Integrality and the Laurent phenomenon for Somos 4 sequences

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Abstract

Somos 4 sequences are a family of sequences defined by a fourth-order quadratic recurrence relation with constant coefficients. For particular choices of the coefficients and the four initial data, such recurrences can yield sequences of integers. Fomin and Zelevinsky have used the theory of cluster algebras to prove that these recurrences also provide one of the simplest examples of the Laurent phenomenon: all the terms of a Somos 4 sequence are Laurent polynomials in the initial data. The integrality of certain Somos 4 sequences has previously been understood in terms of the Laurent phenomenon. However, each of the authors of this paper has independently established the precise correspondence between Somos 4 sequences and sequences of points on elliptic curves. Here we show that these sequences satisfy a stronger condition than the Laurent property, and hence establish a broad set of sufficient conditions for integrality. As a by-product, non-periodic sequences provide infinitely many solutions of an associated quartic Diophantine equation in four variables. The analogous results for Somos 5 sequences are also presented, as well as various examples, including parameter families of Somos 4 integer sequences.

1 Introduction

The study of integer sequences generated by linear recurrences has a long history, and, apart from their intrinsic interest in number theory, nowadays there are many applications of such sequences in computer science and cryptography. However, not so much is known about sequences generated by nonlinear recurrences. In [7] (see e.g. section 1.1.20) it is suggested that

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Somos sequences, which are generated by quadratic recurrence relations of the form

$$S_{n+k}S_n = \sum_{j=1}^{[k/2]} \alpha_j \, S_{n+k-j}S_{n+j}$$

(where the coefficients α_j are constant), are suitable generalizations of linear recurrence sequences, in the sense that they have many analogous properties. In this paper we are mainly concerned with the cases $k = 4$ and $k = 5$, and the problem of determining when such recurrences yield sequences of integers; unlike the situation for linear recurrences, this problem is not so straightforward.

Somos 4 sequences are defined by a fourth-order quadratic recurrence relation of the form

$$A_{n+4}A_n = \alpha \, A_{n+3}A_{n+1} + \beta \, (A_{n+2})^2, \tag{1.1}$$

where α, β are constant coefficients. We also refer to equations like (1.1) as *bilinear* recurrences, by analogy with the Hirota bilinear form of integrable partial differential equations in soliton theory [12], which have discrete (difference equation) counterparts [36]. During an exploration of the combinatorial properties of elliptic theta functions, Michael Somos introduced the sequence defined by (1.1) with coefficients $\alpha = \beta = 1$ and initial data $A_1 = A_2 = A_3 = A_4 = 1$, which begins

$$1, 1, 1, 1, 2, 3, 7, 23, 59, 314, 1529, 8209, 83313, \ldots \tag{1.2}$$

(and in this case the sequence extends symmetrically backwards as $\ldots, 59, 23, 7, 3, 2, 1, 1, 1, 1, 2, \ldots$). It is a remarkable fact that although the recurrence defining the sequence (1.2) is rational (because one must divide by A_n in order to obtain A_{n+4}), this sequence consists entirely of integers. In what follows we are mainly concerned with Somos 4 sequences with integer coefficients and initial data, and we establish a broad set of sufficient conditions for integrality of such sequences. However, it is most convenient to work with sequences over $\mathbb{C}$ and indicate when our results reduce to sequences over $\mathbb{Q}$ or to integer sequences. Given four adjacent non-zero values, the recurrence (1.1) can be iterated either forwards or backwards, so it is natural to define the sequences for $n \in \mathbb{Z}$. (In fact, if a zero is encountered then the sequence can be analytically continued through it, either by using the fact that the recurrence (1.1) has the singularity confinement property of [10], or by making use of the explicit formula for the iterates given in (1.11) below.)

The particular sequence (1.2) above is sometimes referred to as *the* Somos 4 sequence [29], and the first published proof of its integrality appears in the survey by Gale [9]. However, following [24] we will refer to (1.2) as the Somos (4) sequence, in order to distinguish it from other Somos 4 sequences defined by recurrences of the general form (1.1). Such sequences originally arose in the theory of elliptic divisibility sequences (EDS), which were introduced by Morgan Ward [33, 34], who considered integer sequences satisfying a recurrence of the form

$$W_{n+4}W_n = (W_2)^2\, W_{n+3}\, W_{n+1} - W_1\, W_3\, (W_{n+2})^2, \tag{1.3}$$

these being a special case of (1.1) with

$$\alpha = (W_2)^2, \qquad \beta = -W_1\, W_3. \tag{1.4}$$

The form of the recurrence (1.3) defines an antisymmetric sequence, so that $W_n = -W_{-n}$. Ward showed [33] that if the initial data are taken to be

$$W_0 = 0, \quad W_1 = 1, \quad W_2, W_3, W_4 \in \mathbb{Z} \qquad \text{with} \quad W_2 | W_4, \tag{1.5}$$

then it turns out that the subsequent terms of the sequence are all integers satisfying the divisibility property

$$W_n | W_m \qquad \text{whenever} \qquad n|m. \tag{1.6}$$

In this sense, EDS constitute a nonlinear generalization of Lucas sequences. Furthermore, Ward demonstrated that the terms of a generic EDS defined by (1.3) correspond to the multiples $[n]P$ of a point P on an elliptic curve E, and the nth term in the sequence can be written explicitly in terms of the Weierstrass sigma function associated with the curve E, as

$$W_n = \frac{\sigma(n\kappa)}{\sigma(\kappa)^{n^2}}. \tag{1.7}$$

Somos 4 sequences, and EDS in particular, are of considerable interest to number theorists due to the way that primitive prime divisors appear [5, 6]. In that and in other respects, they have a lot in common with linear recurrence sequences (see [7], chapters 10 & 11). Somos 4 sequences have also arisen quite recently in a different guise, providing one of the simplest examples of the combinatorial structures appearing in Fomin and Zelevinsky's theory of cluster algebras. More precisely, the following result was proved in [8].

Theorem 1.1. The Laurent property for Somos 4 sequences: *For a Somos 4 sequence defined by a fourth-order recurrence (1.1) with coefficients α, β and initial data A_1, A_2, A_3, A_4, all of the terms in the sequence are Laurent polynomials in these initial data whose coefficients are in $\mathbb{Z}[\alpha,\beta]$, so that $A_n \in \mathbb{Z}[\alpha,\beta,A_1^{\pm1},A_2^{\pm1},A_3^{\pm1},A_4^{\pm1}]$ for all $n \in \mathbb{Z}$.*

The above result is only one of a series of analogous statements for a wide variety of recurrences in one and more dimensions to which the machinery of cluster algebras was applied in [8]. The integrality of the Somos (4) sequence

(1.2), and also of any other sequence defined by the recurrence (1.1) with coefficients α, $\beta \in \mathbb{Z}$ and initial values $A_1 = A_2 = A_3 = A_4 = 1$, is an immediate consequence of the Laurent property. However, there are many other examples of Somos 4 sequences that consist entirely of integers, but for which the Laurent phenomenon alone is insufficient to deduce integrality. For instance, consider the Somos 4 sequence generated by

$$A_{n+4}A_n = 1331 A_{n+3}A_{n+1} + 119790 A_{n+2}^2 \tag{1.8}$$

with the initial values

$$A_1 = 1, \quad A_2 = 3, \quad A_3 = 121, \quad A_4 = 177023.$$

This turns out to be an integer sequence, i.e. $A_n \in \mathbb{Z}$ for all $n \in \mathbb{Z}$, but Theorem 1.1 is not enough to show this; it follows from Theorem 3.1 below, which relies on the connection between Somos 4 sequences and elliptic curves.

In the rest of the paper we shall make use of the fact that the terms A_n in a Somos 4 sequence correspond to a sequence of points $Q + [n]P$ on an associated elliptic curve E, which was proved by each of us independently, in [31] and [13] respectively. Our results can be combined and summarized as follows.

