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Computer Science at Kent

New eta-reduction and Church-Rosser

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New eta-reduction and Church-Rosser

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Abstract This paper introduces a new eta-reduction rule for λ -calculus with dependent types and prove the property of Church-Rosser.

1 Introduction

For the simple λ -calculus, the property of Church-Rosser w.r.t. β and η -reduction holds, no matter a term is well-typed or not, and the methods of proving this property vary [Bar84,Tak95,Nip01]. Some systems such as Edinburgh Logical Framework [HHP87,HHP92] have β -reduction only, and the proof method is known as that of Tait and Martin-Löf. However, for some dependently typed systems such as Martin-Löf's Logical Framework [ML84,NPS90,Luo94], the story becomes very different because ill-typed terms may not have this property w.r.t. β and η -reduction. The proofs for such systems are very difficult because one has to prove that “well-typed terms have the property of Church-Rosser” [Gog94]. In this paper, a new eta-reduction is introduced in order to give an easy proof of the property of Church-Rosser, and the property does not rely on whether terms are well-typed or not. The new eta-reduction is sufficient for our understanding of eta-reduction in dependent type systems, in the sense that a property that holds w.r.t. the old one also holds w.r.t. the new one.

In Section 2, we give counter examples to demonstrate the problem with the property of Church-Rosser w.r.t. the β -reduction and the old η -reduction. In Section 3, new eta-reduction η' and its one-step reduction are introduced, and the relations between the new and the old (normal) eta-reduction are discussed. In Section 4, we prove the property of Church-Rosser w.r.t. β and η' -reduction, which holds without the condition of well-typedness. The conclusion is given in the last section.

2 Old reduction rules

In this section, we present a basic definition of terms and kinds, and the old β and eta-reduction rules. Counter examples are also given to demonstrate the problem with the property of Church-Rosser.

Definition 1. (Terms and Kinds)

- *Term*
 1. a variable is a term,
 2. $\lambda x : K.M$ is a term if x is a variable, K is a kind and M is a term,
 3. MN is a term if M and N are terms.
- *Kind*
 1. Type is a kind,
 2. $El(M)$ is a kind if M is a term,
 3. $(x : K_1)K_2$ is a kind if K_1 and K_2 are kinds.

Remark 1. Terms and kinds are mutually and recursively defined.

Notation: Following the tradition, Λ denotes the set of all terms and Π the set of all kinds. We shall use M, N, P, Q, R for arbitrary terms, and K for an arbitrary kind, and x, y, z for arbitrary variables. We also write A for $El(A)$ when no confusion may occur.

The old (normal) reduction rules:

$$\begin{aligned}(\lambda x : K.M)N &\longrightarrow_{\beta} [N/x]M \\ \lambda x : K.Mx &\longrightarrow_{\eta} M \quad x \notin FV(M)\end{aligned}$$

Example 1. (Counter examples) The property of Church-Rosser may not hold for ill-typed terms. The following examples will show the problem.

$$\begin{aligned}\lambda x : A.(\lambda y : B.y)x &\longrightarrow_{\beta} \lambda x : A.x \\ \lambda x : A.(\lambda y : B.y)x &\longrightarrow_{\eta} \lambda y : B.y \\ \\ (\lambda z : (y : B)B.\lambda x : A.zx)(\lambda y : B.y) &\longrightarrow_{\beta} \lambda x : A.(\lambda y : B.y)x \\ &\longrightarrow_{\beta} \lambda x : A.x \\ \\ (\lambda z : (y : B)B.\lambda x : A.zx)(\lambda y : B.y) &\longrightarrow_{\eta} (\lambda z : (y : B)B.z)(\lambda y : B.y) \\ &\longrightarrow_{\beta} \lambda y : B.y\end{aligned}$$

where A and B are distinct variables. There is no common reduct for $\lambda x : A.x$ and $\lambda y : B.y$.

3 New eta-reduction rule

In this section, we introduce a new eta-reduction rule and its one-step reduction, and discuss the relations between the old and new.

Definition 2. (*Base terms*)

- a variable is a base term,
- MN is a base term if M is a base term.

