_1 + q_2$ is somewhat more formidable, being given in terms of a certain quadrature as

$$q_1 + q_2 = (c_1 + c_2)(t - t_0) + \int f(T)dT + \text{const.} \tag{30}$$

The integrand f is

$$f(T) = \frac{2(c_1^2 - c_2^2)\Big((c_1 - c_2)^2 + 8c_1c_2\cosh^2 T\Big)}{(c_1 - c_2)^4 + 16c_1c_2(c_1 + c_2)^2\cosh^4 T},$$

and the quadrature can be performed explicitly by partial fractions in $\tanh(T)$, but the answer is omitted here. From (29) and (30) it is apparent that the peakons exchange speeds under the interaction, without a head-on collision, so that $q_1 \sim c_2 t$, $q_2 \sim c_1 t$ as $t \to \infty$. They also undergo a phase shift, which is described by the asymptotics of the term $\int f(T)dT$ in (30), but the precise formula is rather unwieldy and will be presented elsewhere.

5. Qiao's equation

As we already mentioned, our interest in peakon equations with cubic nonlinearity began with Qiao's equation (7), which can also be written as

$$m_t + (u^2 - u_x^2)m_x + 2u_x m^2 = 0. \tag{31}$$

Qiao presented a 2×2 Lax pair for this equation given by the linear system $\mathbf{\Psi}_x = \mathbf{U}\mathbf{\Psi}$, $\mathbf{\Psi}_t = \mathbf{V}\mathbf{\Psi}$ with

$$\begin{aligned} \mathbf{U} &= \begin{pmatrix} -\frac{1}{2} & \frac{1}{2}m\lambda \\ -\frac{1}{2}m\lambda & \frac{1}{2} \end{pmatrix}, \\ \mathbf{V} &= \begin{pmatrix} \lambda^{-2} + \frac{1}{2}(u^2 - u_x^2) & -\lambda^{-1}(u - u_x) - \frac{1}{2}m\lambda(u^2 - u_x^2) \\ \lambda^{-1}(u + u_x) + \frac{1}{2}m\lambda(u^2 - u_x^2) & -\lambda^{-2} - \frac{1}{2}(u^2 - u_x^2) \end{pmatrix}. \end{aligned} \tag{32}$$

Qiao also found a bi-Hamiltonian structure for his equation, namely

$$m_t = \tilde{\mathcal{B}}_1 \frac{\delta \tilde{H}}{\delta m} = \tilde{\mathcal{B}}_2 \frac{\delta H_1}{\delta m} \tag{33}$$

where

$$\tilde{\mathcal{B}}_1 = -4D_x m D_x^{-1} m D_x, \quad \tilde{\mathcal{B}}_2 = -2(D_x - D_x^3), \tag{34}$$

and $\tilde{H}$, H_1 are the same as the conserved quantities for (2) given in section 3 above. (In Qiao's original papers the quantity H_0, proportional to $\tilde{H}$ here, is out by a factor of 2, while the quantity denoted H_1 in [20] is missing the u_x^4 term.) Note that the first operator in (34) is of the form (17) with $b = 1$, and the compatibility of these Hamiltonian structures can be proved by a slight extension of a result in [13].

If we apply the reciprocal transformation

$$dX = \frac{m}{2}dx - \frac{1}{2}m(u^2 - u_x^2)dt, \quad dT = dt$$

to Qiao's equation (7) then we find the pair of equations

$$(m^{-2})_T = -2u_X, \quad (mu_X)_X = 4(u/m - 1). \tag{35}$$

By transforming the Lax pair given by (32) and writing a scalar linear problem for ψ_1, the first component of $\mathbf{\Psi}$, we find that the X part is

$$\psi_{1,XX} + (v_X - v^2)\psi_1 = -\lambda^2\psi, \quad v = m^{-1},$$

which is the Schrödinger equation corresponding to the spectral problem for KdV, and the expression $v_X - v^2$ is the standard Miura map from modified KdV. The corresponding time evolution is

$$\psi_{1,T} = -\frac{1}{\lambda^2}\Big(a\psi_{1,X} - \frac{1}{2}a_x\psi_1\Big), \quad a = u - mu_X/2,$$

from which it is clear that the pair of equations (35) corresponds to a negative flow in the (modified) KdV hierarchy.

Qiao has noted that the equation (7) does not have standard peakons of the form $u = ce^{-|x-ct|}$. The general travelling wave solution for this equation can be solved in terms of an elliptic integral, and some interesting wave shapes have been found in [20] in cases where this integral reduces to expressions in hyperbolic functions. However, here we should like to point out that, at least formally, peakons of the form $u = \pm\sqrt{c}e^{-|x-ct|}$ (just as found for (2) above) do provide solutions of Qiao's equation. From the equation in the form (31) it is clear that if m is given by a delta function then the m^2 terms do not make sense. However, if we take travelling waves $u = u(z)$, $z = x - ct$ and integrate (7) along the z axis against an arbitrary test function φ, and then perform an integration by parts, we find

$$\int m\Big(u^2 - (u')^2 - c\Big)\varphi'(z)\,dz = 0. \tag{36}$$

For the peakon $u(z) = \sqrt{c}\,e^{-|z|}$ we have $u'(z) = -\sqrt{c}\,\mathrm{sgn}(z)e^{-|z|}$ and $m(z) = 2\sqrt{c}\,\delta(z)$, and this satisfies (36) as long as we assume the usual convention that $\mathrm{sgn}(0) = 0$. A more careful derivation could be carried out along the lines of [19]. The equations for N peakons should be extremely degenerate, since $b = 1$ and $G(x)$ is proportional to $\mathrm{sgn}(x)$ in the bracket (23), so p_j are constant and the amplitudes of the peakons do not change. The same conclusion is reached by integrating (7) against a test function and performing integration by parts.

It seems that peakon equations with cubic nonlinearity have several novel features compared with the Camassa-Holm and Degasperis-Procesi equations, and there are many more things to be revealed by further study.

Acknowledgements. We are very grateful to Vladimir Novikov for sharing his latest classification results with us. AH thanks Darryl Holm for suggesting integrating by parts with Qiao's equation, and is very grateful to Alexander Strohmaier for his remarks about distributional solutions.

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