Theorem 1.2. *If (A_n) is a Somos 4 sequence defined by a recurrence (1.1) with coefficients α, β, then the quantity*

$$T = \frac{A_{n-1}\,A_{n+2}}{A_n\,A_{n+1}} + \alpha\left(\frac{(A_n)^2}{A_{n-1}\,A_{n+1}} + \frac{(A_{n+1})^2}{A_n\,A_{n+2}}\right) + \frac{\beta\,A_n\,A_{n+1}}{A_{n-1}\,A_{n+2}} \tag{1.9}$$

is independent of n, and for $\alpha \neq 0$ the sequence corresponds to a sequence of points $Q + [n]P$ on an elliptic curve E with j-invariant

$$j = \frac{(T^4 - 8\beta T^2 - 24\alpha^2 T + 16\beta^2)^3}{\alpha^4(\beta T^4 + \alpha^2 T^3 - 8\beta^2 T^2 - 36\alpha^2\beta T + 16\beta^3 - 27\alpha^4)} \tag{1.10}$$

Moreover, the general term of the sequence can be written in the form

$$A_n = \hat{a}\,\hat{b}^n\,\frac{\sigma(z_0 + n\kappa)}{\sigma(\kappa)^{n^2}}, \tag{1.11}$$

where $\sigma(z) = \sigma(z; g_2, g_3)$ denotes the Weierstrass sigma function associated with the curve E written in the canonical form

$$E: \qquad y^2 = 4x^3 - g_2 x - g_3, \tag{1.12}$$

and the six parameters $g_2, g_3, \hat{a}, \hat{b}, z_0, \kappa \in \mathbb{C}$ are determined uniquely (up to a choice of sign) from the two coefficients α, β and the four initial data A_1, A_2, A_3, A_4. The (x, y) coordinates of the corresponding sequence of points on the curve E are given by $Q + [n]P = (\,\wp(z_0 + n\kappa)\,,\ \wp'(z_0 + n\kappa)\,)$.

Remarks. Given four non-zero initial data A_1, A_2, A_3, A_4, the quantity T in (1.9) can be calculated as

$$T = \frac{(A_1)^2\,(A_4)^2 + \alpha((A_2)^3\,A_4 + A_1\,(A_3)^3) + \beta\,(A_2)^2\,(A_3)^2}{A_1\,A_2\,A_3\,A_4}. \tag{1.13}$$

(For a discussion of zeros in Somos sequences, see [31].) The translation invariant was first found in [31], and also appeared in [14], where it was denoted J, and written in terms of $f_n = A_{n-1}\,A_{n+1}/(A_n)^2$ as

$$T = f_n\,f_{n+1} + \alpha\left(\frac{1}{f_n} + \frac{1}{f_{n+1}}\right) + \frac{\beta}{f_n\,f_{n+1}}. \tag{1.14}$$

(Note that the equation $T =$ constant defines a quartic curve in the (f_n, f_{n+1}) plane: this curve is birationally equivalent to the curve E given in (1.12).) In the paper [13], a more complicated invariant was used, namely the quantity

$$\lambda = (T^2/4 - \beta)/(3\alpha). \tag{1.15}$$

We can also use the equation (1.1) to write T as a more compact expression in terms of five adjacent terms of the sequence, that is

$$T = \frac{A_{n-1}\,A_{n+2}}{A_n\,A_{n+1}} + \frac{\alpha\,A_n^2}{A_{n-1}\,A_{n+1}} + \frac{A_{n-2}\,A_{n+1}}{A_{n-1}\,A_n} \tag{1.16}$$

The explicit formulae for the invariants g_2, g_3 of the curve (1.12) in terms of α, β and T are given in section 2 below, along with a summary of other results that are needed here. It is worth mentioning that the above theorem still holds in the degenerate case when the bracketed expression in the numerator of (1.10) vanishes, so that $j = \infty$ and the curve E becomes singular; in that case the formula (1.11) is written in terms of hyperbolic functions (or, when $g_2 = g_3 = 0$, in terms of rational functions).

The connection between Somos 4 and Somos 5 sequences and elliptic curves was previously understood in unpublished work of several number theorists: see Zagier's discussion of Somos 5 [37], the results of Elkies quoted in [2], and the unpublished work of Nelson Stephens mentioned in [31], where alternative Weierstrass models for the elliptic curve E are used. For another approach using birationally equivalent quartic curves and continued fractions, see [19]. A brief history of Somos sequences can be found on Propp's Somos sequence site [22].

2 Companion elliptic divisibility sequences

The purpose of this section is to present some facts about Somos 4 sequences and their relationship with elliptic curves and division polynomials which will be useful for what follows. Given a Somos 4 sequence with coefficients α,

β and initial data A_1, A_2, A_3, A_4 we can immediately compute the translation invariant T given by (1.9), and then the invariants for the associated curve (1.12) can be calculated directly from the formulae

$$g_2 = \frac{T^4 - 8\beta T^2 - 24\alpha^2 T + 16\beta^2}{12\alpha^2}, \tag{2.1}$$

$$g_3 = -\frac{T^6 - 12\beta T^4 - 36\alpha^2 T^3 + 48\beta^2 T^2 + 144\alpha^2\beta T + 216\alpha^4 - 64\beta^3}{216\alpha^3}, \tag{2.2}$$

while $j = 1728 g_2^3/(g_2^3 - 27g_3^2)$ yields the expression (1.10) for the j-invariant. (Upon substituting for λ as in (1.15), the formulae (2.1) and (2.2) are seen to be equivalent to the formulae given in [13].)

The backward iterates A_0, A_{-1} and A_{-2} are obtained by applying (1.1) in the reverse direction, so that the quantities

$$f_{-1} = A_{-2}\, A_0/(A_{-1})^2, \quad f_0 = A_{-1}\, A_1/(A_0)^2, \quad f_1 = A_0\, A_2/(A_1)^2,$$

can be calculated, and then the associated sequence of points on the curve (1.12) is given by

$$\begin{aligned} Q + [n]P &= \Big(\wp(z_0)\,,\, \wp'(z_0)\Big) + [n]\Big(\wp(\kappa)\,,\, \wp'(\kappa)\Big) \\ &= \Big(\lambda - f_0\,,\, (f_0)^2(f_1 - f_{-1})/\sqrt{\alpha}\Big) + [n]\Big(\lambda\,,\, \sqrt{\alpha}\Big) \end{aligned} \tag{2.3}$$

with λ as in (1.15). Up to the overall ambiguity in the sign of $\sqrt{\alpha} = \wp'(\kappa)$, which corresponds to the freedom to send $\kappa \to -\kappa$, $z_0 \to -z_0$ on $\mathrm{Jac}(E)$, the coefficients and the translation invariant are given as elliptic functions of κ by

$$\alpha = \wp'(\kappa)^2, \quad \beta = \wp'(\kappa)^2\Big(\wp(2\kappa) - \wp(\kappa)\Big), \quad T = \wp''(\kappa). \tag{2.4}$$

Elliptic divisibility sequences (EDS) were originally defined by Morgan Ward [33] as sequences of integers satisfying the property (1.6) as well as the family of recurrences

$$W_{n+m}\, W_{n-m} = \begin{vmatrix} W_m W_{n-1} & W_{m-1} W_n \\ W_{m+1} W_n & W_m W_{n+1} \end{vmatrix}, \tag{2.5}$$

for all m, n, and they were called *proper* EDS if $W_0 = 0$, $W_1 = 1$ and $W_2 W_3 \neq 0$. Here we define a *generalized* EDS to be a sequence $\{\, W_n \,\}_{n\in\mathbb{Z}}$ specified by four non-zero initial data

$$W_1 = 1, \quad W_2, W_3, W_4 \in \mathbb{C}^*$$

together with the recurrence (1.3), of which immediate consequences are that $W_0 = 0$ and the sequence is consistently extended to negative indices so that $W_{-n} = -W_n$ and (1.3) holds for all $n \in \mathbb{Z}$. We may then make the observation that a (generalized) EDS is just a special type of Somos 4 sequence, and thus it follows immediately from Theorem 1.2 (i.e. the formula (1.11) with $\hat{a} = \hat{b} = 1$ and $z_0 = 0$) that the terms of the EDS are given by the explicit expression (1.7) in terms of the Weierstrass sigma function. The whole family of recurrences (2.5) then follows as a direct consequence of this explicit formula together with the three-term equation for the sigma function (see §20.53 in [35]). If we further assume that we have integer initial data satisfying (1.5), then the integrality of the subsequent terms and the divisibility property (1.6) can be proved by induction using the relations (2.5). For other special properties of integer EDS we refer the reader to Ward's papers [33, 34] and to the more recent works [5, 6, 7, 25, 28, 31].