Notation: We shall use \overline{R} for R_1, R_2, \dots, R_n for some $n \geq 0$, and $M\overline{R}$ for $(\dots((MR_1)R_2)\dots R_n)$, and $x\overline{R}$ for an arbitrary base term.

Definition 3. (*Head of terms*)

- $head(x) = x$,
- $head(\lambda x : K.M) = \lambda x : K.M$,
- $head(MN) = head(M)$.

Lemma 1. *If a term is not a base term, then the head of the term is of the form $\lambda x : K.M$.*

New redex for eta-reduction

There are two different forms of redexes: $(\lambda x : K.M)N$ and $\lambda x : K.(y\overline{N})x$ when $x \notin FV(y\overline{N})$. The reduction rules for these redexes are the following.

$$\begin{aligned} (\lambda x : K.M)N &\longrightarrow_{\beta} [N/x]M \\ \lambda x : K.(y\overline{N})x &\longrightarrow_{\eta'} y\overline{N} \quad x \notin FV(y\overline{N}) \end{aligned}$$

The one-step β and η' -reduction are defined in Figure 1.

Remark 2. We have the following remarks.

- The one-step β -reduction is the same as usual, but the redex for the new eta-reduction rule (η') has a more restricted form (*i.e.* $y\overline{N}$ is a base term). For the term $\lambda x : A.(\lambda y : B.y)x$ in Example 1, the new eta-reduction rule (η') cannot be applied because $\lambda y : B.y$ is not a base term.
- It is not the case that one has to do all the β -reductions first, then do η' -reductions when reducing a term. For example, we have

$$x(\lambda y : B.zy)((\lambda x : A.x)a) \longrightarrow_{\eta'} xz((\lambda x : A.x)a) \longrightarrow_{\beta} xza$$

One-step β -reduction:

$$\begin{array}{c}
\overline{(\lambda x : K.M)N \longrightarrow_{\beta} [N/x]M} \\
\\
\frac{M \longrightarrow_{\beta} M'}{MN \longrightarrow_{\beta} M'N} \quad \frac{N \longrightarrow_{\beta} N'}{MN \longrightarrow_{\beta} MN'} \\
\\
\frac{M \longrightarrow_{\beta} M'}{\lambda x : K.M \longrightarrow_{\beta} \lambda x : K.M'} \quad \frac{K \longrightarrow_{\beta} K'}{\lambda x : K.M \longrightarrow_{\beta} \lambda x : K'.M} \\
\\
\frac{M \longrightarrow_{\beta} M'}{El(M) \longrightarrow_{\beta} El(M')} \\
\\
\frac{K_1 \longrightarrow_{\beta} K'_1}{(x : K_1)K_2 \longrightarrow_{\beta} (x : K'_1)K_2} \quad \frac{K_2 \longrightarrow_{\beta} K'_2}{(x : K_1)K_2 \longrightarrow_{\beta} (x : K_1)K'_2}
\end{array}$$

One-step η' -reduction:

$$\begin{array}{c}
\frac{x \notin FV(y\bar{N})}{\lambda x : K.(y\bar{N})x \longrightarrow_{\eta'} y\bar{N}} \\
\\
\frac{M_i \longrightarrow_{\eta'} M'_i \quad n \geq 1}{xM_1 \dots M_i \dots M_n \longrightarrow_{\eta'} xM_1 \dots M'_i \dots M_n} \\
\\
\frac{M \longrightarrow_{\eta'} M'}{\lambda x : K.M \longrightarrow_{\eta'} \lambda x : K.M'} \quad \frac{K \longrightarrow_{\eta'} K'}{\lambda x : K.M \longrightarrow_{\eta'} \lambda x : K'.M} \\
\\
\frac{M \longrightarrow_{\eta'} M'}{El(M) \longrightarrow_{\eta'} El(M')} \\
\\
\frac{K_1 \longrightarrow_{\eta'} K'_1}{(x : K_1)K_2 \longrightarrow_{\eta'} (x : K'_1)K_2} \quad \frac{K_2 \longrightarrow_{\eta'} K'_2}{(x : K_1)K_2 \longrightarrow_{\eta'} (x : K_1)K'_2}
\end{array}$$

Figure 1. One-step reduction

Notations: Let \longrightarrow_R be one-step R -reduction. We denote $\longrightarrow_{\overline{R}}$ as the reflexive closure of \longrightarrow_R , and \twoheadrightarrow_R as the reflexive, transitive closure of \longrightarrow_R , and $=_R$ as the R -convertibility.

Lemma 2. *If $x \notin FV(M)$ and $M \twoheadrightarrow_{\beta\eta'} N$, then $x \notin FV(N)$.*

In some cases, although there is a η' -redex inside a term, this term may not have one-step η' -reduction. For instance, $(\lambda x : A.yx)z$ only has β -reduction although $\lambda x : A.yx$ is a η' -redex. However, we have the following lemma.

Lemma 3. *If there is a η' -redex inside M , then M has a one-step reduction (i.e. either β or η' -reduction).*

Proof. By induction on M .

- If M itself is a η' -redex, then the statement is true.
- If $M \equiv xN_1\dots N_n$ and the redex is inside N_i , then by induction hypothesis, N_i has either β or η' -reduction. By the one-step reduction rules in Figure 1, M has a one-step reduction.
- If $M \equiv \lambda x : K.N$ and the redex is inside K or N , then by induction hypothesis, K or N has either β or η' -reduction. By the one-step reduction rules, M has a one-step reduction.
- If $M \equiv (\lambda x : K.P)N_1\dots N_n$ ($n \geq 1$), then M has at least a one-step β -reduction.
- Other cases for kinds are trivial.

Relations between the old and new

Theorem 1. *A normal form w.r.t. $\beta\eta'$ is also a normal form w.r.t. $\beta\eta$, and vice versa.*

Proof. We proceed the proof by contradiction. An interesting case is to prove that, a η -redex contains a β or η' -redex. Then by Lemma 3, the whole term has a one-step reduction.

Suppose $\lambda x : K.Mx$ is a η -redex, i.e. $x \notin FV(M)$. If M is a base term, then $\lambda x : K.Mx$ is a η' -redex. If M is not a base term, then by Lemma 1, the head of M must be of the form $\lambda x : K'.N$. So, Mx must contain a β -redex. \square

Lemma 4. *A normal form (w.r.t. $\beta\eta$ or $\beta\eta'$) is either of the form $\lambda y : K'.N$ or of the form $y\overline{N}$ (a base term).*

Conjecture 1. If $x \notin FV(M)$ and $\lambda x : K.Mx$ is well-typed, then $\lambda x : K.Mx =_{\beta\eta'} M$.

Proof. We only give an informal proof here since the well-typedness has not been defined yet. We are going to prove that the property of Church-Rosser w.r.t. $\beta\eta'$ -reduction holds for any term, no matter it is well-typed or not. So well-typedness is out of scope of the paper, but the typing inference rules can be found in Appendix. We assume that well-typed terms have good properties such as strong normalisation.

Since $\lambda x : K.Mx$ is well-typed, it is strongly normalising, and so is M . We first reduce M to a normal form M' . By Lemma 4, M' is either of the form $\lambda y : K'.N$ or of the form $y\bar{N}$.

If $M' \equiv y\bar{N}$, then we have

$$\begin{aligned} \lambda x : K.Mx &=_{\beta\eta'} \lambda x : K.M'x \\ &\equiv \lambda x : K.(y\bar{N})x \\ &=_{\eta'} y\bar{N} \\ &=_{\beta\eta'} M \end{aligned}$$

Note that $x \notin FV(y\bar{N})$ by Lemma 2.