If we are working with an elliptic curve E over $\mathbb{C}$, then the terms $W_n = \sigma(n\kappa)/\sigma(\kappa)^{n^2}$ of a generalized EDS are essentially just values of the the division polynomials of E (see Exercises 24 and 33 in chapter 20 of [35], and chapter II of [17]), possibly up to a prefactor of the form γ^{1-n^2} (see Lemma 7 in [28]). Each W_n is an elliptic function of κ that can be written as a polynomial in the variables $(\lambda, \mu) = (\wp(\kappa), \wp'(\kappa))$ with coefficients in $\mathbb{Q}[g_2, g_3]$ (cf. Exercise 3.7 in [26], or chapter II in [17]). For our purposes, in order to make use of the EDS that is the "companion" of a Somos 4 sequence, we will need to write the W_n as polynomials in a slightly different set of variables (Theorem 2.2 below).

Given any Somos 4 sequence satisfying a bilinear recurrence (1.1) with coefficients α, β, there is a natural companion EDS associated with it, which can be defined in two different ways [21, 14].

Definition 1. Algebraic definition of companion EDS: *For a Somos 4 sequence, compute the translation invariant according to (1.9), and then the companion EDS $\{W_n\}_{n\in\mathbb{Z}}$ is the (generalized) EDS that satisfies the same fourth-order recurrence (1.1) with initial values specified to be*

$$W_1 = 1, \quad W_2 = -\sqrt{\alpha}, \quad W_3 = -\beta, \quad W_4 = \mathcal{I}\sqrt{\alpha}, \tag{2.6}$$

where

$$\mathcal{I} = \alpha^2 + \beta T. \tag{2.7}$$

(Note that the coefficients of the Somos 4 sequence are then given in terms of the first three terms of its companion EDS by (1.4).)

Definition 2. Analytic definition of companion EDS: *For a Somos 4 sequence, according to Theorem 1.2 the terms A_n are given explicitly by the formula (1.11) for suitable parameters $g_2, g_3, \hat{a}, \hat{b}, z_0, \kappa$, and then the terms W_n of the companion EDS are given in terms of the Weierstrass sigma function by the formula (1.7) with the same parameters g_2, g_3, κ.*

In his original memoir [33], Morgan Ward showed that an EDS admits two equivalent definitions, one algebraic and the other analytic. By construction, the translation invariant for the Somos 4 sequence and that for its companion EDS have the same value $T = \wp''(\kappa)$, and, upon comparing (1.4) with (2.4), the algebraic and analytic definitions of the companion EDS are easily seen to be equivalent over $\mathbb{C}$; see also the Remarks after equations (1.18) and (2.14) in [14]. The introduction of the companion EDS is very natural in the light of the following result due to one of us with van der Poorten [21].

Theorem 2.1. *The terms A_n of a Somos 4 sequence satisfy the Hankel determinant formulae*

$$(W_1)^2 A_{n+m}A_{n-m} = \begin{vmatrix} W_m A_{n+1} & W_{m-1}A_n \\ W_{m+1}A_n & W_m A_{n-1} \end{vmatrix} \tag{2.8}$$

and

$$W_1 W_2 A_{n+m+1}A_{n-m} = \begin{vmatrix} W_{m+1} A_{n+2} & W_{m-1}A_n \\ W_{m+2}A_{n+1} & W_m A_{n-1} \end{vmatrix} \tag{2.9}$$

for all $m, n \in \mathbb{Z}$, where W_m are the terms of the companion EDS.

Proof. For a purely algebraic proof of (2.8) and (2.9) see [21]. For an analytic proof based on the three-term equation for the sigma function see Corollary 1.2 and Corollary 1.3 in [14]. Note that although $W_1 = 1$ we have included the W_1 terms on the left hand sides above to illustrate the homogeneity of these expressions. $\square$

It is clear that a (generalized) EDS is its own companion, and Ward's formula (2.5) is the special case of (2.8) when $A_n = W_n$. Theorem 2.1 is our main tool for proving integrality properties of Somos 4 sequences. To begin with we can use it to derive a property of EDS which is apparently new.

Theorem 2.2. Polynomial representation for elliptic divisibility sequences: *For a (generalized) EDS defined by the fourth-order recurrence (1.3) with initial data $W_1 = 1$, $W_2 = -\sqrt{\alpha}$, $W_3 = -\beta$, $W_4 = \mathcal{I}\sqrt{\alpha}$, the terms in the sequence satisfy $W_{2n-1} \in \mathbb{Z}[\alpha^2, \beta, \mathcal{I}]$ and $W_{2n} \in \sqrt{\alpha}\ \mathbb{Z}[\alpha^2, \beta, \mathcal{I}]$ for all $n \in \mathbb{Z}$.*

Proof. Since $W_{-n} = -W_n$ we need only consider $n > 0$, and then the proof is by induction on n. Clearly the result is true for the initial data W_1, W_2, W_3, W_4, and the next terms are

$$W_5 = -\alpha^2\mathcal{I} + \beta^3, \qquad W_6 = -\sqrt{\alpha}(\mathcal{I}^2 + \alpha^2\mathcal{I} - \beta^3)\beta,$$

so the appearance of α^2 is evident. Now setting $n = m+1$ and $n = m+2$ in Ward's formula (2.5), and putting in the initial values W_1 and W_2, yields

$$W_{2m+1} = (W_m)^3 W_{m+2} - (W_{m+1})^3 W_{m-1} \tag{2.10}$$

and

$$W_{2m+2} = \frac{(W_{m+2})^2 W_{m+1} W_{m-1} - (W_m)^2 W_{m+3} W_{m+1}}{\sqrt{\alpha}} \tag{2.11}$$

respectively, which correspond to well known identities for division polynomials - see e.g. Exercise 3.7 in [26]. Each term on the right hand side of (2.10) is a product of either four even index terms or four odd index terms, and hence by the inductive hypothesis, for $m \geq 3$, $W_{2m+1} \in \mathbb{Z}[\alpha^2, \beta, \mathcal{I}]$. Similarly for W_{2m+2}, both terms in the numerator on the right hand side of (2.11) consist of a product of two odd index and two even index terms, and so lie in $\alpha\, \mathbb{Z}[\alpha^2, \beta, \mathcal{I}]$ by the inductive hypothesis; dividing out by $\sqrt{\alpha}$ in the denominator gives the required result. $\square$

Remarks. Taking the Weierstrass model

$$Y^2 = X^3 + AX + B, \tag{2.12}$$

which is equivalent to (1.12) via $x = X$, $y = 2Y$, $g_2 = -4A$, $g_3 = -4B$, it is known that the division polynomials corresponding to the multiples $[n](X, Y)$ are elements of $\mathbb{Z}[A, B, X, Y^2]$ for n odd and of $2Y\, \mathbb{Z}[A, B, X, Y^2]$ for n even. Hence using the equation (2.12) to eliminate Y^2, and dividing by $2Y$ where necessary, gives polynomials in $\mathbb{Z}[A, B, X]$ (see Exercise 3.7 in [26], or [17], chapter II). Thus we see that the result of Theorem 2.2 corresponds to a different choice of basis for the division polynomials: after suitable rescaling by $2Y$ for n even, they are polynomials in the variables

$$\alpha^2 = 16Y^4, \qquad \beta = A^2 - 6X^2 A - 12XB - 3X^4,$$

$$\mathcal{I} = 2A^3 + 10X^2 A^2 - (10X^4 - 8XB)A - 2X^6 - 40X^3 B + 16B^2$$

with integer coefficients. As far as we are aware, this is a new result about division polynomials. The quantities $\beta, \mathcal{I}$ and (via (2.12)) α^2 are themselves elements of $\mathbb{Z}[A, B, X]$.