If $M' \equiv \lambda y : K'.N$, because $\lambda x : K.(\lambda y : K'.N)x$ is well-typed, we have $K =_{\beta\eta'} K'$. Then we have

$$\begin{aligned} \lambda x : K.Mx &=_{\beta\eta'} \lambda x : K.M'x \\ &\equiv \lambda x : K.(\lambda y : K'.N)x \\ &=_{\beta\eta'} \lambda x : K'.(\lambda y : K'.N)x \\ &=_{\beta} \lambda x : K'.[x/y]N \\ &=_{\alpha} \lambda y : K'.N \\ &=_{\beta\eta'} M \end{aligned}$$

□

4 Proof of Church-Rosser

In this section, we give some important definitions such as Parallel Reduction and Complete Development for β -reduction, and prove the property of Church-Rosser.

Definition 4. (*Parallel reduction for β*) The parallel reduction, denoted by \Rightarrow_{β} , is defined inductively as follows.

1. $x \Rightarrow_\beta x$,
2. $\lambda x : K.M \Rightarrow_\beta \lambda x : K'.M'$ if $K \Rightarrow_\beta K'$ and $M \Rightarrow_\beta M'$,
3. $MN \Rightarrow_\beta M'N'$ if $M \Rightarrow_\beta M'$ and $N \Rightarrow_\beta N'$,
4. $(\lambda x : K.M)N \Rightarrow_\beta [N'/x]M'$ if $M \Rightarrow_\beta M'$ and $N \Rightarrow_\beta N'$,
5. $Type \Rightarrow_\beta Type$,
6. $El(M) \Rightarrow_\beta El(M')$ if $M \Rightarrow_\beta M'$,
7. $(x : K_1)K_2 \Rightarrow_\beta (x : K'_1)K'_2$ if $K_1 \Rightarrow_\beta K'_1$ and $K_2 \Rightarrow_\beta K'_2$.

Based on the inductive definition of \Rightarrow_β , we have the following lemma.

Lemma 5. *We have the following properties, where M and M' represent arbitrary terms or kinds, and N and N' are arbitrary terms.*

1. $M \Rightarrow_\beta M$.
2. If $M \longrightarrow_\beta M'$ then $M \Rightarrow_\beta M'$.
3. If $M \Rightarrow_\beta M'$ then $M \rightarrow_\beta M'$.
4. If $M \Rightarrow_\beta M'$ and $N \Rightarrow_\beta N'$ then $[N/x]M \Rightarrow_\beta [N'/x]M'$.

Proof. For (1), (3) and (4), by induction on M ; for (2), by induction on the context of the redex. We only prove the most difficult case (4) here.

1. If $M \equiv x$, then $x \Rightarrow_\beta x \equiv M'$ and hence

$$[N/x]M \equiv N \Rightarrow_\beta N' \equiv [N'/x]M'$$

2. If $M \equiv y \neq x$, then $y \Rightarrow_\beta y \equiv M'$ and hence

$$[N/x]M \equiv y \Rightarrow_\beta y \equiv [N'/x]M'$$

3. If $M \equiv \lambda y : K.P$, then $\lambda y : K.P \Rightarrow_\beta \lambda y : K'.P' \equiv M'$ and $K \Rightarrow_\beta K'$ and $P \Rightarrow_\beta P'$. By induction hypothesis, we have $[N/x]K \Rightarrow_\beta [N'/x]K'$ and $[N/x]P \Rightarrow_\beta [N'/x]P'$. Hence

$$[N/x]M \equiv \lambda y : [N/x]K.[N/x]P \Rightarrow_\beta [N'/x]M'$$

4. If $M \equiv PQ$, then there are two sub-cases.

- (a) $PQ \Rightarrow_\beta P'Q' \equiv M'$ and $P \Rightarrow_\beta P'$ and $Q \Rightarrow_\beta Q'$. By induction hypothesis, we have $[N/x]P \Rightarrow_\beta [N'/x]P'$ and $[N/x]Q \Rightarrow_\beta [N'/x]Q'$. Hence $[N/x]M \Rightarrow_\beta [N'/x]M'$.
- (b) $PQ \equiv (\lambda y : K.R)Q \Rightarrow_\beta [Q'/y]R' \equiv M'$ and $R \Rightarrow_\beta R'$ and $Q \Rightarrow_\beta Q'$. By induction hypothesis, we have $[N/x]R \Rightarrow_\beta [N'/x]R'$ and $[N/x]Q \Rightarrow_\beta [N'/x]Q'$. Hence