Theorems 2.1 and 2.2 together imply a stronger version of the Laurent property for Somos 4 sequences, which lead us to establish a fairly weak set of criteria for integrality in the next section.

3 Proof of the strong Laurent property

With the above results at hand, we are now able to state our main result concerning a strong version of the Laurent property for Somos 4 sequences.

Theorem 3.1. The strong Laurent property for Somos 4 sequences: *Consider a Somos 4 sequence defined by a fourth-order recurrence (1.1) with coefficients α, β, initial data A_1, A_2, A_3, A_4, and translation invariant T as* *in (1.13). The subsequent terms of the sequence are elements of the ring $\mathcal{R}$ of polynomials generated by these coefficients and initial data, as well as A_1^{-1} and the quantity $\mathcal{I} = \alpha^2 + \beta T$. In other words,*

$$A_n \in \mathcal{R} = \mathbb{Z}[\alpha, \beta, \mathcal{I}, A_1^{\pm 1}, A_2, A_3, A_4]$$

for all $n \geq 1$.

Proof. Setting $n = m+1$ in each of the Hankel determinant formulae (2.8) and (2.9) and substituting $W_1 = 1$, $W_2 = -\sqrt{\alpha}$ leads to

$$A_{2m+1} = \frac{(W_m)^2 A_m A_{m+2} - W_{m+1} W_{m-1} A_{m+1}^2}{A_1} \tag{3.1}$$

and

$$A_{2m+2} = \frac{W_{m+2} W_{m-1} A_{m+1} A_{m+2} - W_m W_{m+1} A_m A_{m+3}}{\sqrt{\alpha} A_1}. \tag{3.2}$$

The proof that $A_n \in \mathcal{R}$ for $n \geq 1$ then proceeds by induction, using the identities (3.1) and (3.2) for odd/even n respectively. For odd $n = 2m+1$ it is clear from Theorem 2.2 that, on the right hand side of (3.1) each of the terms W_m^2 and $W_{m+1}W_{m-1} \in \mathbb{Z}[\alpha, \beta, \mathcal{I}]$, and by the inductive hypothesis the other terms in (3.1) are in $\mathcal{R}$. Similarly, for even $n = 2m+2$, on the right hand side of (3.2) it is clear that both $W_{m+2}W_{m-1}/\sqrt{\alpha}$ and $W_m W_{m+1}/\sqrt{\alpha} \in \mathbb{Z}[\alpha, \beta, \mathcal{I}]$, and the result follows. □

Remarks. From (1.13) it is clear that $T \in \mathbb{Z}[\,\alpha, \beta, A_1^{\pm 1}, A_2^{\pm 1}, A_3^{\pm 1}, A_4^{\pm 1}\,]$, so $\mathcal{I} \in \mathbb{Z}[\,\alpha, \beta, A_1^{\pm 1}, A_2^{\pm 1}, A_3^{\pm 1}, A_4^{\pm 1}\,]$. Similarly, applying Theorem 3.1 to the reversed sequence $A_n^* = A_{5-n}$ starting from $A_1^* = A_4$ gives $A_n \in \mathbb{Z}[\alpha, \beta, \mathcal{I}, A_1, A_2, A_3, A_4^{\pm 1}]$ for all $n \leq 4$. Hence it follows that for all $n \in \mathbb{Z}$, the terms $A_n \in \mathbb{Z}[\alpha, \beta, A_1^{\pm 1}, A_2^{\pm 1}, A_3^{\pm 1}, A_4^{\pm 1}]$, so Theorem 1.1 is a consequence of Theorem 3.1. However, note that as it stands, Theorem 3.1 is unidirectional (it only applies for $n \geq 1$), whereas Theorem 1.1 applies to both positive and negative n.

Corollary 3.2. Integrality criteria for Somos 4 sequences: *Suppose that a Somos 4 sequence is defined by the recurrence (1.1) for coefficients $\alpha, \beta \in \mathbb{Z}$, and initial values $A_1 = \pm 1$ with non-zero $A_2, A_3, A_4 \in \mathbb{Z}$. If βT is an integer, where T is the translation invariant given by (1.13), then $A_n \in \mathbb{Z}$ for all $n \geq 1$. If it further holds that the backwards iterates $A_0, A_{-1}, A_{-2} \in \mathbb{Z}$, then $A_n \in \mathbb{Z}$ for all $n \in \mathbb{Z}$.*

Proof of Corollary 3.2. For $n \geq 1$ the integrality is obvious, since $\mathcal{I} = \alpha^2 + \beta T$ and α, $\beta \in \mathbb{Z}$. If the three backward iterates adjacent to A_1 are also integers, then to get integrality for $n < 1$ it suffices to apply Theorem 3.1 to the reversed sequence $B_n = A_{2-n}$, and then $B_n \in \mathbb{Z}$ for $n \geq 1$ and the result follows. □

Corollary 3.3. *Given $\alpha, \beta \in \mathbb{Z}$ and $T \in \mathbb{Q}$ with $\beta T \in \mathbb{Z}$, suppose that the quartic equation*

$$s^2v^2 + \alpha\,(su^3 + t^3v) + \beta\,t^2u^2 = T\,stuv \tag{3.3}$$

has a solution of the form $(s,t,u,v) = (A_1, A_2, A_3, A_4)$, with $A_1 = \pm 1$ and non-zero $A_2, A_3, A_4 \in \mathbb{Z}$. Then provided that the orbit of this set of initial data under (1.1) is non-periodic, it produces infinitely many integer solutions of the Diophantine equation (3.3).

Remarks. Note that the equation (3.3) is homogeneous in (s,t,u,v), so it is really the solutions with $\gcd(s,t,u,v) = 1$ that are the interesting ones. For the sequence (1.2) it is easy to show by induction that $\gcd(A_n, A_{n+1}, A_{n+2}, A_{n+3}) = 1$ for all n; but in general for sequences where this holds for $n = 1$ then it need not be so for all n, particularly when $\gcd(\alpha,\beta) \neq 1$; see Example 3.4 for instance. Nevertheless, in such cases it can still be often be checked that one does get infinitely many distinct solutions satisfying $\gcd(s,t,u,v) = 1$.

Proof of Corollary 3.3. From Theorem 1.2 we see that the equation (3.3) is just a rewriting of (1.9). Since T is a conserved quantity for (1.1), for all n the quadruple $(s,t,u,v) = (A_n, A_{n+1}, A_{n+2}, A_{n+3})$ lies on the quartic threefold defined by (3.3), and by Corollary 3.2 this is a quadruple of integers for $n \geq 1$. As long as the orbit of the initial quadruple (A_1, A_2, A_3, A_4) is not periodic (which would correspond to $P \in E$ being a torsion point), these quadruples are all distinct. $\square$

Example 3.4. *Consider the sequence defined by*

$$A_{n+4}A_n = 11^3 A_{n+3}A_{n+1} + 90 \cdot 11^3 A_{n+2}^2$$

with initial values $A_1 = 1, A_2 = 3, A_3 = 11^2, A_4 = 11^3 \cdot 7 \cdot 19$; this is just the example previously mentioned with (1.8). This sequence extends in both directions, thus:

$$\ldots, 2498287, 1221, 7, 1, 3, 121, 177023, 2460698229, \ldots \tag{3.4}$$

In this case we have integer coefficients $\alpha = 1331$, $\beta = 119790$, and with $n = 2$ in the formula (1.9) from Theorem 1.2 we calculate $T = 869$, so certainly $\beta T \in \mathbb{Z}$. Also we have $A_1 = 1$, and from (3.4) we see that on each side of this, the three adjacent iterates (A_2, A_3, A_4 and A_0, A_{-1}, A_{-2} respectively) are all integers, so all the conditions of Corollary 3.2 are met and this sequence consists entirely of integers. It is easy to see that the terms of the sequence grow rapidly, such that $\log A_n \sim Cn^2$ with $C \approx 1.5$. So the sequence cannot be periodic, and hence (by Corollary 3.3) it yields infinitely many solutions of the Diophantine equation

$$s^2v^2 + 1331(su^3 + t^3v) + 119790t^2u^2 = 869stuv.$$

The preceding example admits a rather broad generalization, as follows.

Example 3.5. *Pick three integers a, d, e, and choose an ordered pair of integer factors b, c such that*

$$a^3\,d + e^2 = b\,c. \tag{3.5}$$

Now take the Somos 4 recurrence of the form

$$A_{n+4}A_n = e^3\,A_{n+3}A_{n+1} + ade^3\,(A_{n+2})^2, \tag{3.6}$$

with the initial conditions $A_1 = 1$, $A_2 = a$, $A_3 = e^2$, $A_4 = ce^3$. This extends in both directions as

$$\ldots, e^3(b^2d + e(b+d)), ae(b+d), b, 1, a, e^2, ce^3, ae^6(c+de), \ldots \tag{3.7}$$

so that the three adjacent iterates both to the left and to the right of A_1 are integers. Thus we have $\alpha = e^3$, $\beta = ade^3 \in \mathbb{Z}$ and from (1.9) we calculate $\beta T = de^4(a^3 + be + c) \in \mathbb{Z}$, which means that Corollary 3.2 applies and $A_n \in \mathbb{Z}$ for all $n \in \mathbb{Z}$.

Remarks. The preceding example can be interpreted in several different ways. Working over $\mathbb{C}$, the equation (3.5) defines an affine variety V in $\mathbb{C}^5$, and then by Theorem 1.2 there is a family of elliptic curves fibred over V, with the sequence (3.7) corresponding to a sequence of points on each fibre. Note also that by setting $X = -ad$, $Y = de$, $D = bcd^2$ we get

$$Y^2 = X^3 + D, \tag{3.8}$$

so that the variety V itself admits a fibration with all generic fibres ($bcd \neq 0$) being isomorphic over $\mathbb{C}$ to the same elliptic curve (since all the curves (3.8) are isomorphic for $D \neq 0$). Alternatively, we can consider the function field generated by the variables a, b, c, d, e subject to the relation (3.5). In that case, Theorem 3.1 implies that the recurrence (3.6) generates a sequence of polynomials in the ring $\mathbb{Z}[a, b, c, d, e]/\sim$, where $\sim$ is the corresponding equivalence relation. Note also that Example 3.4 is the particular case $a = 3$, $b = 7$, $c = 133$, $d = 30$, $e = 11$, while the original Somos (4) sequence given by (1.2) is the case $a = 1$, $b = 2$, $c = 1$, $d = 1$, $e = 1$. Generically (provided that $b \neq c$ and $e \neq 1$) a different ordering of the factors b, c in (3.5) not only gives a different sequence (3.7) but also corresponds to a sequence of points on a different (non-isomorphic) elliptic curve.

We should point out that the sufficient criteria for integrality in Corollary 3.2 are not necessary conditions. To see this, it is enough to consider the fact that Somos 4 sequences are invariant under the two-parameter abelian group of gauge transformations defined by

$$A_n \longrightarrow \tilde{A}_n = \tilde{a}\,\tilde{b}^n\,A_n, \qquad \tilde{a}, \tilde{b} \in \mathbb{C}^*, \tag{3.9}$$

in the sense that $\tilde{A}_n$ satisfies the same bilinear recurrence (1.1) as does A_n. Furthermore applying (3.9) leaves the the translation invariant T the same, and hence the curve E in Theorem 1.2 is preserved by the action of this gauge group. Moreover, if $A_n \in \mathbb{Z}$ for all n and the transformation (3.9) is applied with $\tilde{a}$, $\tilde{b} \in \mathbb{Z}^*$, then integrality is preserved for $n \geq 1$, and if we fix $\tilde{b} = 1$ then $\tilde{A}_n \in \mathbb{Z}$ for all n, and $|\tilde{A}_n| \geq 2$ for $|\tilde{a}| \geq 2$.

As well as the gauge transformations (3.9), Somos 4 sequences exhibit an additional *covariance* property: under the action of the group of transformations given by

$$A_n \longrightarrow \hat{A}_n = \hat{c}^{n^2} A_n, \qquad \hat{c} \in \mathbb{C}^*, \tag{3.10}$$

the terms $\hat{A}_n$ of the transformed sequence satisfy a Somos 4 recurrence, which is of the same form (1.1) but with the coefficients and translation invariant rescaled according to

$$\hat{\alpha} = \hat{c}^6 \alpha, \qquad \hat{\beta} = \hat{c}^8 \beta, \qquad \hat{T} = \hat{c}^4 T, \tag{3.11}$$

so that the j-invariant (1.10) is preserved. For each $\hat{c}$, the transformation (3.10) just corresponds to homothety acting on the Jacobian: $z \in \mathrm{Jac}(E)$ is mapped to $\hat{z} = \hat{c}^{-1} z \in \mathrm{Jac}(\hat{E})$. It is clear that applying the transformation (3.10) with $\hat{c} \in \mathbb{Z}^*$ preserves integrality of Somos 4 sequences, as does the transformation

$$A_n \longrightarrow A_n^\dagger = (A_k)^{(n-k)^2-1} A_n$$

which inserts a 1 at the kth term of the sequence. All transformations of this type, that include elements of the group (3.10) with $\hat{c} \in \mathbb{Z}^*$, have the effect of increasing the discriminant of the associated curve E when $|\hat{c}| > 1$.

Corollary 3.2 only applies when one of the terms in a given sequence is ± 1 and sufficiently many adjacent terms are integers. If this is not the case, then one might try to engineer it to be so by applying a combination of the transformations (3.9) and (3.10), in order to minimize the size of the discriminant as much as possible while preserving the requirement that α, β, $\mathcal{I} \in \mathbb{Z}$ as well the the integrality of a given set of adjacent terms. However, without the requirement that one of the terms of the sequence should be ± 1, we can present a different set of sufficient criteria for integrality.

Theorem 3.6. *If $\alpha, \beta \in \mathbb{Z}$ and eight successive terms $A_{-2}, A_{-1} \ldots, A_4, A_5$ are integers, and if*

$$\gcd(\alpha, \beta) = \gcd(A_1, A_2) = \gcd(\alpha, A_0, A_2) = \gcd(\alpha, A_1, A_3) = 1,$$

then the terms A_n of a Somos 4 sequence are integers for all $n \in \mathbb{Z}$.

Proof. Suppose that a prime p appears in the denominator of some term, and suppose that $p \not| A_1$. Then by Theorem 3.1, p divides the denominator of $\mathcal{I}$, and hence it must divide the denominator of $T \in \mathbb{Q}$. Setting $n = 1$ and $m = 3$ in the result (2.8) from Theorem 2.1, we have

$$A_{-2}A_4 - \beta^2 A_0 A_2 = \mathcal{I}\alpha(A_1)^2 \in \mathbb{Z}$$

by the integrality assumptions, whence $p|\alpha$; and then $p \not| \beta$ because $\gcd(\alpha, \beta) = 1$. Thus, making use of the recurrence (1.1) gives

$$A_{-1}A_3 = \alpha A_0 A_2 + \beta(A_1)^2 \equiv \beta(A_1)^2 \not\equiv 0 \bmod p$$

But from the expression (1.16) with $n = 0$ and $n = 2$ we have that $p|A_{-1}A_0A_1$ and $p|A_1A_2A_3$ respectively. It follows that p divides $\gcd(A_0, A_2)$ and hence $\gcd(\alpha, A_0, A_2)$. But $\gcd(\alpha, A_0, A_2) = 1$, contradicting the initial assumption that $p \not| A_1$.