$$[N/x]M \Rightarrow_\beta [(N'/x)Q']/y([N'/x]R') \equiv [N'/x]M'$$

5. If $M \equiv Type$, then $Type \Rightarrow_{\beta} Type \equiv M'$ and hence $[N/x]M \Rightarrow_{\beta} [N'/x]M'$.
6. If $M \equiv El(P)$, then $El(P) \Rightarrow_{\beta} El(P') \equiv M'$ and $P \Rightarrow_{\beta} P'$. By induction hypothesis, we have $[N/x]P \Rightarrow_{\beta} [N'/x]P'$. Hence $[N/x]M \Rightarrow_{\beta} [N'/x]M'$.
7. If $M \equiv (y : K_1)K_2$, then $(y : K_1)K_2 \Rightarrow_{\beta} (y : K'_1)K'_2 \equiv M'$ and $K_1 \Rightarrow_{\beta} K'_1$ and $K_2 \Rightarrow_{\beta} K'_2$. By induction hypothesis, we have $[N/x]K_1 \Rightarrow_{\beta} [N'/x]K'_1$ and $[N/x]K_2 \Rightarrow_{\beta} [N'/x]K'_2$. Hence $[N/x]M \Rightarrow_{\beta} [N'/x]M'$. \square

From Lemma 5 (1), (2) and (3), we know that \rightarrow_{β} is the reflexive, transitive closure of \Rightarrow_{β} . Therefore, to prove the property of Church-Rosser w.r.t. β -reduction, it suffices to show the ‘‘diamond property’’ of \Rightarrow_{β} , *i.e.* if $M \Rightarrow_{\beta} N_1$ and $M \Rightarrow_{\beta} N_2$ then there is a N_3 such that $N_1 \Rightarrow_{\beta} N_3$ and $N_2 \Rightarrow_{\beta} N_3$. But we will prove a stronger statement in Lemma 6.

Definition 5. (Complete development for β) We define a map $cd :: \Lambda \cup \Pi \rightarrow \Lambda \cup \Pi$.

1. $cd(x) = x$
2. $cd(MN) = \begin{cases}xcd(N) & \text{if } M \equiv x \\cd(PQ)cd(N) & \text{if } M \equiv PQ \\[cd(N)/x]cd(R) & \text{if } M \equiv \lambda x : K.R\end{cases}$
3. $cd(\lambda x : K.M) = \lambda x : cd(K).cd(M)$
4. $cd(Type) = Type$
5. $cd(El(M)) = El(cd(M))$
6. $cd((x : K_1)K_2) = (x : cd(K_1))cd(K_2)$

Having defined parallel reduction \Rightarrow_{β} and complete development cd , we prove the following lemma.

Lemma 6. *If $M \Rightarrow_{\beta} N$ then $N \Rightarrow_{\beta} cd(M)$.*

Proof. By induction on M .

1. If $M \equiv x$, then $x \Rightarrow_{\beta} x \equiv N = cd(M)$.
2. If $M \equiv \lambda x : K.P$, then $\lambda x : K.P \Rightarrow_{\beta} \lambda x : K'.P' \equiv N$ and $K \Rightarrow_{\beta} K'$ and $P \Rightarrow_{\beta} P'$. By induction hypothesis, we have $K' \Rightarrow_{\beta} cd(K)$ and $P' \Rightarrow_{\beta} cd(P)$. Hence

$$N \Rightarrow_{\beta} \lambda x : cd(K).cd(P) = cd(M)$$

3. If $M \equiv xP$, then $xP \Rightarrow_\beta xQ \equiv N$ and $P \Rightarrow_\beta Q$. By induction hypothesis, we have $Q \Rightarrow_\beta cd(P)$. Hence