Now by repeating the above argument with each A_n replaced by A_{n+1}, it is clear that if p is a prime which appears in the denominator of some term, and if p is coprime to A_2, then p divides $\gcd(\alpha, A_1, A_3)$. But then, since $\gcd(\alpha, A_1, A_3) = 1$, we must have also that $p|A_2$. Thus we require that $p|\gcd(A_1, A_2)$ and the result follows by contradiction. $\square$

It has been known for some time that Somos 5 sequences, defined by bilinear recurrences of the form

$$\tau_{n+3}\tau_{n-2} = \tilde{\alpha}\,\tau_{n+2}\tau_{n-1} + \tilde{\beta}\,\tau_{n+1}\tau_n, \tag{3.12}$$

are also related to sequences of points on elliptic curves [37, 19, 21] (see also the unpublished results of Elkies quoted in [2]). The complete solution of the initial value problem for the general Somos 5 sequence, in terms of the Weierstrass sigma function, was first presented in [14]. It turns out that Somos 5 sequences have two algebraically independent analogues of the translation invariant, but for our purposes here we will only need the quantity $\tilde{J}$ from [14] given in terms of five initial data by

$$\tilde{J} = \frac{\tau_4\tau_1}{\tau_3\tau_2} + \frac{\tau_5\tau_2}{\tau_4\tau_3} + \tilde{\alpha}\left(\frac{\tau_3\tau_2}{\tau_4\tau_1} + \frac{\tau_4\tau_3}{\tau_5\tau_2}\right) + \tilde{\beta}\frac{(\tau_3)^2}{\tau_5\tau_1}. \tag{3.13}$$

Rather than summarizing all the results of [14], we will just state our main theorem concerning the Laurent property and refer to some of these results as needed in the proof.

Theorem 3.7. The strong Laurent property for Somos 5 sequences: *For a Somos 5 sequence defined by a fifth-order recurrence (3.12) with coefficients $\tilde{\alpha}$, $\tilde{\beta}$ and initial data $\tau_1, \tau_2, \tau_3, \tau_4, \tau_5$, the quantity*

$$\tilde{J} = \frac{\tau_{n+1}\tau_{n-2}}{\tau_n\tau_{n-1}} + \frac{\tau_{n+2}\tau_{n-1}}{\tau_{n+1}\tau_n} + \tilde{\alpha}\left(\frac{\tau_n\tau_{n-1}}{\tau_{n+1}\tau_{n-2}} + \frac{\tau_{n+1}\tau_n}{\tau_{n+2}\tau_{n-1}}\right) + \tilde{\beta}\,\frac{(\tau_n)^2}{\tau_{n+2}\tau_{n-2}} \tag{3.14}$$

is independent of n. The subsequent terms of the sequence are elements of the ring of polynomials generated by these coefficients and initial data, as well as τ_1^{-1}, τ_2^{-1} and the quantity $\tilde{\mathcal{I}} = \tilde{\beta} + \tilde{\alpha}\tilde{J}$. In other words,

$$\tau_n \in \mathbb{Z}[\tilde{\alpha}, \tilde{\beta}, \tilde{\mathcal{I}}, \tau_1^{\pm 1}, \tau_2^{\pm 1}, \tau_3, \tau_4, \tau_5]$$

for all $n \geq 1$.

Proof. The fact that $\tilde{J}$ given by (3.14) is independent of n is part of Theorem 2.5 in [14], where it is rewritten in terms of the quantity

$$h_n = \tau_{n+2}\tau_{n-1}/(\tau_{n+1}\tau_n).$$

According to Corollary 2.10 in [14] (or the comments in Section 7 of [21]), each Somos 5 sequence also has a companion EDS associated with it, with terms denoted a_n there, such that the analogue of (2.9), given by

$$a_1 a_2 \tau_{n+m+1}\tau_{n-m} = \begin{vmatrix} a_{m+1}\tau_{n+2} & a_{m-1}\tau_n \\ a_{m+2}\tau_{n+1} & a_m\tau_{n-1} \end{vmatrix} \tag{3.15}$$

holds for all m, $n \in \mathbb{Z}$. Using further results of that Theorem 2.5, we can define this companion EDS by the Somos 4 recurrence

$$a_{n+4}a_n = \tilde{\mu}^2\, a_{n+3}\, a_{n+1} - \tilde{\alpha}\,(a_{n+2})^2$$

with initial data $a_1 = 1$, $a_2 = -\tilde{\mu}$, $a_3 = \tilde{\alpha}$, $a_4 = \tilde{\mu}\tilde{\beta}$, where

$$\tilde{\mu} = (\tilde{\beta} + \tilde{\alpha}\tilde{J})^{1/4}. \tag{3.16}$$

It follows from Theorem 2.2, making the replacements $\sqrt{\alpha} \to \tilde{\mu}$, $\beta \to -\tilde{\alpha}$, $\mathcal{I} \to \tilde{\beta}$, that a_{2n+1} and $a_{2n}/\tilde{\mu}$ are both elements of $\mathbb{Z}[\tilde{\mu}^4, \tilde{\alpha}, \tilde{\beta}]$, so by (3.16), noting that $\tilde{\mathcal{I}} = \tilde{\mu}^4$ we have $a_{2n+1} \in \mathbb{Z}[\tilde{\alpha}, \tilde{\beta}, \tilde{\mathcal{I}}]$ and $a_{2n} \in \tilde{\mu}\,\mathbb{Z}[\tilde{\alpha}, \tilde{\beta}, \tilde{\mathcal{I}}]$ for all n. Setting $n = m + 1$ and $n = m + 2$ in (3.15) yields two identities for τ_n which for even/odd index n have respectively τ_1 and τ_2 in the denominator. Thereafter the proof proceeds similarly to that of Theorem 3.1. $\square$

Remarks. Similarly to the Remarks after Corollary 3.2, the Laurent property for Somos 5 sequences, as proved in [8], is a consequence of Theorem 3.7. Also, the latter has two immediate corollaries which are the respective analogues of 3.2 and 3.3.

Corollary 3.8. Integrality criteria for Somos 5 sequences: *Suppose that a Somos 5 sequence is defined by the recurrence (3.12) for coefficients $\tilde{\alpha}, \tilde{\beta} \in \mathbb{Z}$, and initial values $\tau_1 = \pm 1$, $\tau_2 = \pm 1$, with non-zero $\tau_3, \tau_4, \tau_5 \in \mathbb{Z}$. If the quantity $\tilde{\alpha}\tilde{J}$ is an integer, with $\tilde{J}$ given by (3.13), then $\tau_n \in \mathbb{Z}$ for all $n \geq 1$. If it further holds that the backward iterates $\tau_0, \tau_{-1}, \tau_{-2} \in \mathbb{Z}$, then $\tau_n \in \mathbb{Z}$ for all $n \in \mathbb{Z}$.*

Corollary 3.9. *Given $\tilde{\alpha},\tilde{\beta}\in\mathbb{Z}$ and $\tilde{J}\in\mathbb{Q}$ with $\tilde{\alpha}\tilde{J}\in\mathbb{Z}$, suppose that the quintic equation*

$$(\tilde{s}\tilde{w}+\tilde{\alpha}\,\tilde{u}^2)(\tilde{s}\tilde{v}^2+\tilde{t}^2\tilde{w})+\tilde{\beta}\,\tilde{t}\tilde{u}^3\tilde{v}=\tilde{J}\,\tilde{s}\tilde{t}\tilde{u}\tilde{v}\tilde{w} \qquad (3.17)$$

has an integer solution of the form $(\tilde{s},\tilde{t},\tilde{u},\tilde{v},\tilde{w})=(\tau_1,\tau_2,\tau_3,\tau_4,\tau_5)$, with $|\tau_1|=1$, $|\tau_2|=1$ and non-zero $\tau_3,\tau_4,\tau_5\in\mathbb{Z}$. Then provided that the orbit of this set of initial data under (3.12) is non-periodic, it produces infinitely many integer solutions of the Diophantine equation (3.17).

Example 3.10. *Take $\tilde{\alpha}=14641=11^4$, $\tilde{\beta}=1771561=11^6$ in (3.12) with initial data $847,8,1,1,33$. From (3.14) we calculate $\tilde{J}=627$, hence $\tilde{\alpha}\tilde{J}\in\mathbb{Z}$, and the sequence extends on either side of these values as*

$$\ldots,805255,847,8,1,1,33,6655,19487171,\ldots, \qquad (3.18)$$

so all the conditions of Corollary 3.8 hold. Thus $\tau_n\in\mathbb{Z}$ for all $n\in\mathbb{Z}$. By Corollary 3.9, for these values of $\tilde{\alpha}$, $\tilde{\beta}$, $\tilde{J}$ the sequence (3.18) gives infinitely many quintuples of integer solutions of the corresponding Diophantine equation, of the form (3.17).

Remarks. It is shown in [14] (Proposition 2.8) that the subsequences of a Somos 5 sequence consisting of the even/odd index terms respectively satisfy the same Somos 4 recurrence relation. If we index the terms of the preceding example such that $\tau_1=8$, $\tau_2=1$, $\tau_3=1$, $\tau_4=33$ etc. then we find that the even index subsequence of (3.18) is essentially the same as the sequence (3.4) in Example 3.4, up to applying transformations of the form (3.9) and (3.10). To be precise, we have $\tau_{2n}=11^{\mathcal{Q}(n)}A_n$ where $\mathcal{Q}(n)=(n-1)(3n-4)/2$. Indeed, both sequences correspond to a sequence of points on the same elliptic curve with j-invariant (from (1.10))

$$j=\frac{5^3\cdot 23^6\cdot 1013^3}{2^4\cdot 11^7\cdot 17^2\cdot 37\cdot 1069}.$$

It is clear from the denominator that the curve has bad reduction mod11, and all terms of the sequence reduce to 0 mod 11 apart from the central terms $8,1,1$.

Note that so far all of our criteria for integrality have been stated for Somos recurrences with integer coefficients. However, even when $\alpha,\beta\in\mathbb{Z}$ the associated curve (1.12) must be defined over $\mathbb{Q}$, or more generally over $\mathbb{Q}(\sqrt{\hat{\alpha}})$ where $\hat{\alpha}$ is the square-free part of α, in order to consider the sequence of points (2.3). Hence it is natural to define Somos sequences over $\mathbb{Q}$, which leads to the question of whether integer sequences can arise for non-integer α or $\beta\in\mathbb{Q}$. The following example illustrates that this is indeed the case.

Example 3.11. *Consider the Somos 4 sequence defined by (1.1) with $\alpha=-1/2$, $\beta=1$ and $A_1=1,A_2=-2,A_3=2,A_3=1$. This sequence has terms*

$$\ldots,2,10,-4,2,3,1,-2,2,1,5,2,12,-26,34,236,352,-1912,\ldots, \qquad (3.19)$$

and consists entirely of integers.

The sequence (3.19) was presented in the Robbins (bilinear) forum [22] by Michael Somos, who found that it should correspond to a sequence of points on an elliptic curve E with j-invariant

$$j = \frac{3^3 \cdot 19051^3}{2^{17} \cdot 1721}; \tag{3.20}$$

this was confirmed by Elkies [1]. Although Theorem 3.1 does not apply directly in this case, we show in the next section how a slight modification of the methods used previously provides a proof of integrality for a one-parameter family of Somos 4 sequences that generalizes Example 3.11.

4 One-parameter family of integer sequences

In this section we consider a one-parameter family of sequences

$$\ldots, 1, -N, N, 1, 1+N^2, N, N^3+2N, -N^4-2N^2-2, N^4+4N^2+2, \ldots \tag{4.1}$$

which is defined by the Somos 4 recurrence (1.1) with coefficients and initial data given by

$$\alpha = -1/N, \quad \beta = 1, \quad A_1 = A_4 = 1, \quad A_2 = -A_3 = -N. \tag{4.2}$$

When $N = 2$ this is the sequence (3.19) in Example 3.11 above. The integrality properties of the sequence (4.1) do not follow from Theorem 3.1, because the parameter α is non-integer for integer $N \neq \pm 1$. Nevertheless, we are able to prove that the sequence (4.1) consists entirely of polynomials in N with integer coefficients, and so the integrality of this sequence when $N \in \mathbb{Z}$ follows immediately. From (1.9) we compute the translation invariant $T = -N^2 - 1/N^2$, and hence from (2.7) we have $\mathcal{I} = -N^2$. Thus we see that $\mathcal{I} \in \mathbb{Z}$ for $N \in \mathbb{Z}$, which leads to special properties for the companion EDS associated with (4.1).

Theorem 4.1. *For the companion EDS associated with (4.1), defined by the fourth order recurrence (1.3) with initial data* $W_1 = 1$, $W_2 = -iN^{-1/2}$, $W_3 = -1$, $W_4 = -iN^{3/2}$, *the terms in the sequence satisfy* $W_{2n-1} \in \mathbb{Z}[N^4]$, $W_{4n} \in iN^{3/2}\,\mathbb{Z}[N^4]$ *and* $W_{4n+2} \in iN^{-1/2}\,\mathbb{Z}[N^4]$ *for all* $n \in \mathbb{Z}$.

Proof. This follows the same pattern as the proof of Theorem 2.2, using the two identities (2.10) and (2.11), except that in (2.11) it is necessary to consider the cases of odd/even m separately. □

[1] See `http://www.math.wisc.edu/~propp/somos/elliptic`

Theorem 4.2. *Consider the Somos 4 sequence defined by the recurrence (1.1) with coefficients $\alpha = -1/N$, $\beta = 1$ and initial data $A_1 = 1$, $A_2 = -N$, $A_3 = N$, $A_4 = 1$. This sequence satisfies $A_n \in \mathbb{Z}[N]$ for all $n \in \mathbb{Z}$. Moreover, $A_n \in \mathbb{Z}[N^2]$ whenever $n \equiv 0$ or 1 mod 4, while $A_n \in N\,\mathbb{Z}[N^2]$ whenever $n \equiv 2$ or 3 mod 4.*

Proof. This proceeds by induction for $n \geq 1$, analogously to the proof of Theorem 3.1, using the two identities (3.1) and (3.2), except that each of these formulae must be considered separately for odd/even m to get the different properties of A_n as n varies mod 4. We omit further details, except to mention that to cover $n < 1$ it suffices to consider the reversed sequence $\hat{A}_n = A_{5-n}$ and then proceed as before. $\square$

Corollary 4.3. *For all $N \in \mathbb{Z}$, the sequence (4.1) defined by the Somos 4 recurrence (1.1) with coefficients and initial data given by (4.2) is an integer sequence.*

Thus we see that the sequence (3.19) is the particular case $N = 2$ of the family of integer sequences given by (4.1) with $N \in \mathbb{Z}$. It is then interesting to consider the family of elliptic curves corresponding to these sequences. By Theorem 1.2, using the formulae (2.1), (2.2) and (2.3) from section 2, for each $N \neq 0$ the sequence (4.1) is associated with the sequence of points

$$\left(-\frac{(N^4-1)^2}{12N^3} - \frac{N}{(N^2-1)^2}, \frac{2iN\sqrt{N}}{(N^2-1)^3} \right) + [n] \left(-\frac{(N^4-1)^2}{12N^3}, \frac{i}{\sqrt{N}} \right)$$

on the curve

$$y^2 = 4x^3 - g_2(N)\,x - g_3(N) \tag{4.3}$$

with invariants

$$g_2(N) = \frac{N^{16} - 4N^{12} + 30N^8 + 20N^4 + 1}{12N^6},$$

$$g_3(N) = \frac{N^{24} - 6N^{20} + 51N^{16} - 56N^{12} + 195N^8 + 30N^4 + 1}{216N^9},$$

and j-invariant

$$j(N) = \frac{(N^{16} - 4N^{12} + 30N^8 + 20N^4 + 1)^3}{N^{16}(N^{12} - 5N^8 + 39N^4 + 2)}. \tag{4.4}$$

(The latter reproduces the value (3.20) when $N = 2$.)