$$N \Rightarrow_\beta xcd(P) = cd(M)$$

4. If $M \equiv (PQ)R$, then $(PQ)R \Rightarrow_\beta M'R' \equiv N$ and $PQ \Rightarrow_\beta M'$ and $R \Rightarrow_\beta R'$. By induction hypothesis, we have $M' \Rightarrow_\beta cd(PQ)$ and $R' \Rightarrow_\beta cd(R)$. Hence

$$N \Rightarrow_\beta cd(PQ)cd(R) = cd(M)$$

5. If $M \equiv (\lambda x : K.P)Q$, then there are two sub-cases.

- (a) $(\lambda x : K.P)Q \Rightarrow_\beta (\lambda x : K'.P')Q' \equiv N$ and $K \Rightarrow_\beta K'$, $P \Rightarrow_\beta P'$ and $Q \Rightarrow_\beta Q'$. By induction hypothesis, we have $P' \Rightarrow_\beta cd(P)$ and $Q' \Rightarrow_\beta cd(Q)$. Hence

$$N \equiv (\lambda x : K'.P')Q' \Rightarrow [cd(Q)/x]cd(P) = cd(M)$$

- (b) $(\lambda x : K.P)Q \Rightarrow_\beta [Q'/x]P' \equiv N$ and $P \Rightarrow_\beta P'$ and $Q \Rightarrow_\beta Q'$. By induction hypothesis, we have $P' \Rightarrow_\beta cd(P)$ and $Q' \Rightarrow_\beta cd(Q)$. By Lemma 5 (4), we have $[Q'/x]P' \Rightarrow_\beta [cd(Q)/x]cd(P)$. Hence $N \Rightarrow_\beta cd(M)$.

6. If $M \equiv Type$, then $Type \Rightarrow_\beta Type \equiv N = cd(M)$.

7. If $M \equiv El(P)$, then $El(P) \Rightarrow_\beta El(P') \equiv N$ and $P \Rightarrow_\beta P'$. By induction hypothesis, we have $P' \Rightarrow_\beta cd(P)$. Hence

$$N \Rightarrow_\beta El(cd(P)) = cd(M)$$

8. If $M \equiv (x : K_1)K_2$, then $(x : K_1)K_2 \Rightarrow_\beta (x : K'_1)K'_2 \equiv N$ and $K_1 \Rightarrow_\beta K'_1$ and $K_2 \Rightarrow_\beta K'_2$. By induction hypothesis, we have $K'_1 \Rightarrow_\beta cd(K_1)$ and $K'_2 \Rightarrow_\beta cd(K_2)$. Hence

$$N \Rightarrow_\beta (x : cd(K_1))cd(K_2) = cd(M)$$

□

Theorem 2. (Church-Rosser for β) For an arbitrary term or kind M , if $M \twoheadrightarrow_\beta N_1$ and $M \twoheadrightarrow_\beta N_2$ then there is a N_3 such that $N_1 \twoheadrightarrow_\beta N_3$ and $N_2 \twoheadrightarrow_\beta N_3$.

Proof. By Lemma 6 and the fact that \twoheadrightarrow_β is the transitive closure of \Rightarrow_β .

Lemma 7. For an arbitrary term or kind M , if $M \longrightarrow_{\eta'} N_1$ and $M \longrightarrow_{\eta'} N_2$ then there is a N_3 such that $N_1 \longrightarrow_{\eta'} N_3$ and $N_2 \longrightarrow_{\eta'} N_3$.

Proof. By induction on M and analysing different cases of one-step η' -reduction. One interesting case is the following.

$M \equiv \lambda x : K.(y\bar{Q})x \rightarrow_{\eta'} y\bar{Q} \equiv N_1$ and $M \rightarrow_{\eta'} \lambda x : K'.(y\bar{Q})x \equiv N_2$. Then, let $N_3 \equiv N_1$ and we have $N_1 \rightarrow_{\eta'}^{\bar{}} N_3$ and $N_2 \rightarrow_{\eta'}^{\bar{}} N_3$.