Strictly speaking the recurrence for (4.1) does not make sense when $N = 0$, but the sequence (4.1) can still be defined by setting $N = 0$ in each of these polynomials. It can be seen that when $N = 0$ the values of these polynomials are given by

$$A_{4m} = (-1)^{m-1} 2^{m(m-1)/2}, \; A_{4m+1} = 2^{m(m-1)/2}, \; A_{4m-1} = A_{4m+2} = 0 \tag{4.5}$$

for all m. The case $N = 1$ is also special, because in that case the base point $Q = \infty$, and we have the sequence

$$1, -1, 1, 1, 2, 1, 3, -5, 7, 4, 23, 29, 59, -129, 314, \ldots \tag{4.6}$$

with $A_0 = 0$, which corresponds to the multiples $[n]P$ of the point $P = (0, i)$ on the curve

$$y^2 = 4x^3 - 4x - 1, \tag{4.7}$$

for which $j = 2^{12} \cdot 3^3/37$. In that case, the sequence (4.1) is almost identical to its companion EDS; more precisely, we find that $A_n = (-i)^{n-1} W_n$ for all n. Over $\mathbb{C}$ (sending $x \to -x$, $y \to iy$), the curve (4.7) is isomorphic to the curve associated with the Somos (4) sequence (1.2) as found in [13], which is not surprising when one observes that the terms of (1.2) are precisely the odd index terms of the sequence (4.6).

The equation (4.3) can be thought of as defining a curve $E/\mathbb{C}(N)$. Letting N vary this gives an elliptic surface fibred over $\mathbb{P}^1$: with the exception of a finite set of values, each value of N defines a non-singular elliptic curve (see Chapter III in [27]). The singular fibres correspond to the zeros of the denominator of $j(N)$, i.e. $N = 0$ and the zeros of $N^{12} - 5N^8 + 39N^4 + 2$, as well as the fibre at infinity. If we set $S = N^4$, then it is clear from (4.4) that $j(N) \in \mathbb{C}(S)$. It turns out that over $\mathbb{C}(\sqrt{N})$, the rational elliptic surface (4.3) is birationally equivalent to the Weierstrass model

$$\hat{y}^2 + 8S\,\hat{y} = \hat{x}^3 + (S-1)^2\,\hat{x}^2 - 8S(S+1)\,\hat{x} \tag{4.8}$$

which is defined over $\mathbb{C}(S)$. In terms of the model (4.8) the sequence (4.1) corresponds to the sequence of points

$$\hat{Q} + [n]\hat{P} = \left(\frac{4S}{(\sqrt{S}-1)^2}\,,\ -4S + \frac{8S\sqrt{S}}{(\sqrt{S}-1)^2} \right) + [n]\Big(\,0\,,\,0\Big)$$

defined over the extended function field $\mathbb{C}(\sqrt{S})$.

5 Conclusions

Using the connection between Somos 4 sequences and their associated companion EDS, together with the identities in Theorem 2.1, we have shown that these sequences possess a stronger variant of the Laurent property in [8], which is expressed by means of the conserved quantity $\mathcal{I}$, given by the formula (2.7) in terms of T, the translation invariant (1.9). This strong Laurent property has produced a set of sufficient criteria for integrality of Somos 4 sequences with integer coefficients, and we have given a similar set of criteria for Somos 5 sequences as well.

The fact that every Somos 4 sequence admits a conserved quantity rests on the link with shifts by multiples $[n]P$ of a point P on an elliptic curve E [31], or equivalently with a discrete linear flow on the Jacobian $\mathrm{Jac}(E)$. This can also be understood from the connection with integrable maps [32]: via the subsitution $f_n = A_{n+1}\,A_{n-1}/(A_n)^2$, the bilinear recurrence (1.1) yields the second-order difference equation

$$f_{n+1}(f_n)^2 f_{n-1} = \alpha f_n + \beta,$$

which has the conserved quantity (1.14), preserves the symplectic form $\omega = (f_{n-1}f_n)^{-1}\, df_{n-1}\wedge df_n$, and is a degenerate case of the family of maps studied by Quispel, Roberts and Thompson [23].

There is another aspect of the Laurent property, that was stated as a conjecture in [8], which is that for the octahedron recurrence (also known as the discrete Hirota equation [36], an integrable partial difference equation) the corresponding Laurent polynomials in the initial data and coefficients have all positive coefficients, and furthermore all these coefficients are 1. Speyer has given a combinatorial proof of this conjecture, using perfect matchings of certain graphs [30], and Speyer and Carroll have also used combinatorics to prove the analogous result for the four-term cube recurrence (also known as the Hirota-Miwa equation). Since the Somos 4 recurrence can be obtained as a reduction of the discrete Hirota equation, Speyer notes that his result implies that the Laurent polynomials in $\mathbb{Z}[\alpha, \beta, A_1^{\pm 1}, A_2^{\pm 1}, A_3^{\pm 1}, A_4^{\pm 1}]$ for Somos 4 have all positive coefficients. Whether this result has any interpretation in terms of elliptic curves is not clear.

Although our most general integrality criteria in Section 3 were formulated for Somos 4 recurrences with integer coefficients, the family of sequences (4.1) illustrates the fact that α and β need not be integers for integer sequences to result. The sequence of polynomials (4.1) provides further insights into the integer sequences corresponding to $N \in \mathbb{Z}$. In particular, consider the case of integer $N > 2$, and suppose $p > 2$ is a prime with $p|N$. In that case, due to the factor of N^{16} in the denominator of the j-invariant, the curve (4.3) - or equivalently (4.8) - has bad reduction at the prime p. Furthermore, by Theorem 4.2 it is clear that $A_{4m+2} \equiv A_{4m+3} \equiv 0 \bmod p$, while from the values (4.5) of these polynomials at $N = 0$ it follows that $p \not| A_{4m}$ and $p \not| A_{4m+1}$ for all m. This means that the distance, or gap length [24], between successive multiples of the prime p appearing in the sequence alternates between one and three. This is in contrast with Robinson's numerical observations for the Somos (4) sequence (1.2), which led him to the conjecture that for this particular sequence, the gap length should be *constant* for any prime p [24]. The p-adic properties of EDS have been considered recently by Silverman [28], and we expect that similar considerations will lead to a better understanding of prime divisors in Somos 4 sequences.

Certain higher order Somos sequences have also appeared in connection with division polynomials [3], higher order integrable maps [1] and divisor sequences for hyperelliptic curves [18, 20]. Based on naïve counting arguments, in [13] one of us made the conjecture that all Somos sequences should arise from appropriate divisor sequences on such curves. In the forum [22], Elkies has given much stronger arguments to the contrary, based on a conjectured theta function formula for the iterates. Moreover, the Laurent property is failed for Somos 8 recurrences; and for the sequence generated by the recurrence

$$S_{n+8}S_n = S_{n+7}S_{n+1} + S_{n+6}S_{n+2} + S_{n+5}S_{n+3} + (S_{n+4})^2$$

with initial values $S_j = 1$ for $j = 1, \ldots, 8$ the logarithmic heights of the rational iterates have exponential growth, so that $\log h(S_n) \sim Kn$ as $n \to \infty$ with $K \approx 0.23$; this growth is incompatible with a discrete linear flow on an Abelian variety. Recently, Halburd has proposed that polynomial growth of logarithmic heights should be an integrability criterion for birational maps [11].

One of us has found that for Somos 6 (and also Somos 7) recurrences, there are two independent conserved quantities, analogous to the translation invariant (1.9) for Somos 4. For these recurrences of sixth and seventh order, it is possible to given an explicit formula for the iterates in terms of two-variable theta functions, or equivalently in terms of the Kleinian sigma functions for an associated curve of genus two. Recently, Kanayama has derived multiplication formulae for genus two sigma functions [15]. One of us has pointed out that the statement of Proposition 3 in [15] is incorrect, but Kanayama has shown us a corrected version of this result [16], which we have used to derive the solutions of Somos 6 and Somos 7 recurrences. The full description of the genus two case is the subject of further investigation.
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