Corollary 1. *For an arbitrary term or kind M , if $M \rightarrow_{\eta'}^{\bar{}} N_1$ and $M \rightarrow_{\eta'}^{\bar{}} N_2$ then there is a N_3 such that $N_1 \rightarrow_{\eta'}^{\bar{}} N_3$ and $N_2 \rightarrow_{\eta'}^{\bar{}} N_3$.*

Theorem 3. (Church-Rosser for η') *For an arbitrary term or kind M , if $M \rightarrow_{\eta'} N_1$ and $M \rightarrow_{\eta'} N_2$ then there is a N_3 such that $N_1 \rightarrow_{\eta'} N_3$ and $N_2 \rightarrow_{\eta'} N_3$.*

Proof. By Corollary 1 and the fact that \rightarrow_{η} is the transitive closure of $\rightarrow_{\eta'}^{\bar{}}$.

Lemma 8. *For an arbitrary term or kind M , if $M \rightarrow_{\beta} N_1$ and $M \rightarrow_{\eta'} N_2$ then there is a N_3 such that $N_1 \rightarrow_{\eta'} N_3$ and $N_2 \rightarrow_{\beta}^{\bar{}} N_3$.*

Proof. By induction on M and analysing different cases of one-step η' and β -reduction. One interesting case is the following.

$M \equiv \lambda x : K.(y\bar{Q})x \rightarrow_{\eta'} y\bar{Q} \equiv N_2$ and $M \rightarrow_{\beta} \lambda x : K'.(y\bar{Q})x \equiv N_1$. Then, let $N_3 \equiv N_2$ and we have $N_1 \rightarrow_{\eta'} N_3$ and $N_2 \rightarrow_{\beta}^{\bar{}} N_3$.

Theorem 4. (Commutation for $\beta\eta'$) *For an arbitrary term or kind M , if $M \rightarrow_{\beta} N_1$ and $M \rightarrow_{\eta'} N_2$ then there is a N_3 such that $N_1 \rightarrow_{\eta'} N_3$ and $N_2 \rightarrow_{\beta} N_3$.*

Proof. By Lemma 8.

Theorem 5. (Church-Rosser for $\beta\eta'$) *For an arbitrary term or kind M , if $M \rightarrow_{\beta\eta'} N_1$ and $M \rightarrow_{\beta\eta'} N_2$ then there is a N_3 such that $N_1 \rightarrow_{\beta\eta'} N_3$ and $N_2 \rightarrow_{\beta\eta'} N_3$.*

Proof. By Theorem 2, 3 and 4.

5 Conclusion

For the λ -calculus defined in Definition 1, the property of Church-Rosser w.r.t. β and η -reduction is not an easy matter, unlike the systems with β -reduction only or the simple λ -calculus with β and η -reduction. In this paper, we introduce a new eta-reduction η' and its one-step reduction. The property of Church-Rosser holds for β and η' -reduction, without the condition of well-typedness.

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Appendix

Inference rules for a dependently typed logical framework

$$\begin{array}{c}
\frac{}{\langle \rangle \text{ valid}} \qquad \frac{\Gamma \vdash K \text{ kind} \quad x \notin FV(\Gamma)}{\Gamma, x : K \text{ valid}} \\
\frac{\Gamma \text{ valid}}{\Gamma \vdash \text{Type kind}} \qquad \frac{\Gamma \vdash A : \text{Type}}{\Gamma \vdash El(A) \text{ kind}} \\
\frac{\Gamma \vdash K \text{ kind} \quad \Gamma, x : K \vdash K' \text{ kind}}{\Gamma \vdash (x : K)K' \text{ kind}} \\
\frac{\Gamma, x : K, \Gamma' \text{ valid}}{\Gamma, x : K, \Gamma' \vdash x : K} \\
\frac{\Gamma, x : K \vdash k : K'}{\Gamma \vdash \lambda x : K. k : (x : K)K'} \\
\frac{\Gamma \vdash k : K \quad \Gamma \vdash K' \text{ kind}}{\Gamma \vdash k : K'} \quad (K =_{\beta\eta'} K') \\
\frac{\Gamma \vdash f : (x : K)K' \quad \Gamma \vdash k : K}{\Gamma \vdash f(k) : [k/x]K'}
\end{array}$$